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SOLID CORES AND SOLID HULLS OF WEIGHTED BERGMAN SPACES

JOSÉ BONET,[1*] WOLFGANG LUSKY,[2] and JARI TASKINEN[3]

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ABSTRACT. We determine the solid hull for $2 < p < \infty$ and the solid core for $1 < p < 2$ of weighted Bergman spaces $A^p_\mu, 1 < p < \infty$, of analytic functions on the disk and on the whole complex plane, for a very general class of nonatomic positive bounded Borel measures μ. New examples are presented. Moreover, we show that the space $A^p_\mu, 1 < p < \infty$, is solid if and only if the monomials are an unconditional basis of this space.

1. Introduction and preliminaries

Consider $R = 1$ or $R = \infty$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We study holomorphic functions $f : R \cdot \mathbb{D} \to \mathbb{C}$, where $R \cdot \mathbb{D} = \mathbb{D}$ if $R = 1$ and $R \cdot \mathbb{D} = \mathbb{C}$ if $R = \infty$. Let $\hat{f}(k)$ be the Taylor coefficients of f; that is, $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$. We take a nonatomic positive bounded Borel measure μ on $[0, R[$ such that $\mu([r, R[) > 0$ for every $r > 0$ and $\int_0^R r^n \, d\mu(r) < \infty$ for all $n > 0$. Put, for $1 \le p < \infty$,

$$\|f\|_p = \Big(\frac{1}{2\pi} \int_0^R \int_0^{2\pi} \big| f(re^{i\varphi}) \big|^p \, d\varphi \, d\mu(r) \Big)^{1/p},$$

and let

$$A^p_\mu = \big\{ f : R \cdot \mathbb{D} \to \mathbb{C} : f \text{ holomorphic with } \|f\|_p < \infty \big\}.$$

We call A^p_μ a *weighted Bergman space*.

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*Corresponding author.
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Let $H(R\cdot\mathbb{D})$ be the space of all holomorphic functions on $R\cdot\mathbb{D}$, and let $A\subset H(R\cdot\mathbb{D})$ be a subspace containing the polynomials. We want to study the *solid core*

$$s(A)=\left\{f\in A: g\in A \text{ for all holomorphic } g \text{ with } \left|\hat{g}(k)\right|\le\left|\hat{f}(k)\right| \text{ for all } k\right\}$$

and the *solid hull*

$$S(A)=\left\{g: R\cdot\mathbb{D}\to\mathbb{C}: g \text{ holomorphic, there is } f\in A \text{ with } \left|\hat{g}(k)\right|\le\left|\hat{f}(k)\right| \text{ for all } k\right\}.$$

We call A *solid* if $A=S(A)$. In the first four sections we consider $A=A^p_\mu$, while in Section 5 we include the case where A consists of weighted sup-norm spaces of holomorphic functions.

The solid hull and core of spaces of analytic functions has been investigated by many authors. We refer the reader to the recent books [6] and [13] and the many references therein. For example, [6] presented a characterization of solid hulls and cores of A^p_μ where $d\mu(r)=(1-r)^\alpha\,dr$ for some $\alpha>0$ and $R=1$.

Originally, our main interest was to replace the "standard weights" $(1-r)^\alpha$ by weights of the form $v_{a,b}(r)=\exp(-a/(1-r)^b)$ for some $a>0$ and $b>0$, which are of a completely different nature and require different methods, and hence to consider $d\mu(r)=v_{a,b}(r)\,dr$. We wanted to extend to weighted Bergman spaces the results of [1] and [2], works which were entirely devoted to this class of weights $v_{a,b}$ in connection with weighted sup-norms. In the present article we give a characterization of solid hulls of A^p_μ if $2<p<\infty$ and solid cores of A^p_μ if $1<p<2$ in our main Theorem 2.1 for much more general μ which, under some mild additional assumptions (Corollary 3.2), results in the explicit computation of many examples including $v(r)=\exp(-a/(1-r)^b)$ for $R=1$ and $v(r)=\exp(-r)$ for $R=\infty$ (see Corollaries 3.4 and 3.5). Finally, Sections 4 and 5 are dedicated to Bergman spaces A^p_μ and weighted sup-norm spaces H^∞_v which are themselves solid. We give examples for this situation in connection with holomorphic functions over the complex plane and show that this can never happen for holomorphic functions over the unit disk. The main results are Theorem 4.1, which states that A^p_μ is solid if and only if the monomials $(z^n)_{n=0}^\infty$ are an unconditional basis of A^p_μ, and Theorem 5.2, which ensures that H^∞_v is solid if and only if $(z^n)_{n=0}^\infty$ is a Schauder basis of the closure H^0_v of the polynomials in H^∞_v.

For a holomorphic g and $0<r$, we define

$$M_p(g,r)=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|g(re^{i\varphi})\right|^p\,d\varphi\right)^{1/p}$$

and $P_ng(z)=\sum_{k=0}^n\hat{g}(k)z^k$. It is well known that, for $1<p<\infty$, there are universal constants $c_p>0$ with $M_p(P_ng,r)\le c_pM_p(g,r)$, where c_p does not depend on g, n, or r. Moreover, we have $\lim_{n\to\infty}M_p(g-P_ng,r)=0$. Hence, we obtain

$$\|P_nf\|_p\le c_p\|f\|_p \quad \text{for all } f\in A^p_\mu \text{ and all } n \qquad \text{and} \qquad \lim_{n\to\infty}\|f-P_nf\|_p=0.$$

In particular, we see that the monomials $z \mapsto z^n$, $n = 0, 1, 2, \ldots$ form a Schauder basis of A^p_μ if $1 < p < \infty$. (Details can be found in [4] and [14].) In the rest of the article $[r]$ denotes the largest integer less than or equal to $r > 0$.

2. Main general result

The main result of this section is Theorem 2.1 below. There are relevant earlier, related works. For example, in Theorem 4.1 of [12], Pavlović established a useful norm in blocks for certain weighted Bergman spaces. (See also earlier work by Mateljević and Pavlović [11].)

Theorem 2.1. *Assume that there are constants $d_1, d_2 > 0$, and $\omega_n > 0, n = 1, 2, \ldots$, numbers $0 \le l_1 < l_2 < \cdots$, and radii $s_1 < s_2 < \cdots$ such that, for every $f \in A^p_\mu$,*

$$d_1 \|f\|_p \le \Big(\sum_{n=1}^{\infty} \omega_n^p M_p^p \big((P_{[l_{n+1}]} - P_{[l_n]}) f, s_n \big) \Big)^{1/p} \le d_2 \|f\|_p. \tag{2.1}$$

(a) *If $2 < p < \infty$, then*

$$S(A^p_\mu) = \Big\{ g : R \cdot \mathbb{D} \to \mathbb{C} : \\ g \text{ holomorphic with } \sum_{n=1}^{\infty} \omega_n^p \Big(\sum_{k=[l_n]+1}^{[l_{n+1}]} \big|\hat{g}(k)\big|^2 s_n^{2k} \Big)^{p/2} < \infty \Big\}.$$

