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THE GENERAL FUBINI THEOREM IN COMPLETE BORNOLOGICAL LOCALLY CONVEX SPACES

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Abstract. The Fubini theorem for the generalized Dobrakov integral in complete bornological locally convex topological vector spaces is proven.

1. Introduction

It is well known, in contrast with the scalar case, that the product of two vector measures need not always exist. This problem has been studied in several papers, where some conditions for the existence of the product of vector measures have been given, see [30] for further references. In [27] the problem of the existence of the product measure in the context of locally convex spaces for bilinear integrals is solved in general. The bornological character of the bilinear integration theory presented therein shows the fitness of making a development of bilinear integration theory in the context of the complete bornological locally convex spaces. Note here the paper of Ballvé and Jiménez Guerra [2] where we can find a list of reference papers to this problem. Also, see [8, 9, 11, 29] for further reading on product of vector-valued measures.

Concerning the Fubini theorem, the first author in [17] generalized the Dobrakov integral to complete bornological locally convex spaces. The sense of this seemingly complicated theory is that, at the present, this is the only known integration theory which completely generalizes the Dobrakov integral to a class

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of non-metrizable locally convex topological vector spaces. Integration of vector valued functions by operator-valued measures, especially the Dobrakov integration technique, has its applications e.g. in study of hybrid systems and optimal control [1], quantum measurement [25], Wiener processes [28], etc.

This paper is, in fact, a continuation of the paper [20], where the construction of bornological product measure is given and a Fubini-type theorem is stated. In Section 2 we recall necessary notions from [14, 15, 16, 17]. For the purpose to prove the general Fubini theorem for bornological product measures the questions on existence and measurability of a partial integral are solved in Section 3. All these results lead up to Section 4 where the complete proof of general Fubini theorem is given.

2. Preliminaries

For basic notions of bornology and the description of the theory of complete bornological locally convex spaces (C. B. L. C. S., for short) see [23, 24, 26].

Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ be Hausdorff C. B. L. C. S. over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex numbers $\mathbb{C}$, equipped with the bornologies $\mathfrak{B}_{\mathbf{X}}$, $\mathfrak{B}_{\mathbf{Y}}$, $\mathfrak{B}_{\mathbf{Z}}$.

One of the equivalent definitions of C. B. L. C. S. is to define these spaces as the inductive limits of Banach spaces. We say that the basis $\mathcal{U}$ of bornology $\mathfrak{B}_{\mathbf{X}}$ has the *vacuum vector* [1] $U_0 \in \mathcal{U}$, if $U_0 \subset U$ for every $U \in \mathcal{U}$. Let the bases $\mathcal{U}$, $\mathcal{W}$, $\mathcal{V}$ be chosen to consist of all $\mathfrak{B}_{\mathbf{X}}$-, $\mathfrak{B}_{\mathbf{Y}}$-, $\mathfrak{B}_{\mathbf{Z}}$-bounded Banach disks in $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ with vacuum vectors $U_0 \in \mathcal{U}$, $U_0 \neq \{0\}$, $W_0 \in \mathcal{W}$, $W_0 \neq \{0\}$, $V_0 \in \mathcal{V}$, $V_0 \neq \{0\}$, respectively. Recall that *a (separable) Banach disk* in $\mathbf{X}$ is a set $U \in \mathfrak{B}_{\mathbf{X}}$ which is closed, absolutely convex and the linear span $\mathbf{X}_U$ of which is a (separable) Banach space. So, the space $\mathbf{X}$ is an inductive limit of Banach spaces $\mathbf{X}_U$, $U \in \mathcal{U}$, i.e.,

$$\mathbf{X} = \operatorname*{injlim}_{U \in \mathcal{U}} \mathbf{X}_U,$$

see [24], where $\mathcal{U}$ is directed by inclusion (analogously for $\mathbf{Y}$ and $\mathcal{W}$, $\mathbf{Z}$ and $\mathcal{V}$, respectively).

Since $\mathbf{X}_U$, $U \in \mathcal{U}$, in the definition of C. B. L. C. S. is a Banach space, it is enough to deal with sequences instead of nets and therefore we introduce the following bornological convergence in the sense of Mackey. We say that a sequence $(\mathbf{x}_n)_1^\infty$ of elements from $\mathbf{X}$ converges bornologically (or, it is Mackey convergent) with respect to the bornology $\mathfrak{B}_{\mathbf{X}}$ with the basis $\mathcal{U}$ to $\mathbf{x} \in \mathbf{X}$, shortly $\mathcal{U}$*-converges*, if there exists $U \in \mathcal{U}$, such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that $(\mathbf{x}_n - \mathbf{x}) \in \varepsilon U$ for every $n \geq n_0$. We write $\mathbf{x} = \mathcal{U}\text{-}\lim_{n \to \infty} \mathbf{x}_n$. To be more precise, we will sometimes call this the U-convergence of elements from $\mathbf{X}$ to show explicitly which $U \in \mathcal{U}$ we have in the mind.

2.1. Operator spaces.

On $\mathcal{U}$ *the lattice operations* are defined as follows. For $U_1, U_2 \in \mathcal{U}$ we have: $U_1 \wedge U_2 = U_1 \cap U_2$, and $U_1 \vee U_2 = \operatorname{acs}(U_1 \cup U_2)$, where

[1] in literature we can find also as terms as the *ground state* or *marked element* or *fiducial vector* or *mother wavelet* depending on the context

acs denotes the topological closure of the absolutely convex span of the set; analogously for $\mathcal{W}$ and $\mathcal{V}$. For $(U_1, W_1, V_1), (U_2, W_2, V_2) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$, we write $(U_1, W_1, V_1) \ll (U_2, W_2, V_2)$ if and only if $U_1 \subset U_2$, $W_1 \supset W_2$, and $V_1 \supset V_2$.

We use Φ, Ψ, Γ to denote the *classes of all functions* $\mathcal{U} \to \mathcal{W}$, $\mathcal{W} \to \mathcal{V}$, $\mathcal{U} \to \mathcal{V}$ *with orders* $<_\Phi$, $<_\Psi$, $<_\Gamma$ defined as follows: for $\varphi_1, \varphi_1 \in \Phi$ we write $\varphi_1 <_\Phi \varphi_2$ whenever $\varphi_1(U) \subset \varphi_2(U)$ for every $U \in \mathcal{U}$ (analogously for $<_\Psi$, $<_\Gamma$ and $\mathcal{W} \to \mathcal{V}$, $\mathcal{U} \to \mathcal{V}$, respectively).

Denote by $L(\mathbf{X}, \mathbf{Y})$ the space of all continuous linear operators $L : \mathbf{X} \to \mathbf{Y}$. We suppose $L(\mathbf{X}, \mathbf{Y}) \subset \Phi$. Analogously, $L(\mathbf{Y}, \mathbf{Z}) \subset \Psi$ and $L(\mathbf{X}, \mathbf{Z}) \subset \Gamma$. The bornologies $\mathfrak{B}_\mathbf{X}$, $\mathfrak{B}_\mathbf{Y}$, $\mathfrak{B}_\mathbf{Z}$ are supposed to be stronger than the corresponding von Neumann bornologies, i.e., the vector operations on the spaces $L(\mathbf{X}, \mathbf{Y})$, $L(\mathbf{Y}, \mathbf{Z})$, $L(\mathbf{X}, \mathbf{Z})$ are compatible with the topologies, and the bornological convergence implies the topological convergence. For a more detailed explanation of the topological and bornological methods of functional analysis in connection with operators see [32].

2.2. **Set functions.** Let T and S be two non-void sets. Let Δ and ∇ be two δ-rings of subsets of sets T and S, respectively. If $\mathcal{A}$ is a system of subsets of the set T, then $\sigma(\mathcal{A})$ (resp. $\delta(\mathcal{A})$) denotes the σ-ring (resp. δ-ring) generated by the system $\mathcal{A}$. Set $\Sigma = \sigma(\Delta)$ and $\Xi = \sigma(\nabla)$. We use χ_E to denote the characteristic function of the set E. By $p_U : \mathbf{X} \to [0, \infty]$ we denote the Minkowski functional of the set $U \in \mathcal{U}$, i.e., $p_U(\mathbf{x}) = \inf_{\mathbf{x} \in \lambda U} |\lambda|$ (if U does not absorb $\mathbf{x} \in \mathbf{X}$, we put $p_U(\mathbf{x}) = +\infty$). Similarly, p_W and p_V indicate the Minkowski functionals of the sets $W \in \mathcal{W}$ and $V \in \mathcal{V}$, respectively.

For every $(U, W) \in \mathcal{U} \times \mathcal{W}$, denote by $\hat{\mathbf{m}}_{U,W} : \Sigma \to [0, \infty]$ *a* (U, W)*-semivariation* of a charge (= finitely additive measure) $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ given by

$$\hat{\mathbf{m}}_{U,W}(E) = \sup p_W \left(\sum_{i=1}^{I} \mathbf{m}(E \cap E_i) \mathbf{x}_i \right), \quad E \in \Sigma,$$

where the supremum is taken over all finite sets $\{\mathbf{x}_i \in U, i = 1, 2, \ldots, I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1, 2, \ldots, I\}$. For $\{E_i \in \Delta, i = 1, 2, \ldots, I\}$ by [3, Corollary 3 of Proposition 9, § 1] we get $E \cap E_i \in \Delta$ for $E \in \Sigma$, and hence $\hat{\mathbf{m}}_{U,W}(E)$ is well defined. Note that this result does not hold if Σ is the σ-algebra generated by Δ. It is well-known that $\hat{\mathbf{m}}_{U,W}$ is a submeasure, i.e., a monotone, subadditive set function, and $\hat{\mathbf{m}}_{U,W}(\emptyset) = 0$. The family $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}} = \{\hat{\mathbf{m}}_{U,W};\ (U, W) \in \mathcal{U} \times \mathcal{W}\}$ is said to be the $(\mathcal{U}, \mathcal{W})$-semivariation of $\mathbf{m}$.

For every $(U, W) \in \mathcal{U} \times \mathcal{W}$, denote by $\|\mathbf{m}\|_{U,W} : \Sigma \to [0, \infty]$ *a scalar* (U, W)*-semivariation* of a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ defined as

$$\|\mathbf{m}\|_{U,W}(E) = \sup \left\| \sum_{i=1}^{I} \lambda_i \mathbf{m}(E \cap E_i) \right\|_{U,W}, \quad E \in \Sigma,$$

where $\|L\|_{U,W} = \sup_{\mathbf{x} \in U} p_W(L(\mathbf{x}))$ and the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\ |\lambda_i| \leq 1, i = 1, 2, \ldots, I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1, 2, \ldots, I\}$. Note that the scalar (U, W)-semivariation $\|\mathbf{m}\|_{U,W}$ is also a submeasure.

