Banach J. Math. Anal. 13 (2019), no. 2, 359–385
https://doi.org/10.1215/17358787-2018-0042
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

COMPLETE SYSTEMS OF UNITARY INVARIANTS FOR SOME CLASSES OF 2-ISOMETRIES

AKASH ANAND,[1] SAMEER CHAVAN,[1] ZENON JAN JABŁOŃSKI,[2] and
JAN STOCHEL[2*]

Dedicated to the memory of Professor Ronald G. Douglas

Communicated by I. B. Jung

Abstract. We characterize the unitary equivalence of 2-isometric operators satisfying the so-called *kernel condition*. This relies on a model for such operators built on operator-valued unilateral weighted shifts and on a characterization of the unitary equivalence of operator-valued unilateral weighted shifts in a fairly general context. We also provide a complete system of unitary invariants for 2-isometric weighted shifts on rooted directed trees satisfying the kernel condition. This is formulated purely in the language of graph theory—namely, in terms of certain generation branching degrees. Finally, we study the membership of the Cauchy dual operators of 2-isometries in classes $C_{0\cdot}$ and $C_{\cdot 0}$.

1. Introduction

We begin by defining the basic concepts which will be discussed in this article. Let $\mathcal{H}$ be a (complex) Hilbert space, and let $\boldsymbol{B}(\mathcal{H})$ stand for the C^*-algebra of all bounded linear operators on $\mathcal{H}$. We say that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is

(i) *hyponormal* if $T^*T - TT^* \geq 0$,
(ii) *subnormal* if it has a normal extension in a possibly larger Hilbert space,
(iii) 2-*hyperexpansive* if $I - 2T^*T + T^{*2}T^2 \leq 0$,

Received Aug. 21, 2018; Accepted Nov. 29, 2018.

[*] Corresponding author.

2010 *Mathematics Subject Classification*. Primary 47B20; Secondary 47B49, 47B37.

Keywords. 2-isometry, kernel condition, complete system of unitary invariants, weighted shift on a directed tree, Cauchy dual operator, $C_{0\cdot}$ and $C_{\cdot 0}$ classes.

(iv) 2-*isometric* if $I - 2T^*T + T^{*2}T^2 = 0$.

Subnormal operators are hyponormal (see [15, Proposition II.4.2]) and 2-isometries are 2-hyperexpansive, but none of these implications can be reversed (see [15, Exercise 3, p. 50] and [23, Lemma 6.1], respectively). Moreover, hyponormal operators which are 2-hyperexpansive are isometric (see [23, Theorem 3.4]). The theory of subnormal and hyponormal operators was initiated by Halmos [18]. The notion of a 2-isometry was invented by Agler [1], while the concept of a 2-hyperexpansive operator goes back to Richter [32] (see also [4, Remark 2]). The *Cauchy dual operator* T' of a left-invertible operator T is defined by $T' = T(T^*T)^{-1}$. This concept is due to Shimorin [35]. The basic relationship between 2-hyperexpansions and hyponormal operators via the Cauchy dual transform is as follows (see [36, Section 5] and [11, Theorem 2.9]):

If $T \in \boldsymbol{B}(\mathcal{H})$ is a 2-hyperexpansive operator, then T is left-invertible and T' is a hyponormal contraction. (1.1)

In a recent paper [3], the present authors solved the Cauchy dual subnormality problem in the negative by showing that there are 2-isometric operators T whose Cauchy dual operators T' are not subnormal. One of the ideas behind the construction of such counterexamples relies on perturbing the so-called *kernel condition* in the context of weighted shifts on directed trees (see [22] for more information on this class of operators). Recall from [3] that $T \in \boldsymbol{B}(\mathcal{H})$ satisfies the kernel condition if

$$T^*T(\ker T^*) \subseteq \ker T^*. \tag{1.2}$$

It was proved in [3, Theorem 6.5] that if $\mathscr{T}$ is a rooted directed tree and $S_{\boldsymbol{\lambda}}$ is a 2-isometric weighted shift on $\mathscr{T}$ with nonzero weights which satisfies the perturbed kernel condition, then the Cauchy dual operator $S'_{\boldsymbol{\lambda}}$ of $S_{\boldsymbol{\lambda}}$ is subnormal if and only if $S_{\boldsymbol{\lambda}}$ satisfies the kernel condition. Further, it was shown in [3, Theorem 3.3] that the Cauchy dual operator T' of a 2-isometry T satisfying the kernel condition is always subnormal. This can in turn be derived from a model theorem for 2-isometries satisfying the kernel condition (see [3, Theorem 2.5]). The model itself is built on operator-valued unilateral weighted shifts and is the starting point of the present investigation. It is worth mentioning that there are Dirichlet-type models for cyclic analytic 2-isometries and for finitely multicyclic 2-isometries given by Richter [33, Theorem 5.1] and by Agler and Stankus [2, Theorems 3.49, 8.32], respectively. Richter used his model to characterize unitary equivalence of cyclic analytic 2-isometries (see [33, Theorem 5.2]; see also [2, Theorem 8.30] for the case of pure cyclic 2-isometries). As far as we know, there are no models for arbitrary 2-isometries.

This article is organized as follows. In Section 2, looking for a complete system of unitary invariants for 2-isometries satisfying the kernel condition, we first discuss the question of unitary equivalence of operator-valued unilateral weighted shifts in the general context. This class of operators was investigated by Lambert [26]. An essential progress in their study, also relevant for our present work, was done in [21]. As opposed to the previous approaches, ours does not require the operator weights to be even quasi-invertible. We only assume that they have dense range. We provide a characterization of unitary equivalence of such operators (see Theorem 2.3). Under some carefully chosen constraints, we obtain a characterization of their unitary equivalence, which resembles that for scalar weighted shifts (see Theorem 2.4; see also [34, Theorem 1] for the scalar case). We conclude this section by characterizing the unitary equivalence of orthogonal sums (of arbitrary cardinality) of injective unilateral weighted shifts (see Theorem 2.7). In passing, we draw the reader's attention to [5], where the so-called *block shifts* generalizing operator-valued unilateral weighted shifts were studied.

In Section 3, using the model for 2-isometries satisfying the kernel condition (see [3, Theorem 2.5]), we answer the question of when two such operators are unitarily equivalent (see Theorem 3.3, Lemma 1.1). We also answer the question of when a completely nonunitary 2-isometry satisfying the kernel condition is unitarily equivalent to an orthogonal sum of scalar unilateral weighted shifts (see Theorem 3.4). This enables us to show that each finitely multicyclic completely nonunitary 2-isometry satisfying the kernel condition is a finite orthogonal sum of weighted shifts (see Corollary 3.7). As a consequence, the adjoint of any such operator is in the Cowen–Douglas class (see [12, Corollary 3.7] for a more general result). We refer the reader to [16] for the definition of the Cowen–Douglas class.

In Section 4, we investigate 2-isometric weighted shifts on directed trees satisfying the condition (4.4), which in general is stronger than the kernel condition itself. However, they coincide in the case when the directed tree is leafless and the weights of the weighted shift under consideration are nonzero (see [3, Lemma 5.6]). Example 4.2 shows that the fact that a weighted shift on a rooted directed tree is completely nonunitary (see [3, Lemma 5.3(viii)]) is no longer true for weighted shifts on rootless directed trees even though they are isometric and nonunitary. Theorem 4.5 provides a model for 2-isometric weighted shifts on rooted directed trees that satisfy the condition (4.4). These operators are modeled by orthogonal sums of inflations of unilateral weighted shifts whose weights come from a single 2-isometric unilateral weighted shift. What is more, the additive exponent of the kth inflation that appears in the orthogonal decomposition (4.6) is equal to $\mathsf{j}_k^{\mathscr{T}}$, the kth generation branching degree of the underlying graph $\mathscr{T}$. This allows us to answer the question of when two such operators are unitarily equivalent by using $\mathsf{j}_k^{\mathscr{T}}$ (see Theorem 4.6). We end this section by showing that there are two unitarily equivalent 2-isometric weighted shifts on nongraph-isomorphic directed trees with nonzero weights which satisfy the kernel condition (see Example 4.8).

We conclude the article with an Appendix which contains related topics. Here we begin by explicitly calculating another unitary invariant: the strong operator topology limit $\mathsf{A}_{T'}$ of the sequence $\{T'^{*n}T'^{n}\}_{n=1}^{\infty}$ for two classes of 2-isometries T; namely, quasi-Brownian isometries and 2-isometries satisfying the kernel condition (see Lemma A.1). We next show that the Cauchy dual operator T' of a 2-isometry T is of class $C_{\cdot 0}$ if and only if T is completely nonunitary. Under the additional assumption that T satisfies the kernel condition, the Cauchy dual operator T' is of class $C_{0\cdot}$ if and only if $G(\{1\}) = 0$ or, equivalently, if and only if T is completely nonunitary and $E(\{1\}) = 0$, where G and E are the spectral measures of T^*T and the zeroth weight W_0 of the model operator W for T, respectively (see Theorem A.3). Note that nonisometric quasi-Brownian isometries do not satisfy the kernel condition (see [3, Corollary 4.6]) and their Cauchy dual operators are never of class $C_{0\cdot}$ (see Proposition A.5(i)).

Now we fix notation and terminology. Let $\mathbb{C}$ stand for the set of complex numbers. Denote by $\mathbb{N}$, $\mathbb{Z}_+$, and $\mathbb{R}_+$ the sets of positive integers, nonnegative integers, and nonnegative real numbers, respectively. Given a set X, we write $\operatorname{card} X$ for the cardinality of X and denote by $\chi_\varDelta$ the characteristic function of a subset $\varDelta$ of X. The σ-algebra of all Borel subsets of a topological space X is denoted by $\mathfrak{B}(X)$. In this article, Hilbert spaces are assumed to be complex and operators are assumed to be linear. Let $\mathcal{H}$ be a Hilbert space. As usual, we denote by $\dim \mathcal{H}$ the orthogonal dimension of $\mathcal{H}$. If $f \in \mathcal{H}$, then $\langle f \rangle$ stands for the linear span of the singleton of f. Given another Hilbert space $\mathcal{K}$, we denote by $\boldsymbol{B}(\mathcal{H}, \mathcal{K})$ the Banach space of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$. The kernel, range, and modulus of an operator $T \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ are denoted by $\ker T$, $\operatorname{ran} T$, and $|T|$, respectively. We abbreviate $\boldsymbol{B}(\mathcal{H}, \mathcal{H})$ to $\boldsymbol{B}(\mathcal{H})$ and regard $\boldsymbol{B}(\mathcal{H})$ as a C^*-algebra. Its unit, which is the identity operator on $\mathcal{H}$, is denoted here by $I_{\mathcal{H}}$, or simply by I if no ambiguity arises. We write $\sigma(T)$ for the spectrum of $T \in \boldsymbol{B}(\mathcal{H})$. Given $T \in \boldsymbol{B}(\mathcal{H})$ and a cardinal number $\mathfrak{n}$, we set $\mathcal{H}^{\oplus \mathfrak{n}} = \bigoplus_{j \in J} \mathcal{H}_j$ and $T^{\oplus \mathfrak{n}} = \bigoplus_{j \in J} T_j$ with $\mathcal{H}_j = \mathcal{H}$ and $T_j = T$ for all $j \in J$, where J is an index set of cardinality $\mathfrak{n}$. We call $\mathcal{H}^{\oplus \mathfrak{n}}$ and $T^{\oplus \mathfrak{n}}$ the $\mathfrak{n}$-*fold inflation* of $\mathcal{H}$ and T, respectively. We adhere to the convention that $\mathcal{H}^{\oplus 0} = \{0\}$ and $T^{\oplus 0} = 0$. If S and T are Hilbert space operators which are unitarily equivalent, then we write $S \cong T$.

We say that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is *completely nonunitary* (resp., *pure*) if there is no nonzero reducing closed vector subspace $\mathcal{L}$ of $\mathcal{H}$ such that the restriction $T|_{\mathcal{L}}$ of T to $\mathcal{L}$ is a unitary (resp., a normal) operator. Following [33], we call T *analytic* if $\bigcap_{n=1}^\infty T^n(\mathcal{H}) = \{0\}$. Note that any analytic operator is completely nonunitary. It is well known that any operator $T \in \boldsymbol{B}(\mathcal{H})$ has a unique orthogonal decomposition $T = N \oplus P$ such that N is a normal operator and P is a pure operator (see [29, Corollary 1.3]). We will refer to N and P as the *normal* and *pure* parts of T, respectively. The following fact can be deduced from [29, Corollary 1.3].

Lemma 1.1. *Operators $T_1 \in \boldsymbol{B}(\mathcal{H}_1)$ and $T_2 \in \boldsymbol{B}(\mathcal{H}_2)$ are unitarily equivalent if and only if their corresponding normal and pure parts are unitarily equivalent.*

2. Unitary equivalence of operator-valued unilateral weighted shifts

In this section, the question of the unitary equivalence of operator-valued unilateral weighted shifts is revisited. First, we give a necessary and sufficient condition for two such operators whose weights have dense range to be unitarily equivalent (see Theorem 2.3). This result generalizes in particular [26, Corollary 3.3] in which weights are assumed to be invertible. If weights are more regular, where the regularity does not refer to invertibility, then the characterization of unitary equivalence takes on a much simpler form (see Theorem 2.4, Corollary 2.5). As an application, we answer the question of when two orthogonal sums of uniformly bounded families of injective unilateral weighted shifts are unitarily equivalent (see Theorem 2.7). We begin by proving a criterion for the modulus of a finite product of bounded operators to be equal to the product of their moduli.

Lemma 2.1. *Let $n \geq 2$ be an integer. Suppose that $A_1, \ldots, A_n \in \boldsymbol{B}(\mathcal{H})$ are such that $|A_i|$ commutes with A_j whenever $i < j$. Then*

(i) *the operators $|A_1|, \ldots, |A_n|$ mutually commute,*
(ii) $|A_1 \cdots A_n|^2 = |A_1|^2 \cdots |A_n|^2$,
(iii) $|A_1 \cdots A_n| = |A_1| \cdots |A_n|$.

