Banach J. Math. Anal. 12 (2018), no. 1, 68–84
https://doi.org/10.1215/17358787-2017-0036
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

DAUGAVET PROPERTY AND SEPARABILITY IN BANACH SPACES

ABRAHAM RUEDA ZOCA

Communicated by D. Werner

ABSTRACT. We give a characterization of the separable Banach spaces with the Daugavet property which is applied to study the Daugavet property in the projective tensor product of an L-embedded space with another nonzero Banach space. The former characterization also motivates the introduction and short study of two indices related to the Daugavet property.

1. INTRODUCTION

A Banach space X is said to have the *Daugavet property* if every rank 1 operator $T : X \longrightarrow X$ satisfies the equality

$$\|T + I\| = 1 + \|T\|, \tag{1.1}$$

where I denotes the identity operator. The previous equality is known as the *Daugavet equation* because Daugavet proved in [10] that every compact operator on $\mathcal{C}([0,1])$ satisfies (1.1). Since then, many examples of Banach spaces with the Daugavet property have appeared, such as $\mathcal{C}(K)$ for a compact Hausdorff and perfect topological space K, $L_1(\mu)$ and $L_\infty(\mu)$ for a nonatomic measure μ, and the space of Lipschitz functions $\mathrm{Lip}(M)$ over a metrically convex space M (see [19], [22], [27], [28] and the references therein for details). Moreover, in [22] (resp., [27]) a characterization of the Daugavet property in terms of the geometry of the slices (resp., nonempty weakly open subsets) of B_X appeared (see Theorem 2.2 for a formal statement).

Received Dec. 9, 2016; Accepted Jan. 25, 2017.
First published online Sep. 21, 2017.
2010 *Mathematics Subject Classification.* Primary 46B04; Secondary 46B28.
Keywords. Daugavet property, projective tensor product, separable Banach spaces, L-embedded Banach spaces, thickness.

In [28, Section 6] Werner posed as an open problem how the Daugavet property is preserved by injective or projective tensor products. Kadets, Kalton, and Werner [20, Corollary 4.3] give an example of a 2-dimensional complex Banach space Y such that $L_\infty^{\mathbb{C}}([0,1]) \widehat{\otimes}_\pi Y$ fails the Daugavet property (see [23, Remark 3.13] for real counterexamples failing to fulfill much weaker requirements than the Daugavet property). Concerning positive results, we only know of those in [7], where the authors, making strong use of the theory of centralizer and function module representation of Banach spaces, proved that the projective tensor product of a Banach space without minimal L-summands and another nonzero Banach space has the Daugavet property. However, to the best of our knowledge, the problem of whether the Daugavet property is preserved by projective tensor products from both factors is still open.

Motivated by this problem and by the recent techniques revealed in [23, Section 4] for the analysis of octahedrality in projective tensor products, in Section 3 we will introduce a characterization of the Daugavet property in separable Banach spaces in terms of coverings of weakly open subsets of the unit ball which will be used to prove the two main results of the article. On the one hand, we prove in Theorem 3.7 that, in the presence of the metric approximation property, the Daugavet property is inherited by taking the projective tensor product of separable L-embedded Banach spaces. On the other hand, we prove in Proposition 3.8 that the hypothesis of separability can be eliminated whenever we are dealing with preduals of JBW*-triples with the Daugavet property. In Section 4, motivated by Lemma 3.1 and the thickness index introduced by Whitley in [29], we introduce two indices that quantitatively measure how far a Banach space is from having the Daugavet property. We will also study the interrelation of these indices with the Daugavet equation and some stability results concerning ℓ_p-sums and inheritance to subspaces. We finish in Section 5 with some remarks and open questions.