Let $\mathbf{X}'$, $\mathbf{Y}'$ be the topological duals of $\mathbf{X}$,$\mathbf{Y}$, respectively. For every $y' \in \mathbf{Y}'$, $U \in \mathcal{U}$ and $E \in \Sigma$ we define the *U-variation* of the charge $y'\mathbf{m} : \Delta \to \mathbf{X}'$ by the equation

$$\mathbf{var}_U(y'\mathbf{m}, E) = \sup \sum_{i=1}^{I} |(y'\mathbf{m})(E \cap E_i)\mathbf{x}_i| \,,$$

where the supremum is taken over all finite pairwise disjoint sets $E_i \in \Delta$ and over all finite sets of elements $\mathbf{x}_i \in U$, $i = 1, 2, \dots, I$. Note that the (U, W)-semivariation of $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ may be expressed in the form

$$\hat{\mathbf{m}}_{U,W}(E) = \sup_{y' \in W^0} \mathbf{var}_U(y'\mathbf{m}, E), \quad E \in \Sigma, \tag{2.1}$$

where $W^0 \in \mathbf{Y}'$ denotes the absolute polar of the set $W \in \mathcal{W}$, see [14].

Analogously, we may define a (W, V)-semivariation $\hat{\mathbf{n}}_{W,V} : \Xi \to [0, \infty]$, and a scalar (W, V)-semivariation $\|\mathbf{n}\|_{W,V} : \Xi \to [0, \infty]$ of a charge $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$. For a more detailed description of the basic $L(\mathbf{X}, \mathbf{Y})$-measure set structures when both $\mathbf{X}$ and $\mathbf{Y}$ are C. B. L. C. S. see [14].

Let $\nu : \mathcal{A} \to [0, \infty]$ be a set function on a system $\mathcal{A}$ of subsets of a non-empty set Ω. We say that *ν is continuous on $\mathcal{A}$* if $\nu(E_n) \to 0$ for any sequence $(E_n)_1^\infty$ of sets from $\mathcal{A}$, such that $E_n \searrow \emptyset$ (i.e., $E_n \supset E_{n+1}$ for each $n \in \mathbb{N}$ and $\bigcap_{n \in \mathbb{N}} E_n = \emptyset$).

Definition 2.1. Let $(U, W) \in \mathcal{U} \times \mathcal{W}$. Denote by

(a) $\Delta_{U,W}$ the greatest δ-subring of Δ of subsets of finite (U, W)-semivariation $\hat{\mathbf{m}}_{U,W}$ and $\Delta_{\mathcal{U},\mathcal{W}} = \{\Delta_{U,W}; (U, W) \in \mathcal{U} \times \mathcal{W}\}$ the lattice with the order given with inclusions of $U \in \mathcal{U}$ and $W \in \mathcal{W}$, respectively;

(b) $\Delta^u_{U,W}$ the greatest δ-subring of Δ on which the restriction $\mathbf{m}_{U,W} : \Delta^u_{U,W} \to L(\mathbf{X}_U, \mathbf{Y}_W)$ of the measure $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is uniformly countable additive with $\mathbf{m}_{U,W}(E) = \mathbf{m}(E)$ for $E \in \Delta^u_{U,W}$ and $\Delta^u_{\mathcal{U},\mathcal{W}} = \{\Delta^u_{U,W}; (U, W) \in \mathcal{U} \times \mathcal{W}\}$ the lattice with the order given with inclusions of $U \in \mathcal{U}$ and $W \in \mathcal{W}$, respectively;

(c) $\Delta^c_{U,W}$ the greatest δ-subring of Δ where $\hat{\mathbf{m}}_{U,W}$ is continuous and $\Delta^c_{\mathcal{U},\mathcal{W}} = \{\Delta^c_{U,W}; (U, W) \in \mathcal{U} \times \mathcal{W}\}$ the lattice with the order given with inclusions of $U \in \mathcal{U}$ and $W \in \mathcal{W}$, respectively.

Analogously for $(W, V) \in \mathcal{W} \times \mathcal{V}$ we define $\nabla_{W,V}$, $\nabla^u_{W,V}$, $\nabla^c_{W,V}$, and $\nabla_{\mathcal{W},\mathcal{V}}$, $\nabla^u_{\mathcal{W},\mathcal{V}}$, $\nabla^c_{\mathcal{W},\mathcal{V}}$. Clearly, the lattices $\Delta^c_{\mathcal{U},\mathcal{W}}$, $\Delta^u_{\mathcal{U},\mathcal{W}}$ are sublattices of $\Delta_{\mathcal{U},\mathcal{W}}$. Concerning the continuity on $\Delta_{U,W}$, $\nabla_{W,V}$, see [31].

Denote by $\Delta_{U,W} \otimes \nabla_{W,V}$ the smallest δ-ring containing all rectangles $A \times B$, $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, where $(U, W) \in \mathcal{U} \times \mathcal{W}$, $(W, V) \in \mathcal{W} \times \mathcal{V}$. If $\mathcal{D}_1$, $\mathcal{D}_2$ are two δ-rings of subsets of T, S, respectively, then obviously $\sigma(\mathcal{D}_1 \otimes \mathcal{D}_2) = \sigma(\mathcal{D}_1) \otimes \sigma(\mathcal{D}_2)$. For every $E \in \delta(\mathcal{D}_1 \otimes \mathcal{D}_2)$ there exist $A \in \mathcal{D}_1$, $B \in \mathcal{D}_2$, such that $E \subset A \times B$. For $E \subset T \times S$, $s \in S$, put

$$E^s = \{t \in T; \, (t, s) \in E\}.$$

2.3. **Basic convergences of functions.** Let $\beta_{\mathcal{U},\mathcal{W}}$ be a lattice of submeasures $\beta_{U,W} : \Sigma \to [0,\infty]$, $(U,W) \in \mathcal{U}\times\mathcal{W}$, where the lattice operations $\wedge$, $\vee$ are defined as follows

$$\begin{aligned} \beta_{U_2,W_2} \wedge \beta_{U_3,W_3} &= \beta_{U_2\wedge U_3, W_2\vee W_3}, \\ \beta_{U_2,W_2} \vee \beta_{U_3,W_3} &= \beta_{U_2\vee U_3, W_2\wedge W_3}, \end{aligned}$$

for $(U_2,W_2),(U_3,W_3) \in \mathcal{U}\times\mathcal{W}$, (e.g. $\beta_{\mathcal{U},\mathcal{W}} = \hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$).

For $(U,W) \in \mathcal{U}\times\mathcal{W}$ denote by $\mathcal{O}(\beta_{U,W}) = \{N \in \Sigma;\ \beta_{U,W}(N) = 0\}$. The set $N \in \Sigma$ is called *$\beta_{\mathcal{U},\mathcal{W}}$-null* if there exists a couple $(U,W) \in \mathcal{U}\times\mathcal{W}$, such that $N \in \mathcal{O}(\beta_{U,W})$. We say that an assertion holds *$\beta_{\mathcal{U},\mathcal{W}}$-almost everywhere*, shortly *$\beta_{\mathcal{U},\mathcal{W}}$-a.e.*, if it holds everywhere except in a $\beta_{\mathcal{U},\mathcal{W}}$-null set. A set $E \in \Sigma$ is said to be of *finite submeasure* $\beta_{\mathcal{U},\mathcal{W}}$ if there exists a couple $(U,W) \in \mathcal{U}\times\mathcal{W}$, such that $\beta_{U,W}(E) < +\infty$.

Definition 2.2. Let $E \in \Sigma$, $R \in \mathcal{U}$ and $(U,W) \in \mathcal{U}\times\mathcal{W}$. We say that *a sequence $(\mathbf{f}_n : T \to \mathbf{X})_1^\infty$ of functions (R,E)-converges $\beta_{U,W}$-a.e. to a function $\mathbf{f} : T \to \mathbf{X}$* if $\lim\limits_{n\to\infty} p_R(\mathbf{f}_n(t) - \mathbf{f}(t)) = 0$ for every $t \in E \setminus N$, where $N \in \mathcal{O}(\beta_{U,W})$.

We say that *a sequence $(\mathbf{f}_n : T \to \mathbf{X})_1^\infty$ of functions $(\mathcal{U},E)$-converges $\beta_{\mathcal{U},\mathcal{W}}$-a.e. to a function $\mathbf{f} : T \to \mathbf{X}$* if there exist $R \in \mathcal{U}$, $(U,W) \in \mathcal{U}\times\mathcal{W}$, such that the sequence $(\mathbf{f}_n)_1^\infty$ of functions (R,E)-converges $\beta_{U,W}$-a.e. to $\mathbf{f}$. We write $\mathbf{f} = \mathcal{U}\text{-}\lim\limits_{n\to\infty} \mathbf{f}_n\ \beta_{\mathcal{U},\mathcal{W}}$-a.e.

Definition 2.3. Let $E \in \Sigma$, $R \in \mathcal{U}$ and $(U,W) \in \mathcal{U}\times\mathcal{W}$. We say that *a sequence $(\mathbf{f}_n : T \to \mathbf{X})_1^\infty$ of functions (R,E)-converges uniformly to a function $\mathbf{f} : T \to \mathbf{X}$*, if $\lim\limits_{n\to\infty} \|\mathbf{f}_n - \mathbf{f}\|_{E,R} = 0$, where $\|\mathbf{f}\|_{E,R} = \sup\limits_{t\in E} p_R(\mathbf{f}(t))$.

We say that *a sequence $(\mathbf{f}_n : T \to \mathbf{X})_1^\infty$ of functions (R,E)-converges $\beta_{U,W}$-almost uniformly to a function $\mathbf{f} : T \to \mathbf{X}$* if for every $\varepsilon > 0$ there exists a set $N \in \Sigma$, such that $\beta_{U,W}(N) < \varepsilon$ and the sequence $(\mathbf{f}_n)_1^\infty$ of functions $(R, E\setminus N)$-converges uniformly to $\mathbf{f}$.

We say that *a sequence $(\mathbf{f}_n : T \to \mathbf{X})_1^\infty$ of functions $(\mathcal{U},E)$-converges $\beta_{\mathcal{U},\mathcal{W}}$-almost uniformly to a function $\mathbf{f} : T \to \mathbf{X}$*, if there exist $R \in \mathcal{U}$, $(U,W) \in \mathcal{U}\times\mathcal{W}$, such that the sequence $(\mathbf{f}_n)_1^\infty$ of functions (R,E)-converges $\beta_{U,W}$-almost uniformly to $\mathbf{f}$.

For a more detail explanation of described convergences of functions in C. B. L. C. S. and relations among them see [13].

2.4. **Measure structures.** For $(U,W) \in \mathcal{U}\times\mathcal{W}$ we say that a charge $\mathbf{m}$ is of *σ-finite (U,W)-semivariation* if there exist sets $E_n \in \Delta_{U,W}$, $n \in \mathbb{N}$, such that $T = \bigcup_{n=1}^\infty E_n$. For $\varphi \in \Phi$ we say that a charge $\mathbf{m}$ is of *σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation* if for every $U \in \mathcal{U}$ the charge $\mathbf{m}$ is of *σ-finite $(U,\varphi(U))$-semivariation.*

Definition 2.4. We say that a charge $\mathbf{m}$ is *of σ_Φ-finite $(\mathcal{U},\mathcal{W})$-semivariation* if there exists a function $\varphi \in \Phi$, such that $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation.

Let $W \in \mathcal{W}$. We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a *(W,σ)-additive vector measure* if μ is a $\mathbf{Y}_W$-valued (countable additive) vector measure.

Definition 2.5. We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a $(\mathcal{W}, \sigma)$-*additive vector measure* if there exists $W \in \mathcal{W}$, such that μ is a (W, σ)-additive vector measure.