Proof. (i) Fix integers $i, j \in \{1, \ldots, n\}$ such that $i < j$. Since $|A_i|A_j = A_j|A_i|$, and thus $|A_i|A_j^* = A_j^*|A_i|$, we see that $|A_i||A_j|^2 = |A_j|^2|A_i|$. Hence $|A_i||A_j| = |A_j||A_i|$, which proves (i).

(ii) By our assumption and (i), we have

$$\begin{aligned} |A_1 \cdots A_n|^2 &= A_n^* \cdots A_2^*|A_1|^2 A_2 \cdots A_n \\ &= |A_1|^2 A_n^* \cdots A_3^*|A_2|^2 A_3 \cdots A_n \\ &\ \ \vdots \\ &= |A_1|^2 \cdots |A_n|^2. \end{aligned} \tag{2.1}$$

(iii) It follows from (2.1) and (i) that

$$|A_1 \cdots A_n|^2 = \big(|A_1| \cdots |A_n|\big)^2.$$

Applying the square root lemma (see [31, Theorem VI.9]) and the fact that the product of commuting positive bounded operators is positive, we conclude that (iii) holds. □

Let us recall the definition of an operator-valued unilateral weighted shift. Suppose that $\mathcal{M}$ is a *nonzero* Hilbert space. Denote by $\ell^2_{\mathcal{M}}$ the Hilbert space of all vector sequences $\{h_n\}_{n=0}^\infty \subseteq \mathcal{M}$ such that $\sum_{n=0}^\infty \|h_n\|^2 < \infty$ equipped with the standard inner product

$$\big\langle \{g_n\}_{n=0}^\infty, \{h_n\}_{n=0}^\infty \big\rangle = \sum_{n=0}^\infty \langle g_n, h_n\rangle, \quad \{g_n\}_{n=0}^\infty, \{h_n\}_{n=0}^\infty \in \ell^2_{\mathcal{M}}.$$

Let $\{W_n\}_{n=0}^\infty \subseteq \boldsymbol{B}(\mathcal{M})$ be a uniformly bounded sequence of operators. Then the operator $W \in \boldsymbol{B}(\ell^2_{\mathcal{M}})$ defined by

$$W(h_0, h_1, \ldots) = (0, W_0 h_0, W_1 h_1, \ldots), \quad (h_0, h_1, \ldots) \in \ell^2_{\mathcal{M}},$$

is called an *operator-valued unilateral weighted shift* with weights $\{W_n\}_{n=0}^\infty$. It is easy to verify that

$$W^*(h_0, h_1, \ldots) = (W_0^* h_1, W_1^* h_2, \ldots), \quad (h_0, h_1, \ldots) \in \ell^2_{\mathcal{M}}, \tag{2.2}$$

$$W^*W(h_0, h_1, \ldots) = (W_0^* W_0 h_0, W_1^* W_1 h_1, \ldots), \quad (h_0, h_1, \ldots) \in \ell^2_{\mathcal{M}}. \tag{2.3}$$

If each weight W_n of W is an invertible (resp., a positive) element of the C^*-algebra $\boldsymbol{B}(\mathcal{M})$, then we say that W is an operator-valued unilateral weighted shift with *invertible* (resp., *positive*) weights. Putting $\mathcal{M} = \mathbb{C}$, we arrive at the well-known notion of a unilateral weighted shift in $\ell^2_{\mathbb{C}} = \ell^2$.

From now on, we assume that $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are nonzero Hilbert spaces and that $W^{(1)} \in \boldsymbol{B}(\ell^2_{\mathcal{M}^{(1)}})$ and $W^{(2)} \in \boldsymbol{B}(\ell^2_{\mathcal{M}^{(2)}})$ are operator-valued unilateral weighted shifts with weights $\{W_n^{(1)}\}_{n=0}^\infty \subseteq \boldsymbol{B}(\mathcal{M}^{(1)})$ and $\{W_n^{(2)}\}_{n=0}^\infty \subseteq \boldsymbol{B}(\mathcal{M}^{(2)})$, respectively. Below, under the assumption that the weights of $W^{(1)}$ have dense range, we characterize bounded operators which intertwine $W^{(1)}$ and $W^{(2)}$ (see [26, Lemma 2.1] for the case of invertible weights).

Lemma 2.2. *Suppose that each operator $W_n^{(1)}$, $n \in \mathbb{Z}_+$, has dense range. Let $A \in \boldsymbol{B}(\ell^2_{\mathcal{M}^{(1)}}, \ell^2_{\mathcal{M}^{(2)}})$ be an operator with the matrix representation $[A_{i,j}]_{i,j=0}^\infty$, where $A_{i,j} \in \boldsymbol{B}(\mathcal{M}^{(1)}, \mathcal{M}^{(2)})$ for all $i, j \in \mathbb{Z}_+$. Then the following two conditions are equivalent:*

(i) *$AW^{(1)} = W^{(2)}A$,*

(ii) *A is lower triangular, that is, $A_{i,j} = 0$ whenever $i < j$, and*

$$A_{i,j}W_{j-1}^{(1)} \cdots W_0^{(1)} = W_{i-1}^{(2)} \cdots W_{i-j}^{(2)} A_{i-j,0}, \quad i \geq j \geq 1. \tag{2.4}$$

Proof. Denote by $\delta_{i,j}$ the Kronecker delta function. Since $W^{(k)}$ has the matrix representation $[\delta_{i,j+1}W_j^{(k)}]_{i,j=0}^\infty$ for $k = 1, 2$, we see that (i) holds if and only if $A_{i,j+1}W_j^{(1)} = W_{i-1}^{(2)}A_{i-1,j}$ for all $i, j \in \mathbb{Z}_+$ (with the convention that $W_{-1}^{(2)} = 0$ and $A_{-1,j} = 0$ for $j \in \mathbb{Z}_+$). Hence, (i) holds if and only if the following equations hold:

$$A_{0,j} = 0, \quad j \in \mathbb{N}, \tag{2.5}$$

$$A_{i+1,j+1}W_j^{(1)} = W_i^{(2)}A_{i,j}, \quad i, j \in \mathbb{Z}_+. \tag{2.6}$$

(i) ⇒ (ii) By induction, we infer from (2.6) that

$$A_{i+k,j+k}W_{j+k-1}^{(1)} \cdots W_j^{(1)} = W_{i+k-1}^{(2)} \cdots W_i^{(2)} A_{i,j}, \quad i, j \in \mathbb{Z}_+, k \in \mathbb{N}. \tag{2.7}$$

This and (2.5) combined with the assumption that each $W_n^{(1)}$ has dense range, imply that A is lower triangular. It is a matter of routine to show that (2.7) implies (2.4).

(ii) ⇒ (i) Since A is lower triangular and (2.4) holds, it remains to show that (2.6) is valid whenever $i \geq j \geq 1$. Applying (2.4) again, we get

$$\begin{aligned} A_{i+1,j+1}W_j^{(1)}(W_{j-1}^{(1)} \cdots W_0^{(1)}) &= W_i^{(2)}(W_{i-1}^{(2)} \cdots W_{i-j}^{(2)} A_{i-j,0}) \\ &= W_i^{(2)} A_{i,j}(W_{j-1}^{(1)} \cdots W_0^{(1)}). \end{aligned}$$

Since each operator $W_n^{(1)}$ has dense range, we conclude that $A_{i+1,j+1}W_j^{(1)} = W_i^{(2)}A_{i,j}$. This completes the proof. □

The question of when the operators $W^{(1)}$ and $W^{(2)}$ whose weights have dense range are unitarily equivalent is answered by the following theorem (see [26, Corollary 3.3] for the case of invertible weights).

Theorem 2.3. *Suppose that for any $k = 1, 2$ and every $n \in \mathbb{Z}_+$, the operator $W_n^{(k)}$ has dense range. Then the following two conditions are equivalent:*

(i) $W^{(1)} \cong W^{(2)}$,

(ii) *there exists a unitary isomorphism* $U_0 \in \boldsymbol{B}(\mathcal{M}^{(1)}, \mathcal{M}^{(2)})$ *such that*

$$|W^{(1)}_{[i]}| = U_0^*|W^{(2)}_{[i]}|U_0, \quad i \in \mathbb{N}, \tag{2.8}$$

where $W^{(k)}_{[i]} = W^{(k)}_{i-1} \cdots W^{(k)}_0$ *for* $i \in \mathbb{N}$ *and* $k = 1, 2$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $U \in \boldsymbol{B}(\ell^2_{\mathcal{M}^{(1)}}, \ell^2_{\mathcal{M}^{(2)}})$ is a unitary isomorphism such that $UW^{(1)} = W^{(2)}U$ and $[U_{i,j}]_{i,j=0}^\infty$ is the matrix representation of U, where $\{U_{i,j}\}_{i,j=0}^\infty \subseteq \boldsymbol{B}(\mathcal{M}^{(1)}, \mathcal{M}^{(2)})$. It follows from Lemma 2.2 that the operator U is lower triangular. Since $U^* = U^{-1}$ is a unitary isomorphism with the corresponding matrix representation $[(U_{j,i})^*]_{i,j=0}^\infty$ and $U^*W^{(2)} = W^{(1)}U^*$, we infer from Lemma 2.2 that U^* is lower triangular. In other words, $U_{i,j} = 0$ whenever $i \neq j$. Since U is a unitary isomorphism, we deduce that for any $i \in \mathbb{Z}_+$, $U_i := U_{i,i}$ is a unitary isomorphism. It follows from (2.4) that

$$U_i W^{(1)}_{[i]} = W^{(2)}_{[i]} U_0, \quad i \in \mathbb{N}.$$

This yields

$$|W^{(1)}_{[i]}|^2 = (W^{(1)}_{[i]})^* U_i^* U_i W^{(1)}_{[i]} = U_0^* |W^{(2)}_{[i]}|^2 U_0, \quad i \in \mathbb{N}.$$

Applying the square root lemma implies (2.8).

(ii) $\Rightarrow$ (i) In view of (2.8), we have

$$\|W^{(1)}_{[i]} f\| = \big\||W^{(1)}_{[i]}| f\big\| = \big\||W^{(2)}_{[i]}| U_0 f\big\| = \|W^{(2)}_{[i]} U_0 f\|, \quad f \in \mathcal{M}^{(1)}, i \in \mathbb{N}. \tag{2.9}$$

By our assumption, for any $k = 1, 2$ and every $i \in \mathbb{N}$, the operator $W^{(k)}_{[i]}$ has dense range. Hence, by (2.9), for every $i \in \mathbb{N}$, there exists a unique unitary isomorphism $U_i \in \boldsymbol{B}(\mathcal{M}^{(1)}, \mathcal{M}^{(2)})$ such that

$$U_i W^{(1)}_{[i]} = W^{(2)}_{[i]} U_0, \quad i \in \mathbb{N}.$$

Set $U = \bigoplus_{i=0}^\infty U_i$. Applying Lemma 2.2 to $A = U$, we get $UW^{(1)} = W^{(2)}U$, which completes the proof. □

Under additional assumptions on weights, the above characterization of unitary equivalence of $W^{(1)}$ and $W^{(2)}$ can be substantially simplified.

Theorem 2.4. *Suppose that for any* $k = 1, 2$ *and every* $n \in \mathbb{Z}_+$, $\ker W^{(1)}_n = \{0\}$, *the operator* $W^{(k)}_n$ *has dense range and* $|W^{(k)}_n|$ *commutes with* $W^{(k)}_m$ *whenever* $m < n$. *Then the following two conditions are equivalent:*

(i) $W^{(1)} \cong W^{(2)}$,

(ii) *there exists a unitary isomorphism* $U_0 \in \boldsymbol{B}(\mathcal{M}^{(1)}, \mathcal{M}^{(2)})$ *such that*

$$|W^{(1)}_n| = U_0^* |W^{(2)}_n| U_0, \quad n \in \mathbb{Z}_+. \tag{2.10}$$

Proof. (i) ⇒ (ii) It follows from Theorem 2.3 that there exists a unitary isomorphism $U_0 \in \boldsymbol{B}(\mathcal{M}^{(1)}, \mathcal{M}^{(2)})$ such that (2.8) holds. We will show that (2.10) is valid. The case of $n = 0$ follows directly from (2.8) with $i = 1$. Suppose now that $n \in \mathbb{N}$. Then, by Lemma 2.1 and (2.8), we have

$$\begin{aligned} |W_n^{(1)}||W_{[n]}^{(1)}| = |W_{[n+1]}^{(1)}| &= U_0^*|W_{[n+1]}^{(2)}|U_0 \\ &= U_0^*|W_n^{(2)}|U_0U_0^*|W_{[n]}^{(2)}|U_0 \\ &= U_0^*|W_n^{(2)}|U_0|W_{[n]}^{(1)}|. \end{aligned} \tag{2.11}$$

Since $W_{[n]}^{(1)}$ is injective, we deduce that the operator $|W_{[n]}^{(1)}|$ has dense range. Hence, by (2.11), $|W_n^{(1)}| = U_0^*|W_n^{(2)}|U_0$.

(ii) ⇒ (i) It follows from Lemma 2.1 that

$$|W_{[i]}^{(k)}| = |W_{i-1}^{(k)}| \cdots |W_0^{(k)}|, \quad i \in \mathbb{N}, k = 1, 2.$$

Hence, by (2.10) and Lemma 2.1, we have

$$|W_{[i]}^{(1)}| = \big(U_0^*|W_{i-1}^{(2)}|U_0\big) \cdots \big(U_0^*|W_0^{(2)}|U_0\big) = U_0^*|W_{[i]}^{(2)}|U_0, \quad i \in \mathbb{N}.$$

In view of Theorem 2.3, $W^{(1)} \cong W^{(2)}$. This completes the proof. □

Corollary 2.5. *Suppose that for $k = 1, 2$, $\{W_n^{(k)}\}_{n=0}^\infty$ are injective diagonal operators with respect to the same orthonormal basis of $\mathcal{M}^{(k)}$. Then $W^{(1)} \cong W^{(2)}$ if and only if Theorem 2.4(ii) is satisfied.*

Remark 2.6. First, it is easily verifiable that Theorem 2.4 remains true if instead of assuming that the operators $\{W_n^{(1)}\}_{n=0}^\infty$ are injective, we assume that the operators $\{W_n^{(2)}\}_{n=0}^\infty$ are injective. Second, the assumption that the operators $\{W_n^{(1)}\}_{n=0}^\infty$ are injective was used only in the proof of the implication (i) ⇒ (ii) of Theorem 2.4. Third, Theorem 2.4(ii) implies that the operators $\{W_n^{(1)}\}_{n=0}^\infty$ are injective if and only if the operators $\{W_n^{(2)}\}_{n=0}^\infty$ are injective.

