Banach J. Math. Anal. 12 (2018), no. 1, 240–257
https://doi.org/10.1215/17358787-2017-0030
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

WEIGHTED BANACH SPACES OF LIPSCHITZ FUNCTIONS

A. JIMÉNEZ-VARGAS

Communicated by K. Jarosz

Abstract. Given a pointed metric space X and a weight v on $\widetilde{X}$ (the complement of the diagonal set in $X \times X$), let $\mathrm{Lip}_v(X)$ and $\mathrm{lip}_v(X)$ denote the Banach spaces of all scalar-valued Lipschitz functions f on X vanishing at the basepoint such that $v\Phi(f)$ is bounded and $v\Phi(f)$ vanishes at infinity on $\widetilde{X}$, respectively, where $\Phi(f)$ is the de Leeuw's map of f on $\widetilde{X}$, under the weighted Lipschitz norm. The space $\mathrm{Lip}_v(X)$ has an isometric predual $\mathcal{F}_v(X)$ and it is proved that $(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*}) = (\mathcal{F}_v(X)^*, \tau_c)$ and $\mathcal{F}_v(X) = ((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_c)$, where τ_{bw^*} denotes the bounded weak* topology and τ_c the topology of uniform convergence on compact sets. The linearization of the elements of $\mathrm{Lip}_v(X)$ is also tackled. Assuming that X is compact, we address the question as to when $\mathrm{Lip}_v(X)$ is canonically isometrically isomorphic to $\mathrm{lip}_v(X)^{**}$, and we show that this is the case whenever $\mathrm{lip}_v(X)$ is an M-ideal in $\mathrm{Lip}_v(X)$ and the so-called *associated weights* $\widetilde{v}_L$ and $\widetilde{v}_l$ coincide.

Introduction

In his paper [9], J. A. Johnson deals briefly with the spaces $L(X, \rho)$ of functions satisfying a very general Lipschitz-type condition. For any set X and any nonnegative function ρ on $X \times X$, $L(X, \rho)$ denotes the Banach space of all bounded scalar-valued functions f on X such that $|f(x) - f(y)| \leq C\rho(x, y)$ for some $C \geq 0$ and all $(x, y) \in X \times X$, under the norm $\|f\| = \max\{\|f\|_\rho, \|f\|_\infty\}$, where $\|f\|_\rho$ is the infimum of all the constants C for which such a domination holds and where $\|f\|_\infty$ is the supremum norm of f. Apparently, his unique reason for presenting those spaces was to show that *generalized Lipschitz spaces* such as spaces of func-

Received Apr. 12, 2017; Accepted Jul. 7, 2017.
First published online Dec. 19, 2017.
2010 *Mathematics Subject Classification.* Primary 46E15; Secondary 46A20.
Keywords. Lipschitz function, little Lipschitz function, duality, weighted Banach space.

tions satisfying a modulus of continuity condition (see [6]) could be considered as *Lipschitz spaces*. In this article, we will introduce the *weighted Banach spaces of Lipschitz functions* which are closely related to the spaces $L(X,\rho)$. The *classical Lipschitz spaces* are special cases of *weighted Lipschitz spaces* so that in a sense our approach is contrary to that of J. A. Johnson.

Let (X,d) be a pointed metric space with a basepoint denoted by e, and let E be a Banach space over the field $\mathbb{K}$ of real or complex numbers. A real-valued function v on the set $\widetilde{X} := \{(x,y) \in X \times X \colon x \neq y\}$ is said to be a *weight* on $\widetilde{X}$ if it is (strictly) positive and continuous.

The *weighted Lipschitz space* $\mathrm{Lip}_v(X,E)$ is the Banach space of all E-valued Lipschitz functions f on X for which $f(e)=0$ such that

$$\sup\Big\{v(x,y)\frac{\|f(x)-f(y)\|}{d(x,y)} \colon (x,y) \in \widetilde{X}\Big\} < \infty,$$

endowed with the *weighted Lipschitz norm*:

$$\mathrm{Lip}_v(f) = \sup\Big\{v(x,y)\frac{\|f(x)-f(y)\|}{d(x,y)} \colon (x,y) \in \widetilde{X}\Big\}.$$

The *weighted little Lipschitz space* $\mathrm{lip}_v(X,E)$ is the closed linear subspace of $\mathrm{Lip}_v(X,E)$ consisting of all those functions f with the property that, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$v(x,y)\frac{\|f(x)-f(y)\|}{d(x,y)} < \varepsilon$$

whenever $0 < d(x,y) < \delta$. In the case $E = \mathbb{K}$, we will write simply $\mathrm{Lip}_v(X)$ and $\mathrm{lip}_v(X)$.

The *Lipschitz space* $\mathrm{Lip}_0(X)$ is the Banach space of all scalar-valued Lipschitz functions f on X for which $f(e)=0$ with the *Lipschitz norm*

$$\mathrm{Lip}(f) = \sup\Big\{\frac{|f(x)-f(y)|}{d(x,y)} \colon (x,y) \in \widetilde{X}\Big\},$$

and the *little Lipschitz space* $\mathrm{lip}_0(X)$ is the closed linear subspace of $\mathrm{Lip}_0(X)$ formed by all functions f such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < d(x,y) < \delta$, then $|f(x)-f(y)|/d(x,y) < \varepsilon$. ($\mathrm{Lip}_0$ and lip_0 spaces have been intensively studied by N. Weaver [15].) Thus $\mathrm{Lip}_v(X)$ may be regarded as all functions f in $\mathrm{Lip}_0(X)$ such that the quotient $|f(x)-f(y)|/d(x,y)$ satisfies a growth condition of order $O(1/v(x,y))$, while $\mathrm{lip}_v(X)$ are those functions for which $|f(x)-f(y)|/d(x,y)$ has a growth rate of order $o(1/v(x,y))$.

The study of Lip_v spaces is new and interesting. Furthermore, each space $\mathrm{Lip}_0(X)$ can be canonically identified with the space $\mathrm{Lip}_v(X)$ by taking the weight $v = 1_{\widetilde{X}}$ (the function constantly 1 on $\widetilde{X}$), and so our results can be applied to Lip_0 spaces.

Although the weights are arbitrary, we can take weights on $\widetilde{X}$ involving the metric structure of (X,d) as, for example, $v = \phi \circ d$, where ϕ is a continuous injective function from $[0,\infty)$ into itself vanishing at 0. Each weight $v = \phi \circ d$ becomes a metric on X if ϕ is in addition subadditive and strictly increasing. Such a function ϕ is called a *modulus function*. The most natural examples of modulus functions are $\phi(t) = t^p$, $\phi(t) = \max\{t, t^p\}$, and $\phi(t) = \ln(1 + t^p)$ for $t \geq 0$ with $0 < p \leq 1$. Moreover, the function $\phi(t) = t/(1 + t)$ ($t \geq 0$) is also a modulus function, and the composition of two modulus functions is again also a modulus function. Our purpose in this paper is to study the duality theory of Lip_v spaces.

We now describe the contents of its two sections. In Section 1, the problems concerning preduality and linearization of $\mathrm{Lip}_v(X)$ are tackled. We analyze the proof of the Ng–Dixmier theorem [14] and show that $\mathrm{Lip}_v(X)$ is isometrically isomorphic under the evaluation map to the dual of the closed subspace $\mathcal{F}_v(X)$ of $\mathrm{Lip}_v(X)^*$ consisting of all linear functionals γ on $\mathrm{Lip}_v(X)$ such that the restriction of γ to the closed unit ball $B(\mathrm{Lip}_v(X))$ of $\mathrm{Lip}_v(X)$ is continuous for the topology τ_p of pointwise convergence on X. This approach permits us to describe the closed unit ball of $\mathcal{F}_v(X)$ as the closed convex balanced hull in $\mathrm{Lip}_v(X)^*$ of the so-called *weighted Lipschitz evaluation functionals*.

We also give a process of linearization of the elements of $\mathrm{Lip}_v(X)$ which is a linearizing construction stronger than a predual space, and we characterize the space $\mathcal{F}_v(X)$ by the following universal property: $\mathcal{F}_v(X)$ contains X through the Dirac map $\delta_X \colon x \mapsto \delta_x$, and for every Banach space E and every map $f \in \mathrm{Lip}_v(X, E)$ there is a unique bounded linear operator $T_f \colon \mathcal{F}_v(X) \to E$ such that $T_f \circ \delta_X = f$. Furthermore, $\|T_f\| = \mathrm{Lip}_v(f)$. A result due to N. Weaver [15, Theorem 2.2.4], justifies our study.

Viewing $\mathrm{Lip}_v(X)$ as the dual of $\mathcal{F}_v(X)$, we study the bounded weak* topology τ_{bw^*} on $\mathrm{Lip}_v(X)$ and state some topological identifications that may be of independent interest. Namely, we prove that $(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*}) = (\mathcal{F}_v(X)^*, \tau_c)$ and $\mathcal{F}_v(X) = ((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_c)$, where τ_c denotes the topology of uniform convergence on compact sets.

In Section 2, we deal the biduality problem as to when $\mathrm{Lip}_v(X)$ is naturally isometrically isomorphic to the bidual of $\mathrm{lip}_v(X)$ for pointed compact metric spaces X, and we show that this is the case whenever $\mathrm{lip}_v(X)$ is an *M-ideal* in $\mathrm{Lip}_v(X)$ and the so-called *associated weights* $\widetilde{v}_L$ and $\widetilde{v}_l$ coincide. Our method of proof in this section will be an adaptation of the reasoning used by Bierstedt and Summers [3] and Boyd and Rueda [4] to study the biduals of weighted Banach spaces of analytic functions. This adaptation to the context of Lipschitz functions is far from being immediate and requires the previous study on the bounded weak* topology of $\mathrm{Lip}_v(X)$.