Let $W \in \mathcal{W}$ and $(\nu_n : \Sigma \to \mathbf{Y})_1^\infty$ be a sequence of (W, σ)-additive vector measures. If for every $\varepsilon > 0$, $E \in \Sigma$ with $p_W(\nu_n(E)) < +\infty$ for each $n \in \mathbb{N}$, and $E_i \in \Sigma$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i, j \in \mathbb{N}$, there exists $J_0 \in \mathbb{N}$, such that for every $J \geq J_0$,

$$p_W\left(\nu_n\left(\bigcup_{i=J+1}^{\infty} E_i \cap E\right)\right) < \varepsilon$$

uniformly for every $n \in \mathbb{N}$, then we say that the sequence of measures $(\nu_n)_1^\infty$ is *uniformly* (W, σ)-*additive on* Σ, see [17].

Definition 2.6. We say that the family of measures $\nu_n : \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, is *uniformly* $(\mathcal{W}, \sigma)$-*additive on* Σ if there exists $W \in \mathcal{W}$, such that the family of measures ν_n, $n \in \mathbb{N}$, is uniformly (W, σ)-additive on Σ.

The following definition generalizes the notion of the σ-additivity of an operator-valued measure in the strong operator topology in Banach spaces, see [4], to C. B. L. C. S.

Definition 2.7. Let $\varphi \in \Phi$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a σ_φ-*additive measure* if $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation, and for every $A \in \Delta_{U,\varphi(U)}$ the charge $\mathbf{m}(A \cap \cdot)\mathbf{x} : \Sigma \to \mathbf{Y}$ is a $(\varphi(U), \sigma)$-additive measure for every $\mathbf{x} \in \mathbf{X}_U$, $U \in \mathcal{U}$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a σ_Φ-*additive measure* if there exists $\varphi \in \Phi$ such that $\mathbf{m}$ is a σ_φ-additive measure.

In what follows $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ are supposed to be operator-valued σ_Φ- and σ_Ψ-additive measures, respectively.

2.5. **An integral in C. B. L. C. S..** We use $\mathcal{M}_{\Delta,\mathcal{U}}$ to denote *the space of all* $(\Delta, \mathcal{U})$-*measurable functions*, i.e., the largest vector space of functions $\mathbf{f} : T \to \mathbf{X}$ with the property: there exists $R \in \mathcal{U}$ such that for every $U \in \mathcal{U}$, $U \supset R$, and $\delta > 0$

$$\{t \in T;\, p_U(\mathbf{f}(t)) \geq \delta\} \in \Sigma.$$

Definition 2.8. A function $\mathbf{f} : T \to \mathbf{X}$ is called Δ-*simple* if $\mathbf{f}(T)$ is a finite set and $\mathbf{f}^{-1}(\mathbf{x}) \in \Delta$ for every $\mathbf{x} \in \mathbf{X} \setminus \{0\}$. Let $\mathcal{S}$ denote the space of all Δ-simple functions.

For $(U, W) \in \mathcal{U} \times \mathcal{W}$ a function $\mathbf{f} : T \to \mathbf{X}$ is said to be $\Delta_{U,W}$-*simple* if $\mathbf{f} = \sum_{i=1}^{I} \mathbf{x}_i \chi_{E_i}$, where $\mathbf{x}_i \in \mathbf{X}_U$, $E_i \in \Delta_{U,W}$, such that $E_i \cap E_j = \emptyset$, for $i \neq j$, $i, j = 1, 2, \ldots, I$. The space of all $\Delta_{U,W}$-simple functions is denoted by $\mathcal{S}_{U,W}$.

A function $\mathbf{f} \in \mathcal{S}$ is said to be $\Delta_{\mathcal{U},\mathcal{W}}$-*simple* if there exists a couple $(U, W) \in \mathcal{U} \times \mathcal{W}$, such that $\mathbf{f} \in \mathcal{S}_{U,W}$. The space of all $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions is denoted by $\mathcal{S}_{\mathcal{U},\mathcal{W}}$.

It may be proved that $\mathcal{M}_{\Delta,\mathcal{U}} \supset \mathcal{F}_\Delta$, where $\mathcal{F}_\Delta$ is the set of functions $\mathbf{f} : T \to \mathbf{X}$, such that there exists a sequence $(\mathbf{f}_n)_1^\infty$ of $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions $\mathcal{U}$-converging on T to $\mathbf{f}$. Elements of $\mathcal{F}_\Delta$ are called $\Delta_{\mathcal{U},\mathcal{W}}$-measurable functions (or measurable in the sense of Dobrakov, see [4]).

Let $(U,W) \in \mathcal{U} \times \mathcal{W}$. For every $E \in \Sigma$ and $\mathbf{f} \in \mathcal{S}_{U,W}$, as usual, we define the integral by the formula

$$\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} = \sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i,$$

where $\mathbf{f} = \sum_{i=1}^{I} \mathbf{x}_i \chi_{E_i}$, $\mathbf{x}_i \in \mathbf{X}_U$ and $E_i \in \Delta_{U,W}$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i,j = 1,2,\dots,I$. Note that for the function $\mathbf{f} \in \mathcal{S}_{U,W}$ the integral $\int_\cdot \mathbf{f}\,\mathrm{d}\mathbf{m}$ is a (W,σ)-additive measure on Σ.

Theorem 2.9. *([17, Theorem 3.8]) Let $\mathbf{m}$ be a σ-additive measure and $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$. If there exists a sequence $(\mathbf{f}_n)_1^\infty$ of $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions, such that*

(a) $\mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-a.e.,*

(b) *the integrals $\int_\cdot \mathbf{f}_n\,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W},\sigma)$-additive measures on Σ,*

then the limit $\nu(E,\mathbf{f}) = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m}$ exists uniformly in $E \in \Sigma$.

Definition 2.10. A function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$ is said to be $\Delta_{\mathcal{U},\mathcal{W}}$*-integrable* if there exists a sequence $(\mathbf{f}_n)_1^\infty$ of $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions, such that

(a) $\mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e.,

(b) $\int_\cdot \mathbf{f}_n\,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W},\sigma)$-additive measures on Σ.

Let $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ denote *the family of all $\Delta_{\mathcal{U},\mathcal{W}}$-integrable functions*. Then *the integral* of a function $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ on a set $E \in \Sigma$ is defined by the equality

$$\mathbf{y}_E = \int_E \mathbf{f}\,\mathrm{d}\mathbf{m} = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m}.$$

Theorem 2.11. *([17, Theorem 4.2]) Let $\nu(E,\mathbf{f}) = \int_E \mathbf{f}\,\mathrm{d}\mathbf{m}$, $E \in \Sigma$ and $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$. Then $\nu(\cdot,\mathbf{f}) : \Sigma \to \mathbf{Y}$ is a $(\mathcal{W},\sigma)$-additive measure.*

The following theorem gives a criterium of integrability of a $(\Delta,\mathcal{U})$-measurable function.

Theorem 2.12. *([17, Theorem 4.3]) A function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$ is $\Delta_{\mathcal{U},\mathcal{W}}$-integrable if and only if there exists a sequence $(\mathbf{f}_n)_1^\infty$ of $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions, such that*

(a) *$(\mathcal{U},E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. to $\mathbf{f}$, and*

(b) *the limit $\mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m} = \nu(E)$ exists for every $E \in \Sigma$.*

In this case $\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m}$ for every set $E \in \Sigma$ and this limit is uniform on Σ.

On integrable functions and further results related to the generalized Dobrakov integral in C. B. L. C. S. see [18] and [19].

Definition 2.13. A function $\mathbf{h} : T \to \mathbf{X}$ is said to be $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-null* if there exists a $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-null set $N \in \Sigma$ such that $\{t \in T; \mathbf{h}(t) \neq 0\} \subset N$.

A function $\mathbf{f} : T \to \mathbf{X}$ is said to be $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-essentially $\Delta_{\mathcal{U},\mathcal{W}}$-measurable* (resp. $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-essentially $\Delta_{\mathcal{U},\mathcal{W}}$-integrable*) if $\mathbf{f} = \mathbf{g} + \mathbf{h}$, where $\mathbf{g}$ is $\Delta_{\mathcal{U},\mathcal{W}}$-measurable (resp. $\Delta_{\mathcal{U},\mathcal{W}}$-integrable) and $\mathbf{h}$ is $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-null. In the case $\mathbf{f}$ is a $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-essentially

$\Delta_{\mathcal{U},\mathcal{W}}$-integrable function, we extend the integral of $\mathbf{f}$ defining $\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} = \int_E \mathbf{g}\,\mathrm{d}\mathbf{m}$ for each $E \in \Sigma$.

Clearly, this integration theory extends with obvious modifications to $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-essentially $\Delta_{\mathcal{U},\mathcal{W}}$-measurable (integrable) functions. Note that the range of an $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-null (hence also of an $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-essentially $\Delta_{\mathcal{U},\mathcal{W}}$-measurable) function need not be separable.

2.6. **Bornological product measures.** A bornological product measure was introduced in [20]. Here we recall its definition.

Definition 2.14. We say that a (bornological) *product measure* of a σ_Φ-additive measure $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and σ_Ψ-additive measure $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ exists on $\Delta \otimes \nabla$ (we write $\mathbf{m} \otimes \mathbf{n} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$) if there exists one and only one σ_Γ-additive measure $\mathbf{m} \otimes \mathbf{n} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$, such that

$$(\mathbf{m} \otimes \mathbf{n})(A \times B)\mathbf{x} = \mathbf{n}(B)\mathbf{m}(A)\mathbf{x}$$

for every $\mathbf{x} \in \mathbf{X}_U$, $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, where there exists $\gamma \in \Gamma$, $\varphi \in \Phi$, $\psi \in \Psi$, such that $\gamma = \psi \circ \varphi$ and $V \subseteq \psi(W)$, $W \subseteq \varphi(U)$, $\gamma(U) \subset \psi(\varphi(U))$.

For more results on bornological product measures and related Fubini-type theorem see [20] and [21].

3. Measurability of the partial integral

Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. According to the example before [6, Theorem 6] it is clear that in the general Fubini theorem we must assume that for a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function $\mathbf{f} : T \times S \to \mathbf{X}_U$ the function $t \mapsto \mathbf{f}(t, s)$, $t \in T$, must be $\Delta_{U,W}$-integrable for all $s \in S$. Since a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function is a pointwise (bornological) limit of $\Delta_{U,W} \otimes \nabla_{W,V}$-simple functions, from [12, Theorem A, § 34] and from the fact that $\Delta_{U,W}$-measurable functions are from the closure of pointwise bornological limits it follows that the function $\mathbf{f}(\cdot, s)$ is $\Delta_{U,W}$-measurable for each $s \in S$. This guarantees that $\mathbf{f} : T \times S \to \mathbf{X}_U$ is $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable.

Recall now the following useful notion, see [19].