We are now in a position to characterize the unitary equivalence of two orthogonal sums of uniformly bounded families of injective unilateral weighted shifts.

Theorem 2.7. *Suppose that for $k = 1, 2$, $\varOmega_k$ is a nonempty set and $\{S_\omega^{(k)}\}_{\omega \in \varOmega_k} \subseteq \boldsymbol{B}(\ell^2)$ is a uniformly bounded family of injective unilateral weighted shifts. Then the following two conditions are equivalent:*

(i) $\bigoplus_{\omega \in \varOmega_1} S_\omega^{(1)} \cong \bigoplus_{\omega \in \varOmega_2} S_\omega^{(2)}$,

(ii) *there exists a bijection $\varPhi \colon \varOmega_1 \to \varOmega_2$ such that $S_{\varPhi(\omega)}^{(2)} = S_\omega^{(1)}$ for all $\omega \in \varOmega_1$.*

Proof. (i) ⇒ (ii) For $k = 1, 2$, we denote by $\mathcal{H}^{(k)}$ the Hilbert space in which the orthogonal sum $T^{(k)} := \bigoplus_{\omega \in \varOmega_k} S_\omega^{(k)}$ acts, and we choose an orthonormal basis $\{e_{\omega,n}^{(k)}\}_{\omega \in \varOmega_k, n \in \mathbb{Z}_+}$ of $\mathcal{H}^{(k)}$ such that $T^{(k)} e_{\omega,n}^{(k)} = \lambda_{\omega,n}^{(k)} e_{\omega,n+1}^{(k)}$ for all $\omega \in \varOmega_k$ and $n \in \mathbb{Z}_+$, where $\lambda_{\omega,n}^{(k)}$ are nonzero complex numbers. Clearly, the space $\bigoplus_{n \in \mathbb{Z}_+} \langle e_{\omega,n}^{(k)} \rangle$ reduces $T^{(k)}$ to an operator which is unitarily equivalent to $S_\omega^{(k)}$ for all $w \in \varOmega_k$ and $k = 1, 2$.

Assume that $T^{(1)} \cong T^{(2)}$. First, we note that there is no loss of generality in assuming that $\varOmega_1 = \varOmega_2 =: \varOmega$ because, due to $(T^{(1)})^* \cong (T^{(2)})^*$, we have

$$\begin{aligned}\operatorname{card} \varOmega_1 &= \dim\Big(\bigoplus_{\omega\in\varOmega_1} \ker(S_\omega^{(1)})^*\Big)\\ &= \dim \ker(T^{(1)})^*\\ &= \dim \ker(T^{(2)})^* = \operatorname{card} \varOmega_2.\end{aligned}$$

In turn, by [34, Corollary 1], we can assume that $\lambda_{\omega,n}^{(k)} > 0$ for all $\omega \in \varOmega$, $n \in \mathbb{Z}_+$ and $k = 1, 2$. For $k = 1, 2$, we denote by $\mathcal{M}^{(k)}$ the orthogonal sum $\bigoplus_{\omega\in\varOmega} \langle e_{\omega,0}^{(k)}\rangle$ and we denote by $W^{(k)}$ the operator-valued unilateral weighted shift on $\ell^2_{\mathcal{M}^{(k)}}$ with weights $\{W_n^{(k)}\}_{n=0}^\infty \subseteq \boldsymbol{B}(\mathcal{M}^{(k)})$ uniquely determined by the following equations:

$$W_n^{(k)} e_{\omega,0}^{(k)} = \lambda_{\omega,n}^{(k)} e_{\omega,0}^{(k)}, \quad \omega \in \varOmega, n \in \mathbb{Z}_+, k = 1, 2.$$

($W^{(k)}$ is well-defined because $\|T^{(k)}\| = \sup_{n\in\mathbb{Z}_+} \sup_{\omega\in\varOmega} \lambda_{\omega,n}^{(k)} = \sup_{n\in\mathbb{Z}_+} \|W_n^{(k)}\|$.) We claim that $T^{(k)} \cong W^{(k)}$ for $k = 1, 2$. Indeed, for $k = 1, 2$, there exists a unique unitary isomorphism $V_k \in \boldsymbol{B}(\mathcal{H}^{(k)}, \ell^2_{\mathcal{M}^{(k)}})$ such that

$$V_k e_{\omega,n}^{(k)} = \big(\underset{\langle 0\rangle}{0}, \ldots, 0, \underset{\langle n\rangle}{e_{\omega,0}^{(k)}}, 0, \ldots\big), \quad \omega \in \varOmega, n \in \mathbb{Z}_+.$$

It is a matter of routine to show that $V_k T^{(k)} e_{\omega,n}^{(k)} = W^{(k)} V_k e_{\omega,n}^{(k)}$ for all $\omega \in \varOmega$, $n \in \mathbb{Z}_+$ and $k = 1, 2$. This implies the claimed unitary equivalence. As a consequence, we see that $W^{(1)} \cong W^{(2)}$. Hence, by Corollary 2.5, there exists a unitary isomorphism $U_0 \in \boldsymbol{B}(\mathcal{M}^{(1)}, \mathcal{M}^{(2)})$ such that

$$U_0 W_n^{(1)} = W_n^{(2)} U_0, \quad n \in \mathbb{Z}_+. \tag{2.12}$$

Given $k, l \in \{1, 2\}$ and $\omega_0 \in \varOmega$, we set

$$\varOmega_{\omega_0}^{(k,l)} = \{\omega \in \varOmega \colon \lambda_{\omega,n}^{(k)} = \lambda_{\omega_0,n}^{(l)} \ \forall n \in \mathbb{Z}_+\} = \{\omega \in \varOmega \colon S_\omega^{(k)} = S_{\omega_0}^{(l)}\}.$$

Our next goal is to show that

$$\operatorname{card} \varOmega_{\omega_0}^{(1,1)} = \operatorname{card} \varOmega_{\omega_0}^{(2,1)}, \quad \omega_0 \in \varOmega. \tag{2.13}$$

For this, fix $\omega_0 \in \varOmega$. It follows from the injectivity of U_0 that

$$\begin{aligned}U_0\Big(\bigcap_{n=0}^\infty \ker(\lambda_{\omega_0,n}^{(1)} I - W_n^{(1)})\Big) &= \bigcap_{n=0}^\infty U_0\big(\ker(\lambda_{\omega_0,n}^{(1)} I - W_n^{(1)})\big)\\ &\overset{(2.12)}{=} \bigcap_{n=0}^\infty \ker(\lambda_{\omega_0,n}^{(1)} I - W_n^{(2)}).\end{aligned} \tag{2.14}$$

Since

$$\ker(\lambda_{\omega_0,n}^{(1)} I - W_n^{(k)}) = \bigoplus_{\substack{\omega\in\varOmega\colon\\ \lambda_{\omega,n}^{(k)} = \lambda_{\omega_0,n}^{(1)}}} \langle e_{\omega,0}^{(k)}\rangle, \quad n \in \mathbb{Z}_+, k = 1, 2,$$

and consequently

$$\bigcap_{n=0}^{\infty} \ker(\lambda_{\omega_0,n}^{(1)} I - W_n^{(k)}) = \bigoplus_{\omega \in \varOmega_{\omega_0}^{(k,1)}} \langle e_{\omega,0}^{(k)} \rangle, \quad k = 1, 2,$$

we deduce that

$$\begin{aligned} \operatorname{card} \varOmega_{\omega_0}^{(1,1)} &= \dim \bigoplus_{\omega \in \varOmega_{\omega_0}^{(1,1)}} \langle e_{\omega,0}^{(1)} \rangle \\ &= \dim \bigcap_{n=0}^{\infty} \ker(\lambda_{\omega_0,n}^{(1)} I - W_n^{(1)}) \\ &\overset{(2.14)}{=} \dim \bigcap_{n=0}^{\infty} \ker(\lambda_{\omega_0,n}^{(1)} I - W_n^{(2)}) = \operatorname{card} \varOmega_{\omega_0}^{(2,1)}. \end{aligned}$$

Hence, the condition (2.13) holds. Since by (2.12), $U_0^* W_n^{(2)} = W_n^{(1)} U_0^*$ for all $n \in \mathbb{Z}_+$, we infer from (2.13) that

$$\operatorname{card} \varOmega_{\omega_0}^{(2,2)} = \operatorname{card} \varOmega_{\omega_0}^{(1,2)}, \quad \omega_0 \in \varOmega. \tag{2.15}$$

Using the equivalence relations $\mathcal{R}_k \subseteq \varOmega \times \varOmega$, $k = 1, 2$, defined by

$$\omega \mathcal{R}_k \omega' \iff S_\omega^{(k)} = S_{\omega'}^{(k)}, \quad \omega, \omega' \in \varOmega, k, l \in \{1, 2\},$$

and combining (2.13) with (2.15) we obtain (ii).

(ii) $\Rightarrow$ (i) This implication is obvious. □

3. Unitary equivalence of 2-isometries satisfying the kernel condition

In view of the well-known characterizations of the unitary equivalence of normal operators (see, e.g., [6, Chapter 7]), Lemma 1.1 reduces the question of unitary equivalence of 2-isometries satisfying the kernel condition to the consideration of pure operators in this class. By Theorem 3.2 below, a 2-isometry satisfying the kernel condition is pure if and only if it is unitarily equivalent to an operator-valued unilateral weighted shift W on $\ell^2_{\mathcal{M}}$ with weights $\{W_n\}_{n=0}^{\infty}$ defined by (3.2). Our first goal is to give necessary and sufficient conditions for two such operators to be unitarily equivalent (see Theorem 3.3). Next, we discuss the question of when a pure 2-isometry satisfying the kernel condition is unitarily equivalent to an orthogonal sum of unilateral weighted shifts (see Theorem 3.4). This enables us to answer the question of whether all finitely multicyclic pure 2-isometries satisfying the kernel condition are necessarily finite orthogonal sums of weighted shifts (see Corollary 3.7).

Before stating a model theorem for pure 2-isometries satisfying the kernel condition, we list some basic properties of the sequence $\{\xi_n\}_{n=0}^{\infty}$ of self-maps of the interval $[1, \infty)$ which are defined by

$$\xi_n(x) = \sqrt{\frac{1 + (n+1)(x^2 - 1)}{1 + n(x^2 - 1)}}, \quad x \in [1, \infty), n \in \mathbb{Z}_+. \tag{3.1}$$

Lemma 3.1. *Let $\{\xi_n\}_{n=0}^\infty$ be as in (3.1). Then the following hold:*

- (i) *ξ_0 is the identity map,*
- (ii) *$\xi_{m+n} = \xi_m \circ \xi_n$ for all $m, n \in \mathbb{Z}_+$,*
- (iii) *$\xi_n(1) = 1$ for all $n \in \mathbb{Z}_+$,*
- (iv) *$\xi_n(x) > \xi_{n+1}(x) > 1$ for all $x \in (1,\infty)$ and $n \in \mathbb{Z}_+$,*
- (v) *if $\{\zeta_n\}_{n=0}^\infty$ is a sequence of self-maps of $[1,\infty)$ such that ζ_0 is the identity map and $\zeta_{n+1} = \sqrt{\frac{2\zeta_n^2-1}{\zeta_n^2}}$ for all $n \in \mathbb{Z}_+$, then $\zeta_n = \xi_n$ for all $n \in \mathbb{Z}_+$.*

The following model theorem, which is a part of [3, Theorem 2.5], classifies (up to unitary equivalence) pure 2-isometries satisfying the kernel condition.

Theorem 3.2. *If $\mathcal{H} \neq \{0\}$ and $T \in \boldsymbol{B}(\mathcal{H})$, then the following are equivalent:*

- (i) *T is an analytic 2-isometry satisfying the kernel condition,*
- (ii) *T is a completely nonunitary 2-isometry satisfying the kernel condition,*
- (iii) *T is a pure 2-isometry satisfying the kernel condition,*
- (iv) *T is unitarily equivalent to an operator-valued unilateral weighted shift W on $\ell^2_{\mathcal{M}}$ with weights[1] $\{W_n\}_{n=0}^\infty$ given by*

$$\left.\begin{array}{l} W_n = \displaystyle\int_{[1,\infty)} \xi_n(x) E(dx), \quad n \in \mathbb{Z}_+, \\ \text{where } E \text{ is a compactly supported } \boldsymbol{B}(\mathcal{M})\text{-valued Borel spectral} \\ \text{measure on the interval } [1,\infty). \end{array}\right\} \tag{3.2}$$

Now we answer the question of when two pure 2-isometries satisfying the kernel condition are unitarily equivalent. We refer the reader to [22, Section 2.2] (resp., [6, Chapter 7]) for necessary information on the diagonal operators (resp., the spectral type and the multiplicity function of a selfadjoint operator, which is a complete system of its unitary invariants).