Notation. Given Banach spaces E and F, we denote by $\mathcal{L}(E, F)$ the Banach space of all continuous linear mappings from E into F with the canonical norm of operators. As usual, E^* stands for $\mathcal{L}(E, \mathbb{K})$, J_E for the canonical injection from E into E^{**}, $B(E)$ for the closed unit ball of E, $S(E)$ for the unit sphere of E, and $\mathrm{Ext}(B(E))$ for the set of extreme points of $B(E)$. Given $M \subset E$ and $N \subset E^*$, M° and $N_\circ$ denote the polar set of M in E^* and the prepolar set of N in E, respectively. The bipolar of M is the set $(M^\circ)_\circ$. We will denote by $\overline{\mathrm{lin}}(M)$ and $\overline{\mathrm{aco}}(M)$ the closed linear hull and the closed convex balanced hull of M in E, respectively. For a locally convex space (F, τ), we denote by $(F, \tau)'$ the space of all continuous linear mappings of (F, τ) into $\mathbb{K}$.

1. Preduality and linearization of weighted Lipschitz spaces

Let X be a pointed metric space and let v be a weight on $\widetilde{X}$. The *topology* τ_p *of pointwise convergence on* X is the locally convex topology on $\mathrm{Lip}_v(X)$ generated by the seminorms of the form

$$|f|_G = \sup\{|f(x)|: x \in G\}, \quad f \in \mathrm{Lip}_v(X),$$

where G ranges over the family of all finite subsets of X. According to [11, Chapter 7, Theorem 1], $B(\mathrm{Lip}_v(X))$ is τ_p-compact because it is pointwise closed in $\mathbb{K}^X$ and, for each point $x \in X$, the set $\{f(x): f \in B(\mathrm{Lip}_v(X))\}$ has a compact closure. Therefore $\mathrm{Lip}_v(X)$ is a dual Banach space by the Dixmier–Ng theorem [14, Theorem 1].

Let $\mathcal{F}_v(X)$ be the space of all linear functionals γ on $\mathrm{Lip}_v(X)$ such that the restriction of γ to $B(\mathrm{Lip}_v(X))$ is τ_p-continuous. An application of the proof of the Dixmier–Ng theorem to the space $\mathrm{Lip}_v(X)$ yields the following.

Theorem 1.1. *Let X be a pointed metric space and let v be a weight on $\widetilde{X}$. Then $\mathcal{F}_v(X)$ is a closed subspace of the space $\mathrm{Lip}_v(X)^*$ equipped with the norm*

$$\|\gamma\| = \sup\{|\gamma(f)|: f \in B(\mathrm{Lip}_v(X))\} \quad (\gamma \in \mathcal{F}_v(X)),$$

and the evaluation map $P_X: \mathrm{Lip}_v(X) \to \mathcal{F}_v(X)^*$, defined by

$$P_X(f)(\gamma) = \gamma(f) \quad (f \in \mathrm{Lip}_v(X), \gamma \in \mathcal{F}_v(X)),$$

is an isometric isomorphism from $(\mathrm{Lip}_v(X), \mathrm{Lip}_v)$ onto $\mathcal{F}_v(X)^*$. Moreover, $B(\mathcal{F}_v(X)) = P_X(B(\mathrm{Lip}_v(X)))_\circ$ with respect to the duality $(\mathcal{F}_v(X), \mathcal{F}_v(X)^*)$.

For each $(x,y) \in \widetilde{X}$, let us define the *weighted Lipschitz evaluation functional at* (x,y) by

$$\delta^v_{(x,y)} = v(x,y)\frac{\delta_x - \delta_y}{d(x,y)},$$

where δ_x is the *evaluation functional at* x given by $\delta_x(f) = f(x)$ for all $f \in \mathrm{Lip}_v(X)$.

Corollary 1.2. *Let X be a pointed metric space and let v be a weight on $\widetilde{X}$. We have*

(i) $B(\mathcal{F}_v(X)) = \overline{\mathrm{aco}}(\{\delta^v_{(x,y)}: (x,y) \in \widetilde{X}\}) \subset \mathrm{Lip}_v(X)^*$,

(ii) $\mathcal{F}_v(X) = \overline{\mathrm{lin}}(\{\delta_x: x \in X\}) \subset \mathrm{Lip}_v(X)^*$.

Proof. (i) It is easy to check that each functional δ_x with $x \in X$, defined on $\mathrm{Lip}_v(X)$, belongs to $\mathcal{F}_v(X)$, and hence so does every functional $\delta^v_{(x,y)}$ with $(x,y) \in \widetilde{X}$. Since P_X maps $\mathrm{Lip}_v(X)$ onto $\mathcal{F}_v(X)^*$ by Theorem 1.1, we have

$$\begin{aligned} P_X(B(\mathrm{Lip}_v(X))) &= \{P_X(f): f \in \mathrm{Lip}_v(X), |\delta^v_{(x,y)}(f)| \le 1, \forall (x,y) \in \widetilde{X}\} \\ &= \{P_X(f): f \in \mathrm{Lip}_v(X), |P_X(f)(\delta^v_{(x,y)})| \le 1, \forall (x,y) \in \widetilde{X}\} \\ &= \{F \in \mathcal{F}_v(X)^*: |F(\delta^v_{(x,y)})| \le 1, \forall (x,y) \in \widetilde{X}\} \\ &= \{\delta^v_{(x,y)}: (x,y) \in \widetilde{X}\}^\circ, \end{aligned}$$

and therefore $P_X(B(\mathrm{Lip}_v(X)))_\circ = (\{\delta^v_{(x,y)} \colon (x,y) \in \widetilde{X}\}^\circ)_\circ$. Since

$$P_X\big(B\big(\mathrm{Lip}_v(X)\big)\big)_\circ = B\big(\mathcal{F}_v(X)\big)$$

by Theorem 1.1, the equality in (i) follows by the bipolar theorem.

(ii) From (i) we infer that $\mathcal{F}_v(X)$ is the closed linear hull in $\mathrm{Lip}_v(X)^*$ of the set $\{\delta^v_{(x,y)} \colon (x,y) \in \widetilde{X}\}$. Then the equality in (ii) follows since the linear hulls of this set and the set $\{\delta_x \colon x \in X\}$ coincide. Note that

$$\delta_x = \delta_x - \delta_e = \big(d(x,e)/v(x,e)\big)\delta^v_{(x,e)} \quad (x \in X, x \neq e). \qquad \square$$

We now make some comments about Corollary 1.2.

Remark 1.3.

(1) By (i), $\mathcal{F}_v(X)$ consists of all linear functionals γ on $\mathrm{Lip}_v(X)$ of the form

$$\gamma(f) = \sum_{n=1}^{\infty} \alpha_n v(x_n, y_n) \frac{f(x_n) - f(y_n)}{d(x_n, y_n)}$$

for all $f \in \mathrm{Lip}_v(X)$, where $\{\alpha_n\}_{n\in\mathbb{N}} \in \ell_1$ and $\{(x_n, y_n)\}_{n\in\mathbb{N}} \in \widetilde{X}^{\mathbb{N}}$.

(2) The preceding item (ii) shows that $\mathcal{F}_v(X)$ is separable whenever X is separable.

(3) Another proof of the equality in (ii): if the closed linear hull in $\mathrm{Lip}_v(X)^*$ of the set $\{\delta_x \colon x \in X\}$ is not equal to $\mathcal{F}_v(X)$, then the Hahn–Banach theorem gives us a nonzero functional $P_X(f) \in \mathcal{F}_v(X)^*$ for some $f \in \mathrm{Lip}_v(X)$ such that $P_X(f)(\delta_x) = f(x) = 0$ for all $x \in X$, which yields a contradiction.

We next present a process for linearizing the elements of $\mathrm{Lip}_v(X, E)$.

Theorem 1.4. *Let X be a pointed metric space, and let v be a weight on $\widetilde{X}$.*

(i) *The Dirac map $\delta_X \colon x \mapsto \delta_x$ is in $\mathrm{Lip}_v(X, \mathcal{F}_v(X))$ and $\mathrm{Lip}_v(\delta_X) \leq 1$.*

(ii) *Universal property: For each Banach space E and each map $f \in \mathrm{Lip}_v(X, E)$, there is a unique operator $T_f \in \mathcal{L}(\mathcal{F}_v(X), E)$ such that $T_f \circ \delta_X = f$; that is, the diagram*

$$\begin{tikzcd} X \arrow[d, "\delta_X"'] \arrow[dr, "f"] & \\ \mathcal{F}_v(X) \arrow[r, dashed, "T_f"'] & E \end{tikzcd}$$

commutes. Furthermore, $\|T_f\| = \mathrm{Lip}_v(f)$.