Definition 3.1. Let $(U, W) \in \mathcal{U} \times \mathcal{W}$. If $\mathbf{g} : T \to \mathbf{X}_U$ is a $\Delta_{U,W}$-measurable function, then the *$L^1_{U,W}$-gauge of the function* $\mathbf{g}$ *on the set* $E \in \Sigma$, denoted by $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E)$, is a non-negative not necessarily finite number defined by the equality

$$\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E) = \sup\left\{ p_W\left(\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} \right) \right\},$$

where the supremum is taken over all $\mathbf{f} \in \mathcal{S}_{U,W}$, such that $p_U(\mathbf{f}(t)) \le p_U(\mathbf{g}(t))$ for each $t \in E$. The $L^1_{U,W}$-gauge of the function $\mathbf{g}$ is then defined by

$$\hat{\mathbf{m}}_{U,W}(\mathbf{g}, T) = \sup_{E \in \Sigma} \hat{\mathbf{m}}_{U,W}(\mathbf{g}, E).$$

Let us denote by $L^1_{U,W}(\mathbf{m})$ the space of all $\Delta_{U,W}$-integrable functions with the bounded and continuous seminorm $\hat{\mathbf{m}}_{U,W}(\cdot, E)$. Analogously, for $(W,V) \in \mathcal{W} \times \mathcal{V}$ we define the $L^1_{W,V}$-gauge $\hat{\mathbf{n}}_{W,V}(\cdot, F)$ and the space $L^1_{W,V}(\mathbf{n})$.

Let $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function and let $\mathbf{f}(\cdot, s)$ be $\Delta_{U,W}$-integrable for each $s \in S$. In this part of paper we investigate the theory of $\nabla_{W,V}$-measurability and the $\hat{\mathbf{n}}_{W,V}$-essential $\nabla_{W,V}$-measurability of the partial integral $\mathbf{g}_E$,

$$\mathbf{g}_E(s) = \int_{E^s} \mathbf{f}(\cdot, s)\,\mathrm{d}\mathbf{m}, \quad s \in S,\ E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V}).$$

In this part we also obtain results for $\nabla_{W,V}$-measurability of the function $\mathbf{h}_E$,

$$\mathbf{h}_E(s) = \hat{\mathbf{m}}_{U,W}(\mathbf{f}(\cdot, s), E^s), \quad s \in S,$$

and other results which will be important for the proof of general Fubini theorem in Section 4. In what follows $\mathbf{g}_E$ and $\mathbf{h}_E$ always denote the above stated functions.

Theorem 3.2. *Let $(U,W,V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$ and $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta^c_{U,W} \otimes \nabla_{W,V}$-measurable function. Then for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{h}_E$ is $\nabla_{W,V}$-measurable.*

Proof. Let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and let $(\mathbf{f}_n)_1^\infty$ be a sequence of $\Delta^c_{U,W} \otimes \nabla_{W,V}$-simple functions, such that $\mathbf{f}_n(t,s) \to \mathbf{f}(t,s)$ and $p_U(\mathbf{f}_n(t,s)) \nearrow p_U(\mathbf{f}(t,s))$ for each $(t,s) \in T \times S$. According to [5, Theorem 4] we get

$$\hat{\mathbf{m}}_{U,W}(\mathbf{f}(\cdot, s), E^s) = \sup_{y' \in W^0} \int_{E^s} p_U(\mathbf{f}(\cdot, s))\,\mathrm{d}\mathbf{var}_U(y'\mathbf{m}, \cdot)$$

for each $s \in S$. The same equality also holds for each $\mathbf{f}_n$, $n \in \mathbb{N}$. Then the Fatou lemma yields

$$\hat{\mathbf{m}}_{U,W}(\mathbf{f}(\cdot, s), E^s) = \lim_{n\to\infty} \hat{\mathbf{m}}_{U,W}(\mathbf{f}_n(\cdot, s), E^s)$$

for each $s \in S$. Therefore it is enough to prove that the theorem holds for each $\Delta^c_{U,W} \otimes \nabla_{W,V}$-simple function $\mathbf{f} : T \times S \to \mathbf{X}_U$.

Let $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta^c_{U,W} \otimes \nabla_{W,V}$-simple function of the form $\mathbf{f} = \sum_{i=1}^r \mathbf{x}_i \chi_{E_i}$, where $\mathbf{x}_i \in \mathbf{X}_U$, $E_i \in \Delta^c_{U,W} \otimes \nabla_{W,V}$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i,j = 1,2,\ldots,r$, and let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$. Since $\Delta^c_{U,W} \otimes \nabla_{W,V} \cap \sigma(\Delta_{U,W} \otimes \nabla_{W,V}) = \Delta^c_{U,W} \otimes \nabla_{W,V}$, and since $E_i \in \Delta^c_{U,W} \otimes \nabla_{W,V}$, $i = 1,2,\ldots,r$, then we may suppose without loss of generality that $E \in \Delta^c_{U,W} \otimes \nabla_{W,V}$.

Choose $A \in \Delta^c_{U,W}$ and $B \in \nabla_{W,V}$, such that $E \subset A \times B$. Let $\mathbf{x} \in U$, and $\mathbf{k} : T \to \mathbf{X}_U$ be a $\Delta^c_{U,W}$-simple function defined by $\mathbf{k} = (\sum_{i=1}^r p_U(\mathbf{x}_i)) \cdot \mathbf{x}\chi_A$. Obviously, $\mathbf{k} \in L^1_{U,W}(\mathbf{m})$, cf. [19, Theorem 3.8(c)]. Let us denote by $\mathcal{R}$ a ring of all finite unions of pairwise disjoint rectangles $C \times D$, $C \in \Delta^c_{U,W}$, $D \in \nabla_{W,V}$, cf. [12, Theorem E, § 33]. If $F_i \in \mathcal{R} \cap (A \times B)$ for $i = 1,2,\ldots,r$, then for $\mathbf{g} = \sum_{i=1}^r \mathbf{x}_i \chi_{F_i}$ the function $s \mapsto \hat{\mathbf{m}}_{U,W}(\mathbf{g}(\cdot, s), A)$, $s \in S$, is clearly $\nabla_{W,V}$-measurable.

Denote by $\mathcal{M}_1$ a class of all sets $F_1 \in \Delta^c_{U,W} \otimes \nabla_{W,V} \cap (A \times B)$ for which the function $s \mapsto \hat{\mathbf{m}}_{U,W}(\mathbf{g}(\cdot, s), A)$, $s \in S$, is $\nabla_{W,V}$-measurable provided $\mathbf{g} = \sum_{i=1}^r \mathbf{x}_i \chi_{F_i}$ and $F_2, \ldots, F_r \in \mathcal{R} \cap (A \times B)$. Then $\mathcal{R} \cap (A \times B) \subset \mathcal{M}_1$ and since $p_U(\mathbf{g}(t,s)) \leq p_U(\mathbf{g}_0(t))$ for each $(t,s) \in T \times S$, then $\mathcal{M}_1$ is a monotone class of sets by the Lebesgue dominated convergence theorem, see [5, Theorem 17].

So, $\mathcal{M}_1 = \Delta^c_{U,W} \otimes \nabla_{W,V} \cap (A \times B)$ by [12, Theorem B, § 6]. Similarly, if $\mathcal{M}_2$ is a class of all sets $F_2 \in \Delta^c_{U,W} \otimes \nabla_{W,V} \cap (A \times B)$ for which the function $s \mapsto \hat{\mathbf{m}}_{U,W}(\mathbf{g}(\cdot, s), A)$, $s \in S$, is $\nabla_{W,V}$-measurable provided $\mathbf{g} = \sum_{i=1}^{r} \mathbf{x}_i \chi_{F_i}$, $F_1 \in \mathcal{M}_1$ and $F_3, \dots, F_r \in \mathcal{R} \cap (A \times B)$, then $\mathcal{M}_2 = \Delta^c_{U,W} \otimes \nabla_{W,V} \cap (A \times B)$. Continuing in this way we get

$$\mathcal{M}_r = \Delta^c_{U,W} \otimes \nabla_{W,V} \cap (A \times B),$$

which completes the proof of theorem. □

Recall that $A \subset W^0$ is called *norming set* for $\mathbf{Y}_W$ if $p_U(\mathbf{y}) = \sup_{y' \subset A} |\langle \mathbf{y}, \mathbf{y'} \rangle|$ for each $\mathbf{y} \in \mathbf{Y}_W$, $W \in \mathcal{W}$, cf. [22, Definition 2.8.1]. Note that separable Banach spaces and their duals have countable norming sets.

Theorem 3.3. *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$ and $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function. Then for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{h}_E$ is $\nabla_{W,V}$-measurable.*

Proof. Let $y'_n \in W^0$, $n \in \mathbb{N}$, be a countable norming set and let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$. Then by [5, Theorem 4] holds

$$\mathbf{h}_E(s) = \hat{\mathbf{m}}_{U,W}(\mathbf{f}(\cdot, s), E^s) = \sup_{n \in \mathbb{N}} \int_{E^s} p_U(\mathbf{f}(\cdot, s)) \, \mathrm{d}\mathbf{var}_U(y'_n \mathbf{m}, \cdot)$$

for each $s \in S$. Therefore by [12, Theorem A, § 20] it is enough to prove $\Delta_{U,W}$-measurability of the function

$$s \mapsto \int_E p_U(\mathbf{f}(\cdot, s)) \, \mathrm{d}\mathbf{var}_U(y'_n \mathbf{m}, \cdot), \quad s \in S,$$

for each $n \in \mathbb{N}$. But it follows directly from Theorem 3.2 since by assumption the function $\mathbf{f}$ is $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable and $\mathbf{var}_U(y'_n \mathbf{m}, \cdot)$ is a σ-additive finite measure on $\Delta_{U,W}$ for each $n \in \mathbb{N}$. □

Theorem 3.4. *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. If $\mathbf{f} : T \times S \to \mathbf{X}_U$ is a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function and $\mathbf{f}(\cdot, s) \in L^1_{U,W}(\mathbf{m})$ for each $s \in S$, then for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the functions $\mathbf{g}_E$ and $\mathbf{h}_E$ are $\nabla_{W,V}$-measurable. Moreover, if the product measure $\mathbf{m}_{U,W} \otimes \mathbf{n}_{W,V}$ exists on $\Delta_{U,W} \otimes \nabla_{W,V}$ and if $\mathbf{h}_{T \times S} \in L^1_{W,V}(\mathbf{n})$, then $\mathbf{f} \in L^1_{U,V}(\mathbf{m} \otimes \mathbf{n})$.*

Proof. Let $(\mathbf{f}_n)_1^\infty$ be a sequence of $\Delta_{U,W} \otimes \nabla_{W,V}$-simple functions on $T \times S$, such that $\mathbf{f}_n(t, s) \to \mathbf{f}(t, s)$ and $p_U(\mathbf{f}_n(t, s)) \nearrow p_U(\mathbf{f}(t, s))$ for each $(t, s) \in T \times S$. Then clearly $\mathbf{f}_n(\cdot, s) \in L^1_{U,W}(\mathbf{m})$ for each $n \in \mathbb{N}$ and each $s \in S$. Thus, $\mathbf{f}$ is $\Delta^c_{U,W} \otimes \nabla_{W,V}$-measurable. Then by Theorem 3.2 the function $\mathbf{h}_E$ is $\nabla_{W,V}$-measurable for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$. Further, by the Lebesgue dominated convergence theorem, see [5, Theorem 17], we have

$$\hat{\mathbf{m}}_{U,W}(\mathbf{f}(\cdot, s) - \mathbf{f}_n(\cdot, s), T) \to 0$$

for each $s \in S$. Let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and put

$$\mathbf{g}_{n,E}(s) = \int_{E^s} \mathbf{f}_n(\cdot, s) \, \mathrm{d}\mathbf{m}, \quad s \in S, \; n \in \mathbb{N}.$$

According to [21, Lemma 2.5(i)] the functions $\mathbf{g}_{n,E}$, $n \in \mathbb{N}$, are $\nabla_{W,V}$-measurable. Using [19, Lemma 3.13] we get

$$p_W(\mathbf{g}_{n,E}(s) - \mathbf{g}_E(s)) \le \hat{\mathbf{m}}_{U,W}(\mathbf{f}_n(\cdot, s) - \mathbf{f}(\cdot, s), T) \to 0$$

as $n \to \infty$. So, $\mathbf{g}_{n,E}(s) \to \mathbf{g}_E(s)$ for each $s \in S$, which proves $\nabla_{W,V}$-measurability of $\mathbf{g}_E$, because $\nabla_{W,V}$-measurable functions are closed with respect to bornological limits of sequences.