(b) *If $1 < p < 2$, then*

$$s(A^p_\mu) = \Big\{ g : R \cdot \mathbb{D} \to \mathbb{C} : \\ g \text{ holomorphic with } \sum_{n=1}^{\infty} \omega_n^p \Big(\sum_{k=[l_n]+1}^{[l_{n+1}]} \big|\hat{g}(k)\big|^2 s_n^{2k} \Big)^{p/2} < \infty \Big\}.$$

Theorem 2.1 is proved below. Before presenting the proof, we point out that condition (2.1) can be realized for any given μ. Indeed, fix $\beta > 16 \cdot 3^{p-1}(1+2^p)c_p^p + 2$, and use induction to obtain $0 = l_1 < l_2 < l_3 < \cdots$ and $0 \le s_1 < s_2 < \cdots < R$ with

$$\int_0^{s_n} r^{l_n p}\, d\mu = \beta \int_{s_n}^R r^{l_n p}\, d\mu \qquad \text{and} \qquad \int_0^{s_n} r^{l_{n+1} p}\, d\mu = \frac{1}{\beta} \int_{s_n}^R r^{l_{n+1} p}\, d\mu. \tag{2.2}$$

Instead of starting with $n = 1$, we can just as well start the induction with $n = n_0$, for example, for some $n_0 \ge 0$ (with $l_1 = 0$ and arbitrary s_1) and restrict the preceding relations to all $n \ge n_0$. Moreover, put

$$\omega_n = \Big(\int_0^{s_n} \Big(\frac{r}{s_n}\Big)^{l_n p} d\mu + \int_{s_n}^R \Big(\frac{r}{s_n}\Big)^{l_{n+1} p} d\mu \Big)^{1/p}.$$

Then there are constants $d_1, d_2 > 0$ such that, for every $f \in A^p_\mu$,

$$d_1 \|f\|_p \le \Big(\sum_{n=1}^{\infty} \omega_n^p M_p^p \big((P_{[l_{n+1}]} - P_{[l_n]}) f, s_n \big) \Big)^{1/p} \le d_2 \|f\|_p.$$

This was shown in [5] for $p = 1$ and in [10] for $1 < p < \infty$ and $R = 1$, but with some slight modifications the proofs carry over to the case $R = \infty$.

Example 2.2.

(i) Let $d\mu(r) = dr$ where $R = 1$. Then we obtain

$$l_n = \frac{1}{p}(a^{n-1} - 1) \qquad \text{and} \qquad s_n = \Big(\frac{\beta}{\beta+1}\Big)^{a^{1-n}} \quad \text{where } a = \frac{\log(\beta+1)}{\log(1+\beta) - \log(\beta)}.$$

This can be easily verified using the definition (starting with $n = 0$) and induction.

(ii) Let $d\mu(r) = r^\alpha\, dr$ for some $\alpha > 0$ and $R = 1$. With example (i) and $l_n p + \alpha = (a^{n-1} - 1)$, where a is the number in (i), we obtain

$$l_n = \frac{1}{p}(a^{n-1} - 1) - \frac{\alpha}{p} \qquad \text{and} \qquad s_n = \Big(\frac{\beta}{\beta+1}\Big)^{a^{1-n}}$$

for $n \geq 2$ with $l_1 = 0$ and $s_1 = \beta/(\beta + 1)$.

Now we turn to the proof of Theorem 2.1. Let $f : R \cdot \mathbb{D} \to \mathbb{C}$ be holomorphic. Recall that $\hat{f}(n)r^n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\varphi})e^{-in\varphi}\, d\varphi$ for each $0 < r < R$ and each $n = 0, 1, 2, \ldots$. For $g(re^{i\varphi}) = r^{n(p-1)}e^{-in\varphi}/(\int_0^R r^{np}\, d\mu)^{1-1/p}$, we have

$$\big|\hat{f}(n)\big|\Big(\int_0^R r^{np}\, d\mu\Big)^{1/p} = \frac{1}{2\pi}\Big|\int_0^R \int_0^{2\pi} f(re^{i\varphi})g(re^{i\varphi})\, d\varphi\, d\mu\Big| \leq \|f\|_p.$$

In the following, we make use of the Khintchine inequality (see [7, Theorem 2.b.3.]); that is, for arbitrary b_k and n we have

$$A_p\Big(\sum_{k=1}^n |b_k|^2\Big)^{1/2} \leq \Big(\frac{1}{2^n}\sum_{\theta_k=\pm 1}\Big|\sum_{k=1}^n b_k\theta_k\Big|^p\Big)^{1/p} \leq B_p\Big(\sum_{k=1}^n |b_k|^2\Big)^{1/2},$$

where A_p, B_p are universal constants not depending on n. (The summation in the central expression runs over the 2^n different possibilities of the change of signs.)

Conclusion of the proof of Theorem 2.1. For a holomorphic function g put

$$\alpha(g) = \Big(\sum_{n=1}^\infty \omega_n^p M_p^p\big((P_{[l_{n+1}]} - P_{[l_n]})f, s_n\big)\Big)^{1/p}.$$

As assumed, $\alpha(\cdot)$ is equivalent to $\|\cdot\|_p$. Moreover, let

$$\gamma(g) = \Big(\sum_{n=1}^\infty \omega_n^p\Big(\sum_{k=[l_n]+1}^{[l_{n+1}]} \big|\hat{g}(k)\big|^2 s_n^{2k}\Big)^{p/2}\Big)^{1/p},$$

and let $V = \{g : R \cdot \mathbb{D} \to \mathbb{C} : g \text{ holomorphic with } \gamma(g) < \infty\}$. Recall Parseval's identity, which implies that

$$M_2^2\big((P_{[l_{n+1}]} - P_{[l_n]})f, s_n\big) = \sum_{k=[l_n]+1}^{[l_{n+1}]} \big|\hat{g}(k)\big|^2 s_n^{2k}.$$

Proof of (a). Let $g \in S(A^p_\mu)$. Then there is $f \in A^p_\mu$ with $|\hat{g}(k)| \le |\hat{f}(k)|$ for all k. If $2 < p < \infty$, then

$$\gamma(g) \le \gamma(f) \le \alpha(f) \le d_2 \|f\|_p < \infty.$$

Hence $g \in V$.

Now let $g \in V$. Put $\Delta_n = \{+1, -1\}^{[l_{n+1}]-[l_n]}$. For $\Theta_n = (\theta_{[l_n]+1}, \dots, \theta_{[l_{n+1}]}) \in \Delta_n$, put

$$g_{\Theta_n}(\varphi) = \sum_{k=[l_n]+1}^{[l_{n+1}]} \theta_k \hat{g}(k) s_n^k e^{ik\varphi} \qquad \text{and} \qquad g_n(\varphi) = \sum_{k=[l_n]+1}^{[l_{n+1}]} \hat{g}(k) s_n^k e^{ik\varphi}.$$

Let $\tilde{\Theta}_n$ be such that

$$M_p(g_{\tilde{\Theta}_n}, s_n) \le \Big(\frac{1}{2^{[l_{n+1}]-[l_n]}} \sum_{\Theta_n \in \Delta_n} M_p^p(g_{\Theta_n}, s_n) \Big)^{1/p}.$$

The Khintchine inequality yields

$$M_p(g_{\tilde{\Theta}_n}, s_n) \le B_p M_2(g_n, s_n).$$

Put $h = \sum_n g_{\tilde{\Theta}_n}$. Then, by the preceding estimates,

$$d_1 \|h\|_p \le \alpha(h) \le B_p \gamma(g) < \infty.$$

Hence $h \in A^p_\mu$. Since by definition $|\hat{h}(k)| = |\hat{g}(k)|$ for all k, we obtain $g \in S(A^p_\mu)$.