Theorem 3.3. *Suppose that $W \in \boldsymbol{B}(\ell^2_{\mathcal{M}})$ is an operator valued unilateral weighted shift with weights $\{W_n\}_{n=0}^\infty$ given by*

$$W_n = \int_{[1,\infty)} \xi_n(x) E(dx), \quad n \in \mathbb{Z}_+,$$

where $\{\xi_n\}_{n=0}^\infty$ are as in (3.1) and E is a compactly supported $\boldsymbol{B}(\mathcal{M})$-valued Borel spectral measure on $[1,\infty)$. Let $(\widetilde{W}, \widetilde{\mathcal{M}}, \{\widetilde{W}_n\}_{n=0}^\infty, \widetilde{E})$ be any other such system. Then the following conditions are equivalent:

- (i) *$W \cong \widetilde{W}$,*
- (ii) *$W_0 \cong \widetilde{W}_0$,*
- (iii) *the spectral types and the multiplicity functions of W_0 and $\widetilde{W}_0$ coincide,*
- (iv) *the spectral measures E and $\widetilde{E}$ are unitarily equivalent.*

Moreover, if the operators W_0 and $\widetilde{W}_0$ are diagonal, then (ii) holds if and only if

- (v) *$\dim\ker(\lambda I - W_0) = \dim\ker(\lambda I - \widetilde{W}_0)$ for all $\lambda \in \mathbb{C}$.*

---

[1] Note that the sequence $\{W_n\}_{n=0}^\infty \subseteq \boldsymbol{B}(\mathcal{M})$ defined by (3.2) is uniformly bounded, and consequently $W \in \boldsymbol{B}(\ell^2_{\mathcal{M}})$.

Proof. Since $\xi_0(x) = x$ for all $x \in [1,\infty)$, E and $\widetilde{E}$ are the spectral measures of W_0 and $\widetilde{W_0}$, respectively. Hence, the conditions (ii) and (iv) are equivalent. That (ii) and (iii) are equivalent follows from [6, Theorem 7.5.2]. Note that $\{W_n\}_{n=0}^\infty$ are commuting positive bounded operators such that $W_n \geq I$ for all $n \in \mathbb{Z}_+$. The same is true for $\{\widetilde{W}_n\}_{n=0}^\infty$. Therefore, W and $\widetilde{W}$ satisfy the assumptions of Theorem 2.4.

(i) ⇒ (ii) This is a direct consequence of Theorem 2.4.

(iv) ⇒ (i) If $UE = \widetilde{E}U$, where $U \in \boldsymbol{B}(\mathcal{M}, \widetilde{\mathcal{M}})$ is a unitary isomorphism, then by [6, Theorem 5.4.9] $UW_n = \widetilde{W}_n U$ for $n \in \mathbb{Z}_+$. Hence, by Theorem 2.4, $W \cong \widetilde{W}$.

It is a simple matter to show that if the operators W_0 and $\widetilde{W_0}$ are diagonal, then the conditions (ii) and (v) are equivalent. This completes the proof. □

It follows from Theorems 3.2 and 3.3 that the spectral type and the multiplicity function of the spectral measure of W_0 form a complete system of unitary invariants for completely nonunitary 2-isometries satisfying the kernel condition.

Theorem 3.4 below answers the question of when a completely nonunitary 2-isometry satisfying the kernel condition is unitarily equivalent to an orthogonal sum of unilateral weighted shifts. In the case when $\ell^2_{\mathcal{M}}$ is a separable Hilbert space, this result can in fact be deduced from [26, Theorem 3.9]. There are two reasons why we have decided to include the proof of Theorem 3.4. First, our result is stated for Hilbert spaces which are not assumed to be separable. Second, an essential part of the proof of Theorem 3.4 will be used later in the proof of Theorem 4.5.

Theorem 3.4. *Let $W \in \boldsymbol{B}(\ell^2_{\mathcal{M}})$ be an operator-valued unilateral weighted shift with weights $\{W_n\}_{n=0}^\infty$ given by*

$$W_n = \int_{[1,\infty)} \xi_n(x) E(dx), \quad n \in \mathbb{Z}_+, \tag{3.3}$$

where $\{\xi_n\}_{n=0}^\infty$ are as in (3.1) and E is a compactly supported $\boldsymbol{B}(\mathcal{M})$-valued Borel spectral measure on $[1,\infty)$. Then the following conditions are equivalent:

(i) *$W \cong \bigoplus_{j\in J} S_j$, where S_j are unilateral weighted shifts,*
(ii) *W_0 is a diagonal operator.*

Moreover, if (i) holds, then the index set J is of cardinality $\dim\ker W^*$.

Proof. (ii) ⇒ (i) Since W_0 is a diagonal operator and $W_0 \geq I$, there exist an orthonormal basis $\{e_j\}_{j\in J}$ of $\mathcal{M}$ and a system $\{\lambda_j\}_{j\in J} \subseteq [1,\infty)$ such that

$$W_0 e_j = \lambda_j e_j, \quad j \in J.$$

By (2.2), $\dim\ker W^* = \dim \mathcal{M} =$ the cardinality of J. Note that E, which is the spectral measure of W_0, is given by

$$E(\varDelta) f = \sum_{j\in J} \chi_{\varDelta}(\lambda_j) \langle f, e_j\rangle e_j, \quad f \in \mathcal{M}, \varDelta \in \mathfrak{B}\big([1,\infty)\big). \tag{3.4}$$

Let S_j be the unilateral weighted shift in ℓ^2 with weights $\{\xi_n(\lambda_j)\}_{n=0}^\infty$. By [23, Lemma 6.1, Proposition 6.2], S_j is a 2-isometry such that $\|S_j\| = \lambda_j$ for every

$j \in J$. Since $\sup_{j\in J} \lambda_j < \infty$, we see that $\bigoplus_{j\in J} S_j \in \boldsymbol{B}((\ell^2)^{\oplus \mathfrak{n}})$, where $\mathfrak{n}$ is the cardinal number of J. Define the operator $V\colon \ell^2_{\mathcal{M}} \to (\ell^2)^{\oplus\mathfrak{n}}$ by

$$\big(V(h_0, h_1, \ldots)\big)_j = \big(\langle h_0, e_j\rangle, \langle h_1, e_j\rangle, \ldots\big), \quad j \in J, (h_0, h_1, \ldots) \in \ell^2_{\mathcal{M}}.$$

Since for every $(h_0, h_1, \ldots) \in \ell^2_{\mathcal{M}}$,

$$\sum_{j\in J}\sum_{n=0}^{\infty}\big|\langle h_n, e_j\rangle\big|^2 = \sum_{n=0}^{\infty}\sum_{j\in J}\big|\langle h_n, e_j\rangle\big|^2 = \sum_{n=0}^{\infty}\|h_n\|^2 = \big\|(h_0, h_1, \ldots)\big\|^2,$$

the operator V is an isometry. Note that for all $j, k \in J$ and $m \in \mathbb{Z}_+$,

$$\big(V(\underset{\langle 0\rangle}{0}, \ldots, 0, \underset{\langle m\rangle}{e_k}, 0, \ldots)\big)_j = \begin{cases} (0, 0, \ldots) & \text{if } j \neq k, \\ (\underset{\langle 0\rangle}{0}, \ldots, 0, \underset{\langle m\rangle}{1}, 0, \ldots) & \text{if } j = k, \end{cases}$$

which means that the range of V is dense in $(\ell^2)^{\oplus\mathfrak{n}}$. Thus V is a unitary isomorphism. It follows from (3.3) that

$$W_n e_j = \int_{[1,\infty)} \xi_n(x) E(dx) e_j \overset{(3.4)}{=} \xi_n(\lambda_j) e_j, \quad j \in J, n \in \mathbb{Z}_+. \tag{3.5}$$

This implies that

$$\begin{aligned} VW(h_0, h_1, \ldots) &= \big\{\big(0, \langle W_0 h_0, e_j\rangle, \langle W_1 h_1, e_j\rangle, \ldots\big)\big\}_{j\in J} \\ &\overset{(3.5)}{=} \big\{\big(0, \xi_0(\lambda_j)\langle h_0, e_j\rangle, \xi_1(\lambda_j)\langle h_1, e_j\rangle, \ldots\big)\big\}_{j\in J} \\ &= \big\{S_j\big(V(h_0, h_1, \ldots)\big)_j\big\}_{j\in J} \\ &= \Big(\bigoplus_{j\in J} S_j\Big) V(h_0, h_1, \ldots), \quad (h_0, h_1, \ldots) \in \ell^2_{\mathcal{M}}. \end{aligned}$$

(i) ⇒ (ii) Suppose that $W \cong \bigoplus_{j\in J} S_j$, where S_j are unilateral weighted shifts. Since W is a 2-isometry, so is S_j for every $j \in J$. Hence S_j is injective for every $j \in J$. As a consequence, there is no loss of generality in assuming that the weights of S_j are positive (see [34, Corollary 1]). By [23, Lemma 6.1(ii)], for every $j \in J$ there exists $\lambda_j \in [1, \infty)$ such that $\{\xi_n(\lambda_j)\}_{n=0}^{\infty}$ are weights of S_j. Let $\widetilde{\mathcal{M}}$ be a Hilbert space such that $\dim \widetilde{\mathcal{M}}$ = the cardinality of J, let $\{\tilde{e}_j\}_{j\in J}$ be an orthonormal basis of $\widetilde{\mathcal{M}}$, and let $\widetilde{E}$ be a $\boldsymbol{B}(\widetilde{\mathcal{M}})$-valued Borel spectral measure on $[1, \infty)$ given by

$$\widetilde{E}(\varDelta) f = \sum_{j\in J} \chi_\varDelta(\lambda_j)\langle f, \tilde{e}_j\rangle \tilde{e}_j, \quad f \in \widetilde{\mathcal{M}}, \varDelta \in \mathfrak{B}\big([1, \infty)\big).$$

Since by [23, Proposition 6.2], $\sup_{j\in J} \lambda_j = \sup_{j\in J} \|S_j\| < \infty$, the spectral measure $\widetilde{E}$ is compactly supported in $[1, \infty)$. Define the sequence $\{\widetilde{W}_n\}_{n=0}^{\infty} \subseteq \boldsymbol{B}(\widetilde{\mathcal{M}})$ by

$$\widetilde{W}_n = \int_{[1,\infty)} \xi_n(x) \widetilde{E}(dx), \quad n \geq 0.$$

Note that the sequence $\{\widetilde{W}_n\}_{n=0}^{\infty}$ is uniformly bounded (see footnote 1). Clearly, $\widetilde{W}_0 \tilde{e}_j = \lambda_j \tilde{e}_j$ for all $j \in J$, which means that $\widetilde{W}_0$ is a diagonal operator. Denote by $\widetilde{W}$ the operator-valued unilateral weighted shift on $\ell^2_{\widetilde{\mathcal{M}}}$ with weights $\{\widetilde{W}_n\}_{n=0}^\infty$. It follows from the proof of the implication (ii) $\Rightarrow$ (i) that $\widetilde{W} \cong \bigoplus_{j\in J} S_j$. Hence $W \cong \widetilde{W}$. By Theorem 3.3, W_0 is a diagonal operator. □

Remark 3.5. Regarding Theorem 3.4, it is worth noting that if $\dim\ker W^* \leq \aleph_0$ and W_0 is diagonal, then W can be modeled by a weighted composition operator on an L^2-space over a σ-finite measure space (use [10, Section 2.3(g)] and an appropriately adapted version of [9, Corollary C.2]).

Recall that for a given operator $T \in B(\mathcal{H})$, the smallest cardinal number $\mathfrak{n}$ for which there exists a closed vector subspace $\mathcal{N}$ of $\mathcal{H}$ such that $\dim\mathcal{N} = \mathfrak{n}$ and $\mathcal{H} = \bigvee_{n=0}^\infty T^n(\mathcal{N})$ is called the *order of multicyclicity* of T. If the order of multicyclicity of T is finite, then T is called *finitely multicyclic*. As shown in Lemma 3.6 below, the order of multicyclicity of a completely nonunitary 2-isometry can be calculated explicitly (in fact, the proof of Lemma 3.6 contains more information). Lemma 3.6(i) appeared in [20, Proposition 1(i)] with a slightly different definition of the order of multicyclicity and a different proof. Lemma 3.6(ii) is covered by [11, Lemma 2.19(b)] in the case of finite multicyclicity. In fact, the proof of Lemma 3.6(ii), which is given below, works for analytic operators having Wold-type decomposition in the sense of Shimorin [35].

Lemma 3.6. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be an operator. Then*

(i) *the order of multicyclicity of T is greater than or equal to* $\dim\ker T^*$,
(ii) *if T is a completely nonunitary 2-isometry, then the order of multicyclicity of T is equal to* $\dim\ker T^*$.

Proof. (i) Let $\mathcal{N}$ be a closed vector subspace of $\mathcal{H}$ such that $\mathcal{H} = \bigvee_{n=0}^\infty T^n(\mathcal{N})$, and let $P \in \boldsymbol{B}(\mathcal{H})$ be the orthogonal projection of $\mathcal{H}$ onto $\ker T^*$. Clearly, $\ker T^* \perp T^n(\mathcal{H})$ for all $n \in \mathbb{N}$. If $f \in \ker T^* \ominus \overline{P(\mathcal{N})}$, then

$$\langle f, T^0 h\rangle = \langle f, Ph\rangle = 0, \quad h \in \mathcal{N},$$

which together with the previous statement yields $f \in (\bigvee_{n=0}^\infty T^n(\mathcal{N}))^\perp = \{0\}$. Hence $\overline{P(\mathcal{N})} = \ker T^*$. As a consequence, the operator $P|_{\mathcal{N}}\colon \mathcal{N} \to \ker T^*$ has dense range, which implies that $\dim\ker T^* \leq \dim\mathcal{N}$ (see [19, Problem 56]). This gives (i).

(ii) Since, by [35, Theorem 3.6], $\mathcal{H} = \bigvee_{n=0}^\infty T^n(\ker T^*)$, we see that the order of multicyclicity of T is less than or equal to $\dim\ker T^*$. This combined with (i) completes the proof. □

The following result generalizes the remarkable fact that a finitely multicyclic completely nonunitary isometry is unitarily equivalent to an orthogonal sum of finitely many unilateral unweighted shifts (see [25, Proposition 2.4]).