For the second statement we have to prove that $L^1_{U,V}$-gauge $(\widehat{\mathbf{m} \otimes \mathbf{n}})_{U,V}(\mathbf{f}, \cdot)$ is continuous on $\sigma(\Delta_{U,W} \otimes \nabla_{W,V})$. Let $(E_k)_1^\infty$ be a sequence of sets from $\sigma(\Delta_{U,W} \otimes \nabla_{W,V})$, such that $E_k \searrow \emptyset$. The assumption $\mathbf{f}(\cdot, s) \in L^1_{U,W}(\mathbf{m})$ for each $s \in S$ and the Lebesgue dominated convergence theorem implies that $\mathbf{h}_{E_k}(s) \to 0$ for each $s \in S$. Then from $\mathbf{h}_{T\times S} \in L^1_{W,V}(\mathbf{n})$ and the Lebesgue dominated convergence theorem again we get $\hat{\mathbf{n}}_{W,V}(\mathbf{h}_{E_k}, S) \to 0$. Then [20, Theorem 2.6] yields

$$(\widehat{\mathbf{m} \otimes \mathbf{n}})_{U,V}(\mathbf{f}, E_k) \le \hat{\mathbf{n}}_{W,V}(\mathbf{h}_{E_k}, S) \to 0,$$

which completes the proof. □

Theorem 3.5. *Let $(U, W, V) \in \mathcal{U}\times\mathcal{W}\times\mathcal{V}$. Let $\mathbf{f} : T\times S \to \mathbf{X}_U$ be a $\Delta^c_{U,W}\otimes\nabla_{W,V}$-measurable function and for each $s \in S$ the function $t \mapsto \mathbf{f}(t, s)$, $t \in T$, be $\Delta_{U,W}$-integrable. Then for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{g}_E$ is $\nabla_{W,V}$-measurable.*

Proof. Put $F = \{(t, s) \in T \times S;\ \mathbf{f}(t, s) \neq 0\}$. Then $F \in \sigma(\Delta^c_{U,W} \otimes \nabla_{W,V})$, and therefore there exist sets $A \in \sigma(\Delta^c_{U,W})$ and $B \in \sigma(\nabla_{W,V})$ such that $F \subset A \times B$. Choose a sequence $(A_n)_1^\infty$ of sets from $\Delta^c_{U,W}$, such that $A_n \nearrow A$. Obviously,

$$F_n = \{(t, s) \in T \times S;\ p_U(\mathbf{f}(t, s)) < n\} \in \sigma(\Delta^c_{U,W} \otimes \nabla_{W,V})$$

and $F_n \nearrow F$, $n \in \mathbb{N}$. Now it is easy to see that

$$H_n = (A_n \times B) \cap F_n \in \Delta^c_{U,W} \otimes \sigma(\nabla_{W,V}),$$

also $H_n \nearrow F$, and $\mathbf{f}(\cdot, s)\chi_{H_n} \in L^1_{U,W}(\mathbf{m})$ for each $n \in \mathbb{N}$ and each $s \in S$. Then by Theorem 3.4 the functions $\mathbf{g}_{n,E}$,

$$\mathbf{g}_{n,E}(s) = \int_{E^s} \mathbf{f}(\cdot, s)\chi_{H_n}(\cdot, s)\,\mathrm{d}\mathbf{m}, \quad n \in \mathbb{N},\ s \in S,\ E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$$

are $\nabla_{W,V}$-measurable. Since $\Delta_{U,W}$-integrability of function $t \mapsto \mathbf{f}(t, s)$, $t \in T$, for each $s \in S$ implies that $\mathbf{g}_E(s) = \lim_{n\to\infty} \mathbf{g}_{n,E}(s)$ for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and each $s \in S$, the theorem is proved. □

Theorem 3.6. *Let $(U, W, V) \in \mathcal{U}\times\mathcal{W}\times\mathcal{V}$. Let $\mathbf{f} : T\times S \to \mathbf{X}_U$ be a $\Delta_{U,W}\otimes\nabla_{W,V}$-measurable function and let for each $s \in S$ the function $\mathbf{f}(\cdot, s)$ be $\Delta_{U,W}$-integrable. Then for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{g}_E$ is weakly $\nabla_{W,V}$-measurable, i.e., for each $y' \in W^0$ the function $y'\mathbf{g}_E$ is $\nabla_{W,V}$-measurable. Therefore, if $\mathbf{Y}_W$ is separable, then $\mathbf{g}_E$ is $\nabla_{W,V}$-measurable for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$.*

Proof. Let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and $y' \in W^0$. Then

$$y'\mathbf{g}_E(s) = \int_{E^s} \mathbf{f}(\cdot, s)\,\mathrm{d}y'\mathbf{m}$$

for each $s \in S$ (see the paragraph after Theorem 3.5 in [19]) and we have

$$\mathbf{var}_U(y'\mathbf{m}, A) = \widehat{y'\mathbf{m}}(A) \leq p_{W^0}(y') \cdot \hat{\mathbf{m}}_{U,W}(A) < +\infty$$

for each $A \in \Delta_{U,W}$. Thus, $\widehat{y'\mathbf{m}}$ is continuous on $\Delta_{U,W}$ and by Theorem 3.5 $y'\mathbf{g}_E$ is $\nabla_{W,V}$-measurable.

For the proof of the second statement see [22, Theorem 3.5.3]. $\square$

Theorem 3.7. *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Let $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function and for each $s \in S$ the function $\mathbf{f}(\cdot, s)$ be $\Delta_{U,W}$-integrable. If*

$$\mathbf{f}_n = \sum_{i=1}^{r_n} \mathbf{x}_{n,i} \chi_{E_{n,i}}, \ \mathbf{x}_{n,i} \in \mathbf{X}_U, \ E_{n,i} \in \Delta_{U,W} \otimes \nabla_{W,V}, \ i = 1, 2, \ldots, r_n, \ n \in \mathbb{N},$$

is a sequence of $\Delta_{U,W} \otimes \nabla_{W,V}$-simple functions such that $\mathbf{f}_n(t, s) \to \mathbf{f}(t, s)$ for each $(t, s) \in T \times S$ and if $\mathbf{X}^1_U$ is the closed span of

$$\mathbf{X}_0 = \bigcup_{n=1}^{\infty} \sum_{i=1}^{r_n} \mathbf{x}_{n,i}$$

in $\mathbf{X}_U$, then for each $s \in S$ the function $\mathbf{f}(\cdot, s)$ is integrable with respect to the restriction of the measure $\mathbf{m}^1_{U,W} : \Delta_{U,W} \to L(\mathbf{X}^1_U, \mathbf{Y}_W)$ and the set of all finite sums of the form $\sum_{j=1}^{r} \mathbf{m}(A_j)\mathbf{x}_j$, $A_j \in \Delta_{U,W}$, $\mathbf{x}_j \in \mathbf{X}_0$, $j = 1, 2, \ldots, r$, is a dense subset of a set

$$\left\{ \int_A \mathbf{f}(\cdot, s) \, \mathrm{d}\mathbf{m}; \ A \in \sigma(\Delta_{U,W}), s \in S \right\}$$

of the space $\mathbf{Y}_W$.

Proof. Under the assumptions of theorem (see also proofs of convergence theorems in [18]) for each $s \in S$ there exist a sequence $(F_k(s))_1^\infty$ of sets from $\Delta_{U,W}$, a set $N(s) \in \sigma(\Delta_{U,W})$, and a subsequence $(n_k(s))_1^\infty$ of natural numbers, such that

$$\lim_{k\to\infty} \int_A \mathbf{f}_{n_k(s)}(\cdot, s) \chi_{F_k(s) \cup N(s)}(\cdot, s) \, \mathrm{d}\mathbf{m} = \int_A \mathbf{f}(\cdot, s) \, \mathrm{d}\mathbf{m}$$

uniformly with respect to $A \in \sigma(\Delta_{U,W})$. It remains to observe that for each $s \in S$ integrals on the left-hand side of the last equality are of the form $\sum_{j=1}^{r} \mathbf{m}(A_j)\mathbf{x}_j$ with $A_j \in \Delta_{U,W}$, $\mathbf{x}_j \in \mathbf{X}_0$, $j = 1, \ldots, r$. Note that the (U, W)-semivariation of the restricted measure $\mathbf{m}^1_{U,W} : \Delta_{U,W} \to L(\mathbf{X}^1_U, \mathbf{Y}_W)$ is less than or equal to the (U, W)-semivariation of $\mathbf{m}_{U,W} : \Delta_{U,W} \to L(\mathbf{X}_U, \mathbf{Y}_W)$, hence it is finite on $\Delta_{U,W}$. $\square$

As a direct consequence of theorem we have the following result.

Corollary 3.8. Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Let $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function and let for each $s \in S$ the function $\mathbf{f}(\cdot, s)$ be $\Delta_{U,W}$-integrable. Let $\{\mathbf{m}(A)\mathbf{x}; \ A \in \Delta_{U,W}\}$ be a separable subset of $\mathbf{Y}_W$ for each $\mathbf{x} \in \mathbf{X}_U$. Then

(i) the set $B = \{\int_A \mathbf{f}(\cdot, s) \, \mathrm{d}\mathbf{m}; \ A \in \sigma(\Delta_{U,W}), s \in S\}$ is a separable subset of $\mathbf{Y}_W$; especially we may choose $W \in \mathcal{W}$ such that $\operatorname{span} B = \mathbf{Y}_W$;

(ii) for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{g}_E$ is $\nabla_{W,V}$-measurable.

Theorem 3.9. *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Let $\Delta_{U,W}$ be generated by a countable system of subsets of T, let $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function and for each $s \in S$ the function $\mathbf{f}(\cdot, s)$ be $\Delta_{U,W}$-integrable. Then*

(a) *the set $\left\{\int_A \mathbf{f}(\cdot, s)\,\mathrm{d}\mathbf{m};\ A \in \sigma(\Delta_{U,W}), s \in S\right\}$ is a separable subset of $\mathbf{Y}_W$;*
(b) *for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{g}_E$ is $\nabla_{W,V}$-measurable; and*
(c) *the function v,*

$$v(s) = \sup_{A \in \sigma(\Delta_{U,W})} p_W\left(\int_A \mathbf{f}(\cdot, s)\,\mathrm{d}\mathbf{m}\right), \quad s \in S,$$

is finite-valued and $\nabla_{W,V}$-measurable.