Proof. (i) For any $(x,y) \in \widetilde{X}$, we have $\|\delta_x - \delta_y\| \leq d(x,y)/v(x,y)$ because

$$\|\delta_x - \delta_y\| = \sup\big\{\big|f(x) - f(y)\big| \colon f \in B\big(\mathrm{Lip}_v(X)\big)\big\}.$$

(ii) Let E be a Banach space and let $f \in \mathrm{Lip}_v(X, E)$. Then the map $T_f \colon \mathcal{F}_v(X) \to E^{**}$ defined by

$$T_f(\gamma)(\phi) = \gamma(\phi \circ f) \quad \big(\gamma \in \mathcal{F}_v(X), \phi \in E^*\big)$$

belongs to $\mathcal{L}(\mathcal{F}_v(X), E^{**})$ and

$$\|T_f\| = \sup_{\|\gamma\|\leq 1} \big\|T_f(\gamma)\big\| = \sup_{\|\phi\|\leq 1} \sup_{\|\gamma\|\leq 1} \big|\gamma(\phi \circ f)\big| = \sup_{\|\phi\|\leq 1} \mathrm{Lip}_v(\phi \circ f) \leq \mathrm{Lip}_v(f).$$

Furthermore, if $J_E \colon E \to E^{**}$ is the canonical injection, we have

$$(T_f \circ \delta_X)(x)(\phi) = T_f(\delta_x)(\phi) = \delta_x(\phi \circ f) = \phi\big(f(x)\big) = J_E\big(f(x)\big)(\phi)$$

for every $x \in X$ and $\phi \in E^*$, and hence $T_f \circ \delta_X(x) = J_E(f(x)) \in J_E(E)$ for every $x \in X$. By Corollary 1.2(ii), it follows that $T_f(\gamma) \in J_E(E)$ for every $\gamma \in \mathcal{F}_v(X)$. Now, by identifying $J_E(f(x)) \in J_E(E)$ with $f(x) \in E$, we have $T_f \in \mathcal{L}(\mathcal{F}_v(X), E)$ and $T_f \circ \delta_X = f$. Finally, notice that the conditions $T_f \circ \delta_x = f$ and $\|T_f\| \leq \mathrm{Lip}_v(f)$ imply that $\|T_f\| = \mathrm{Lip}_v(f)$. Assume now that there exists $S_f \in \mathcal{L}(\mathcal{F}_v(X), E)$ such that $S_f \circ \delta_X = f$. Then $S_f(\delta_x) = T_f(\delta_x)$ for all $x \in X$, and taking into account again Corollary 1.2(ii), it follows that $S_f = T_f$. This proves the uniqueness of T_f and completes the proof of the theorem. $\square$

We next prove that the universal property characterizes $\mathcal{F}_v(X)$ uniquely up to an isometric isomorphism.

Corollary 1.5. *Let X be a pointed metric space and let v be a weight on $\widetilde{X}$. If $\mathcal{G}_v(X)$ is a Banach space and β_X is a map in $\mathrm{Lip}_v(X, \mathcal{G}_v(X))$ with $\mathrm{Lip}_v(\beta_X) \leq 1$ satisfying the above universal property, then there exists an isometric isomorphism T_{β_X} from $\mathcal{F}_v(X)$ onto $\mathcal{G}_v(X)$ such that $T_{\beta_X} \circ \delta_X = \beta_X$.*

Proof. There exist bounded linear operators $T_{\beta_X} \colon \mathcal{F}_v(X) \to \mathcal{G}_v(X)$ and $T_{\delta_X} \colon \mathcal{G}_v(X) \to \mathcal{F}_v(X)$ such that $T_{\beta_X} \circ \delta_X = \beta_X$, $T_{\delta_X} \circ \beta_X = \delta_X$, $\|T_{\beta_X}\| = \mathrm{Lip}_v(\beta_X) \leq 1$ and $\|T_{\delta_X}\| = \mathrm{Lip}_v(\delta_X) \leq 1$. Then $T_{\beta_X} \circ T_{\delta_X} \circ \beta_X = \beta_X$ and, since also $\mathrm{Id}_{\mathcal{G}_v(X)} \circ \beta_X = \beta_X$, it follows that $T_{\beta_X} \circ T_{\delta_X} = \mathrm{Id}_{\mathcal{G}_v(X)}$ by the uniqueness of the universal property. Similarly, we obtain that $T_{\delta_X} \circ T_{\beta_X} = \mathrm{Id}_{\mathcal{F}_v(X)}$. It follows that T_{β_X} is bijective and also that $T_{\beta_X}^{-1} = T_{\delta_X}$. Moreover, $\|\gamma\| = \|T_{\delta_X}(T_{\beta_X}(\gamma))\| \leq \|T_{\beta_X}(\gamma)\| \leq \|\gamma\|$ for all $\gamma \in \mathcal{F}_v(X)$. Hence $T_{\beta_X} \colon \mathcal{F}_v(X) \to \mathcal{G}_v(X)$ is an isometric isomorphism. $\square$

Theorem 1.4 permits us to identify the spaces $\mathrm{Lip}_v(X, E)$ and $\mathcal{L}(\mathcal{F}_v(X), E)$.

Corollary 1.6. *Let X be a pointed metric space, let v be a weight on $\widetilde{X}$, and let E be a Banach space. The map $f \mapsto T_f$ is an isometric isomorphism from $\mathrm{Lip}_v(X, E)$ onto $\mathcal{L}(\mathcal{F}_v(X), E)$.*

Proof. For any $\alpha, \beta \in \mathbb{K}$ and $f, g \in \mathrm{Lip}_v(X, E)$, we have

$$(\alpha T_f + \beta T_g) \circ \delta_X = \alpha(T_f \circ \delta_X) + \beta(T_g \circ \delta_X) = \alpha f + \beta g.$$

Now, the uniqueness of $T_{\alpha f + \beta g}$ in $\mathcal{L}(\mathcal{F}_v(X), E)$ satisfying this equality implies that $T_{\alpha f + \beta g} = \alpha T_f + \beta T_g$. Hence the map $f \mapsto T_f$ from $\mathrm{Lip}_v(X, E)$ into $\mathcal{L}(\mathcal{F}_v(X), E)$ is linear. Moreover, this map is an isometry by Theorem 1.4. It remains to check its surjectivity. Given $T \in \mathcal{L}(\mathcal{F}_v(X), E)$, we can define a map $f \colon X \to E$ by $f(x) = T(\delta_x)$ for all $x \in X$. At this point, since

$$\begin{aligned}\big\|f(x) - f(y)\big\| &= \big\|T(\delta_x) - T(\delta_y)\big\| = \big\|T(\delta_x - \delta_y)\big\| \leq \|T\| \|\delta_x - \delta_y\| \\ &\leq \|T\| d(x, y)/v(x, y)\end{aligned}$$

for all $(x,y) \in \widetilde{X}$, the function f is in $\mathrm{Lip}_v(X,E)$. Now, by the universal property, there is a unique operator $T_f \in \mathcal{L}(\mathcal{F}_v(X),E)$ such that $T_f \circ \delta_X = f$, and so $T = T_f$. □

We finish this section establishing some facts about topologies on $\mathrm{Lip}_v(X)$ which will be needed later. We can view $\mathrm{Lip}_v(X)$ as the dual of $\mathcal{F}_v(X)$ by Theorem 1.1, and therefore we can consider other topologies on $\mathrm{Lip}_v(X)$. The *weak* topology* τ_{w^*} is the locally convex topology on $\mathrm{Lip}_v(X)$ defined by the seminorms of the form

$$p_G(f) = \sup\{|\gamma(f)| : \gamma \in G\}, \quad f \in \mathrm{Lip}_v(X),$$

where G varies over the family of all finite subsets of $\mathcal{F}_v(X)$. Since the functionals δ_x on $\mathrm{Lip}_v(X)$ are elements of $\mathcal{F}_v(X)$, we have that τ_{w^*} is larger than τ_p. Moreover, both topologies coincide on $B(\mathrm{Lip}_v(X))$ because $B(\mathrm{Lip}_v(X))$ is τ_{w^*}-compact. The *bounded weak* topology* τ_{bw^*} is the largest topology on $\mathrm{Lip}_v(X)$ which coincides with the topology τ_{w^*} on norm-bounded sets of $\mathcal{F}_v(X)$. The following proposition gathers some properties of the bounded weak* topology on $\mathrm{Lip}_v(X)$ which can be found or deduced in [12].

Proposition 1.7 ([12, Section 2.7])**.** *Let X be a pointed metric space and let v be a weight on $\widetilde{X}$ Then we have the following.*

(1) *τ_{w^*} is smaller than τ_{bw^*}, and τ_{bw^*} is smaller that the norm topology on $\mathrm{Lip}_v(X)$.*
(2) *τ_{bw^*} agrees with the topology of uniform convergence on sequences in $\mathcal{F}_v(X)$ which tend in norm to zero.*
(3) *A subset U of $\mathrm{Lip}_v(X)$ is open in $(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})$ if and only if $U \cap nB(\mathrm{Lip}_v(X))$ is open in $(nB(\mathrm{Lip}_v(X)), \tau_{w^*})$ for each $n \in \mathbb{N}$.*
(4) *A subset of $\mathrm{Lip}_v(X)$ is τ_{bw^*}-bounded if and only if it is norm-bounded.*
(5) *A subset of $\mathrm{Lip}_v(X)$ is τ_{bw^*}-compact if and only if it is norm-bounded and τ_{w^*}-compact.*
(6) *$(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})$ is a complete semi-Montel space.*
(7) *If γ is a linear functional on $\mathrm{Lip}_v(X)$, then γ is τ_{bw^*}-continuous if and only if γ is τ_{w^*}-continuous.*

Let us recall that a locally convex space E is a *(DFC)-space* if there exists a Fréchet space F such that $E = (F', \tau_c)$, where τ_c is the *topology of uniform convergence on compact sets of F*. Moreover, we will denote by τ_b the *topology of uniform convergence on bounded sets.*

Theorem 1.8. *Let X be a pointed metric space, and let v be a weight on $\widetilde{X}$. Then we have the following.*

(i) *$(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})$ is a (DFC)-space. More precisely, $((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_b)$ is a Fréchet space and the evaluation map is a topological isomorphism from $(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})$ onto $(((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_b)', \tau_c)$.*
(ii) *The evaluation map is a topological isomorphism from $\mathcal{F}_v(X)$ onto the space $((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_b)$.*
(iii) *The identity map is a topological isomorphism from $((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_b)$ onto $((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_c)$.*

(iv) *The evaluation map* P_X *is a topological isomorphism from* $(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})$ *onto* $(\mathcal{F}_v(X)^*, \tau_c)$.