Corollary 3.7. *A finitely multicyclic completely nonunitary 2-isometry T satisfying the kernel condition is unitarily equivalent to an orthogonal sum of $\mathfrak{n}$ unilateral weighted shifts, where $\mathfrak{n}$ equals the order of multicyclicity of T. Moreover, for each cardinal number $\mathfrak{n} \geq \aleph_0$ there exists a completely nonunitary 2-isometry* *satisfying the kernel condition, whose order of multicyclicity equals* $\mathfrak{n}$ *and which is not unitarily equivalent to any orthogonal sum of unilateral weighted shifts.*

Proof. Apply Theorem 3.4, Lemma 3.6, and the fact that positive operators in finite-dimensional Hilbert spaces are diagonal but in infinite-dimensional are not necessarily diagonal. ☐

4. Unitary equivalence of 2-isometric weighted shifts on directed trees satisfying the kernel condition

This section provides a model for a 2-isometric weighted shift $S_{\boldsymbol\lambda}$ on a rooted directed tree $\mathscr{T}$ which satisfies (4.4) (see Theorem 4.5). Although the kernel condition is weaker than (4.4), both are equivalent if $\mathscr{T}$ is leafless and the weights of $S_{\boldsymbol\lambda}$ are nonzero. The aforesaid model enables us to classify (up to unitary equivalence) 2-isometric weighted shifts on rooted directed trees satisfying (4.4) in terms of the kth generation branching degree (see Theorem 4.6).

We begin with necessary information on weighted shifts on directed trees. (The reader is referred to [22] for more details on this subject; see also [7], [24], and [28] for very recent developments.) Let $\mathscr{T} = (V, E)$ be a directed tree (if not stated otherwise, V and E stand for the sets of vertices and edges of $\mathscr{T}$, respectively). If $\mathscr{T}$ has a root, then we denote it by ω. We set $V^\circ = V$ if $\mathscr{T}$ is rootless and $V^\circ = V \setminus \{\omega\}$ otherwise. We say that $\mathscr{T}$ is *leafless* if $V = V'$, where $V' := \{u \in V\colon \mathsf{Chi}(u) \neq \emptyset\}$. Given $W \subseteq V$ and $n \in \mathbb{Z}_+$, we set $\mathsf{Chi}^{\langle n\rangle}(W) = W$ if $n = 0$ and $\mathsf{Chi}^{\langle n\rangle}(W) = \mathsf{Chi}(\mathsf{Chi}^{\langle n-1\rangle}(W))$ if $n \geq 1$, where $\mathsf{Chi}(W) = \bigcup_{u\in W}\{v \in V\colon (u, v) \in E\}$. We put $\mathsf{Des}(W) = \bigcup_{n=0}^{\infty} \mathsf{Chi}^{\langle n\rangle}(W)$. Given $v \in V$, we write $\mathsf{Chi}(v) = \mathsf{Chi}(\{v\})$ and $\mathsf{Chi}^{\langle n\rangle}(v) = \mathsf{Chi}^{\langle n\rangle}(\{v\})$. For $v \in V^\circ$, a unique $u \in V$ such that $(u, v) \in E$ is said to be the *parent* of v; we denote it by $\mathsf{par}(v)$. The cardinality of $\mathsf{Chi}(v)$ is called the *degree* of a vertex $v \in V$ and denoted by $\deg v$. Recall that if $\mathscr{T}$ is rooted, then by [22, Corollary 2.1.5], we have

$$V = \bigsqcup_{n=0}^{\infty} \mathsf{Chi}^{\langle n\rangle}(\omega) \quad \text{(the disjoint union).} \tag{4.1}$$

Following [22, p. 67], we define the directed tree $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ by

$$\left.\begin{aligned} V_{\eta,\kappa} &= \{-k\colon k \in J_\kappa\} \cup \{0\} \cup \big\{(i, j)\colon i \in J_\eta, j \in J_\infty\big\}, \\ E_{\eta,\kappa} &= E_\kappa \cup \big\{\big(0, (i, 1)\big)\colon i \in J_\eta\big\} \cup \big\{\big((i, j), (i, j + 1)\big)\colon i \in J_\eta, j \in J_\infty\big\}, \\ E_\kappa &= \big\{(-k, -k + 1)\colon k \in J_\kappa\big\}, \end{aligned}\right\} \tag{4.2}$$

where $\eta \in \{2, 3, 4, \ldots\} \cup \{\infty\}$, $\kappa \in \mathbb{Z}_+ \cup \{\infty\}$ and $J_\iota = \{k \in \mathbb{Z}_+\colon 1 \leq k \leq \iota\}$ for $\iota \in \mathbb{Z}_+ \sqcup \{\infty\}$. The directed tree $\mathscr{T}_{\eta,\kappa}$ is leafless—it has only one branching vertex 0 and $\deg 0 = \eta$. Moreover, it is rooted if $\kappa < \infty$ and rootless if $\kappa = \infty$.

Let $\mathscr{T} = (V, E)$ be a directed tree. In what follows, $\ell^2(V)$ stands for the Hilbert space of square summable complex functions on V equipped with the standard inner product. If W is a nonempty subset of V, then we regard the Hilbert space $\ell^2(W)$ as a closed vector subspace of $\ell^2(V)$ by identifying each $f \in \ell^2(W)$ with the function $\widetilde{f} \in \ell^2(V)$ which extends f and vanishes on the set $V \setminus W$. Note that the set $\{e_u\}_{u\in V}$, where $e_u \in \ell^2(V)$ is the characteristic function of $\{u\}$, is an orthonormal basis of $\ell^2(V)$. Given a system $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$ of complex numbers, we define the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$, called a *weighted shift on $\mathscr{T}$ with weights $\boldsymbol{\lambda}$* (or simply a *weighted shift on $\mathscr{T}$*), as

$$\begin{aligned}\mathscr{D}(S_{\boldsymbol{\lambda}}) &= \big\{f \in \ell^2(V)\colon \varLambda_{\mathscr{T}} f \in \ell^2(V)\big\},\\ S_{\boldsymbol{\lambda}} f &= \varLambda_{\mathscr{T}} f, \quad f \in \mathscr{D}(S_{\boldsymbol{\lambda}}),\end{aligned}$$

where $\mathscr{D}(S_{\boldsymbol{\lambda}})$ stands for the *domain* of $S_{\boldsymbol{\lambda}}$, and $\varLambda_{\mathscr{T}}$ is the mapping defined on complex functions f on V by

$$(\varLambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\mathsf{par}(v)) & \text{if } v \in V^\circ,\\ 0 & \text{if } v \text{ is a root of } \mathscr{T}.\end{cases}$$

Now we collect some properties of weighted shifts on directed trees that are needed in this article (see [22, Propositions 3.1.3, 3.1.8, 3.4.3, 3.5.1]). From now on, we adopt the convention that $\sum_{v\in\emptyset} x_v = 0$.

Lemma 4.1. *Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$. Then*

(i) *e_u is in $\mathcal{D}(S_{\boldsymbol{\lambda}})$ if and only if $\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 < \infty$; if $e_u \in \mathscr{D}(S_{\boldsymbol{\lambda}})$, then $S_{\boldsymbol{\lambda}} e_u = \sum_{v\in\mathsf{Chi}(u)} \lambda_v e_v$ and $\|S_{\boldsymbol{\lambda}} e_u\|^2 = \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2$,*

(ii) *$S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ if and only if $\sup_{u\in V} \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 < \infty$; if this is the case, then $\|S_{\boldsymbol{\lambda}}\|^2 = \sup_{u\in V} \|S_{\boldsymbol{\lambda}} e_u\|^2 = \sup_{u\in V} \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2$.*

Moreover, if $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$, then

(iii)

$$\ker S_{\boldsymbol{\lambda}}^* = \begin{cases} \langle e_\omega\rangle \oplus \bigoplus_{u\in V'} (\ell^2(\mathsf{Chi}(u)) \ominus \langle \boldsymbol{\lambda}^u\rangle) & \textit{if } \mathscr{T} \textit{ is rooted},\\ \bigoplus_{u\in V'} (\ell^2(\mathsf{Chi}(u)) \ominus \langle \boldsymbol{\lambda}^u\rangle) & \textit{otherwise},\end{cases}$$

where $\boldsymbol{\lambda}^u \in \ell^2(\mathsf{Chi}(u))$ is given by $\boldsymbol{\lambda}^u\colon \mathsf{Chi}(u) \ni v \to \lambda_v \in \mathbb{C}$,

(iv) *$|S_{\boldsymbol{\lambda}}| e_u = \|S_{\boldsymbol{\lambda}} e_u\| e_u$ for all $u \in V$.*

According to [3, Lemma 5.3(viii)], bounded weighted shifts on rooted directed trees are completely nonunitary. As shown in Example 4.2 below, this is no longer true for bounded weighted shifts on rootless directed trees even though they are isometric and nonunitary (note that, by [23, Proposition 3.5], 2-isometric bilateral weighted shifts are always unitary).

Example 4.2. Let us consider any isometric weighted shift $S_{\boldsymbol{\lambda}}$ on the directed tree $\mathscr{T}_{\eta,\infty}$ (see (4.2)) with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V_{\eta,\infty}}$, where $\eta \in \{2,3,4,\ldots\} \cup \{\infty\}$ is fixed. This means that $\sum_{i=1}^{\eta} |\lambda_{i,1}|^2 = 1$ and $|\lambda_{i,j}| = |\lambda_{-k}| = 1$ for all $i \in J_\eta$, $j \in J_\infty \setminus \{1\}$ and $k \in \mathbb{Z}_+$. We will show that $S_{\boldsymbol{\lambda}}$ is nonunitary but is not completely nonunitary. For this, by Wold's decomposition theorem (see [37, Theorem 1.1]), it suffices to prove that $\ker S_{\boldsymbol{\lambda}}^* \neq \{0\}$ and $\bigoplus_{n=0}^{\infty} S_{\boldsymbol{\lambda}}^n(\ker S_{\boldsymbol{\lambda}}^*) \neq \ell^2(V_{\eta,\infty})$. In view of Lemma 4.1(iii), we have

$$\ker S_{\boldsymbol{\lambda}}^* = \bigoplus_{v\in V_{\eta,\infty}} \big(\ell^2\big(\mathsf{Chi}(v)\big) \ominus \langle \boldsymbol{\lambda}^v\rangle\big). \tag{4.3}$$

Since $\eta \geq 2$ and $\boldsymbol{\lambda}^v \neq 0$ for all $v \in V_{\eta,\infty}$, we deduce that the only nonzero term in the orthogonal decomposition (4.3) is $\ell^2(\mathsf{Chi}(0)) \ominus \langle \boldsymbol{\lambda}^0 \rangle$. Hence $\ker S_{\boldsymbol{\lambda}}^* \neq \{0\}$ and

$$\bigoplus_{n=0}^{\infty} S_{\boldsymbol{\lambda}}^n(\ker S_{\boldsymbol{\lambda}}^*) \subseteq \chi_{\Omega} \cdot \ell^2(V_{\eta,\infty}) \neq \ell^2(V_{\eta,\infty}),$$

where $\Omega = \bigcup_{n=1}^{\infty} \mathsf{Chi}^{\langle n \rangle}(0)$. This proves our claim.

Remark 4.3. By [3, Remark 5.8, Proposition 5.11], a 2-isometric weighted shift on a rootless directed tree with nonzero weights which satisfies the kernel condition is isometric. Further, if $S_{\boldsymbol{\lambda}}$ is an isometric weighted shift on a rootless directed tree, then by Wold's decomposition theorem, it is (up to unitary equivalence) an orthogonal sum $W \oplus S^{\oplus \mathfrak{n}}$, where W is a unitary operator, S is the isometric unilateral shift of multiplicity 1, and $\mathfrak{n} = \dim \ker S_{\boldsymbol{\lambda}}^*$. In particular, the isometry $S_{\boldsymbol{\lambda}}$ in Example 4.2 is equal to $U \oplus S^{\oplus(\eta-1)}$, where U is the unitary bilateral shift of multiplicity 1.

Recall that a weighted shift $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ on a leafless directed tree $\mathscr{T}$ with nonzero weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$ satisfies the kernel condition if and only if there exists a family $\{\alpha_v\}_{v \in V} \subseteq \mathbb{R}_+$ such that

$$\|S_{\boldsymbol{\lambda}} e_u\| = \alpha_{\mathsf{par}(u)}, \quad u \in V^\circ. \tag{4.4}$$

In general, (4.4) is stronger than the kernel condition (see [3, Lemma 5.6]). In view of [3, Remark 5.8, Proposition 5.10], if $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ is a 2-isometric weighted shift on a rooted directed tree $\mathscr{T}$ with nonzero weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$ which satisfies the kernel condition, then $\mathscr{T}$ is leafless, $\|S_{\boldsymbol{\lambda}} e_v\| =$ constant on $\mathsf{Chi}^{\langle n \rangle}(\omega)$ for every $n \in \mathbb{Z}_+$, and the corresponding sequence of constants forms a sequence of positive weights of a 2-isometric unilateral weighted shift (see [23, Lemma 6.1(ii)]). This suggests the following method of constructing such $S_{\boldsymbol{\lambda}}$'s.