Proof. Without loss of generality we may assume that $\Delta_{U,W}$ is generated by a countable ring $\mathfrak{R} = \{N_n; n \in \mathbb{N}\}$, see [12, Theorem C, § 5].

We will prove (a) and (b) together. With respect to Corollary 3.8 it suffices to show that $\mathbf{Y}_{\mathbf{x}} = \{\mathbf{m}(A)\mathbf{x};\ A \in \Delta_{U,W}\}$ is a separable subset of $\mathbf{Y}_W$ for each $\mathbf{x} \in \mathbf{X}_U$.

Let $\mathbf{x} \in \mathbf{X}_U$. Put $\mathfrak{R}_n = (N_1 \cup \cdots \cup N_n) \cap \mathfrak{R}$, and $\mathfrak{S}_n = \sigma(\mathfrak{R}_n)$, $n \in \mathbb{N}$. Then clearly $\Delta_{U,W} = \delta(\mathfrak{R}) = \bigcup_{n=1}^{\infty} \mathfrak{S}_n$. Let us show that a set $\mathbf{Y}_0$ of all finite sums of the form $\sum_{i=1}^{r} \mathbf{m}(\mathfrak{R}_{n_i})\mathbf{x}$ is dense in $\mathbf{Y}_{\mathbf{x}}$ ($\mathbf{Y}_0$ is clearly separable).

Let $A \in \Delta_{U,W}$. Then there exists an n_A, such that $A \in \mathfrak{S}_{n_A}$. Let $\mathbf{h}_{n_A} : \mathfrak{S}_{n_A} \to \mathbf{Y}_W$ be a control measure for the vector measure $\mathbf{m}(\cdot)\mathbf{x} : \mathfrak{S}_{n_A} \to \mathbf{Y}_W$. Then the desired assertion directly follows from [12, Theorem D, § 13] applied to $\mathbf{h}_{n_A}$ and from the following inequality

$$\begin{aligned} p_W(\mathbf{m}(A_1)\mathbf{x} - \mathbf{m}(A_2)\mathbf{x}) &\leq p_W(\mathbf{m}(A_1 - A_2)\mathbf{x}) + p_W(\mathbf{m}(A_2 - A_1)\mathbf{x}) \\ &\leq 2\|\mathbf{m}(\cdot)\mathbf{x}\|_{U,W}(A_1 \triangle A_2), \quad A_1, A_2 \in \mathfrak{S}_{n_A}. \end{aligned}$$

(c) Since $A \mapsto \int_A \mathbf{f}(\cdot, s)\mathrm{d}\mathbf{m}$, $A \in \sigma(\Delta_{U,W})$, is a (W, σ)-additive vector measure on a σ-ring, then v is a finite-valued measure, see [10, Theorem IV.10.4]. By [10, Theorem IV.10.5] and [12, Theorem D] we get

$$v(s) = \sup_{n \in \mathbb{N}} p_W\left(\int_{N_n} \mathbf{f}(\cdot, s)\,\mathrm{d}\mathbf{m}\right)$$

for each $s \in S$. Thus part (b) and [12, Theorem A, § 20] imply $\nabla_{W,V}$-measurability of v. □

Theorem 3.10. *Let $(U, W) \in \mathcal{U} \times \mathcal{W}$ and $\mathbf{X}_U$ be a separable space. Then for each $A \in \sigma(\Delta_{U,W})$ there exists a σ-additive measure $\lambda_A : \sigma(\Delta_{U,W}) \to [0, \infty)$, such that $C \in \sigma(\Delta_{U,W})$ and*

$$\lambda_A(A \cap C) = 0 \Rightarrow \hat{\mathbf{m}}_{U,W}(A \cap C) = 0.$$

Proof. Let $A \in \sigma(\Delta_{U,W})$ and choose a sequence $(A_n)_1^{\infty}$ of sets from $\Delta_{U,W}$, such that $A_n \nearrow A$. Since

$$\hat{\mathbf{m}}_{U,W}(C) = \sup_{y' \in W^0} \mathbf{var}_U(y'\mathbf{m}, C)$$

for each $C \in \sigma(\Delta_{U,W})$, see (2.1), then

$$\hat{\mathbf{m}}_{U,W}(A \cap C) = \lim_{n\to\infty} \hat{\mathbf{m}}_{U,W}(A_n \cap C)$$

for each $C \in \sigma(\Delta_{U,W})$.

Let us suppose that the theorem is proved for all $A \in \Delta_{U,W}$ and consider σ-additive measures $\lambda_n : \sigma(\Delta_{U,W}) \to [0,\infty)$, such that $C \in \sigma(\Delta_{U,W})$ and $\lambda_n(A_n \cap C) = 0$ implies $\hat{\mathbf{m}}_{U,W}(A_n \cap C) = 0$, $n \in \mathbb{N}$. Put

$$\lambda_A(C) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_n(A_n \cap C)}{1 + \lambda_n(T)}, \quad C \in \sigma(\Delta_{U,W}).$$

Obviously, λ_A has required properties, and therefore it is sufficient to prove the theorem for each $A \in \Delta_{U,W}$.

Let $A \in \Delta_{U,W}$ and $\{\mathbf{x}_k \in \mathbf{X}_U, k \in \mathbb{N}\}$ be a dense subset of $\mathbf{X}_U$. Let for each $k \in \mathbb{N}$

$$\lambda_k : A \cap \sigma(\Delta_{U,W}) \to [0,\infty)$$

be a control measure for vector measure $\mathbf{m}(\cdot)\mathbf{x}_k : A \cap \sigma(\Delta_{U,W}) \to \mathbf{Y}_W$. Then clearly

$$\lambda_A(C) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\lambda_k(A \cap C)}{1 + \lambda_k(A)}, \quad C \in \sigma(\Delta_{U,W}),$$

has the required properties. □

Theorem 3.11. *Let $(U,W,V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Let $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function, for each $s \in S$ the function $\mathbf{f}(\cdot, s)$ be $\Delta_{U,W}$-integrable. If for each $B \in \sigma(\nabla_{W,V})$ there exists a σ-additive measure $\lambda_B : \sigma(\nabla_{W,V}) \to [0,\infty)$, such that*

$$\lambda_B(B \cap D) = 0 \Rightarrow \hat{\mathbf{n}}_{W,V}(B \cap D) = 0, \quad D \in \sigma(\nabla_{W,V}),$$

then for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{g}_E$ is $\hat{\mathbf{n}}_{W,V}$-essentially $\nabla_{W,V}$-measurable.

Proof. Let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$. Consider $A \in \sigma(\Delta_{U,W})$ and $B \in \sigma(\nabla_{W,V})$, such that $E \subset A \times B$ and let $\lambda_B : \sigma(\nabla_{W,V}) \to [0,\infty)$ be the corresponding measure. Let $(\mathbf{f}_n : T \to \mathbf{X}_U)_1^\infty$ be a sequence of $\Delta_{U,W} \otimes \nabla_{W,V}$-simple functions, such that $\mathbf{f}_n(t,s) \to \mathbf{f}(t,s)$ for each $(t,s) \in T \times S$ and let $\mathbf{X}_U^1$ be the closed linear span of the union of their ranges in $\mathbf{X}_U$. By Theorem 3.7 we may replace $\mathbf{X}_U$ by the separable subspace $\mathbf{X}_U^1$. By Theorem 3.10 there exists a σ-additive measure $\mu_A : \sigma(\Delta_{U,W}) \to [0,\infty)$, such that $C \in \sigma(\Delta_{U,W})$ and $\mu_A(A \cap C) = 0 \Rightarrow \hat{\mathbf{m}}^1_{U,W}(A \cap C) = 0$, where $\hat{\mathbf{m}}^1_{U,W}$ is the (U,W)-semivariation of the restricted measure $\mathbf{m}^1_{U,W} : \sigma(\Delta_{U,W}) \to L(\mathbf{X}_U^1, \mathbf{Y}_W)$. Clearly, $\hat{\mathbf{m}}^1_{U,W}(C) \le \hat{\mathbf{m}}_{U,W}(C)$ for each $C \in \sigma(\Delta_{U,W})$. Obviously,

$$F = \bigcup_{n=0}^{\infty} \{(t,s) \in T \times S;\ \mathbf{f}_n(t,s) \neq 0\} \in \sigma(\Delta_{U,W}) \otimes \sigma(\nabla_{W,V}) = \sigma(\Delta_{U,W} \otimes \nabla_{W,V}),$$

where $\mathbf{f}_0 = \mathbf{f}$. Since $\mu_A \otimes \lambda_B : \sigma(\Delta_{U,W} \otimes \nabla_{W,V}) \to [0,\infty)$ is a σ-additive measure, by the Egoroff-Luzin theorem there exists a set $N \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$, $N \subset F$, a sequence $(F_k)_1^\infty$ of set from $\Delta_{U,W} \otimes \nabla_{W,V}$, such that $(\mu_A \otimes \lambda_B)(N) = 0$, $F_k \nearrow F \backslash N$, and a sequence $(\mathbf{f}_n)_1^\infty$ of functions U-converges uniformly to $\mathbf{f}$ on each F_k, $k \in \mathbb{N}$. Clearly,

$$\mathbf{g}_E(s) = \mathbf{g}_{E\cap(F\setminus N)}(s) + \mathbf{g}_{E\cap N}(s) = \lim_{k\to\infty} \mathbf{g}_{E\cap F_k}(s) + \mathbf{g}_{E\cap N}(s)$$

for each $s \in S$. By Theorem 3.7 each function $\mathbf{g}_{E\cap F_k}$, $k \in \mathbb{N}$, is $\nabla_{W,V}$-measurable. Thus, it is sufficient to prove that the function $\mathbf{g}_{E\cap N}$ is $\hat{\mathbf{n}}_{W,V}$-null.

Obviously, $\{s \in S; \mathbf{g}_{E\cap N}(s) \neq 0\} \subset B$. Since

$$0 = (\mu_A \otimes \lambda_B)(A \times B \cap N) = \int_B \mu_A(A \cap N^s)\,\mathrm{d}\lambda_B,$$

there exists a set $D \in \sigma(\nabla_{W,V})$ with $\lambda_B(B \cap D) = 0$ such that $\mu_A(A \cap N^s) = 0$ for each $s \in B \setminus D$, see [12, Theorem A, § 36]. But then $\hat{\mathbf{m}}^1_{U,W}(A \cap N^s) = 0$, and therefore $\mathbf{g}_{E\cap N}(s) = 0$ for each $s \in B \setminus D$. Thus $\{s \in S; \mathbf{g}_{E\cap N}(s) \neq 0\} \subset B \cap D$. However, $\hat{\mathbf{n}}_{W,V}(B \cap D) = 0$, and therefore $\mathbf{g}_{E\cap N}$ is $\hat{\mathbf{n}}_{W,V}$-null, which completes the proof. □

Remark 3.12. Let $(U, W, V) \in \mathcal{U}\times\mathcal{W}\times\mathcal{V}$. Let $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function and for each $s \in S$ the function $\mathbf{f}(\cdot, s)$ be $\Delta_{U,W}$-integrable. Then $\nabla_{W,V}$-measurability of function $\mathbf{g}_E$ for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ depends naturally on the function $\mathbf{f}$. Particularly, if the range of $\mathbf{f}$ is a relatively σ-compact set on $\mathbf{X}_U$, then [18, Theorem 3.2] and [18, Theorem 4.4] (on interchange of limit and integral) immediately imply $\nabla_{W,V}$-measurability of the function $\mathbf{g}_E$ for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$.