Proof of (b). We retain the preceding notation. Let $g \in V$, and let $f : R{\cdot}\mathbb{D} \to \mathbb{C}$ be holomorphic with $|\hat{f}(k)| \le |\hat{g}(k)|$ for all k. Then

$$d_1 \|f\|_p \le \alpha(f) \le \gamma(f) \le \gamma(g) < \infty.$$

This implies that $f \in A^p_\mu$ and hence $g \in s(A^p_\mu)$.

Now let $g \in s(A^p_\mu)$. Let $\tilde{\tilde{\Theta}}_n \in \Delta_n$ be such that

$$\Big(\frac{1}{2^{[l_{n+1}]-[l_n]}} \sum_{\Theta_n \in \Delta_n} M_p^p(g_{\Theta_n}, s_n) \Big)^{1/p} \le M_p(g_{\tilde{\tilde{\Theta}}_n}, s_n).$$

Put $h = \sum_n g_{\tilde{\tilde{\Theta}}_n}$. Then we obtain $|\hat{h}(k)| = |\hat{g}(k)|$ for all k. Hence $h \in A^p_\mu$. The Khintchine inequality together with the choice of $\tilde{\tilde{\Theta}}_n$ yields

$$\gamma(g) = \gamma(h) \le A_p^{-1} \alpha(h) \le d_2 A_p^{-1} \|h\|_p < \infty.$$

We conclude that $g \in V$. $\square$

3. Main examples

Quite often it is very difficult to compute the parameters l_n and s_n in (2.2). Therefore, it is worthwhile to consider special cases which yield an equivalent representation of the norm $\|\cdot\|_p$ satisfying (2.1) and which are easier to compute and cover many examples. To this end, let $v : [0, R[\to]0, \infty[$ be a weight function; that is, let v be continuous, decreasing, and let it satisfy

$$\lim_{r\to R} v(r) = 0 \qquad \text{and} \qquad \sup_r r^n v(r) < \infty \quad \text{for all } n > 0.$$

Moreover, let ν be a nonatomic positive Borel measure on $[0, R[$ such that $\nu([r, R[) > 0$ for every $r > 0$ and such that $\int_0^R r^n v(r)\, d\nu(r) < \infty$ for every $n \geq 0$. Put, for $1 \leq p < \infty$,

$$\|f\|_p = \Big(\int_0^R M_p^p(f, r) v(r)\, d\nu(r)\Big)^{1/p}.$$

Here we consider A^p_μ with $d\mu(r) = v(r)\, d\nu(r)$. Actually, one can relax the conditions on v somewhat. It suffices to require that v be decreasing on $[r_0, R[$ for some $r_0 \in]0, R[$. This follows from the fact that, for $d\tilde\mu = 1_{[r_0,R[}\, d\mu$, the L_p-norms with respect to μ and $\tilde\mu$ are equivalent. Indeed, using the fact that $M_p(f, r)$ is increasing with respect to r for holomorphic functions f, we see that

$$\int_{r_0}^R M_p^p(f, r)\, d\mu(r) \leq \int_0^R M_p^p(f, r)\, d\mu(r) \leq \Big(1 + \frac{\mu([r_0, R[)}{\mu([0, R[)}\Big) \int_{r_0}^R M_p^p(f, r)\, d\mu(r).$$

For any $n > 0$, let $r_n \in [0, R[$ be a point where the function $r \mapsto r^n v(r)$ attains its global maximum. It is easily seen that $r_m < r_n$ if $m < n$. In the following, we assume that

$$\begin{gathered} r_n \text{ is the unique global maximum of } r^n v(r) \text{ for all } n \\ \text{and there are no further local maxima.} \end{gathered} \tag{3.1}$$

For example, this is the case if v is differentiable and v'/v is injective. Assumption (3.1) implies that $r^n v(r)$ is decreasing for $r \geq r_n$. Moreover, we assume that v satisfies the following.

Condition (b_0). There are numbers $1 < b < K$ and $m_1 < m_2 < \cdots$ with $\lim_{n\to\infty} m_n = \infty$ such that

$$b \leq \Big(\frac{r_{m_n}}{r_{m_{n+1}}}\Big)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \leq K.$$

Condition (b_0) is exactly the same as condition (b) in [1], except that the treatment of weighted Banach spaces of analytic functions with sup-norms requires $2 < b < K$. We refer the reader to [1] and [9] for more information and examples related to these conditions.

We take the parameters of condition (b_0) and put

$$I_n = \nu\big([r_{m_n}, r_{m_{n+1}}]\big),$$

and we assume that

$$I_n < \infty \quad \text{for all } n \quad \text{and} \quad \limsup_{n\to\infty} \frac{I_n}{\min(I_{n-1}, I_{n+1})} < b. \tag{3.2}$$

Theorem 3.1. *Let $1 < p < \infty$. Assume that v satisfies condition (b_0) with (3.1) and (3.2). Then there are constants $d_1, d_2 > 0$ with*

$$d_1\|f\|_p \le \Big(\sum_{n=1}^{\infty} M_p^p\big((P_{[m_{n+1}/p]} - P_{[m_n/p]})f, r_{m_n}\big)v(r_{m_n})I_n\Big)^{1/p} \le d_2\|f\|_p \tag{3.3}$$

for all $f \in A^p_\mu$.

In view of (2.1), we can apply Theorem 2.1 with the preceding $l_n = m_n/p$, $\omega_n^p = v(r_{m_n})I_n$, and $s_n = r_{m_n}$.

Corollary 3.2. *Let $d\mu = vd\nu$.*

(a) *If $2 < p < \infty$, then*

$$S(A^p_\mu) = \Big\{g : R\cdot\mathbb{D} \to \mathbb{C} : g \textit{ holomorphic with}$$
$$\sum_{n=1}^{\infty} v(r_{m_n})I_n\Big(\sum_{k=[m_n/p]+1}^{[m_{n+1}/p]} \big|\hat{g}(k)\big|^2 r_{m_n}^{2k}\Big)^{p/2} < \infty\Big\}.$$

(b) *If $1 < p < 2$, then*

$$s(A^p_\mu) = \Big\{g : R\cdot\mathbb{D} \to \mathbb{C} : g \textit{ holomorphic with}$$
$$\sum_{n=1}^{\infty} v(r_{m_n})I_n\Big(\sum_{k=[m_n/p]+1}^{[m_{n+1}/p]} \big|\hat{g}(k)\big|^2 r_{m_n}^{2k}\Big)^{p/2} < \infty\Big\}.$$

Before we prove Theorem 3.1, we present the following examples. They are concrete cases to which Corollary 3.2 applies, thus permitting us to calculate explicitly all the parameters which appear in the solid hull and solid core.