Procedure 4.4. Let $\mathscr{T}$ be a rooted and leafless directed tree. Take a sequence $\{\beta_n\}_{n=0}^{\infty}$ of positive weights of a 2-isometric unilateral weighted shift. By [23, Lemma 6.1(ii)], there exists $x \in [1, \infty)$ such that $\beta_n = \xi_n(x)$ for all $n \in \mathbb{Z}_+$ (the converse statement is true as well). Then, using (4.1) and the following equation (see [8, (2.2.6)])

$$\mathsf{Chi}^{\langle n+1 \rangle}(\omega) = \bigsqcup_{u \in \mathsf{Chi}^{\langle n \rangle}(\omega)} \mathsf{Chi}(u), \quad n \in \mathbb{Z}_+,$$

we can define inductively for every $n \in \mathbb{Z}_+$ the system $\{\lambda_v\}_{v \in \mathsf{Chi}^{\langle n+1 \rangle}(\omega)}$ of complex numbers (not necessarily nonzero) such that $\sum_{w \in \mathsf{Chi}(u)} |\lambda_w|^2 = \beta_n^2$ for all $u \in \mathsf{Chi}^{\langle n \rangle}(\omega)$. Let $S_{\boldsymbol{\lambda}}$ be the weighted shift on $\mathscr{T}$ with the so-constructed weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$. Clearly, in view of Lemma 4.1(i), we have

$$x = \beta_0 = \|S_{\boldsymbol{\lambda}} e_\omega\|.$$

Since the sequence $\{\xi_n(t)\}_{n=0}^{\infty}$ is monotonically decreasing for every $t \in [1, \infty)$ (see Lemma 3.1), we infer from (4.1) and Lemma 4.1(ii) that $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ and $\beta_0 = \|S_{\boldsymbol{\lambda}}\|$. By [3, Proposition 5.10], $S_{\boldsymbol{\lambda}}$ is a 2-isometric weighted shift on $\mathscr{T}$

which satisfies (4.4) for some $\{\alpha_v\}_{v\in V} \subseteq \mathbb{R}_+$. Hence, according to [3, Lemma 5.6], $S_{\boldsymbol{\lambda}}$ satisfies the kernel condition.

We will show in Theorem 4.5 below that a 2-isometric weighted shift on a rooted directed tree which satisfies (4.4) is unitarily equivalent to an orthogonal sum of 2-isometric unilateral weighted shifts with positive weights; the orthogonal sum always contains a *basic* 2-isometric unilateral weighted shift with weights $\{\xi_n(x)\}_{n=0}^\infty$ for some $x \in [1,\infty)$ and a number of inflations of 2-isometric unilateral weighted shifts with weights $\{\xi_n(x)\}_{n=k}^\infty$, where k varies over a (possibly empty) subset of $\mathbb{N}$ (see Remark 4.7).

For $x \in [1,\infty)$, we denote by $S_{[x]}$ the unilateral weighted shift in ℓ^2 with weights $\{\xi_n(x)\}_{n=0}^\infty$, where $\{\xi_n\}_{n=0}^\infty$ is as in (3.1). Given a leafless directed tree $\mathscr{T}$ and $k \in \mathbb{N}$, we define the *kth generation branching degree* $\mathfrak{j}_k^{\mathscr{T}}$ of $\mathscr{T}$ by

$$\mathfrak{j}_k^{\mathscr{T}} = \sum_{u\in\mathsf{Chi}^{\langle k-1\rangle}(\omega)} (\deg u - 1), \quad k \in \mathbb{N}. \tag{4.5}$$

Let us note that the proof of Theorem 4.5(i) relies on the technique involved in the proof of the implication (iii) ⇒ (v) of [3, Theorem 2.5].

Theorem 4.5. *The following two statements hold.*

(i) *Let $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ be a 2-isometric weighted shift on a rooted directed tree $\mathscr{T}$ satisfying (4.4) for some $\{\alpha_v\}_{v\in V} \subseteq \mathbb{R}_+$. Then $\mathscr{T}$ is leafless and $S_{\boldsymbol{\lambda}}$ is unitarily equivalent to the orthogonal sum*

$$S_{[x]} \oplus \bigoplus_{k=1}^{\infty} \big(S_{[\xi_k(x)]}\big)^{\oplus j_k}, \tag{4.6}$$

where $x = \|S_{\boldsymbol{\lambda}} e_\omega\|$ and $j_k = \mathfrak{j}_k^{\mathscr{T}}$ for all $k \in \mathbb{N}$. Moreover, if the weights of $S_{\boldsymbol{\lambda}}$ are nonzero, then $j_k \leq \aleph_0$ for all $k \in \mathbb{N}$.

(ii) *For any $x \in [1,\infty)$ and any sequence of cardinal numbers $\{j_k\}_{k=1}^\infty$, the orthogonal sum (4.6) is unitarily equivalent to a 2-isometric weighted shift $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ on a rooted directed tree $\mathscr{T}$ satisfying (4.4) for some $\{\alpha_v\}_{v\in V} \subseteq \mathbb{R}_+$ such that $x = \|S_{\boldsymbol{\lambda}} e_\omega\|$. Moreover, if $j_k \leq \aleph_0$ for all $k \in \mathbb{N}$, then the weights of $S_{\boldsymbol{\lambda}}$ can be chosen to be positive.*

Proof. (i) First, observe that by [3, Lemma 5.7], $\mathscr{T}$ is leafless. To prove the unitary equivalence part, we show that $S_{\boldsymbol{\lambda}}$ is unitarily equivalent to an operator-valued unilateral weighted shift $\widetilde{W}$ on $\ell^2_{\ker S_{\boldsymbol{\lambda}}^*}$ with weights $\{\widetilde{W}_n\}_{n=0}^\infty$ satisfying the assumptions of Theorem 3.4 and the fact that $\widetilde{W}_0$ is a diagonal operator.

It follows from (4.4) and [3, Lemma 5.6] that $T := S_{\boldsymbol{\lambda}}$ satisfies the kernel condition. By Lemma 4.1(iii), $\ker T^* \neq \{0\}$ and so T is a nonunitary 2-isometry. Hence, by [3, Theorem 2.5], the spaces $\{T^n(\ker T^*)\}_{n=0}^\infty$ are mutually orthogonal. Since, by [3, Lemma 5.3(viii)], T is analytic, we infer from [35, Theorem 3.6] that

$$\ell^2(V) = \bigoplus_{n=0}^{\infty} \mathcal{M}_n, \tag{4.7}$$

where $\mathcal{M}_n := T^n(\ker T^*)$ for $n \in \mathbb{Z}_+$. Given that T is nonunitary and left-invertible, we see that $\mathcal{M}_n$ is a nonzero closed vector subspace of $\ell^2(V)$ and $\varLambda_n := T|_{\mathcal{M}_n} \colon \mathcal{M}_n \to \mathcal{M}_{n+1}$ is a linear homeomorphism for every $n \in \mathbb{Z}_+$. Therefore, by [19, Problem 56], the Hilbert spaces $\mathcal{M}_n$ and $\mathcal{M}_0$ are unitarily equivalent for every $n \in \mathbb{Z}_+$. Set $V_0 = I_{\mathcal{M}_0}$. Let $\varLambda_0 = U_0|\varLambda_0|$ be the polar decomposition of $\varLambda_0$. Then $U_0 \colon \mathcal{M}_0 \to \mathcal{M}_1$ is a unitary isomorphism. Put $V_1 = U_0^{-1} \colon \mathcal{M}_1 \to \mathcal{M}_0$. For $n \geq 2$, let $V_n \colon \mathcal{M}_n \to \mathcal{M}_0$ be any unitary isomorphism. By (4.7), we can define the unitary isomorphism $V \colon \ell^2(V) \to \ell^2_{\mathcal{M}_0}$ by

$$V(h_0 \oplus h_1 \oplus \cdots) = (V_0 h_0, V_1 h_1, \ldots), \quad h_0 \oplus h_1 \oplus \cdots \in \ell^2(V).$$

Let $W \in \boldsymbol{B}(\ell^2_{\mathcal{M}_0})$ be the operator-valued unilateral weighted shift with (uniformly bounded) invertible weights $\{V_{n+1}\varLambda_n V_n^{-1}\}_{n=0}^\infty \subseteq \boldsymbol{B}(\mathcal{M}_0)$. It is a routine matter to verify that $VT = WV$. Therefore, $T = S_{\boldsymbol{\lambda}}$ is unitarily equivalent to W. Since the zeroth weight of W—say, W_0—equals $V_1 \varLambda_0 V_0^{-1}$, we get $W_0 = |\varLambda_0|$. A careful look at the proof of [21, Proposition 2.2] reveals that W is unitarily equivalent to a 2-isometric operator-valued unilateral weighted shift $\widetilde{W}$ on $\ell^2_{\mathcal{M}_0}$ with invertible weights $\{\widetilde{W}_n\}_{n=0}^\infty \subseteq \boldsymbol{B}(\mathcal{M}_0)$ such that $\widetilde{W}_0 = |W_0|$ and $\widetilde{W}_n \cdots \widetilde{W}_0 \geq 0$ for all $n \in \mathbb{Z}_+$. Thus

$$\widetilde{W}_0 = |\varLambda_0|. \tag{4.8}$$

By [32, Lemma 1], $\|\widetilde{W}h\| \geq \|h\|$ for all $h \in \ell^2_{\mathcal{M}_0}$, which yields

$$\|\widetilde{W}_0 h_0\| = \big\|(0, \widetilde{W}_0 h_0, 0, \ldots)\big\| = \big\|\widetilde{W}(h_0, 0, \ldots)\big\| \geq \|h_0\|, \quad h_0 \in \mathcal{M}_0.$$

Hence, by (4.8), $\widetilde{W}_0 \geq I$. This combined with the proof of [21, Theorem 3.3] and Lemma 3.1(v) implies that

$$\widetilde{W}_n = \int_{[1, \|\widetilde{W}_0\|]} \xi_n(x) E(dx), \quad n \in \mathbb{Z}_+,$$

where E is the spectral measure of $\widetilde{W}_0$.

Our next goal is to show that

$$\mathcal{M}_0 \text{ reduces } |S_{\boldsymbol{\lambda}}| \text{ and } \widetilde{W}_0 = |S_{\boldsymbol{\lambda}}||_{\mathcal{M}_0}. \tag{4.9}$$

For this, observe that $S_{\boldsymbol{\lambda}}$ extends the operator $\varLambda_0 \colon \mathcal{M}_0 \to \mathcal{M}_1$ and consequently

$$\langle \varLambda_0^* \varLambda_0 f, g \rangle = \langle S_{\boldsymbol{\lambda}}^* S_{\boldsymbol{\lambda}} f, g \rangle, \quad f, g \in \mathcal{M}_0. \tag{4.10}$$

Knowing that $S_{\boldsymbol{\lambda}}$ satisfies the kernel condition, we infer from (4.10) that $\varLambda_0^* \varLambda_0 = S_{\boldsymbol{\lambda}}^* S_{\boldsymbol{\lambda}}|_{\mathcal{M}_0}$. This means that the orthogonal projection of $\ell^2(V)$ onto $\mathcal{M}_0$ commutes with $S_{\boldsymbol{\lambda}}^* S_{\boldsymbol{\lambda}}$. By the square root lemma, it commutes with $|S_{\boldsymbol{\lambda}}|$ as well, which together with (4.8) implies (4.9).

It follows from (4.1) and Lemma 4.1(iii) that

$$\mathcal{M}_0 = \ker S_{\boldsymbol{\lambda}}^* = \langle e_\omega \rangle \oplus \bigoplus_{k=1}^\infty \mathcal{G}_k, \tag{4.11}$$

where $\mathcal{G}_k = \bigoplus_{u\in\mathsf{Chi}^{\langle k-1\rangle}(\omega)}(\ell^2(\mathsf{Chi}(u))\ominus\langle\boldsymbol{\lambda}^u\rangle)$ for $k\in\mathbb{N}$. In view of Lemma 4.1(iv) and (4.4), we see that $|S_{\boldsymbol{\lambda}}|e_\omega = \|S_{\boldsymbol{\lambda}}e_\omega\|e_\omega$ and

$$|S_{\boldsymbol{\lambda}}|f = \sum_{v\in\mathsf{Chi}(u)} f(v)|S_{\boldsymbol{\lambda}}|e_v = \alpha_u f, \quad f\in\ell^2\big(\mathsf{Chi}(u)\big), u\in V.$$

This combined with (4.9) and [3, Lemma 5.9(iii)] implies that

$$\left.\begin{aligned}&\widetilde{W}_0 \text{ is a diagonal operator,}\\ &\langle e_\omega\rangle \text{ reduces } \widetilde{W}_0 \text{ and } \widetilde{W}_0|_{\langle e_\omega\rangle} = xI_{\langle e_\omega\rangle} \text{ with } x:=\|S_{\boldsymbol{\lambda}}e_\omega\|,\\ &\mathcal{G}_k \text{ reduces } \widetilde{W}_0 \text{ and } \widetilde{W}_0|_{\mathcal{G}_k} = \xi_k(x)I_{\mathcal{G}_k} \text{ for every } k\in\mathbb{N}.\end{aligned}\right\} \tag{4.12}$$

Since 2-isometries are injective and, by Lemma 4.1(i), $\|S_{\boldsymbol{\lambda}}e_u\|^2 = \sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2$, we see that $\boldsymbol{\lambda}^u\neq 0$ for every $u\in V$. As a consequence, we have

$$\dim\mathcal{G}_k = \sum_{u\in\mathsf{Chi}^{\langle k-1\rangle}(\omega)}(\deg u - 1) = \mathrm{j}_k^{\mathscr{T}}, \quad k\in\mathbb{N}. \tag{4.13}$$

Now, following the proof of the implication (ii) ⇒ (i) of Theorem 3.4 and applying (4.11), (4.12), and (4.13), we see that $S_{\boldsymbol{\lambda}}$ is unitarily equivalent to the orthogonal sum (4.6). The “moreover” part is a direct consequence of [22, Proposition 3.1.10].