2. Notation and preliminaries

We will consider only real Banach spaces. Given a Banach space X, we will denote the unit ball and the unit sphere of X by B_X and S_X, respectively. Moreover, given $x \in X$ and $r > 0$, we will denote $B(x,r) = x + rB_X = \{y \in X : \|x-y\| \leq r\}$. We will also denote by X^* the topological dual of X. Given a bounded subset C of X, we will mean by a *slice of* C a set of the following form

$$S(C, x^*, \alpha) := \{x \in C : x^*(x) > \sup x^*(C) - \alpha\},$$

where $x^* \in X^*$ and $\alpha > 0$. If X is a dual Banach space, the previous set will be a w^*-*slice* if x^* belongs to the predual of X. Note that finite intersections of slices of C (resp., of w^*-slices of C) form a basis for the inherited weak (resp., weak-star) topology of C.

According to [16], a Banach space X is said to be an *L-embedded Banach space* if there exists a subspace Z of X^{**} such that $X^{**} = X \oplus_1 Z$. Examples of L-embedded Banach spaces are $L_1(\mu)$ spaces, preduals of von Neumann algebras, duals of M-embedded spaces, or the dual of the disk algebra (see [16, Example IV.1.1] for formal definitions and details).

Given two Banach spaces X and Y, we will denote by $L(X,Y)$ the space of all linear and bounded operators from X to Y, and we will denote by $X\widehat{\otimes}_\pi Y$ the projective tensor product of X and Y. Moreover, we will say that X has the *metric approximation property* if there exists a net of finite-rank and norm 1 operators $S_\alpha : X \longrightarrow X$ such that $S_\alpha(x) \to x$ for all $x \in X$ (see [26] for a detailed treatment of the tensor product theory and approximation properties).

The theory of *almost isometric ideals* will be an essential tool for our results related to the Daugavet property in tensor product spaces. Let Z be a subspace of a Banach space X. We say that Z is an *almost isometric ideal* (*ai-ideal*) in X if X is locally complemented in Z by almost isometries. This means that, for each $\varepsilon > 0$ and for each finite-dimensional subspace $E \subseteq X$, there exists a linear operator $T : E \to Z$ satisfying

(1) $T(e) = e$ for each $e \in E \cap Z$, and
(2) $(1-\varepsilon)\|e\| \leq \|T(e)\| \leq (1+\varepsilon)\|e\|$ for each $e \in E$;

that is, T is a $(1+\varepsilon)$-isometry fixing the elements of E. If the T's satisfy only (1) and the right-hand side of (2), we get the well-known concept of Z being an *ideal* in X (see [14]). Note that the principle of local reflexivity means that X is an ai-ideal in X^{**} for every Banach space X. Moreover, the Daugavet property is inherited by ai-ideals (see [3]). It is known that, given two Banach spaces X and Y and given an ideal Z in X, then $Z\widehat{\otimes}_\pi Y$ is a closed subspace of $X\widehat{\otimes}_\pi Y$ (see, e.g., [25, Theorem 1]). It is also known that whenever X^{**} or Y has the metric approximation property, then $X^{**}\widehat{\otimes}_\pi Y$ is an isometric subspace of $(X\widehat{\otimes}_\pi Y)^{**}$ (see [23, Proposition 2.3] and [25, Theorem 1]). We will freely use these two facts throughout Sections 3 and 5. We will also use the following characterization of ideals in Banach spaces (see [3, Theorem 1.1] and references therein for details).

Theorem 2.1. *Let X be a Banach space and Y be a subspace of X. The following assertions are equivalent.*

(1) *Y is an ideal in X.*
(2) *There exists a Hahn–Banach extension operator, that is, an operator $\varphi : Y^* \longrightarrow X^*$ such that, for every $y^* \in Y^*$ and $y \in Y$, it follows that $\|\varphi(y^*)\| = \|y^*\|$ and that $\varphi(y^*)(y) = y^*(y)$.*

Let X be a Banach space. Whitley [29] defined the following *thickness* index:

$$T_W(X) := \inf\Big\{r > 0 : \exists\{x_1,\ldots,x_n\} \subseteq S_X \text{ with } S_X \subseteq \bigcup_{i=1}^n B(x_i,r)\Big\}.$$