Example 3.3. (i) *$R = 1$ and $d\mu(r) = \exp(-\alpha/(1-r)^\beta)\,dr$ for some $\alpha, \beta > 0$.*

We take $v(r) = \exp(-\alpha/(1-r)^\beta)$ and $d\nu(r) = dr$. The weight v satisfies condition (b_0) with

$$m_n = \beta\Big(\frac{\beta}{\alpha}\Big)^{1/\beta} n^{2+2/\beta} - \beta n^2 \qquad \text{and} \qquad r_{m_n} = 1 - \Big(\frac{\alpha}{\beta}\Big)^{1/\beta}\frac{1}{n^{2/\beta}}$$

and $b = e^1$ (see [1], Theorem 3.1.) Here $I_n = (\alpha/\beta)^{1/\beta}(n^{-2/\beta} - (n+1)^{-2/\beta})$. Hence

$$\lim_{n\to\infty} \frac{I_n}{\min(I_{n-1}, I_{n+1})} = 1.$$

This shows that (3.2) is satisfied. We note that (3.1) holds, too, according to [1]. So we can apply Corollary 3.2.

(ii) *$R = 1$ and $d\mu(r) = (1 - \log(1-r))^{-1}\,dr$.*

Here we take

$$v(r) = 1 - r \qquad \text{and} \qquad d\nu(r) = \frac{dr}{(1-r)(1-\log(1-r))}.$$

Note that $r_m = 1 - 1/(m+1)$ is the only zero of the derivative of $r^m v(r)$. Hence (3.1) is satisfied. If we take $m_n = 9^n$ and hence $r_{m_n} = 1 - 1/(9^n+1)$, then a simple calculation reveals that v satisfies condition (b_0) with $b = 3$. We obtain

$$I_n = \int_{r_{m_n}}^{r_{m_{n+1}}} d\nu = \log\Big(\frac{1+\log(9^{n+1}+1)}{1+\log(9^n+1)}\Big)$$

from which we infer $\lim_{n\to\infty} I_n/\min(I_{n-1}, I_{n+1}) = 1$. This implies (3.2).

(iii) $R = \infty$ *and* $d\mu(r) = e^{-r}\,dr$.

Here we take $v(r) = e^{-r}$, $d\nu(r) = dr$. Note that $r_m = m$ is the only zero of the derivative of $r^m v(r)$. Hence (3.1) is satisfied. Put

$$m_1 = 1 \qquad \text{and} \qquad m_{n+1} = m_n + 2\sqrt{m_n}, \quad n = 1, 2\dots, \text{ and } r_{m_n} = m_n.$$

A simple calculation yields, with

$$\begin{aligned}
&-x - \frac{1}{2}\Big(\frac{x}{1-x}\Big)^2 \le \log(1-x) \le -x \quad \text{if } 0 < x < 1,\\
&\exp\Big(\frac{4\sqrt{m}}{\sqrt{m}+2} - 2\Big) \le \Big(\frac{r_{m_n}}{r_{m_{n+1}}}\Big)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}\\
&\qquad = \exp\Big(m\log\Big(1 - \frac{2}{\sqrt{m}+2}\Big) + 2\sqrt{m}\Big) \le \exp\Big(\frac{4\sqrt{m}}{\sqrt{m}+2}\Big).
\end{aligned}$$

Similarly, with

$$\begin{aligned}
&x - \frac{x^2}{2} \le \log(1+x) \le x \quad \text{for } 0 < x < 1,\\
&\exp\Big(4 - 2\Big(1 + \frac{2}{\sqrt{m}}\Big)\Big) \le \exp\Big((m + 2\sqrt{m})\log\Big(1 + \frac{2}{\sqrt{m}}\Big) - 2\sqrt{m}\Big)\\
&\quad = \Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \le e^4.
\end{aligned}$$

This shows that condition (b_0) holds. Moreover, we easily obtain

$$I_n = 2\sqrt{m_n} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{I_n}{\min(I_{n-1}, I_{n+1})} = 1,$$

which yields (3.2). Observe that in this case we can take $m_n = n^2$ (see Theorem 3.1 in [3]). This fact is not surprising, since one can easily prove by induction that our selection of m_n above satisfies $(n-1)^2 \le m_n \le n^2$ for each n.

Corollary 3.4. *Let* $R = 1$ *and* $d\mu(r) = \exp(-1/(1-r))\,dr$.

(a) *If* $2 < p < \infty$, *then*

$$S(A^p_\mu) = \Big\{g \in H(\mathbb{D}) : \sum_{n=1}^{\infty} e^{-n^2}\Big(\frac{1}{n^3}\Big)\Big(\sum_{k=[n^4/p]+1}^{[(n+1)^4/p]} \big|\hat{g}(k)\big|^2\Big(1 - \frac{1}{n^2}\Big)^{2k}\Big)^{p/2} < \infty\Big\}.$$

(b) *If* $1 < p < 2$, *then*

$$s(A^p_\mu) = \Big\{ g \in H(\mathbb{D}) : \sum_{n=1}^{\infty} e^{-n^2} \Big(\frac{1}{n^3}\Big) \Big(\sum_{k=[n^4/p]+1}^{[(n+1)^4/p]} \big|\hat{g}(k)\big|^2 \Big(1 - \frac{1}{n^2}\Big)^{2k} \Big)^{p/2} < \infty \Big\}.$$

Proof. Example 3.3(i) and Corollary 3.2 yield, with $\alpha = \beta = 1$ and $m_n = n^4 - n^2$,

$$S(A^p_\mu) = \Big\{ g \in H(\mathbb{D}) :$$

$$\sum_{n=1}^{\infty} e^{-n^2} \Big(\frac{1}{n^2} - \frac{1}{(n+1)^2}\Big) \Big(\sum_{k=[(n^4-n^2)/p]+1}^{[((n+1)^4-(n+1)^2)/p]} \big|\hat{g}(k)\big|^2 \Big(1 - \frac{1}{n^2}\Big)^{2k} \Big)^{p/2} < \infty \Big\}$$

if $2 < p < \infty$ and

$$s(A^p_\mu) = \Big\{ g \in H(\mathbb{D}) :$$

$$\sum_{n=1}^{\infty} e^{-n^2} \Big(\frac{1}{n^2} - \frac{1}{(n+1)^2}\Big) \Big(\sum_{k=[(n^4-n^2)/p]+1}^{[((n+1)^4-(n+1)^2)/p]} \big|\hat{g}(k)\big|^2 \Big(1 - \frac{1}{n^2}\Big)^{2k} \Big)^{p/2} < \infty \Big\}$$

if $1 < p < 2$. If we let k run, in the preceding summations, from $[n^4/p] + 1$ to $[(n+1)^4/p]$ instead, then we obtain conditions which are equivalent to the preceding ones and hence characterize again $S(A^p_\mu)$ and $s(A^p_\mu)$. This follows from

$$n^4 - n^2 \le n^4 \le (n+1)^4 - (n+1)^2 \quad \text{for all } n.$$

(Compare this with Lemma 3.2. and Example 3.3(i) in [1].) Then, finally, Corollary 3.4 follows from

$$\Big(\frac{1}{2}\Big)\frac{1}{n^3} \le \frac{1}{n^2} - \frac{1}{(n+1)^2} \le \frac{2}{n^3} \quad \text{for all } n. \qquad \square$$

Corollary 3.5. *Let* $R = \infty$ *and* $d\mu(r) = e^{-r}\, dr$.