Proof. Item (i) follows by applying [13, Theorem 4.1], because in light of the definition of the topology τ_{bw^*}, its properties (2), (3), and (5) in Proposition 1.7, and the Banach–Alaoglu theorem, we have that $(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})$ is a locally convex space and that $\{nB(\mathrm{Lip}_v(X))\}_{n\in\mathbb{N}}$ is an increasing sequence of convex, balanced, and τ_{bw^*}-compact subsets of $\mathrm{Lip}_v(X)$ with the property that a set $U \subset \mathrm{Lip}_v(X)$ is τ_{bw^*}-open whenever $U \cap nB(\mathrm{Lip}_v(X))$ is open in $(nB(\mathrm{Lip}_v(X)), \tau_{\mathrm{bw}^*})$ for every $n \in \mathbb{N}$.

(ii) Using Proposition 1.7(7), it is easily proved that the evaluation map S, defined by

$$S(\gamma)(f) = \gamma(f) \quad \big(f \in \mathrm{Lip}_v(X), \gamma \in \mathcal{F}_v(X)\big),$$

is an algebraic isomorphism from $\mathcal{F}_v(X)$ onto $(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})'$ and that its inverse is the map S^{-1} from $(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})'$ to $\mathcal{F}_v(X)$, given by

$$S^{-1}(\phi)(f) = \phi(f) \quad \big(f \in \mathrm{Lip}_v(X), \phi \in \big(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*}\big)'\big).$$

Let B be a τ_{bw^*}-bounded subset of $\mathrm{Lip}_v(X)$. By Proposition 1.7(4), $B \subset \lambda B(\mathrm{Lip}_v(X))$ for some $\lambda > 0$. Given that $\gamma \in \mathcal{F}_v(X)$, we have that $|S(\gamma)(f)| \leq \lambda\|\gamma\|$ for all $f \in B$, and this proves that S is continuous from $\mathcal{F}_v(X)$ to $((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_b)$. On the other hand, if $\phi \in (\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})'$, then we have that $\phi \in \mathrm{Lip}_v(X)^*$ by Proposition 1.7(1), and therefore $|S^{-1}(\phi)(f)| \leq \|\phi\|$ for all $f \in B(\mathrm{Lip}_v(X))$. Since $B(\mathrm{Lip}_v(X))$ is a τ_{bw^*}-bounded subset of $\mathrm{Lip}_v(X)$ by Proposition 1.7(4), we conclude that S^{-1} is continuous from $((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_b)$ to $\mathcal{F}_v(X)$. This finishes the proof of (ii).

(iii) This follows because $(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})$ is semi-Montel.

(iv) This follows from (i) and (ii). Indeed, the evaluation map, say R, is a topological isomorphism from $(\mathrm{Lip}_0(X), \tau_{\mathrm{bw}^*})$ onto $(((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_b)', \tau_c)$ by (i). Using (ii), it is easy to check that the transpose map S^t is a topological isomorphism from $(((\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})', \tau_b)', \tau_c)$ onto $(\mathcal{F}_v(X)^*, \tau_c)$. Hence $S^t \circ R$ is a topological isomorphism from $(\mathrm{Lip}_v(X), \tau_{\mathrm{bw}^*})$ onto $(\mathcal{F}_v(X)^*, \tau_c)$, and an easy verification shows that $S^t \circ R$ is the evaluation map P_X defined in Theorem 1.1. □

Although Theorem 1.8 is interesting in its own right, we have proved it essentially to deduce the following result, which we will use later.

Corollary 1.9. *Let* X *be a pointed metric space and let* v *be a weight on* $\widetilde{X}$. *If* K *is a compact subset of* $\widetilde{X}$, *then the set*

$$\Big\{f \in B\big(\mathrm{Lip}_v(X)\big) \colon \sup_{(x,y)\in K}\Big[v(x,y)\frac{|f(x)-f(y)|}{d(x,y)}\Big] \leq 1\Big\}$$

is a neighborhood of zero in $(B(\mathrm{Lip}_v(X)), \tau_p)$.

Proof. Define $\sigma \colon \widetilde{X} \to \mathcal{F}_v(X)$ by

$$\sigma(x,y) = v(x,y)\frac{\delta_x - \delta_y}{d(x,y)}, \quad (x,y) \in \widetilde{X}.$$

Since the mappings $(x,y)\mapsto v(x,y)$, $x\mapsto\delta_x$, and $(x,y)\mapsto d(x,y)$ are continuous, so is σ. Then $\sigma(K)$ is a compact subset of $\mathcal{F}_v(X)$ and therefore the polar

$$\sigma(K)^\circ=\Big\{F\in\mathcal{F}_v(X)^*\colon \sup_{(x,y)\in K}\big|F\big(\sigma(x,y)\big)\big|\le 1\Big\}$$

is a neighborhood of zero in $(\mathcal{F}_v(X)^*,\tau_c)$. Then, by Theorem 1.8(iv), the set

$$P_X^{-1}\big(\sigma(K)^\circ\big)=\left\{f\in\mathrm{Lip}_v(X)\colon \sup_{(x,y)\in K}\left[v(x,y)\frac{|f(x)-f(y)|}{d(x,y)}\right]\le 1\right\}$$

is a neighborhood of zero in $(\mathrm{Lip}_v(X),\tau_{\mathrm{bw}^*})$. By Proposition 1.7(3), it follows that

$$B\big(\mathrm{Lip}_v(X)\big)\cap\left\{f\in\mathrm{Lip}_v(X)\colon \sup_{(x,y)\in K}\left[v(x,y)\frac{|f(x)-f(y)|}{d(x,y)}\right]\le 1\right\}$$

is a neighborhood of zero in $(B(\mathrm{Lip}_v(X)),\tau_{w^*})$. The proof is now complete because

$$\big(B\big(\mathrm{Lip}_v(X)\big),\tau_{w^*}\big)=\big(B\big(\mathrm{Lip}_v(X)\big),\tau_p\big). \qquad\square$$

2. Biduality of weighted Lipschitz spaces

Given a pointed compact metric space X and a weight v on $\widetilde{X}$, we address in this section the biduality problem as to when $\mathrm{Lip}_v(X)$ is naturally isometrically isomorphic to $\mathrm{lip}_v(X)^{**}$. Observe that $\widetilde{X}$ is a locally compact metric space. Let $C_b(\widetilde{X})$ be the Banach space of all bounded continuous scalar-valued functions on $\widetilde{X}$ with the supremum norm, and let $C_0(\widetilde{X})$ be its closed subspace of functions which vanish at infinity. We begin noting that the map $\Phi_v\colon \mathrm{Lip}_v(X)\to C_b(\widetilde{X})$, defined by

$$\Phi_v(f)(x,y)=v(x,y)\frac{f(x)-f(y)}{d(x,y)}\quad \big(f\in\mathrm{Lip}_v(X),(x,y)\in\widetilde{X}\big),$$

is an isometric linear embedding from $\mathrm{Lip}_v(X)$ into $C_b(\widetilde{X})$. In the case $v=1_{\widetilde{X}}$, Φ_v is called the *de Leeuw's map* and denoted by Φ, as in [15, p. 34]. We next study the restriction of the map Φ_v to $\mathrm{lip}_v(X)$.

Proposition 2.1. *Let X be a pointed compact metric space and let v be a weight on $\widetilde{X}$. Then the map $\Phi_v|_{\mathrm{lip}_v(X)}$ is an isometric linear embedding from $\mathrm{lip}_v(X)$ into $C_0(\widetilde{X})$. In fact, the supremum $\mathrm{Lip}_v(f)$ is attained for each $f\in\mathrm{lip}_v(X)$. Moreover, $\mathrm{lip}_v(X)=\Phi_v^{-1}(C_0(\widetilde{X}))$ and as a consequence $\mathrm{lip}_v(X)$ is a closed subspace of $\mathrm{Lip}_v(X)$.*

Proof. To prove the first assertion, we only need to check that Φ_v maps $\mathrm{lip}_v(X)$ to $C_0(\widetilde{X})$. Let $f\in\mathrm{lip}_v(X)$ and let $\varepsilon>0$. Then there is a $\delta>0$ such that if $(x,y)\in\widetilde{X}$ and $d(x,y)<\delta$, then $|\Phi_v(f)(x,y)|<\varepsilon$. Clearly, the set $K=\{(x,y)\in X\times X\colon d(x,y)\ge\delta\}$ is a compact subset of $\widetilde{X}$, and if $(x,y)\in\widetilde{X}\setminus K$, then we have that $|\Phi_v(f)(x,y)|<\varepsilon$, and this proves that $\Phi_v(f)\in C_0(\widetilde{X})$. Moreover, we deduce that $\|\Phi_v(f)\|_\infty\le\varepsilon+\|\Phi_v(f|_K)\|_\infty$, and hence that $\|\Phi_v(f)\|_\infty\le\|\Phi_v(f|_K)\|_\infty$; therefore $\|\Phi_v(f)\|_\infty=\|\Phi_v(f|_K)\|_\infty$ and thus $\mathrm{Lip}_v(f)=|\Phi_v(f)(x,y)|$ for some point $(x,y)\in K$.