4. The general Fubini theorem

Lemma 4.1. *Let $(U, W, V) \in \mathcal{U}\times\mathcal{W}\times\mathcal{V}$, and $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function. Then there exist sequences $(A_n)_1^\infty \in \Delta_{U,W}$, $(B_n)_1^\infty \in \nabla_{W,V}$, such that $\mathbf{f}$ is $\delta(\{A_n \times B_n\}_{n\in\mathbb{N}})$-measurable.*

Proof. By definition of a $\Delta_{U,W}\otimes\nabla_{W,V}$-measurable function there exists a sequence $(\mathbf{f}_k)_1^\infty$ of $\Delta_{U,W}\otimes\nabla_{W,V}$-simple functions, such that $\mathbf{f}_k(t, s) \to \mathbf{f}(t, s)$ for each $(t, s) \in T \times S$. Each $\mathbf{f}_k$ is of the form $\mathbf{f}_k = \sum_{i=1}^{r_k} \mathbf{x}_{k,i}\chi_{E_{k,i}}$, where $\mathbf{x}_{k,i} \in \mathbf{X}_U$, $E_{k,i} \in \Delta_{U,W} \otimes \nabla_{W,V}$, $E_{k,i} \cap E_{k,j} = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, r_k$. Since $\Delta_{U,W} \otimes \nabla_{W,V}$ is the smallest δ-ring over all rectangles $A\times B$, $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, the obviously valid δ-version of [12, Theorem D, § 5] implies that for each couple (k, i), $k \in \mathbb{N}$, $i = 1, 2, \ldots, r_k$, there exist sequences $(A_{k,i,j})_{j=1}^\infty \in \Delta_{U,W}$, $(B_{k,i,j})_{j=1}^\infty \in \nabla_{W,V}$, such that

$$E_{k,i} \in \delta(\{A_{k,i,j} \times B_{k,i,j}\}_{j\in\mathbb{N}}).$$

By a suitable enumeration of the countable set

$$\{(k, i, j);\ k \in \mathbb{N},\ i = 1, 2, \ldots, r_k,\ j \in \mathbb{N}\}$$

we immediately get the desired sequences $(A_n)_1^\infty \in \Delta_{U,W}$ and $(B_n)_1^\infty \in \nabla_{W,V}$. □

The next lemma is a direct consequence of the Orlicz-Pettis theorem, see [22, Theorem 3.2.3] and [10, Theorem IV.10.1].

Lemma 4.2. *Let $V \in \mathcal{V}$ and $\mathbf{x}_{n,k} \in \mathbf{Z}_V$, $k, n \in \mathbb{N}$. Let for each $n \in \mathbb{N}$ the series $\sum_{n=1}^{\infty} \mathbf{z}_{n,k}$ be unconditionally V-bornologically convergent (in $\mathbf{Z}_V$), and let for each $I_n \subset \mathbb{N}$ the series $\sum_{n=1}^{\infty} \sum_{k \in I_n} \mathbf{z}_{n,k}$ be unconditionally V-bornologically convergent. Then the series $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \mathbf{z}_{n,k}$ is unconditionally V-bornologically convergent.*

Using the above lemmas we prove the following

Lemma 4.3. *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Let $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function, for each $s \in S$ the function $\mathbf{f}(\cdot, s)$ be $\Delta_{U,W}$-integrable, and for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{g}_E$ be $\nabla_{W,V}$-integrable. Then the set function*

$$E \mapsto \int_S \int_{E^s} \mathbf{f}(\cdot, s)\, \mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n}, \quad E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V}),$$

is a (V, σ)-additive measure on $\sigma(\Delta_{U,W} \otimes \nabla_{W,V})$.

Proof. Let $(E_k)_1^\infty$ be a sequence of pairwise disjoint sets from $\sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and put $E_0 = \bigcup_{k=1}^{\infty} E_k$. We have to show that

$$\int_S \int_{E^s} \mathbf{f}(\cdot, s)\, \mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n} = \sum_{k=1}^{\infty} \int_S \int_{E_k^s} \mathbf{f}(\cdot, s)\, \mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n}$$

in the sense of unconditional V-bornological convergence. By [18, Theorem 4.4] (on interchange of limit and integral) it is enough to show that the series on the right-hand side is unconditionally V-bornologically convergent.

By Lemma 4.1 there exists a countable system $\mathcal{A} \subset \Delta_{U,W}$, such that $E_k \in \sigma(\mathcal{A}) \otimes \sigma(\nabla_{W,V})$ for each $k \in \mathbb{N}$. Choose the sets $A \in \sigma(\mathcal{A})$, $B \in \sigma(\nabla_{W,V})$, such that $E_0 \subset A \times B$, and choose the sequence $(B_n)_1^\infty$ of sets from $\nabla_{W,V}$, such that $B_n \nearrow B$ and $B_0 = \emptyset$. By Theorem 3.9(c) the function v,

$$v(s) = \sup_{A_1 \in \sigma(\mathcal{A})} p_W \left(\int_{A_1 \cap E_0^s} \mathbf{f}(\cdot, s)\, \mathrm{d}\mathbf{m} \right), \quad s \in S,$$

is finite-valued and $\nabla_{W,V}$-measurable. Thus,

$$F_n = \{s \in S;\ 0 \le v(s) < n\} \in \sigma(\nabla_{W,V})$$

for each $n = 0, 1, \ldots$, and $F_n \nearrow$. Put

$$G_n = B_n \cap F_n \setminus B_{n-1} \cap F_{n-1}, \quad n \in \mathbb{N}.$$

Then G_n, $n \in \mathbb{N}$, are pairwise disjoint elements of $\nabla_{W,V}$, and $\bigcup_{n=1}^{\infty} G_n \subset B$. Put

$$\mathbf{z}_{n,k} = \int_{G_n} \int_{E_k^s} \mathbf{f}(\cdot, s)\, \mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n}, \quad n, k \in \mathbb{N}.$$

Using Lemma 4.1 we will show that the series $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mathbf{z}_{n,k}$ is unconditionally V-bornologically convergent, and this will prove the lemma since by [18, Theorem 4.4] we have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mathbf{z}_{n,k}=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\int_{G_n}\int_{E_k^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n}=\sum_{k=1}^{\infty}\int_{S}\int_{E_k^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n}.$$

Therefore, it remains to verify the validity of assumptions of Lemma 4.1.

Let $n\in\mathbb{N}$ be fixed. We will show that for each $z'\in V^0$ the equality

$$\left\langle\int_{G_n}\int_{E_0^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n},z'\right\rangle=\left\langle\sum_{k=1}^{\infty}\int_{G_n}\int_{E_k^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n},z'\right\rangle=\left\langle\sum_{k=1}^{\infty}\mathbf{z}_{n,k},z'\right\rangle$$

holds in the sense of unconditional V-bornological convergence, and so by Orlicz-Pettis theorem we will prove the unconditional V-bornological convergence of $\sum_{k=1}^{\infty}\mathbf{z}_{n,k}$.

Since the function $\mathbf{f}(\cdot,s)$ is $\Delta_{U,W}$-integrable for each $s\in S$, by [18, Theorem 4.4] we immediately get that for each $s\in S$

$$\int_{E_0^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}=\sum_{k=1}^{\infty}\int_{E_k^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}$$

in the sense of unconditional V-bornological convergence. ¿From the definition of the function v it is clear that

$$p_W\left(\sum_{k\in K}\int_{E_k^s}\mathbf{f}(\cdot,s)\mathrm{d}\mathbf{m}\right)\le v(s)$$

for each $s\in S$ and each $K\subset\mathbb{N}$. Thus, for any finite $K\subset\mathbb{N}$ by [19, Lemma 3.3] we have

$$\begin{aligned}&\left|\left\langle\sum_{k\in K}\int_{G_n}\int_{E_k^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n},z'\right\rangle\right|\\
&\le p_{V^0}(z')\cdot p_V\left(\int_{G_n}\left(\sum_{k\in K}\int_{E_k^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\right)\mathrm{d}\mathbf{n}\right)\\
&\le p_{V^0}(z')\cdot\sup_{s\in G_n}p_W\left(\sum_{k\in K}\int_{E_k^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\right)\cdot\hat{\mathbf{n}}_{W,V}(G_n)\\
&\le p_{V^0}(z')\cdot\sup_{s\in G_n}v(s)\cdot\hat{\mathbf{n}}_{W,V}(B_n)\\
&\le p_{V^0}(z')\cdot n\cdot\hat{\mathbf{n}}_{W,V}(B_n)<+\infty.\end{aligned}$$

Therefore, the series

$$\left\langle\sum_{k=1}^{\infty}\int_{G_n}\int_{E_k^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n},z'\right\rangle=\sum_{k=1}^{\infty}\int_{G_n}\int_{E_k^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}(z'\mathbf{n})$$

is unconditionally V-bornologically convergent, thus by [18, Theorem 4.4] we get

$$\sum_{k=1}^{\infty}\langle \mathbf{z}_{n,k}, z'\rangle = \sum_{k=1}^{\infty}\int_{G_n}\int_{E_k^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}(z'\mathbf{n}) = \int_{G_n}\int_{E_0^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}(z'\mathbf{n})$$
$$= \left\langle \int_{G_n}\int_{E_0^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n}, z'\right\rangle,$$

which was to be shown.

Now, let $I_n \subset \mathbb{N}$, $n \in \mathbb{N}$, and put

$$E = \bigcup_{n=1}^{\infty}(T \times G_n) \cap \left(\bigcup_{k\in I_n} E_k\right).$$

Since G_n, $n \in \mathbb{N}$, are pairwise disjoint, the $\nabla_{W,V}$-integrability of $\mathbf{g}_E$ implies that the series

$$\sum_{n=1}^{\infty}\int_{G_n}\int_{\left(\bigcup_{k\in I_n} E_k\right)^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n} = \sum_{n=1}^{\infty}\left(\sum_{k\in I_n}\mathbf{z}_{n,k}\right)$$

is unconditionally V-bornologically convergent, i.e., the assumptions of Lemma 4.1 are satisfied which was to be shown. □

Lemma 4.4. *Let $(U,W) \in \mathcal{U}\times\mathcal{W}$, and $\mathbf{f} : T \to \mathbf{X}_U$ be a $\Delta_{U,W}$-measurable function. Then there exists a σ-additive measure $\lambda : \sigma(\Delta_{U,W}) \to [0,\infty)$, such that $N \in \sigma(\Delta_{U,W})$, $\lambda(N) = 0$ implies that $\mathbf{f}\chi_N$ is $\Delta_{U,W}$-integrable and $\int_N \mathbf{f}\,\mathrm{d}\mathbf{m} = 0$.*