(a) *If* $2 < p < \infty$, *then*

$$S(A^p_\mu) = \Big\{ g \in H(\mathbb{C}) : \sum_{n=1}^{\infty} e^{-n^2} 2n \Big(\sum_{k=[n^2/p]+1}^{[(n+1)^2/p]} \big|\hat{g}(k)\big|^2 n^{2k} \Big)^{p/2} < \infty \Big\}.$$

(b) *If* $1 < p < 2$, *then*

$$s(A^p_\mu) = \Big\{ g \in H(\mathbb{C}) : \sum_{n=1}^{\infty} e^{-n^2} 2n \Big(\sum_{k=[n^2/p]+1}^{[(n+1)^2/p]} \big|\hat{g}(k)\big|^2 n^{2k} \Big)^{p/2} < \infty \Big\}.$$

Proof. This is a consequence of Example 3.3(iii) and Corollary 3.2. $\square$

Lemma 3.6. *Let* $1 \le p < \infty$, *let* $0 < r < s$, *and let* $f(z) = \sum_{m \le j \le n} \alpha_j z^j$ *for some* α_j *and* $0 \le m < n$. *Then we have*

$$\text{(i)} \quad M_p(f, r) \le \Big(\frac{r}{s}\Big)^m M_p(f, s)$$

and

$$\text{(ii)} \quad M_p(f,s) \leq \left(\frac{s}{r}\right)^n M_p(f,r).$$

Proof. Part (i) follows from the fact that, for holomorphic f, the function $M_p(f,\cdot)$ is increasing in r, while part (ii) is Lemma 3.1(i) of [8]. □

Now consider $1 < p < \infty$, and let m_n, I_n satisfy condition (b_0), (3.1), and (3.2).

Lemma 3.7. *Fix k, n, and $r_{m_k} \leq r \leq r_{m_{k+1}}$. Then we have*

$$\text{(i)} \quad \left(\frac{r}{r_{m_n}}\right)^{m_n} \frac{v(r)}{v(r_{m_n})} \leq \left(\frac{1}{b}\right)^{n-k-1} \quad \text{if } k < n$$

and

$$\text{(ii)} \quad \left(\frac{r}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} \leq K\left(\frac{1}{b}\right)^{k-n-1} \quad \text{if } k \geq n.$$

Proof. If $k < n$, then we have

$$\begin{aligned}
&\left(\frac{r}{r_{m_n}}\right)^{m_n} \frac{v(r)}{v(r_{m_n})} \\
&\quad = \left(\frac{r}{r_{m_{k+1}}}\right)^{m_n} \frac{v(r)}{v(r_{m_{k+1}})} \left(\frac{r_{m_{k+1}}}{r_{m_{k+2}}}\right)^{m_n} \frac{v(r_{m_{k+1}})}{v(r_{m_{k+2}})} \cdots \left(\frac{r_{m_{n-1}}}{r_{m_n}}\right)^{m_n} \frac{v(r_{m_{n-1}})}{v(r_{m_n})} \\
&\quad \leq \left(\frac{r}{r_{m_{k+1}}}\right)^{m_{k+1}} \frac{v(r)}{v(r_{m_{k+1}})} \left(\frac{r_{m_{k+1}}}{r_{m_{k+2}}}\right)^{m_{k+2}} \frac{v(r_{m_{k+1}})}{v(r_{m_{k+2}})} \cdots \left(\frac{r_{m_{n-1}}}{r_{m_n}}\right)^{m_n} \frac{v(r_{m_{n-1}})}{v(r_{m_n})} \\
&\quad \leq \left(\frac{1}{b}\right)^{n-k-1}.
\end{aligned}$$

If $k \geq n+1$, then we have

$$\begin{aligned}
&\left(\frac{r}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} \\
&\quad = \left(\frac{r}{r_{m_k}}\right)^{m_{n+1}} \frac{v(r)}{v(r_{m_k})} \left(\frac{r_{m_k}}{r_{m_{k-1}}}\right)^{m_{n+1}} \frac{v(r_{m_k})}{v(r_{m_{k-1}})} \cdots \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \\
&\quad \leq \left(\frac{r}{r_{m_k}}\right)^{m_k} \frac{v(r)}{v(r_{m_k})} \left(\frac{r_{m_k}}{r_{m_{k-1}}}\right)^{m_{k-1}} \frac{v(r_{m_k})}{v(r_{m_{k-1}})} \cdots \left(\frac{r_{m_{n+2}}}{r_{m_{n+1}}}\right)^{m_{n+1}} \frac{v(r_{m_{n-1}})}{v(r_{m_n})} K \\
&\quad \leq K\left(\frac{1}{b}\right)^{k-n-1}.
\end{aligned}$$

Similarly, for $k = n$,

$$\left(\frac{r}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} \leq \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \leq K.$$

□

Now fix $k_0 > 0$ and $0 < \rho < b$ such that

$$\frac{I_n}{\min(I_{n-1}, I_{n+1})} \leq \rho \quad \text{if } k \geq k_0. \tag{3.4}$$

Corollary 3.8. *Let $f_n(z) = \sum_{m_n/p \le j < m_{n+1}/p} \alpha_j z^j$, where $n \ge k_0$. Then, for any $k \ge k_0$ we have*

$$\int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r) v(r)\, d\nu(r) \le c\Big(\frac{\rho}{b}\Big)^{|n-k|} M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n. \tag{3.5}$$

Here $c > 0$ is a universal constant independent of k, n, f_n.

Proof. First let $k < n$. Then Lemmas 3.6(i) and 3.7(i) imply that

$$\begin{aligned}
&\int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r) v(r)\, d\nu(r) \\
&\qquad \le M_p^p(f_n, r_{m_n}) v(r_{m_n}) \int_{r_{m_k}}^{r_{m_{k+1}}} \Big(\frac{r}{r_{m_n}}\Big)^{m_n} \frac{v(r)}{v(r_{m_n})}\, d\nu(r) \\
&\qquad \le c_0 M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n \Big(\prod_{j=k}^{n-1} \frac{I_j}{I_{j+1}}\Big) \Big(\frac{1}{b}\Big)^{|n-k|} \\
&\qquad \le c_1 \Big(\frac{\rho}{b}\Big)^{|n-k|} M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n,
\end{aligned}$$

where c_0, c_1 are universal constants. If $k \ge n$, then we use Lemmas 3.6(ii) and 3.7(ii) to get

$$\begin{aligned}
&\int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r) v(r)\, d\nu(r) \\
&\qquad \le M_p^p(f_n, r_{m_n}) v(r_{m_n}) \int_{r_{m_k}}^{r_{m_{k+1}}} \Big(\frac{r}{r_{m_n}}\Big)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})}\, d\nu(r) \\
&\qquad \le K b M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n \Big(\prod_{j=n}^{k-1} \frac{I_{j+1}}{I_j}\Big) \Big(\frac{1}{b}\Big)^{|n-k|} \\
&\qquad \le c_2 \Big(\frac{\rho}{b}\Big)^{|n-k|} M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n,
\end{aligned}$$

where c_2 is a universal constant. ◻

Conclusion of the proof of Theorem 3.1. Let $f \in A_\mu^p$, say, $f = \sum_n f_n$, where f_n is as in Corollary 3.8. We can assume that $f_n = 0$ for $n \le k_0$ with k_0 as in (3.4).