(ii) Let $\{j_k\}_{k=1}^\infty$ be a sequence of cardinal numbers, and let $x\in[1,\infty)$. Set $T = S_{[x]}\oplus\bigoplus_{k=1}^\infty(S_{[\xi_k(x)]})^{\oplus j_k}$. First, we construct a directed tree $\mathscr{T}$. Without loss of generality, we may assume that the set $\{n\in\mathbb{N}\colon j_n\geq 1\}$ is nonempty. Let $1\leq n_1<n_2<\cdots$ be a (finite or infinite) sequence of positive integers such that

$$\{n\in\mathbb{N}\colon j_n\geq 1\} = \{n_1, n_2, \ldots\}.$$

Then using induction one can construct a leafless directed tree $\mathscr{T}=(V,E)$ with root ω such that each set $\mathsf{Chi}^{\langle n_k-1\rangle}(\omega)$ has exactly one vertex of degree $1+j_{n_k}$ and such that these particular vertices are the only vertices in V of degree greater than 1; clearly, the other vertices of V are of degree 1 (see Figure 1). Note that if $k\geq 3$, then a directed tree with these properties is not unique (up to graph-isomorphism). Using Procedure 4.4, we can find a system $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V^\circ}\subseteq\mathbb{R}_+$ such that $S_{\boldsymbol{\lambda}}\in\boldsymbol{B}(\ell^2(V))$, where $S_{\boldsymbol{\lambda}}$ is a 2-isometry which satisfies (4.4) for some $\{\alpha_v\}_{v\in V}\subseteq\mathbb{R}_+$ and $x=\|S_{\boldsymbol{\lambda}}e_\omega\|$. If additionally $j_n\leq\aleph_0$ for all $n\in\mathbb{N}$, then the weights $\{\lambda_v\}_{v\in V^\circ}$ can be chosen to be positive (see Procedure 4.4). Since

$$j_n = \sum_{u\in\mathsf{Chi}^{\langle n-1\rangle}(\omega)}(\deg u - 1), \quad n\in\mathbb{N},$$

we deduce from (i) that $T\cong S_{\boldsymbol{\lambda}}$. □

Combining Theorem 4.5(i), Theorem 2.7, and Lemma 3.1(iv), we get the following classification theorem.

Theorem 4.6. *For $k=1,2$, let $\mathscr{T}_k=(V_k,E_k)$ be a directed tree with root ω_k, and let $S_{\boldsymbol{\lambda}_k}\in\boldsymbol{B}(\ell^2(V_k))$ be a 2-isometric weighted shift on $\mathscr{T}_k$ with weights $\boldsymbol{\lambda}_k=\{\lambda_{k,v}\}_{v\in V_k^\circ}$ which satisfies (4.4) for some $\{\alpha_{k,v}\}_{v\in V_k}\subseteq\mathbb{R}_+$. Then $S_{\boldsymbol{\lambda}_1}\cong S_{\boldsymbol{\lambda}_2}$ if and only if one of the following conditions holds:*

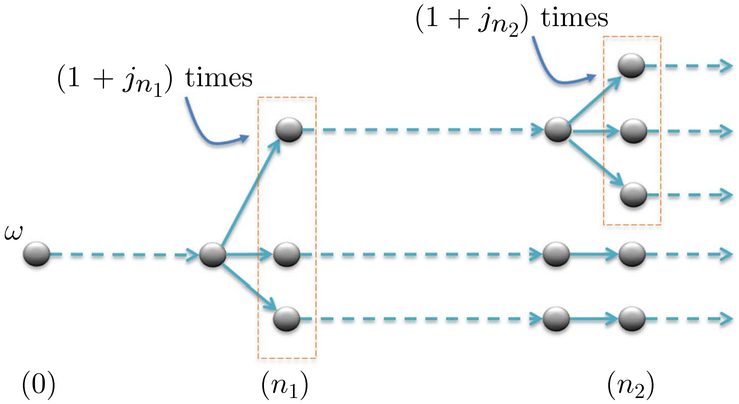

FIGURE 1. An example of a leafless directed tree $\mathscr{T}$ with the properties required in the proof of Theorem 4.5(ii).

(i) $\|S_{\boldsymbol\lambda_1}e_{\omega_1}\| = \|S_{\boldsymbol\lambda_2}e_{\omega_2}\| > 1$ *and* $\mathfrak{j}_n^{\mathscr{T}_1} = \mathfrak{j}_n^{\mathscr{T}_2}$ *for every* $n \in \mathbb{N}$,
(ii) $\|S_{\boldsymbol\lambda_1}e_{\omega_1}\| = \|S_{\boldsymbol\lambda_2}e_{\omega_2}\| = 1$ *and* $\sum_{n=1}^\infty \mathfrak{j}_n^{\mathscr{T}_1} = \sum_{n=1}^\infty \mathfrak{j}_n^{\mathscr{T}_2}$.

It is worth pointing out that, by [3, Remark 5.8, Lemma 5.9(iv)] and Theorem 4.6, the sequence $(\|S_{\boldsymbol\lambda}e_\omega\|, \mathfrak{j}_1^{\mathscr{T}}, \mathfrak{j}_2^{\mathscr{T}}, \mathfrak{j}_3^{\mathscr{T}}, \ldots)$ forms a complete system of unitary invariants for nonisometric 2-isometric weighted shifts $S_{\boldsymbol\lambda}$ on rooted directed trees $\mathscr{T}$ with nonzero weights satisfying the kernel condition. In turn, the quantity $\sum_{n=1}^\infty \mathfrak{j}_n^{\mathscr{T}}$ forms a complete system of unitary invariants for isometric weighted shifts $S_{\boldsymbol\lambda}$ on rooted directed trees $\mathscr{T}$ (see [25, Proposition 2.4]).

Remark 4.7. Let us make a few observations concerning Theorem 4.5(i) (still under the assumptions of this theorem). First, if $S_{\boldsymbol\lambda}$ is not an isometry, then Lemma 3.1(iv) implies that the additive exponent j_k of the inflation $(S_{[\xi_k(x)]})^{\oplus j_k}$ that appears in the orthogonal decomposition (4.6) is maximal for every $k \in \mathbb{N}$. Second, by Lemma 3.1(ii), the weights of $S_{[\xi_k(x)]}$ take the form $\{\xi_n(x)\}_{n=k}^\infty$. Hence, the weights of components of the decomposition (4.6) are built on the weights of a single 2-isometric unilateral weighted shift. Third, in view of Corollary 3.7 and Theorem 4.5(i), general completely nonunitary 2-isometric operators satisfying the kernel condition cannot be modeled by weighted shifts on rooted direct trees. Finally, in view of Procedure 4.4 and Theorem 4.6, there exist two unitarily equivalent 2-isometric weighted shifts on the same rooted directed tree, one with nonzero weights, the other with some zero weights.

Concluding this section, we show that there are unitarily equivalent 2-isometric weighted shifts on nongraph isomorphic directed trees that satisfy (4.4).

Example 4.8. For $k = 1, 2$, let $\mathscr{T}_k = (V_k, E_k)$ be a directed tree with root ω_k as in Figure 2. Clearly, these two directed graphs are not graph-isomorphic. Moreover, we have (see (4.5) for notation)

$$\mathfrak{j}_n^{\mathscr{T}_1} = \mathfrak{j}_n^{\mathscr{T}_2} = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

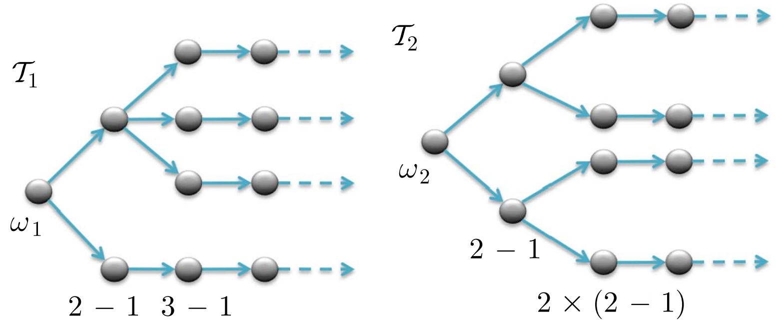

FIGURE 2. Two nongraph isomorphic directed trees used in Example 4.8.

Fix $x \in [1, \infty)$. Using Procedure 4.4, one can construct for $k = 1, 2$, a 2-isometric weighted shift $S_{\boldsymbol{\lambda}_k} \in \boldsymbol{B}(\ell^2(V_k))$ on $\mathscr{T}_k$ with weights $\boldsymbol{\lambda}_k = \{\lambda_{k,v}\}_{v \in V_k^\circ}$ which satisfies (4.4) for some $\{\alpha_{k,v}\}_{v \in V_k} \subseteq \mathbb{R}_+$ and the equation $x = \|S_{\boldsymbol{\lambda}_k} e_{\omega_k}\|$. The above combined with Theorem 4.5(i) implies that

$$S_{\boldsymbol{\lambda}_k} \cong S_{[x]} \oplus S_{[\xi_1(x)]} \oplus (S_{[\xi_2(x)]})^{\oplus 2}, \quad k = 1, 2,$$

and so $S_{\boldsymbol{\lambda}_1} \cong S_{\boldsymbol{\lambda}_2}$. In particular, if $x = 1$, then $S_{\boldsymbol{\lambda}_1}$ and $S_{\boldsymbol{\lambda}_2}$ are unitarily equivalent isometries.

Appendix: When is the Cauchy dual operator in $C_{0\cdot}$ or in $C_{\cdot 0}$?

In this section we assume that $\mathcal{H} \neq \{0\}$. We begin by recalling necessary concepts from [37, Chapter II]. A contraction $S \in \boldsymbol{B}(\mathcal{H})$ is of class $C_{0\cdot}$ (resp., $C_{\cdot 0}$) if $S^n f \to 0$ (resp., $S^{*n} f \to 0$) as $n \to \infty$ for all $f \in \mathcal{H}$. If S is of class $C_{0\cdot}$ and of class $C_{\cdot 0}$, then we say that S is of class C_{00}. Observe that the norm of a contraction which is not of class $C_{0\cdot}$ (or not of class $C_{\cdot 0}$) must equal 1. Clearly, a contraction S is of class $C_{0\cdot}$ if and only if $\mathsf{A}_S = 0$, where A_S stands for the limit in the strong (equivalently, weak) operator topology of the sequence $\{S^{*n} S^n\}_{n=1}^\infty$. That such a limit exists plays a key role in the theory of unitary and isometric asymptotes (see [37, Chapter IX]; see also [17, Theorem 1]). As we know, the Cauchy dual operator T' of a 2-isometry T is always a contraction (see (1.1)), so we can look for an explicit description of $\mathsf{A}_{T'}$. By examining the proof of [3, Corollary 4.6], we can calculate $\mathsf{A}_{T'}$ for two classes of 2-isometries; namely, quasi-Brownian isometries and 2-isometries satisfying the kernel condition. Recall that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is a *quasi-Brownian isometry* if T is a 2-isometry such that $\triangle_T T = \triangle_T^{1/2} T \triangle_T^{1/2}$, where $\triangle_T = T^*T - I$. A quasi-Brownian isometry, called in [27] a $\triangle_T$*-regular* 2*-isometry*, generalizes the notion of a Brownian isometry introduced in [2].

Lemma A.1. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a* 2*-isometry, and let G_T be the spectral measure of T^*T. Then the following assertions hold:*

(i) *if T satisfies the kernel condition, then $\mathsf{A}_{T'} = G_T(\{1\})$,*
(ii) *if T is a quasi-Brownian isometry, then $\mathsf{A}_{T'} = \frac{1}{2} G_T(\{1\}) + (I + T^*T)^{-1}$.*

Before stating the main result of this section, we justify the following fact.

Lemma A.2. *If $T \in \boldsymbol{B}(\mathcal{H})$ is left-invertible and T' is of class $C_{0\cdot}$ or of class $C_{\cdot 0}$, then T is completely nonunitary.*

Proof. First, note the following.

$$\begin{gathered}\textit{If } T \textit{ is left-invertible and } T \textit{ is an orthogonal sum of operators } A \textit{ and } B,\\ \textit{that is, } T = A \oplus B, \textit{ then } A \textit{ and } B \textit{ are left-invertible and } T' = A' \oplus B'.\end{gathered} \tag{A.1}$$

This together with the fact that the Cauchy dual operator of a unitary operator is unitary completes the proof. □

Now, we can prove the main result of this section.

Theorem A.3. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a 2-isometry. Then*

(i) *T' is of class $C_{\cdot 0}$ if and only if T is completely nonunitary.*

Moreover, if T satisfies the kernel condition, then

(ii) *T' is of class $C_{0\cdot}$ if and only if T is completely nonunitary and $E(\{1\}) = 0$, where E is as in Theorem 3.2(iv),*

(iii) *T' is of class $C_{0\cdot}$ if and only if T' is of class C_{00} or, equivalently, if and only if $G_T(\{1\}) = 0$, where G_T is the spectral measure of T^*T.*

Proof. First, observe that if T' is of class $C_{0\cdot}$ or of class $C_{\cdot 0}$, then by Lemma A.2, T is completely nonunitary. Note also that the same conclusion holds if $G_T(\{1\}) = 0$. Indeed, otherwise there exists a nonzero closed vector subspace $\mathcal{M}$ of $\mathcal{H}$ reducing T to a unitary operator. Then $T^*T = I$ on $\mathcal{M}$ and thus 1 is in the point spectrum of T^*T, which implies that $G_T(\{1\}) \neq 0$, which is a contradiction. These two observations show that there is no loss of generality in assuming that T is completely nonunitary.

(i) It is enough to prove that T' is of class $C_{\cdot 0}$ (under the assumption that T is completely nonunitary). Using (A.1), the well-known identity $(T')' = T$ (which holds for any left-invertible operator T) and observing that the Cauchy dual operator of a left-invertible normal operator is normal and a normal 2-isometry is unitary (see [23, Theorem 3.4]), one can deduce from (1.1) that T' is a pure and hyponormal contraction. Since, according to [30, Theorem 3], a pure and hyponormal contraction is of class $C_{\cdot 0}$, we are done.