Finally, we show that if $f \in \mathrm{Lip}_v(X)$ and $\Phi_v(f) \in C_0(\widetilde{X})$, then $f \in \mathrm{lip}_v(X)$. Indeed, let $\varepsilon > 0$. Then there exists a compact set $K \subset \widetilde{X}$ such that $|\Phi_v(f)(x,y)| < \varepsilon$ whenever $(x,y) \in \widetilde{X} \setminus K$. Since $\{K_\delta \colon \delta > 0\}$, where $K_\delta = \{(x,y) \in X \times X \colon d(x,y) \geq \delta\}$, is an exhaustive family of compacts subsets of $\widetilde{X}$, we have that $K \subset K_\delta$ for some $\delta > 0$. Hence, if $(x,y) \in \widetilde{X}$ and $d(x,y) < \delta$, then $|\Phi_v(f)(x,y)| < \varepsilon$ and so $f \in \mathrm{lip}_v(X)$. □

The restriction map $\gamma \mapsto \gamma|_{\mathrm{lip}_v(X)}$ from $\mathcal{F}_v(X)$ to $\mathrm{lip}_v(X)^*$ that we introduce next is the essential tool in our study on the biduality of $\mathrm{Lip}_v(X)$.

Theorem 2.2. *Let X be a pointed compact metric space and let v be a weight on $\widetilde{X}$. Then the restriction map $R_X \colon \mathcal{F}_v(X) \to \mathrm{lip}_v(X)^*$, defined by*

$$R_X(\gamma)(f) = \gamma(f) \quad \big(f \in \mathrm{lip}_v(X), \gamma \in \mathcal{F}_v(X)\big),$$

is a nonexpansive linear surjective map. In fact, for each $\phi \in \mathrm{lip}_v(X)^*$, there exists a $\gamma \in \mathcal{F}_v(X)$ such that $R_X(\gamma) = \phi$ and $\|\gamma\| = \|\phi\|$.

Proof. Since $\mathcal{F}_v(X) \subset \mathrm{Lip}_v(X)^*$, it is clear that R_X is a linear map from $\mathcal{F}_v(X)$ into $\mathrm{lip}_v(X)^*$ satisfying $\|R_X(\gamma)\| \leq \|\gamma\|$ for all $\gamma \in \mathcal{F}_v(X)$. We next prove that R_X is surjective. Take $\phi \in \mathrm{lip}_v(X)^*$. The functional $T \colon \Phi_v(\mathrm{lip}_v(X)) \to \mathbb{K}$, defined by $T(\Phi_v(f)) = \phi(f)$ for all $f \in \mathrm{lip}_v(X)$, is linear, continuous, and $\|T\| = \|\phi\|$. In light of Proposition 2.1, by the Hahn–Banach theorem there exists a continuous linear functional $\widetilde{T} \colon \mathcal{C}_0(\widetilde{X}) \to \mathbb{K}$ such that $\widetilde{T}(\Phi_v(f)) = T(\Phi_v(f))$ for all $f \in \mathrm{lip}_v(X)$ and $\|\widetilde{T}\| = \|T\|$. Now, by the Riesz representation theorem, there exists a finite regular Borel measure μ on $\widetilde{X}$ with total variation $\|\mu\| = \|\widetilde{T}\|$ such that

$$\widetilde{T}(g) = \int_{\widetilde{X}} g \, d\mu$$

for all $g \in \mathcal{C}_0(\widetilde{X})$, and thus

$$\phi(f) = \int_{\widetilde{X}} \Phi_v(f) \, d\mu$$

for all $f \in \mathrm{lip}_v(X)$. If we now define

$$\gamma(f) = \int_{\widetilde{X}} \Phi_v(f) \, d\mu \quad \big(f \in \mathrm{Lip}_v(X)\big),$$

it is clear that $\gamma \in \mathrm{Lip}_v(X)^*$ and $\gamma(f) = \phi(f)$ for all $f \in \mathrm{lip}_v(X)$. Then $\|\gamma\| \geq \|\phi\| = \|\mu\|$. Conversely,

$$\big|\gamma(f)\big| \leq \int_{\widetilde{X}} \big|\Phi_v(f)\big| d|\mu| \leq \big\|\Phi_v(f)\big\|_\infty \|\mu\| = \mathrm{Lip}_v(f)\|\mu\|$$

for all $f \in \mathrm{Lip}_v(X)$, and thus $\|\gamma\| \leq \|\mu\|$. Hence $\|\gamma\| = \|\mu\| = \|\phi\|$. It remains to show that γ is τ_p-continuous on $B(\mathrm{Lip}_v(X))$. Thus, let $\{f_i\}$ be a net in $B(\mathrm{Lip}_v(X))$ which converges pointwise on X to zero. Then $\{\Phi_v(f_i)\}$ converges pointwise on $\widetilde{X}$ to zero and, since $|\Phi_v(f_i)(x,y)| \leq \|\Phi_v(f_i)\|_\infty = \mathrm{Lip}_v(f_i) \leq 1$ for all $i \in I$ and for all $(x,y) \in \widetilde{X}$, it follows that $\{\gamma(f_i)\}$ converges to zero by the Lebesgue bounded convergence theorem. This finishes the proof. □

The double duality results for $\mathrm{Lip}_0(X)$ have an interesting history (see [15, p. 99, Notes 3.3] for an abstract on results obtained about this problem). The identification between $\mathrm{Lip}_0(X)$ and $\mathrm{lip}_0(X)^{**}$ in those results is justly the isometric isomorphism $P_X^{-1} \circ R_X^*$, where $P_X \colon \mathrm{Lip}_0(X) \to \mathcal{F}(X)^*$ is the evaluation, $\mathcal{F}(X)$ is the Lipschitz free Banach space over X, and $R_X \colon \mathcal{F}(X) \to \mathrm{lip}_0(X)^*$ is the restriction map. This fact justifies the following definition.

Definition 2.3. Let X be a pointed compact metric space and let v be a weight on $\widetilde{X}$. We say that $\mathrm{lip}_v(X)^{**}$ is canonically isometrically (topologically) isomorphic to $\mathrm{Lip}_v(X)$ if the map $P_X^{-1} \circ R_X^*$ is an isometric (resp., a topological) isomorphism from $\mathrm{lip}_v(X)^{**}$ onto $\mathrm{Lip}_v(X)$.

Remark 2.4. An easy verifications yields

$$\begin{aligned}
(P_X^{-1} \circ R_X^*)(\phi)(x) &= \delta_x\big((P_X^{-1} \circ R_X^*)(\phi)\big) \\
&= P_X\big((P_X^{-1} \circ R_X^*)(\phi)\big)(\delta_x) \\
&= P_X\big(P_X^{-1}\big(R_X^*(\phi)\big)\big)(\delta_x) \\
&= R_X^*(\phi)(\delta_x) \\
&= \phi\big(R_X(\delta_x)\big) \\
&= \phi(\delta_x|_{\mathrm{lip}_v(X)})
\end{aligned}$$

for any $\phi \in \mathrm{lip}_v(X)^{**}$ and $x \in X$.

Taking into account that $P_X \colon \mathrm{Lip}_v(X) \to \mathcal{F}_v(X)^*$ is always an isometric isomorphism by Theorem 1.1 and that $R_X \colon \mathcal{F}_v(X) \to \mathrm{lip}_v(X)^*$ is an isometric (resp., a topological) isomorphism if and only $R_X^* \colon \mathrm{lip}_v(X)^{**} \to \mathcal{F}_v(X)^*$ is an isometric (resp., a topological) isomorphism, we have the following.

Proposition 2.5. *Let X be a pointed compact metric space and let v be a weight on $\widetilde{X}$. The following are equivalent:*

(i) $\mathrm{lip}_v(X)^{**}$ *is canonically isometrically (topologically) isomorphic to* $\mathrm{Lip}_v(X)$,
(ii) R_X *is an isometric (resp., a topological) isomorphism from $\mathcal{F}_v(X)$ onto* $\mathrm{lip}_v(X)^*$.

Therefore the question as to when $\mathrm{Lip}_v(X)$ is canonically isometrically isomorphic to $\mathrm{lip}_v(X)^{**}$ is the question as to when R_X is an isometric isomorphism. We next characterize when R_X becomes a topological isomorphism.

Proposition 2.6. *Let X be a pointed compact metric space and let v be a weight on $\widetilde{X}$. The following are equivalent.*

(i) $\mathrm{Lip}_v(X)$ *is canonically topologically isomorphic to* $\mathrm{lip}_v(X)^{**}$.
(ii) *There is a constant $\lambda \geq 1$ such that $B(\mathrm{Lip}_v(X))$ is contained in the closure of $\lambda B(\mathrm{lip}_v(X))$ in $(B(\mathrm{Lip}_v(X)), \tau_p)$.*

(iii) *There exists a constant* $\lambda \geq 1$ *such that*

$$\|\gamma\| \leq \lambda \sup\bigl\{\bigl|\gamma(f)\bigr| : f \in B\bigl(\operatorname{lip}_v(X)\bigr)\bigr\} \quad \textit{for all } \gamma \in \mathcal{F}_v(X).$$

Proof. (i) $\Rightarrow$ (ii): If (i) holds, then there exists a $\lambda \geq 1$ such that $\|\gamma\| \leq \lambda \|R_X(\gamma)\|$ for all $\gamma \in \mathcal{F}_v(X)$. This implies that $B(\operatorname{Lip}_v(X))$ is contained in the closure of $\lambda B(\operatorname{lip}_v(X))$ in $(B(\operatorname{Lip}_v(X)), \tau_p)$. Otherwise, by the Hahn–Banach separation theorem, we could find a function $g \in B(\operatorname{Lip}_v(X))$ and a τ_p-continuous linear functional γ on $\operatorname{Lip}_v(X)$ such that $|\gamma(h)| \leq 1$ for all $h \in \lambda B(\operatorname{lip}_v(X))$ and $|\gamma(g)| > 1$. Since $\gamma \in \mathcal{F}_v(X)$ and $\lambda\|R_X(\gamma)\| = \lambda\|\gamma|_{\operatorname{lip}_v(X)}\| \leq 1 < |\gamma(g)| \leq \|\gamma\|$, we would arrive at a contradiction.