To prove the right-hand inequality in Theorem 3.1 we use that $M_p(f_n, r) \le cM_p(f, r)$ for a universal constant independent of r, as well as that, in view of (3.1), $r^{m_n} v(r)$ is decreasing for $r \ge r_{m_n}$. We have

$$\begin{aligned}
&\sum_n M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n \\
&\qquad \le \sum_n \int_{r_{m_n}}^{r_{m_{n+1}}} \Big(\frac{r_{m_n}}{r}\Big)^{m_n} \frac{v(r_{m_n})}{v(r)} M_p^p(f_n, r) v(r)\, d\nu(r) \\
&\qquad \le \sum_n \int_{r_{m_n}}^{r_{m_{n+1}}} \Big(\frac{r_{m_n}}{r_{m_{n+1}}}\Big)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})} M_p^p(f_n, r) v(r)\, d\nu(r)
\end{aligned}$$

$$
\begin{aligned}
&\le K \sum_n \int_{r_{m_n}}^{r_{m_{n+1}}} M_p^p(f_n, r) v(r)\, d\nu(r) \\
&\le c^p K \sum_n \int_{r_{m_n}}^{r_{m_{n+1}}} M_p^p(f, r) v(r)\, d\nu(r) \\
&\le c^p K \|f\|_p^p.
\end{aligned}
$$

This in particular implies that $\sum_n M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n < \infty$.

Now we show the left-hand inequality of Theorem 3.1. Using the Minkowski inequality in the first estimate and Corollary 3.8 in the second one, we obtain

$$
\begin{aligned}
\|f\|_p^p &= \sum_k \int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f, r) v(r)\, d\nu(r) \\
&\le \sum_k \Big(\sum_n \Big(\int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r) v(r)\, d\nu(r) \Big)^{1/p} \Big)^p \\
&\le c_1 \sum_k \Big(\sum_n \Big(\frac{\rho}{b}\Big)^{|n-k|/p} \big(M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n \big)^{1/p} \Big)^p \\
&\le c_2 \sum_k \sum_n \Big(\frac{\rho}{b}\Big)^{|n-k|/p} M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n \\
&\le c_3 \sum_n M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n.
\end{aligned}
$$

Here c_1, c_2, c_3 are universal constants. In the second to last inequality we used the Hölder inequality in the following way. Put $a_n = (M_p^p(f_n, r_{m_n}) v(r_{m_n}) I_n)^{1/p}$. Then

$$
\sum_n \Big(\frac{\rho}{b}\Big)^{|n-k|/p} a_n \le \Big(\sum_n \Big(\frac{\rho}{b}\Big)^{|n-k|/p} a_n^p \Big)^{1/p} \cdot \Big(\sum_n \Big(\frac{\rho}{b}\Big)^{|n-k|/p} \Big)^{1/q},
$$

with $1/p + 1/q = 1$. In the last inequality we interchanged the summation over k and n and utilized $\sup_k \sum_n (\rho/b)^{|n-k|/p} = \sup_n \sum_k (\rho/b)^{|n-k|/p} < \infty$. $\square$

4. Solid Bergman spaces

Recall that a Bergman space A_μ^p is solid if $S(A_\mu^p) = A_\mu^p$.

Theorem 4.1. *Let $1 < p < \infty, p \neq 2$. Then the following are equivalent:*

(i) *A_μ^p is solid,*
(ii) *$s(A_\mu^p) = A_\mu^p$,*
(iii) *the monomials $(z^n)_{n=0}^\infty$ are an unconditional basis of A_μ^p,*
(iv) *the normalized monomials $(z^n/\|z^n\|_p)_{n=0}^\infty$ are equivalent to the unit vector basis of l^p,*
(v) *$\sup_n (l_{n+1} - l_n) < \infty$ for the numbers l_n in (2.1).*

Remark 4.2. If $p = 2$, then the normalized monomials are an orthonormal basis for A_μ^2 and all conditions (i)–(iv) are satisfied.

The following example is relevant in connection with Theorem 4.1.

Example 4.3. Consider $R = \infty$ and $v(r) = \exp(-(\log r)^2)$, $d\nu(r) = dr$. (This is included in Example 2.2 of [9].) Note that v is decreasing on $[1, \infty[$ which suffices in view of the remarks at the beginning of Section 3. We easily see that $r_m = \exp(m/2)$ is the only zero of the derivative of $r^m v(r)$. Hence (3.1) is satisfied. We get, for any $n > 0$ and $m > 0$,

$$\Big(\frac{r_m}{r_n}\Big)^m \frac{v(r_m)}{v(r_n)} = \Big(\frac{r_n}{r_m}\Big)^n \frac{v(r_n)}{v(r_m)} = \exp\Big(\frac{(n-m)^2}{4}\Big).$$

So, if we take $m_n = 4n$, then condition (b_0) is satisfied with $b = e^4$. Moreover, we have $I_n = \exp(2n+2) - \exp(2n)$. An easy calculation shows that (3.2) holds. Hence we can consider (2.1) with $l_n = m_n/p$. Therefore, $\sup_n(l_{n+1} - l_n) = 4/p < \infty$. This means that, for $d\mu(r) = v(r)\,dr$, the Bergman space A^p_μ is solid.

For the preceding example it is essential that $R = \infty$. Indeed, we have the following.

Corollary 4.4. *Let $1 < p < \infty$, $p \neq 2$, and $R = 1$. Then no Bergman space A^p_μ is solid.*

We prove Corollary 4.4 at the end of this section. For the proof of Theorem 4.1 we need the following.

Lemma 4.5. *Let (e_n) be a Schauder basis of a Banach space X with basis projections P_n. For $M \subset \mathbb{N}$, let T_M be the linear (not necessarily continuous) operator defined in the linear span of (e_n) by $T_M e_k = e_k$ if $k \in M$ and $T_M e_k = 0$ otherwise. If the basis (e_n) is not unconditional, then there is $N \subset \mathbb{N}$ such that, for any n, there exists m_n and $0 \neq y \in P_{m_n}X$ with $n\|y\| \leq \|T_N y\|$.*

Proof. If (e_n) is a conditional basis, then there exists an operator of the form T_N which is unbounded on X. Hence there is a sequence $x_k \in X$ with $\|x_k\| = 1$ and $\lim_{k\to\infty} \|T_N x_k\| = \infty$, and we find k_n with $n = n\|x_{k_n}\| < \|T_N x_{k_n}\|$ for all n. Using $T_N P_l = P_l T_N$ for all l, we find m_n such that

$$0 < n\|P_{m_n} x_{k_n}\| \leq \|P_{m_n} T_N x_{k_n}\| = \|T_N P_{m_n} x_{k_n}\| \quad \text{for all } n. \qquad \square$$

In the following we retain the definition of T_N with respect to the monomials (z^n).