(ii) and (iii) Assume that T satisfies the kernel condition. In view of Theorem 3.2, we may further assume that $T = W$, where W is as in (iv) of this theorem. Using Lemma A.1(i), we deduce that W' is of class $C_{0\cdot}$ if and only if $G_W(\{1\}) = 0$. We will show that

$$G_W\big(\{1\}\big) = 0 \qquad \text{if and only if} \qquad E\big(\{1\}\big) = 0. \tag{A.2}$$

Set $\eta = \sup(\operatorname{supp}(E))$. Note that $\eta \in [1, \infty)$. It follows from (2.3) and (3.2) that

$$W^*W = \bigoplus_{j=0}^{\infty} \int_{[1,\eta]} \phi_j(x) E(dx), \tag{A.3}$$

where $\phi_j\colon [1,\eta] \to \mathbb{R}_+$ is given by $\phi_j(x) = \xi_j(x)^2$ for $x \in [1,\eta]$ and $j \in \mathbb{Z}_+$. By Lemma 3.1, $1 \leq \phi_j \leq \eta^2$ for all $j \in \mathbb{Z}_+$. This together with (A.3), [6, Theorem 5.4.10], and the uniqueness part of the spectral theorem implies that

$$G_W(\varDelta) = \bigoplus_{j=0}^{\infty} E\big(\phi_j^{-1}(\varDelta)\big), \quad \varDelta \in \mathfrak{B}\big([1,\eta^2]\big).$$

Since $\phi_j^{-1}(\{1\}) = \{1\}$ for all $j \in \mathbb{Z}_+$, we conclude that (A.2) holds. This together with (i) completes the proof. □

Remark A.4. According to [13, Theorem 3.1], all positive integer powers T'^n of the Cauchy dual operator T' of a 2-hyperexpansive operator $T \in \boldsymbol{B}(\mathcal{H})$ are hyponormal. This immediately implies that if $T \in \boldsymbol{B}(\mathcal{H})$ is a 2-hyperexpansive operator such that T' is of class $C_{0\cdot}$, then T' is of class C_{00}.

Regarding Theorem A.3, note that there exist completely nonunitary 2-isometries satisfying the kernel condition whose Cauchy dual operators are not of class $C_{0\cdot}$. To see this, consider a nonzero Hilbert space $\mathcal{M}$ and a compactly supported $\boldsymbol{B}(\mathcal{M})$-valued Borel spectral measure E on the interval $[1,\infty)$ such that $E(\{1\}) \neq 0$. Then, by Theorems 3.2 and A.3(ii), the operator-valued unilateral weighted shift W on $\ell^2_{\mathcal{M}}$ with weights $\{W_n\}_{n=0}^{\infty}$ defined by (3.2) has all the required properties.

The following proposition shows that unlike the case of 2-isometries satisfying the kernel condition, the Cauchy dual operator of a quasi-Brownian isometry is never of class $C_{0\cdot}$ (see also Lemma A.1(ii)).

Proposition A.5. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a 2-isometry, and let T' be its Cauchy dual operator. Then the following assertions hold:*

(i) *if T is a quasi-Brownian isometry, then for every $n \in \mathbb{Z}_+$,*

$$\|T'^n f\|^2 \geq c_n \|f\|^2, \quad f \in \mathcal{H}, \tag{A.4}$$

where $c_n = \frac{1+\|T\|^{2(1-2n)}}{1+\|T\|^2}$ is the largest constant for which (A.4) holds; in particular, T' is not of class $C_{0\cdot}$ and $\|T'\| = 1$,

(ii) *if T satisfies the kernel condition, then for every $n \in \mathbb{Z}_+$,*

$$\|T'^n f\|^2 \geq c_n \|f\|^2, \quad f \in \mathcal{H}, \tag{A.5}$$

where $c_n = \frac{1}{1+n(\|T\|^2-1)}$ is the largest constant for which (A.5) holds.

Proof. (i) Fix $n \in \mathbb{Z}_+$. Note that T'^n is left-invertible. Denote by $\hat{c}_n$ the largest positive constant for which (A.4) holds. Define $s_n\colon [1,\infty) \to (0,\infty)$ by

$$s_n(x) = \frac{1+x}{1+x^{1-2n}}, \quad x \in [1,\infty).$$

Using [3, Theorem 4.5], the fact that $\sigma(T^*T) \subseteq [1,\infty)$, and the functional calculus (see [14, Theorem VIII.2.6]), we deduce that

$$\begin{aligned}\hat{c}_n &= \frac{1}{\|(T'^{*n}T'^{n})^{-1}\|} = \frac{1}{\|s_n(T^*T)\|} = \frac{1}{\sup_{x\in\sigma(T^*T)} s_n(x)} \\ &= \frac{1}{s_n(\sup\sigma(T^*T))} = \frac{1}{s_n(\|T\|^2)}.\end{aligned}$$

Due to (1.1), the "in particular" part of (i) is now clear.

(ii) Argue as in (i) using [3, Theorem 3.3] in place of [3, Theorem 4.5]. ☐

As a direct consequence of Proposition A.5 and the fact that $\|T\| \geq 1$ for any 2-isometry T (see [32, Lemma 1]), we get

$$\lim_{n\to\infty} c_n = \begin{cases} 0 & \text{if } T \text{ is a 2-isometry satisfying (1.2) and } \|T\| \neq 1, \\ \frac{1}{1+\|T\|^2} & \text{if } T \text{ is a quasi-Brownian isometry and } \|T\| \neq 1. \end{cases}$$

Acknowledgments. A part of this paper was written while the second author visited Jagiellonian University in the summer of 2018. He wishes to thank the faculty and administration of this institution for their warm hospitality.

Jabłoński and Stochel's work was partially supported by NCN (National Science Centre) decision DEC-2013/11/B/ST1/03613.

References

1. J. Agler, *A disconjugacy theorem for Toeplitz operators*, Amer. J. Math. **112** (1990), no. 1, 1–14. Zbl 0708.47020. MR1037599. DOI 10.2307/2374849. 360
2. J. Agler and M. Stankus, *m-isometric transformations of Hilbert spaces, I*, Integral Equations Operator Theory **21** (1995), no. 4, 383–429. Zbl 0836.47008. MR1321694. DOI 10.1007/BF01222016. 360, 380
3. A. Anand, S. Chavan, Z. J. Jabłoński, and J. Stochel, *A solution to the Cauchy dual subnormality problem for* 2*-isometries*, preprint, arXiv:1702.01264v4 [math.FA]. 360, 361, 362, 369, 374, 375, 376, 378, 379, 380, 383
4. A. Athavale, *On completely hyperexpansive operators*, Proc. Amer. Math. Soc. **124** (1996), no. 12, 3745–3752. Zbl 0863.47017. MR1353373. DOI 10.1090/S0002-9939-96-03609-X. 360
5. B. Bagchi and G. Misra, *The homogeneous shifts*, J. Funct. Anal. **204** (2003), no. 2, 293–319. Zbl 1059.47033. MR2017317. DOI 10.1016/S0022-1236(02)00088-5. 361
6. M. S. Birman and M. Z. Solomjak, *Spectral Theory of Selfadjoint Operators in Hilbert Space*, Math. Appl. (Soviet Ser.), D. Reidel, Dordrecht, 1987. Zbl 0744.47017. MR1192782. 368, 369, 370, 382
7. P. Budzyński, P. Dymek, and M. Ptak, *Analytic structure of weighted shifts on directed trees*, Math. Nachr. **290** (2017), no. 11–12, 1612–1629. Zbl 06778801. MR3683793. DOI 10.1002/mana.201500448. 373
8. P. Budzyński, Z. J. Jabłoński, I. B. Jung, and J. Stochel, *Unbounded subnormal weighted shifts on directed trees*, J. Math. Anal. Appl. **394** (2012), no. 2, 819–834. Zbl 1304.47042. MR2927501. DOI 10.1016/j.jmaa.2012.04.074. 375
9. P. Budzyński, Z. J. Jabłoński, I. B. Jung, and J. Stochel, *Unbounded subnormal composition operators in* L^2*-spaces*, J. Funct. Anal. **269** (2015), no. 7, 2110–2164. Zbl 1321.47056. MR3378870. DOI 10.1016/j.jfa.2015.01.024. 372

10. P. Budzyński, Z. J. Jabłoński, I. B. Jung, and J. Stochel, *Unbounded weighted composition operators in L^2-spaces*, Lecture Notes in Math. **2209**, Springer, Cham, 2018. Zbl 06848851. MR3793599. DOI 10.1007/978-3-319-74039-3. 372
11. S. Chavan, *On operators Cauchy dual to 2-hyperexpansive operators*, Proc. Edinb. Math. Soc. (2) **50** (2007), no. 3, 637–652. Zbl 1155.47024. MR2360521. DOI 10.1017/S0013091505001124. 360, 372
12. S. Chavan, *On operators close to isometries*, Studia Math. **186** (2008), no. 3, 275–293. Zbl 1147.47019. MR2403669. DOI 10.4064/sm186-3-6. 361
13. S. Chavan, *An inequality for spherical Cauchy dual tuples*, Colloq. Math. **131** (2013), no. 2, 265–271. Zbl 1296.47006. MR3092455. DOI 10.4064/cm131-2-8. 382
14. J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Grad. Texts in Math. **96**, Springer, New York, 1990. Zbl 0706.46003. MR1070713. 383
15. J. B. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monogr. **36**, Amer. Math. Soc., Providence, 1991. Zbl 0743.47012. MR1112128. DOI 10.1090/surv/036. 360
16. M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math. **141** (1978), no. 3–4, 187–261. Zbl 0427.47016. MR0501368. DOI 10.1007/BF02545748. 361
17. R. G. Douglas, *On the operator equation $S^*XT = X$ and related topics*, Acta Sci. Math. (Szeged) **30** (1969), 19–32. Zbl 0177.19204. MR0250106. 380
18. P. R. Halmos, *Normal dilations and extensions of operators*, Summa Brasil. Math. **2** (1950), 125–134. Zbl 0041.23201. MR0044036. 360
19. P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Grad. Texts in Math. **19**, Springer, New York, 1982. Zbl 0496.47001. MR0675952. 372, 377
20. D. A. Herrero, *On multicyclic operators*, Integral Equations Operator Theory **1** (1978), no. 1, 57–102. Zbl 0394.47001. MR0490887. DOI 10.1007/BF01682740. 372
21. Z. J. Jabłoński, *Hyperexpansive operator valued unilateral weighted shifts*, Glasg. Math. J. **46** (2004), no. 2, 405–416. Zbl 1072.47028. MR2062623. DOI 10.1017/S0017089504001892. 360, 377
22. Z. J. Jabłoński, I. B. Jung, and J. Stochel, *Weighted shifts on directed trees*, Mem. Amer. Math. Soc. **216** (2012), no. 1017. Zbl 1248.47033. MR2919910. DOI 10.1090/S0065-9266-2011-00644-1. 360, 369, 373, 374, 378
23. Z. J. Jabłoński and J. Stochel, *Unbounded 2-hyperexpansive operators*, Proc. Edinb. Math. Soc. (2) **44** (2001), no. 3, 613–629. Zbl 0993.47003. MR1875772. DOI 10.1017/S001309159900139X. 360, 370, 371, 374, 375, 381
24. D. W. Kribs, R. H. Levene, and S. C. Power, *Commutants of weighted shift directed graph operator algebras*, Proc. Amer. Math. Soc. **145** (2017), no. 8, 3465–3480. Zbl 1380.47048. MR3652799. DOI 10.1090/proc/13477. 373
25. C. S. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory*, Birkhäuser, Boston, 1997. Zbl 0918.47013. MR1464436. DOI 10.1007/978-1-4612-1998-9. 372, 379
26. A. Lambert, *Unitary equivalence and reducibility of invertibly weighted shifts*, Bull. Aust. Math. Soc. **5** (1971), 157–173. Zbl 0217.45303. MR0295128. DOI 10.1017/S000497270004702X. 360, 362, 364, 370
27. W. Majdak, M. Mbekhta, and L. Suciu, *Operators intertwining with isometries and Brownian parts of 2-isometries*, Linear Algebra Appl. **509** (2016), 168–190. Zbl 1394.47020. MR3546405. DOI 10.1016/j.laa.2016.07.014. 380
28. R. A. Martínez-Avendaño, *Hypercyclicity of shifts on weighted L^p spaces of directed trees*, J. Math. Anal. Appl. **446** (2017), no. 1, 823–842. Zbl 1346.05032. MR3554758. DOI 10.1016/j.jmaa.2016.08.066. 373
29. B. B. Morrel, *A decomposition for some operators*, Indiana Univ. Math. J. **23** (1973/1974), 497–511. Zbl 0252.47015. MR0343079. DOI 10.1512/iumj.1973.23.23042. 362
30. C. R. Putnam, *Hyponormal contractions and strong power convergence*, Pacific J. Math. **57** (1975), no. 2, 531–538. Zbl 0321.47018. MR0380493. 381

31. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis*, 2nd ed., Academic Press, New York, 1980. Zbl 0459.46001. MR0751959. 363
32. S. Richter, *Invariant subspaces of the Dirichlet shift*, J. Reine Angew. Math. **386** (1988), 205–220. Zbl 0635.46021. MR0936999. DOI 10.1515/crll.1988.386.205. 360, 377, 383
33. S. Richter, *A representation theorem for cyclic analytic two-isometries*, Trans. Amer. Math. Soc. **328**, no. 1 (1991), 325–349. Zbl 0762.47009. MR1013337. DOI 10.2307/2001885. 360, 362
34. A. L. Shields, “Weighted shift operators and analytic function theory” in *Topics in Operator Theory*, Math. Surveys **13**, Amer. Math. Soc., Providence, 1974, 49–128. Zbl 0303.47021. MR0361899. 361, 367, 371
35. S. Shimorin, *Wold-type decompositions and wandering subspaces for operators close to isometries*, J. Reine Angew. Math. **531** (2001), 147–189. Zbl 0974.47014. MR1810120. DOI 10.1515/crll.2001.013. 360, 372, 376
36. S. Shimorin, *Complete Nevanlinna-Pick property of Dirichlet-type spaces*, J. Funct. Anal. **191** (2002), no. 2, 276–296. Zbl 1038.46022. MR1911187. DOI 10.1006/jfan.2001.3871. 360
37. B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, 2nd ed., Universitext, Springer, New York, 2010. Zbl 1234.47001. MR2760647. DOI 10.1007/978-1-4419-6094-8. 374, 380

[1]Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, India.

E-mail address: akasha@iitk.ac.in; chavan@iitk.ac.in

[2]Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków, Poland.

E-mail address: Zenon.Jablonski@im.uj.edu.pl; Jan.Stochel@im.uj.edu.pl