Proof. Let $(\mathbf{f}_n : T \to \mathbf{X}_U)_1^\infty$ be a sequence of $\Delta_{U,W}$-simple functions such that $\mathbf{f}_n(t) \to \mathbf{f}(t)$ for each $t \in T$. To each vector measure $A \mapsto \int_A \mathbf{f}_n\,\mathrm{d}\mathbf{m}$, $A \in \sigma(\Delta_{U,W})$, $n \in \mathbb{N}$, take a control measure $\lambda_n : \sigma(\Delta_{U,W}) \to [0,\infty)$. Now it is sufficient to put

$$\lambda(A) = \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\lambda_n(A)}{1+\lambda_n(T)}, \quad A \in \sigma(\Delta_{U,W}),$$

which has the desired properties. □

Lemma 4.5. *Let $(U,W,V) \in \mathcal{U}\times\mathcal{W}\times\mathcal{V}$. Let $\mathbf{f} : T\times S \to \mathbf{X}_U$ be a $\Delta_{U,W}\otimes\nabla_{W,V}$-measurable function and for each $s \in S$ the function $\mathbf{f}(\cdot,s)$ be $\Delta_{U,W}$-integrable. Then for each set $E \in \sigma(\Delta_{U,W}\otimes\nabla_{W,V})$ the function $\mathbf{g}_E$ is $\nabla_{W,V}$-measurable.*

Proof. Let $E \in \sigma(\Delta_{U,W}\otimes\nabla_{W,V})$. Since the function $\mathbf{f}\chi_E$ is $\Delta_{U,W}\otimes\nabla_{W,V}$-measurable, then by Lemma 4.1 there exists a sequence $(A_n)_1^\infty$ of sets from $\sigma(\Delta_{U,W})$, such that $\mathbf{f}\chi_E$ is $\delta(\{A_1,\dots,A_n,\dots\})\otimes\nabla_{W,V}$-measurable. By [7, Theorem 4] for each $s \in S$ the function $\mathbf{f}\chi_E(\cdot,s)$ is integrable with respect to the restriction

$$\mathbf{m}^* = \mathbf{m} : \delta(\{A_1,\dots,A_n,\dots\}) \to L(\mathbf{X}_U,\mathbf{Y}_W),$$

and there holds

$$\mathbf{g}_E(s) = \int_{E^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m} = \int_{E^s}\mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}^* = \mathbf{g}_E^*(s).$$

Since $\delta(\{A_1,\dots,A_n,\dots\})$ is σ-generated, then $\mathbf{g}_E = \mathbf{g}_E^*$, and by Theorem 3.9(b) is $\nabla_{W,V}$-measurable. □

Finally, now we are able to prove the main theorem of this paper – the general Fubini theorem for the Dobrakov integral in C. B. L. C. S.

Theorem 4.6. *(**Fubini**) Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Let the product measure $\mathbf{m}_{U,W} \otimes \mathbf{n}_{W,V} : \Delta_{U,W} \otimes \nabla_{W,V} \to L(\mathbf{X}_U, \mathbf{Z}_V)$ exist and let $\mathbf{f} : T \times S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function. Further, let for each $s \in S$ the function $\mathbf{f}(\cdot, s)$ be $\Delta_{U,W}$-integrable. Then the following conditions are equivalent:*

(a) *for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{f}$ is $\Delta_{U,W} \otimes \nabla_{W,V}$-integrable;*
(b) *for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{g}_E$ is $\hat{\mathbf{n}}_{W,V}$-essentially $\nabla_{W,V}$-integrable.*

If these conditions are satisfied, then

$$\int_E \mathbf{f} \,\mathrm{d}(\mathbf{m} \otimes \mathbf{n}) = \int_S \int_{E^s} \mathbf{f}(\cdot, s) \,\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{n} \tag{4.1}$$

for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$.

Proof. Without loss of generality we may suppose that for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{g}_E$ is $\nabla_{W,V}$-measurable. Let $(\mathbf{f}_n : T \to \mathbf{X}_U)_1^\infty$ be a sequence of $\Delta_{U,W} \otimes \nabla_{W,V}$-simple functions, such that $\mathbf{f}_n(t, s) \to \mathbf{f}(t, s)$ and $p_U(\mathbf{f}_n(t, s)) \nearrow p_U(\mathbf{f}(t, s))$ for each $(t, s) \in T \times S$. For each vector measure

$$E \mapsto \int_E \mathbf{f}_n \,\mathrm{d}(\mathbf{m} \otimes \mathbf{n}), \quad E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V}),\ n \in \mathbb{N},$$

take a control measure $\lambda_n : \sigma(\Delta_{U,W} \otimes \nabla_{W,V}) \to [0, \infty)$ and put

$$\lambda(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_n(E)}{1 + \lambda_n(T)}, \quad E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V}).$$

Let $\mathbf{X}_U^1$ be the closed linear span of the set $\{\mathbf{f}_n(t, s); (t, s) \in T \times S, n \in \mathbb{N}\}$. Then $\mathbf{X}_U^1$ is a separable Banach space and according to Theorem 3.7 we may replace $\mathbf{X}_U$ by $\mathbf{X}_U^1$. Hence, we may assume that $\mathbf{X}_U$ is a separable Banach space.

Take $A_0 \in \sigma(\Delta_{U,W})$ and $B_0 \in \sigma(\nabla_{W,V})$ such that

$$F = \{(t, s) \in T \times S;\ \mathbf{f}(t, s) \neq 0\} \subset A_0 \times B_0.$$

Then by Theorem 3.10 there exists a σ-additive measure $\nu_{A_0} : \sigma(\Delta_{U,W}) \to [0, \infty)$ such that $C \in \sigma(\Delta_{U,W})$ and $\nu_{A_0}(A_0 \cap C) = 0 \Rightarrow \hat{\mathbf{m}}_{U,W}(A_0 \cap C) = 0$.

Let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$. Then by Lemma 4.5 the function $\mathbf{g}_E$ is $\nabla_{W,V}$-measurable. Therefore by Lemma 4.4 there exists a σ-additive measure $\eta_E : \sigma(\nabla_{W,V}) \to [0, \infty)$ such that $D \in \sigma(\nabla_{W,V})$, $\eta_E(D) = 0$ implies $\mathbf{g}_E \chi_D$ is $\nabla_{W,V}$-integrable and $\int_D \mathbf{g}_E \,\mathrm{d}\mathbf{n} = 0$. Put

$$\mu_E(G) = \lambda(G) + (\nu_{A_0} \otimes \eta_E)(G), \quad G \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V}).$$

Then from the above stated results and [12, Theorem A, § 36] it follows that if $N \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and $\mu_E(N) = 0$, then the function $\mathbf{f}\chi_{N \cap E}$ is $\Delta_{U,W} \otimes \nabla_{W,V}$-integrable, the function $\mathbf{g}_{N \cap E}$ is $\nabla_{W,V}$-integrable and

$$\int_{N \cap E} \mathbf{f} \,\mathrm{d}(\mathbf{m} \otimes \mathbf{n}) = \int_S \mathbf{g}_{N \cap E} \,\mathrm{d}\mathbf{n} = 0.$$

According to the Egoroff-Lusin theorem there is a set $N \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ with $\mu_E(N) = 0$, a sequence $(F_k)_1^\infty$ of sets from $\Delta_{U,W} \otimes \nabla_{W,V}$, such that $F_k \nearrow F \setminus N$, and the sequence $(\mathbf{f}_n)_1^\infty$ of functions U-converging uniformly to $\mathbf{f}$ on each F_k, $k \in \mathbb{N}$. Thus by [20, Theorem 3.1] the function $\mathbf{f}\chi_{E\cap F}$ is $\Delta_{U,W} \otimes \nabla_{W,V}$-integrable for each $k \in \mathbb{N}$, the function $\mathbf{g}_{E\cap F}$ is $\nabla_{W,V}$-integrable and

$$\int_{G\cap E\cap F_k} \mathbf{f}\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n}) = \int_S \mathbf{g}_{G\cap E\cap F_k}\,\mathrm{d}\mathbf{n} = \int_S \int_{(G\cap E\cap F_k)^s} \mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{n} \tag{4.2}$$

for each $G \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$. Since by assumption the function $\mathbf{f}(\cdot,s)$ is $\Delta_{U,W}$-integrable for each $s \in S$, then

$$\begin{aligned} \mathbf{g}_{E\cap F_k}(s) = \int_{(E\cap F_k)^s} \mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m} &\to \int_{[E\cap(F\setminus N)]^s} \mathbf{f}(\cdot,s)\,\mathrm{d}\mathbf{m} \\ &= \mathbf{g}_{E\cap(F\cap N)}(s) = \mathbf{g}_{E\setminus N}(s) \end{aligned} \tag{4.3}$$

for each $s \in S$.

(a)⇒(b) and (4.1). Let us suppose that $\mathbf{f}$ is $\Delta_{U,W} \otimes \nabla_{W,V}$-integrable and $B \in \sigma(\nabla_{W,V})$. Then

$$\begin{aligned} \int_B \mathbf{g}_{E\cap F}\,\mathrm{d}\mathbf{n} = \int_{(A_0\times B)\cap E\cap F_k} \mathbf{f}\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n}) &\to \int_{(A_0\times B)\cap(F\setminus N)\cap E} \mathbf{f}\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n}) \\ &= \int_{(A\times B)\cap E} \mathbf{f}\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n}). \end{aligned} \tag{4.4}$$

Then [18, Theorem 4.4], (4.3) and (4.4) imply that the function $\mathbf{g}_{E\setminus N}$, hence also $\mathbf{g}_E$, is $\nabla_{W,V}$-integrable and thus

$$\int_B \mathbf{g}_E\,\mathrm{d}\mathbf{n} = \int_B \mathbf{g}_{E\setminus N}\,\mathrm{d}\mathbf{n} = \int_{(A_0\times B)\cap E} \mathbf{f}\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n})$$

for each $B \in \sigma(\nabla_{W,V})$. Taking $B = B_0$ we get the equality (4.1).

(b)⇒(a) and (4.1). Let us suppose that for each $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $\mathbf{g}_E$ is $\nabla_{W,V}$-integrable. Take $E = A_0 \times B_0$ in the proof of (a)⇒(b) and (4.1) above. Then for each $k \in \mathbb{N}$ the function $\mathbf{f}\chi_{F_k} = \mathbf{f}\chi_{(A_0\times B_0)\cap F_k}$ is $\Delta_{U,W} \otimes \nabla_{W,V}$-integrable and

$$(\mathbf{f}\chi_{F_k})(t,s) \to (\mathbf{f}\chi_{F\setminus N})(t,s) \tag{4.5}$$

for each $(t,s) \in T \times S$.

Since by Lemma 4.3 the set function

$$G \mapsto \int_S \mathbf{g}_G\,\mathrm{d}\mathbf{n}, \quad G \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V}),$$

is a (V,σ)-additive measure, then by (4.2) we have

$$\begin{aligned} \int_G \mathbf{f}\chi_{F_k}\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n}) &= \int_{(A_0\times B_0)\cap G\cap F_k} \mathbf{f}\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n}) = \int_S \mathbf{g}_{(A_0\times B_0)\cap G\cap F_k}\,\mathrm{d}\mathbf{n} \\ &= \int_S \mathbf{g}_{F_k\cap G}\,\mathrm{d}\mathbf{n} \to \int_S \mathbf{g}_{G\cap(F\setminus N)}\,\mathrm{d}\mathbf{n} = \int_S \mathbf{g}_G\,\mathrm{d}\mathbf{n}. \end{aligned} \tag{4.6}$$

Then [18, Theorem 4.4], (4.5) and (4.6) imply $\Delta_{U,W} \otimes \nabla_{W,V}$-integrability of function $\mathbf{f}$ and the equality (4.1). The theorem is proved. □

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