Lemma 4.6. *Let $1 < p < \infty$, $p \neq 2$, and assume that there are constants $c_n > 0$, $d_n > 0$ with $\sup_n d_n/c_n < \infty$, integers $0 < a_n < b_n < a_{n+1}$, and radii s_n such that, for any $f_n \in A^p_\mu$ with $f_n(z) = \sum_{a_n \leq j \leq b_n} \alpha_j z^j$, we have*

$$c_n M_p(f_n, s_n) \leq \|f_n\|_p \leq d_n M_p(f_n, s_n).$$

If $\sup_n(b_n - a_n) = \infty$, then the monomials are not unconditional in A^p_μ.

Proof. It is well known that the monomials are a conditional basis sequence with respect to the norm $M_p(\cdot, 1)$. So we find $N \subset \mathbb{N}$ and $y_n \in Y_n := \operatorname{span}\{z^j : 0 \leq j \leq m_n\}$ with $M_p(y_n, 1) = 1$ and $n \leq M_p(T_N y_n, 1)$. Find k_n with $b_{k_n} - a_{k_n} > m_n$, put $Y_n = \{z^j : a_{k_n} \leq j \leq b_{k_n}\} \subset A^p_\mu$, and define $S_n : X_n \to Y_n$ by

$$(S_n f)(z) = z^{a_{k_n}} f(z/s_n).$$

Then, according to our assumptions, we have $\|S_n\| \cdot \|S_n^{-1}\| \le d_n/c_n < c$ for some universal constant c. Put $M_n = \{a_{k_n} + j : j \in N, j \le m_n\}$. Then $S_n T_N S_n^{-1} = T_{M_n}|_{X_n}$. If we consider $M = \bigcup_n M_n$, then the preceding shows that T_M is unbounded on A^p_μ. This proves that the system of monomials is conditional in A^p_μ. □

Conclusion of the proof of Theorem 4.1. We have that (i) ⇔ (ii) follows from the definition of solid hull, while (ii) ⇔ (iii) follows from the definition of solid core. (Recall that, in any case, the monomials are a basis of A^p_μ.) Now (iii) and Lemma 4.6 imply (v). Finally, (v) and (2.1) imply (iv), while (iv) trivially implies (iii). □

Proof of Corollary 4.4. Proposition 3.5 of [8] shows that, for $R = 1$, the assumptions of Lemma 4.6 are always satisfied. Hence the system of monomials can never be unconditional. In view of Theorem 4.1, the Bergman space A^p_μ can never be solid. □

5. Solid weighted spaces of entire functions with sup-norms

In this section we consider weighted Banach spaces of analytic functions with sup-norms. The main result of this section, Theorem 5.2, complements Theorem 4.1. This result was announced in Remark 5.6 of [1]. Here, as in Section 3, a continuous weight $v : \mathbb{C} \to]0, \infty[$ is a function satisfying

$$v(z) = v(|z|), \quad z \in \mathbb{C}, \qquad v(r) \ge v(s) \quad \text{if } 0 \le r < s \qquad \text{and}$$
$$\lim_{r\to\infty} r^n v(r) = 0 \quad \text{for all } n \ge 0.$$

We deal with the weighted space H_v^∞ over $\mathbb{C}$, that is,

$$H_v^\infty = \big\{ f : \mathbb{C} \to \mathbb{C} : f \text{ holomorphic}, \|f\|_v := \sup_{z\in\mathbb{C}} |f(z)| v(z) < \infty \big\}.$$

Let H_v^0 be the closure of the polynomials in H_v^∞.

Similarly to the weighted L_p-norms in Sections 3 and 4, one sees that it suffices to require only that $v(r) \ge v(s)$ for $r_0 \le r < s$ and some $r_0 > 0$, since $\|f\|_v$ and $\sup_{r_0 \le |z| < \infty} |f(z)| v(z)$ are equivalent for holomorphic f. Again, for $n > 0$ let $r_n \in [0, \infty[$ be a point where the function $r \mapsto r^n v(r)$ attains its global maximum. The next lemma can be easily proved with induction (which was done in Lemma 5.1 of [9]). The indices m_n are needed in the following.

Lemma 5.1. *For any $b > 2$ there are numbers $0 < m_1 < m_2 < \cdots$ with $\lim_{n\to\infty} m_n = \infty$ and*

$$b = \min\Big(\Big(\frac{r_{m_n}}{r_{m_{n+1}}}\Big)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \Big(\frac{r_{m_{n+1}}}{r_{m_n}}\Big)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \Big).$$

Actually, one can show that Lemma 5.1 works for all $b > 1$, but we need $b > 2$ in the following proof. There are examples of weights on $\mathbb{C}$ such that the monomials $(z^n)_{n=0}^\infty$ are a Schauder basis in the Banach space H_v^0. This is the same as saying that the Taylor series of each element in H_v^0 converges with respect to the weighted sup-norm $\|\cdot\|_v$. In the known examples, in this case, $(z^n/\|z^n\|_v)_{n=0}^\infty$ is equivalent to the unit vector basis of c_0. Moreover, here H_v^∞ is solid. We show that this is always true provided that $(z^n)_{n=0}^\infty$ is a Schauder basis of H_v^0. We also characterize this situation by a property for the indices m_n of Lemma 5.1. Our arguments are similar to those of [8].

Let $h(z) = \sum_{k=0}^\infty b_k z^k$. As before, let P_n be the partial sum operators, that is,

$$(P_n h)(z) = \sum_{k=0}^n b_k z^k.$$

If the monomials are a basis of H_v^0, then $\sup_n \|P_n|_{H_v^0}\| = \sup_n \|P_n|_{H_v^\infty}\| < \infty$. For any k, we have

$$|b_k| \cdot \|z^k\|_v = |b_k| r_k^k v(r_k) = \Big|\frac{1}{2\pi}\int_0^{2\pi} h(r_k e^{i\varphi}) e^{-ik\varphi}\, d\varphi\Big| v(r_k) \le \|h\|_v. \tag{5.1}$$

Moreover, take the numbers m_n of Lemma 5.1 and put

$$(R_n h)(z) = \sum_{k=0}^{m_{n-1}} b_k z^k + \sum_{m_{n-1}<k\le m_n} \frac{[m_n]-k}{[m_n]-[m_{n-1}]} b_k z^k.$$

Finally, put $M_\infty(h,r) = \sup_{|z|=r} |h(z)|$.