(ii) $\Rightarrow$ (iii): Let $\gamma \in \mathcal{F}_v(X)$. Taking into account (ii) and the fact that $(B(\operatorname{Lip}_v(X)), \tau_p) = (B(\operatorname{Lip}_v(X)), \tau_{w^*})$, there exists a constant $\lambda \geq 1$ such that

$$\begin{aligned}\|\gamma\| &= \sup\bigl\{\bigl|\gamma(h)\bigr| : h \in B\bigl(\operatorname{Lip}_v(X)\bigr) \cap \lambda\overline{B\bigl(\operatorname{lip}_v(X)\bigr)}^{\tau_{w^*}}\bigr\} \\ &\leq \sup\bigl\{\bigl|\gamma(h)\bigr| : h \in \lambda\overline{B\bigl(\operatorname{lip}_v(X)\bigr)}^{\tau_{w^*}}\bigr\}.\end{aligned}$$

An easy argument shows that this last supremum agrees with

$$\sup\bigl\{\bigl|\gamma(h)\bigr| : h \in \lambda B\bigl(\operatorname{lip}_v(X)\bigr)\bigr\}$$

and thus we have $\|\gamma\| \leq \lambda \sup\{|\gamma(f)| : f \in B(\operatorname{lip}_v(X))\}$.

(iii) $\Rightarrow$ (i): By Theorem 2.2, R_X is a continuous linear surjective map from $\mathcal{F}_v(X)$ onto $\operatorname{lip}_v(X)^*$. If (iii) holds, then $\|\gamma\| \leq \lambda\|R_X(\gamma)\|$ for all $\gamma \in \mathcal{F}_v(X)$ and, in particular, R_X is injective. Hence R_X is a topological isomorphism and we have (i) by Proposition 2.5. $\square$

We may complete Proposition 2.6 showing, with a similar proof, that the topological isomorphism R_X becomes an isometric isomorphism when the constant λ takes its minimum value 1. Assertion (iii) of that proposition means that $\operatorname{lip}_v(X)$ is a λ-norming subspace of $\mathcal{F}_v(X)^*$. If $\lambda = 1$, then it is called *norming*.

Proposition 2.7. *Let* X *be a pointed compact metric space and let* v *be a weight on* $\widetilde{X}$*. The following are equivalent:*

(i) $\operatorname{Lip}_v(X)$ *is canonically isometrically isomorphic to* $\operatorname{lip}_v(X)^{**}$.
(ii) $B(\operatorname{lip}_v(X))$ *is dense in* $(B(\operatorname{Lip}_v(X)), \tau_p)$.
(iii) $\operatorname{lip}_v(X)$ *is a norming subspace of* $\mathcal{F}_v(X)^*$.

Our next objective is to show that $\operatorname{Lip}_v(X)$ is canonically isometrically isomorphic to $\operatorname{lip}_v(X)^{**}$ only if it is so topologically; the concepts of the M-ideal and associated weights $\widetilde{v}_l$ and $\widetilde{v}_L$ are the tools which will permit us to prove it. Let us begin by recalling that a closed subspace J of a Banach space E is called an *M-ideal* if there is a closed subspace J_0 of the dual space E^* such that E^* is the ℓ_1-sum $J^\perp \oplus_1 J_0$, where $J^\perp$ is the annihilator of J in E^*. (This notion was first investigated by Alfsen and Effros in [1], and it is studied in detail by Harmand, Werner, and Werner in [8]. M-ideals in Lipschitz spaces have been addressed by Berninger and Werner [2] and by Kalton [10].)

Given a weight $v\colon \widetilde{X} \to \mathbb{R}$, define $\widetilde{v}\colon \widetilde{X} \to \mathbb{R}$ by $\widetilde{v}(x,y) = 1/v(x,y)$. It is clear that

$$B\big(\mathrm{lip}_v(X)\big) = \Big\{f \in \mathrm{lip}_v(X)\colon \frac{|f(x)-f(y)|}{d(x,y)} \leq \widetilde{v}(x,y), \forall (x,y) \in \widetilde{X}\Big\},$$
$$B\big(\mathrm{Lip}_v(X)\big) = \Big\{f \in \mathrm{Lip}_v(X)\colon \frac{|f(x)-f(y)|}{d(x,y)} \leq \widetilde{v}(x,y), \forall (x,y) \in \widetilde{X}\Big\}.$$

Let $v_l, v_L\colon \widetilde{X} \to \mathbb{R}$ be given by

$$v_l(x,y) = \sup\Big\{\frac{|f(x)-f(y)|}{d(x,y)}\colon f \in B\big(\mathrm{lip}_v(X)\big)\Big\},$$
$$v_L(x,y) = \sup\Big\{\frac{|f(x)-f(y)|}{d(x,y)}\colon f \in B\big(\mathrm{Lip}_v(X)\big)\Big\},$$

and define $\widetilde{v}_l, \widetilde{v}_L\colon \widetilde{X} \to \mathbb{R}$ by $\widetilde{v}_l(x,y) = 1/v_l(x,y)$ and $\widetilde{v}_L(x,y) = 1/v_L(x,y)$. Then $\widetilde{v}_l$ and $\widetilde{v}_L$ are weights on $\widetilde{X}$ for which $0 < v \leq \widetilde{v}_L \leq \widetilde{v}_l$, and they are called the *weights associated* to v_l and v_L, respectively. Notice that the supremum $\widetilde{v}_L(x,y)$ is attained for each $(x,y) \in \widetilde{X}$.

Note that if R_X is an isometric isomorphism from $\mathcal{F}_v(X)$ onto $\mathrm{lip}_v(X)^*$, then the set

$$\mathcal{G}_v(X) := \big\{\gamma \in \mathcal{F}_v(X)\colon \|\gamma\| = \|\gamma|_{\mathrm{lip}_v(X)}\|\big\}$$

coincides with $\mathcal{F}_v(X)$. However, $\mathcal{G}_v(X)$ is not necessarily a linear space. We next give some conditions under which $\mathcal{G}_v(X)$ is a linear subspace of $\mathcal{F}_v(X)$ and the corresponding versions for $\mathcal{G}_v(X)$ in place of $\mathcal{F}_v(X)$ of the evaluation map $P_X\colon \mathrm{Lip}_v(X) \to \mathcal{F}_v(X)^*$ and the restriction map $R_X\colon \mathcal{F}_v(X) \to \mathrm{lip}_v(X)^*$ are isometric isomorphisms.

Proposition 2.8. *Let X be a pointed compact metric space and let v be a weight on $\widetilde{X}$. Assume that* $\mathrm{lip}_v(X)$ *is an M-ideal in* $\mathrm{Lip}_v(X)$*. Then we have the following.*

(i) *$\mathcal{G}_v(X)$ is a linear subspace of $\mathcal{F}_v(X)$ and the restriction map $S_X\colon \mathcal{G}_v(X) \to \mathrm{lip}_v(X)^*$, defined by*

$$S_X(\gamma)(f) = \gamma(f) \quad \big(f \in \mathrm{lip}_v(X), \gamma \in \mathcal{G}_v(X)\big),$$

is an isometric isomorphism.

(ii) *If we assume in addition that $\widetilde{v}_L = \widetilde{v}_l$, then the evaluation map $Q_X\colon \mathrm{Lip}_v(X) \to \mathcal{G}_v(X)^*$, given by*

$$Q_X(f)(\gamma) = \gamma(f) \quad \big(f \in \mathrm{Lip}_v(X), \gamma \in \mathcal{G}_v(X)\big),$$

is an isometric isomorphism.

Proof. (i) If $\mathrm{lip}_v(X)$ is an M-ideal in $\mathrm{Lip}_v(X)$, then every functional in $\mathrm{lip}_v(X)^*$ has a unique norm-preserving extension to a functional in $\mathrm{Lip}_v(X)^*$ by [8, Chapter I, Proposition 1.12]. Hence $\mathcal{G}_v(X)$ is a linear subspace of $\mathcal{F}_v(X)$. Clearly, S_X is a linear well-defined map from $\mathcal{G}_v(X)$ to $\mathrm{lip}_v(X)^*$. By the very definition of $\mathcal{G}_v(X)$, S_X is an isometry. The surjectivity of S_X follows from the second assertion of Theorem 2.2.

(ii) It is immediate that $Q_X\colon \mathrm{Lip}_v(X) \to \mathcal{G}_v(X)^*$ is linear and continuous (in fact, $\|Q_X(f)\| \leq \mathrm{Lip}_v(f)$ for all $f \in \mathrm{Lip}_v(X)$). To prove that Q_X is injective, let $f, g \in \mathrm{Lip}_v(X)$ and suppose that $Q_X(f) = Q_X(g)$. For each $x \in X$, the functional $\delta_x\colon \mathrm{Lip}_v(X) \to \mathbb{K}$ belongs to $\mathcal{F}_v(X)$ by Corollary 1.2, and

$$\|\delta_x\| = d(x,e)v_L(x,e) = \frac{d(x,e)}{\widetilde{v}_L(x,e)} = \frac{d(x,e)}{\widetilde{v}_l(x,e)} = d(x,e)v_l(x,e) = \|\delta_x|_{\mathrm{lip}_v(X)}\|.$$

Hence $\delta_x \in \mathcal{G}_v(X)$ and $f(x) = Q_X(f)(\delta_x) = Q_X(g)(\delta_x) = g(x)$ for all $x \in X$, and we conclude that $f = g$.