Theorem 5.2. *The following are equivalent:*

(i) $\sup_n (m_{n+1} - m_n) < \infty$ *where m_n are the indices of Lemma 5.1,*
(ii) $(z^n)_{n=0}^\infty$ *is a Schauder basis of H_v^0,*
(iii) $(z^n/\|z^n\|_v)_{n=0}^\infty$ *is equivalent to the unit vector basis of c_0,*
(iv) H_v^∞ *is solid,*
(v) H_v^0 *is solid.*

Proof. Put $V_n = R_n - R_{n-1}$. According to Proposition 5.2 in [9], since we assumed that $b > 2$ in Lemma 5.1, the norms $\|h\|_v$ and $\sup_n \sup_{r_{m_{n-1}} \le r \le r_{m_{n+1}}} M_\infty(V_n h, r) v(r)$ are equivalent. Since Lemma 3.3 in [9] implies that the operators V_n are uniformly bounded on H_v^∞, we obtain constants $c_1 > 0$ and $c_2 > 0$ with

$$c_1 \sup_n \|V_n h\|_v \le \|h\|_v \le c_2 \|V_n h\|_v \quad \text{for all } h \in H_v^\infty. \tag{5.2}$$

(i) $\Rightarrow$ (ii): Observe that, by the definition of V_n, $\dim V_n(H_v^0) = [m_{n+1}] - [m_{n-1}]$. By (i) we obtain $\sup_n \dim V_n(H_v^0) < \infty$. With the definition of P_j and (5.1), we see that $\sup_{j,n} \|P_j|_{V_n(H_v^0)}\| \le \sup_n([m_{n+1}] - [m_{n-1}]) < \infty$. With (5.2) and $P_j V_n = V_n P_j$ for all j and n, we conclude that the projections P_j are uniformly bounded. Hence $(z^n)_{n=0}^\infty$ is a Schauder basis of H_v^0.

(ii) $\Rightarrow$ (i): Assume that (ii) holds. By definition, $V_n(P_{m_{n+1}} - P_{m_{n-1}}) = V_n$. In view of the uniform boundedness of the V_n and (5.2), we obtain constants $c_1' > 0$ and $c_2' > 0$ with

$$c_1' \sup_n \big\|(P_{m_{n+1}} - P_{m_n})h\big\|_v \le \|h\|_v \le c_2' \sup_n \big\|(P_{m_{n+1}} - P_{m_n})h\big\|_v \tag{5.3}$$

for all $h \in H_v^\infty$. Here the first inequality follows from the uniform boundedness of the P_n in view of (ii), while the second inequality follows from (5.2). Let $t_n \in [0, R[$

be such that

$$t_n = r_{m_n} \quad \text{if } b = \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})}$$

and

$$t_n = r_{m_{n+1}} \quad \text{if } b = \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}$$

in Lemma 5.1 Then Corollary 3.2(b) of [9] implies that

$$\big\|(P_{m_{n+1}} - P_{m_n})h\big\|_v \le 2bM_\infty\big((P_{m_{n+1}} - P_{m_n})h, t_n\big)v(t_n).$$

With (5.3) we obtain

$$\begin{aligned} d_1 \sup_n M_\infty\big((P_{m_{n+1}} - P_{m_n})h, t_n\big)v(t_n) \\ \le \|h\|_v \le d_2 \sup_n M_\infty\big((P_{m_{n+1}} - P_{m_n})h, t_n\big)v(t_n) \end{aligned} \tag{5.4}$$

for some constants $d_1 > 0$, $d_2 > 0$ and all $h \in H_v^0$.

It is well known that there are bounded holomorphic functions whose Taylor series do not converge with respect to $M_\infty(\cdot, 1)$. By going over to suitable Cesàro means if necessary, we see that, for each $n \in \mathbb{N}$, there is a polynomial f of degree N and an index $M \le N$ such that

$$M_\infty(f, 1) = 1 \quad \text{but } n \le M_\infty(P_M f, 1).$$

Proceeding by contradiction, assume that (i) does not hold, that is, $\sup_n(m_{n+1} - m_n) = \infty$. Then we find k with dim $(P_{m_{k+1}} - P_{m_k})H_v^0 > N$. Put $h(z) = z^{m_k} f(z)/v(t_k)$. Then, in view of (5.4), we obtain

$$d_1 \le \|h\|_v \le d_2 \qquad \text{and} \qquad \frac{n}{d_2} \le \|P_{M+m_k} h\|_v.$$

This implies that the projections P_j are not uniformly bounded, contradicting assumption (ii). This contradiction implies that $\sup_n(m_{n+1} - m_n) < \infty$, and we have checked that (ii) $\Rightarrow$ (i).

Moreover, if $\sup_n(m_{n+1} - m_n) < \infty$, then (5.4) easily implies that the normalized monomials are equivalent to the unit vector basis of c_0. Hence we have (ii) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (ii) is trivial.

(iii) $\Rightarrow$ (iv): By the preceding we know already that (iii) implies (ii) and hence (5.4). If σ_n is the nth Cesàro mean and $h \in H_v^\infty$, then $\sigma_n h \in H_v^0$. We have $\sigma_n P_j = P_j \sigma_n$ for all n and j. Moreover, $\|\sigma_n h\|_v \le \|h\|_v$ and $\sup_n \|\sigma_n h\|_v = \|h\|_v$. This implies that (5.4) remains valid for all $h \in H_v^\infty$. This together with the fact that $\sup_n(m_{n+1} - m_n) < \infty$ shows that H_v^∞ is solid.

(iv) $\Rightarrow$ (iii): This implication follows from Theorem 5.2 in [1]. (iv) $\Rightarrow$ (v): If $g \in S(H_v^0)$, then by definition and (iii),

$$\lim_{n\to\infty} \hat{g}(n)\|z^n\|_v = 0,$$

which implies by (iii) that $g \in H_v^0$.

(v) $\Rightarrow$ (iv): If $g \in S(H_v^\infty)$, then by definition $\sigma_n g \in S(H_v^0) = H_v^0$ for all n. This implies that $g \in H_v^\infty$. ☐

(In [9], the second author showed that $v(r) = \exp(-(\log r)^2)$, $R = \infty$, satisfies (ii) (and hence all assertions) of Theorem 5.2.)

Note that in the preceding proof we do not use the fact that our functions are defined on $\mathbb{C}$. The arguments work just as well for weighted spaces of holomorphic functions over the unit disk $\mathbb{D}$. However, in this case $\lim_{n\to\infty} r_n = 1$ and this fact together with

$$4 < b^2 \leq \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}-m_n}$$

implies that $\sup_n(m_{n+1} - m_n) = \infty$ (in view of Lemma 5.1, which remains true over $\mathbb{D}$). This means that in the case of holomorphic functions over $\mathbb{D}$ the preceding theorem is empty (cf. Corollary 5.3 in [1]).

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[1]Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, E-46071 Valencia, Spain.
E-mail address: jbonet@mat.upv.es

[2]Institute of Mathematics, University of Paderborn, D-33098 Paderborn, Germany.
E-mail address: lusky@math.upb.de

[3]Department of Mathematics and Statistics, P.O. Box 68, University of Helsinki, 00014 Helsinki, Finland.
E-mail address: jari.taskinen@helsinki.fi