Since each $\gamma \in \mathcal{G}_v(X)$ is τ_p-continuous on $B_{\mathrm{Lip}_v(X)}$, the restriction $Q_X|_{B(\mathrm{Lip}_v(X))}$ is continuous with respect to the relative τ_p-topology and the w^*-topology $\sigma(\mathcal{G}_v(X)^*, \mathcal{G}_v(X))$. Since $B(\mathrm{Lip}_v(X))$ is τ_p-compact, it follows that $Q_X(B(\mathrm{Lip}_v(X)))$ is $\sigma(\mathcal{G}_v(X)^*, \mathcal{G}_v(X))$-compact. Also, $Q_X(B(\mathrm{Lip}_v(X)))$ is convex and balanced. By the bipolar theorem, $Q_X(B(\mathrm{Lip}_v(X))) = (Q_X(B(\mathrm{Lip}_v(X)))_\circ)^\circ$ with respect to the duality $(\mathcal{G}_v(X)^*, \mathcal{G}_v(X))$. In view of

$$Q_X\big(B\big(\mathrm{Lip}_v(X)\big)\big)_\circ = \big\{\gamma \in \mathcal{G}_v(X)\colon \big|\gamma(f)\big| \leq 1, \forall f \in B\big(\mathrm{Lip}_v(X)\big)\big\} = B\big(\mathcal{G}_v(X)\big),$$

we deduce that $(Q_X(B(\mathrm{Lip}_v(X)))_\circ)^\circ = B(\mathcal{G}_v(X)^*)$. Hence $Q_X(B(\mathrm{Lip}_v(X))) = B(\mathcal{G}_v(X)^*)$. This proves that $Q_X\colon \mathrm{Lip}_v(X) \to \mathcal{G}_v(X)^*$ is a surjective isometry. $\square$

Our next result was stated by Kalton in [10, Remark, p. 194] for the case $v = 1_{\widetilde{X}}$ (see [2] for $X = [0,1]$ with the Hölder metric).

Proposition 2.9. *Let X be a pointed compact metric space, let v be a weight on $\widetilde{X}$, and assume that $\mathrm{Lip}_v(X)$ is canonically topologically isomorphic to $\mathrm{lip}_v(X)^{**}$. Then $\mathrm{lip}_v(X)$ is an M-ideal in $\mathrm{Lip}_v(X)$.*

Proof. By Proposition 2.6, there exists $\lambda \geq 1$ such that $B(\mathrm{Lip}_v(X))$ is contained in the closure of $\lambda B(\mathrm{lip}_v(X))$ in $(B(\mathrm{Lip}_v(X)), \tau_p)$. We follow the lines of the proof of [4, Proposition 3.5]. Let $f \in B(\mathrm{Lip}_v(X))$, let $g_1, g_2, g_3 \in B(\mathrm{lip}_v(X))$, and let $\varepsilon > 0$. Let k be the smallest natural with $k \geq \max\{2, (\lambda+1)/\varepsilon\}$. By Proposition 2.1, we can find a compact set $K_1 \subset \widetilde{X}$ such that

$$\big|\Phi_v(g_i)(x,y)\big| < \varepsilon$$

for $i = 1, 2, 3$ and all $(x,y) \in \widetilde{X} \setminus K_1$. By Corollary 1.9, we can find a function $f_1 \in \lambda B(\mathrm{lip}_v(X))$ such that

$$\sup\big\{\big|\Phi_v(f - f_1)(x,y)\big|\colon (x,y) \in K_1\big\} \leq \varepsilon.$$

Since $f_1 \in \mathrm{lip}_v(X)$, we can find a compact set $K_2 \subset \widetilde{X}$ with $K_1 \subset K_2$ such that

$$\big|\Phi_v(f_1)(x,y)\big| < \varepsilon$$

for all $(x,y) \in \widetilde{X} \setminus K_2$. Next, we choose f_2 in $\lambda B(\mathrm{lip}_v(X))$ such that

$$\sup\big\{\big|\Phi_v(f - f_2)(x,y)\big|\colon (x,y) \in K_2\big\} \leq \varepsilon.$$

Continuing this process, we get an increasing chain of k compact sets $K_1 \subset K_2 \subset \cdots \subset K_k$ and functions $(f_j)_{j=1}^k$ in $\lambda B(\mathrm{lip}_v(X))$ such that

$$\big|\Phi_v(f_l)(x,y)\big| < \varepsilon$$

for all $(x,y) \in \widetilde{X} \setminus K_{l+1}$, $l = 1, \dots, k-1$, and

$$\sup\left\{\left|\Phi_v(f - f_l)(x,y)\right| : (x,y) \in K_l\right\} \leq \varepsilon, \quad l = 1, \dots, k.$$

Then, for $1 \leq l \leq k$, we have

$$\left|\Phi_v(f - f_l)(x,y)\right| \leq 1 + \varepsilon$$

for all $(x,y) \in \widetilde{X}$ with the possible exception of those (x,y) in $K_{l+1} \setminus K_l$. Define $h = (1/k)\sum_{j=1}^{k} f_j$ and let $(x,y) \in \widetilde{X}$. If $(x,y) \in K_1$, then since

$$\left|\Phi_v(f - f_l)(x,y)\right| \leq \varepsilon$$

for $1 \leq l \leq k$, we have

$$\left|\Phi_v(f - g_i - h)(x,y)\right| \leq 1 + \varepsilon$$

for $i = 1, 2, 3$. For (x,y) in $K_{l+1} \setminus K_l$, we have

$$\left|\Phi_v(g_i)(x,y)\right| < \varepsilon$$

for $i = 1, 2, 3$ and

$$\left|\Phi_v(f - f_j)(x,y)\right| \leq 1 + \varepsilon$$

for all $1 \leq j \leq k$ with the possible exception of l. As $\mathrm{Lip}_v(f - f_l) \leq \lambda + 1$, for $(x,y) \in K_{l+1} \setminus K_l$ and $i = 1, 2, 3$ we have

$$\begin{aligned}
&\left|\Phi_v(f - g_i - h)(x,y)\right| \\
&\quad\leq \sum_{j=1, j\neq l}^{k} \frac{1}{k}\left|\Phi_v(f - f_j)(x,y)\right| + \frac{1}{k}\left|\Phi_v(f - f_l)(x,y)\right| + \left|\Phi_v(g_i)(x,y)\right| \\
&\quad\leq \frac{k-1}{k}(1+\varepsilon) + \frac{1}{k}(\lambda + 1) + \varepsilon \leq 1 + 3\varepsilon.
\end{aligned}$$

We have proved that $\mathrm{lip}_v(X)$ satisfies the (restricted) 3-ball property and therefore $\mathrm{lip}_v(X)$ is an M-ideal in $\mathrm{Lip}_v(X)$ by [8, Theorem 2.2]. □

We now can connect the information obtained in Propositions 2.6 and 2.7 as follows.

Theorem 2.10. *Let X be a pointed compact metric space and let v be a weight on $\widetilde{X}$. Then the following are equivalent:*

(i) $\mathrm{Lip}_v(X)$ *is canonically isometrically isomorphic to* $\mathrm{lip}_v(X)^{**}$*,*
(ii) $\mathrm{Lip}_v(X)$ *is canonically topologically isomorphic to* $\mathrm{lip}_v(X)^{**}$*,*
(iii) $\mathrm{lip}_v(X)$ *is an M-ideal in* $\mathrm{Lip}_v(X)$ *and* $\widetilde{v}_l = \widetilde{v}_L$*.*

Proof. (i) ⇒ (ii): This is immediate.

(ii) ⇒ (i): If (ii) is true, then $R_X \colon \mathcal{F}_v(X) \to \mathrm{lip}_v(X)^*$ is a topological isomorphism by Proposition 2.5. Moreover, $\mathrm{lip}_v(X)$ is an M-ideal in $\mathrm{Lip}_v(X)$ by Proposition 2.9, and therefore $S_X \colon \mathcal{G}_v(X) \to \mathrm{lip}_v(X)^*$ is an isometric isomorphism by Proposition 2.8. We now claim that $\mathcal{F}_v(X) \subset \mathcal{G}_v(X)$. Indeed, let $\gamma \in \mathcal{F}_v(X)$. Then $R_X(\gamma) \in \mathrm{lip}_v(X)^*$ and therefore $R_X(\gamma) = S_X(\phi)$ for some $\phi \in \mathcal{G}_v(X)$. As $\mathcal{G}_v(X) \subset \mathcal{F}_v(X)$ and $R_X|_{\mathcal{G}_v(X)} = S_X$, it follows that $R_X(\phi) = S_X(\phi) = R_X(\gamma)$, which implies that $\gamma = \phi$ and thus that $\gamma \in \mathcal{G}_v(X)$. Therefore $\mathcal{F}_v(X) = \mathcal{G}_v(X)$

and $R_X = S_X$. Hence $R_X\colon \mathcal{F}_v(X) \to \mathrm{lip}_v(X)^*$ is an isometric isomorphism, and we obtain (i) by Proposition 2.5.

(i) ⇒ (iii): If (i) holds, then $B(\mathrm{lip}_v(X))$ is dense in $(B(\mathrm{Lip}_v(X)), \tau_p)$ by Proposition 2.7. We first prove that $\widetilde{v}_l$ coincides with $\widetilde{v}_L$. Let $(x, y) \in \widetilde{X}$. It is clear that $v_l(x, y) \le v_L(x, y)$. To prove the reverse inequality, let $f \in B(\mathrm{Lip}_v(X))$ and let $\varepsilon > 0$. By Corollary 1.9, there exists $g \in B(\mathrm{lip}_v(X))$ such that $|\Phi_v(x, y)(f - g)| < \varepsilon$. If follows that

$$\frac{|f(x) - f(y)|}{d(x, y)} < \frac{\varepsilon}{v(x, y)} + \frac{|g(x) - g(y)|}{d(x, y)} \le \frac{\varepsilon}{v(x, y)} + v_l(x, y).$$

Passing to the supremum, we arrive at $v_L(x, y) \le (\varepsilon/v(x, y)) + v_l(x, y)$. By the arbitrariness of ε, we conclude that $v_L(x, y) \le v_l(x, y)$ and thus that $v_L(x, y) = v_l(x, y)$. Hence $\widetilde{v}_l(x, y) = \widetilde{v}_L(x, y)$. On the other hand, $\mathrm{lip}_v(X)$ is an M-ideal in $\mathrm{Lip}_v(X)$ by Proposition 2.9.

(iii) ⇒ (i): By Proposition 2.8, $Q_X\colon \mathrm{Lip}_v(X) \to \mathcal{G}_v(X)^*$ is an isometric isomorphism. Since that is also true of $P_X\colon \mathrm{Lip}_v(X) \to \mathcal{F}_v(X)^*$ by Theorem 1.1, it follows that $P_X \circ Q_X^{-1}\colon \mathcal{G}_v(X)^* \to \mathcal{F}_v(X)^*$ is an isometric isomorphism.

On the other hand, $S_X\colon \mathcal{G}_v(X) \to \mathrm{lip}_v(X)^*$ is an isometric isomorphism by Proposition 2.8. Since $\mathrm{lip}_v(X)^*$ is a separable dual space by Remark 1.3 and Theorem 2.2, it has the Radon–Nikodým property. Hence $\mathcal{G}_v(X)$ also has that property. By [7, p. 144], $\mathcal{G}_v(X)$ is the unique isometric predual of $\mathcal{G}_v(X)^*$. In fact, $\mathcal{G}_v(X)$ is strongly unique (see [5, Definition 2.2.29]) and therefore the map $(P_X \circ Q_X^{-1})^*\colon \mathcal{F}_v(X)^{**} \to \mathcal{G}_v(X)^{**}$ carries $J_{\mathcal{F}_v(X)}(\mathcal{F}_v(X))$ onto $J_{\mathcal{G}_v(X)}(\mathcal{G}_v(X))$. For any $f \in \mathrm{Lip}_v(X)$ and $\gamma \in \mathcal{F}_v(X)$, a simple calculation gives

$$\begin{aligned}(P_X \circ Q_X^{-1})^*\big(J_{\mathcal{F}_v(X)}(\gamma)\big)\big(Q_X(f)\big) &= \big(J_{\mathcal{F}_v(X)}(\gamma) \circ P_X \circ Q_X^{-1}\big)\big(Q_X(f)\big)\\ &= J_{\mathcal{F}_v(X)}(\gamma)\big(P_X(f)\big)\\ &= P_X(f)(\gamma) = \gamma(f).\end{aligned}$$

Hence $\mathcal{F}_v(X) = \mathcal{G}_v(X)$. Indeed, let $\gamma \in \mathcal{F}_v(X)$. Then $(P_X \circ Q_X^{-1})^*(J_{\mathcal{F}_v(X)}(\gamma)) = J_{\mathcal{G}_v(X)}(\phi)$ for some $\phi \in \mathcal{G}_v(X)$, and therefore

$$\gamma(f) = (P_X \circ Q_X^{-1})^*\big(J_{\mathcal{F}_v(X)}(\gamma)\big)\big(Q_X(f)\big) = J_{\mathcal{G}_v(X)}(\phi)\big(Q_X(f)\big) = Q_X(f)(\phi) = \phi(f)$$

for all $f \in \mathrm{Lip}_v(X)$; that is, $\gamma = \phi \in \mathcal{G}_v(X)$ as desired. Therefore $R_X = S_X$. Hence R_X is an isometric isomorphism from $\mathcal{F}_v(X)$ onto $\mathrm{lip}_v(X)^*$ and we have (i) by Proposition 2.5. □

We finish the paper with an application on extreme points.

Corollary 2.11. *Let X be a pointed compact metric space and let v be a weight on $\widetilde{X}$. Suppose that $\mathrm{Lip}_v(X)$ is canonically isometrically isomorphic to $\mathrm{lip}_v(X)^{**}$. Then every extreme point of $B(\mathcal{F}_v(X))$ if of the form $\lambda\delta^v_{(x,y)}$ for $\lambda \in S(\mathbb{K})$ and $(x, y) \in \widetilde{X}$.*

Proof. Let $\gamma \in \mathrm{Ext}(B(\mathcal{F}_v(X))$. Hence $R_X(\gamma) \in \mathrm{Ext}(B(\mathrm{lip}_v(X)^*))$ by Proposition 2.5. Since $\Phi_v|_{\mathrm{lip}_v(X)}$ is an isometric linear embedding from $\mathrm{lip}_v(X)$ into $C_0(\widetilde{X})$ by Proposition 2.1, we have that there exists $F \in \mathrm{Ext}(B(C_0(\widetilde{X})^*)$ such that

$(\Phi_v|_{\mathrm{lip}_v(X)})^*(F) = R_X(\gamma)$ by [15, Lemma 2.5.1]. By the Arens–Kelley theorem, F is of the form $\lambda\psi_{(x,y)}$ where $\lambda \in S(\mathbb{K})$ and $\psi_{(x,y)}$ is the evaluation functional at a point $(x,y) \in \widetilde{X}$ defined on $C_0(\widetilde{X})$. An easy verification gives

$$\begin{aligned}\gamma(f) &= R_X(\gamma)(f)\\ &= (\Phi_v|_{\mathrm{lip}_v(X)})^*(\lambda\psi_{(x,y)})(f)\\ &= \lambda\psi_{(x,y)}\big(\Phi_v(f)\big)\\ &= \lambda\Phi_v(f)(x,y)\\ &= \lambda v(x,y)\frac{\delta_x - \delta_y}{d(x,y)}(f)\end{aligned}$$

for all $f \in \mathrm{lip}_v(X)$, and thus γ has the desired form. □

Acknowledgments. The author's work was partially supported by the Ministerio de Economía, Industria, y Competitividad (MINECO) grant MTM2014-58984-P, by the European Regional Development Fund (ERDF), and by Junta de Andalucía grant FQM-194.

References

1. E. M. Alfsen and E. G. Effros, *Structure in real Banach spaces, I*, Ann. of Math. (2) **96** (1972), 98–173. Zbl 0248.46019. MR0352946. DOI 10.2307/1970895. 251
2. H. Berninger and D. Werner, *Lipschitz spaces and M-ideals*, Extracta Math. **18** (2003), no. 1, 33–56. Zbl 1042.46007. MR1989296. 251, 253
3. K. D. Bierstedt and W. H. Summers, *Biduals of weighted Banach spaces of analytic functions*, J. Aust. Math. Soc. **54** (1993), no. 1, 70–79. Zbl 0801.46021. MR1195659. 242
4. C. Boyd and P. Rueda, *The biduality problem and M-ideals in weighted spaces of holomorphic functions*, J. Convex Anal. **18** (2011), no. 4, 1065–1074. Zbl 1257.32007. MR2917867. 242, 253
5. H. G. Dales, F. K. Dashiell, Jr., A.T.-M. Lau, and D. Strauss, *Banach Spaces of Continuous Functions as Dual Spaces*, CMS Books Math., Springer, Cham, 2016. Zbl 1368.46003. MR3588279. 255
6. R. B. Fraser, *Banach spaces of functions satisfying a modulus of continuity condition*, Studia Math. **32** (1969), 277–283. Zbl 0177.40901. MR0244747. 241
7. G. Godefroy, "Existence and uniqueness of isometric preduals: A survey" in *Banach Space Theory (Iowa City, 1987)*, Contemp. Math. **85**, Amer. Math. Soc., Providence, 1989, 131–193. Zbl 0674.46010. MR0983385. 255
8. P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. **1547**, Springer, Berlin, 1993. Zbl 0789.46011. MR1238713. 251, 252, 254
9. J. A. Johnson, *Banach spaces of Lipschitz functions and vector-valued Lipschitz functions*, Trans. Amer. Math. Soc. **148** (1970), 147–169. Zbl 0194.43603. MR0415289. DOI 10.2307/1995044. 240
10. N. J. Kalton, *Spaces of Lipschitz and Hölder functions and their applications*, Collect. Math. **55** (2004), no. 2, 171–217. Zbl 1069.46004. MR2068975. 251, 253
11. J. L. Kelley, *General Topology*, Grad. Texts in Math. **27**. Springer, New York, 1975. Zbl 0306.54002. MR0370454. 243
12. R. E. Megginson, *An Introduction to Banach Space Theory*, Grad. Texts in Math. **183**, Springer, New York, 1998. Zbl 0910.46008. MR1650235. DOI 10.1007/978-1-4612-0603-3. 246

13. J. Mujica, *A Banach–Dieudonné theorem for germs of holomorphic functions*, J. Funct. Anal. **57** (1984), no. 1, 31–48. Zbl 0546.46021. MR0744918. DOI 10.1016/0022-1236(84)90099-5. 247
14. K. F. Ng, *On a theorem of Dixmier*, Math. Scand. **29** (1971), 279–280. Zbl 0243.46023. MR0338740. 242, 243
15. N. Weaver, *Lipschitz Algebras*, World Scientific, River Edge, NJ, 1999. Zbl 0936.46002. MR1832645. 241, 242, 248, 250, 256

Departamento de Matemáticas, Universidad de Almería, 04120 Almería, Spain.
E-mail address: ajimenez@ual.es