In [9] it was proved that $T_W(X)$ is equal to

$$T(X) := \inf\Big\{r > 0 : \exists\{x_1,\ldots,x_n\} \subseteq S_X \text{ with } B_X \subseteq \bigcup_{i=1}^n B(x_i,r)\Big\}$$

whenever X is infinite-dimensional. Moreover, it is known that $1 \leq T(X) \leq 2$ whenever X is an infinite-dimensional Banach space (see [29, Lemma 2]). Furthermore, if X is a separable Banach space, the condition $T(X) = 2$ can be characterized in terms of the Daugavet equation. Indeed, it is proved in [21] that

$T(X) = 2$ if and only if there exists a 1-norming subspace $Y \subseteq X^*$ such that the equation $\|T + I\| = 1 + \|T\|$ holds true for every rank 1 operator $T : X \to X$ of the form $T = y^* \otimes x$ such that $x \in X$ and that $y^* \in Y$.

Related to the thickness index in Banach spaces is the concept of *octahedral norms*. According to [12], a Banach space X has an octahedral norm if, for every finite-dimensional subspace $Y \subseteq X$ and every $\varepsilon > 0$, there exists $x \in S_X$ such that

$$\|y + \lambda x\| \geq (1 - \varepsilon)\big(\|y\| + |\lambda|\big)$$

holds for every $\lambda \in \mathbb{R}$ and every $y \in Y$. It is known (see [12]) that X has an octahedral norm if and only if $T(X) = 2$. If, in addition, X is separable, it is known (see [13, Lemma 9.1]) that X has an octahedral norm if and only if there exists $u \in S_{X^{**}}$ such that

$$\|x + u\| = 1 + \|x\|$$

holds for every $x \in X$.

Finally, we will state the following characterization of the Daugavet property, proved in [22, Lemma 2.1] and [27, Lemma 2.2], which will freely be used throughout the text.

Theorem 2.2. *Let X be a Banach space. The following assertions are equivalent.*

(1) *X has the Daugavet property.*
(2) *For every $x \in S_X$, every $\varepsilon > 0$, and every slice S of B_X there exists $y \in S$ such that $\|x + y\| > 2 - \varepsilon$.*
(3) *For every $x \in S_X$, every $\varepsilon > 0$, and every nonempty weakly open subset W of B_X there exists $y \in W$ such that $\|x + y\| > 2 - \varepsilon$.*
(4) *For every $x^* \in S_{X^*}$, every $\varepsilon > 0$, and every w^*-slice S of B_{X^*} there exists $y^* \in S$ such that $\|x^* + y^*\| > 2 - \varepsilon$.*
(5) *For every $x^* \in S_{X^*}$, every $\varepsilon > 0$, and every nonempty weakly-star open subset W of B_{X^*} there exists $y^* \in W$ such that $\|x^* + y^*\| > 2 - \varepsilon$.*

Note that the preceding theorem implies that Banach spaces with the Daugavet property have an octahedral norm (see [22, Lemma 2.8] for details).

3. The Daugavet property in separable Banach spaces and applications

It is known (see [13, Lemma 9.1]) that a Banach space X has an octahedral norm if and only if $T(X) = 2$, which is in turn equivalent to the fact that whenever there exist $x_1, \ldots, x_n \in X$ such that $B_X \subseteq \bigcup_{i=1}^n B(x_i, r_i)$, then there exists $i \in \{1, \ldots, n\}$ such that $B_X \subseteq B(x_i, r_i)$. We wonder whether a similar statement can be established for the Daugavet property. The following lemma will characterize the above property in terms of a thickness condition. In order to see that, we will introduce a bit of notation. According to [13], given a Banach space X, the *ball topology*, denoted by b_X, is defined as the coarsest topology on X so that every closed ball is closed in b_X. As a consequence, a basis for the topology b_X is formed by the sets of the form

$$X \setminus \bigcup_{i=1}^n B(x_i, r_i),$$

where $x_1, \ldots, x_n$ are elements of X and $r_1, \ldots, r_n$ are positive numbers.

Lemma 3.1. *Let X be a Banach space. The following assertions are equivalent.*

(1) *X has the Daugavet property.*
(2) *Given a nonempty relatively weakly open set W of B_X, it follows that, whenever there exist $x_1, \ldots, x_n \in X$ such that $W \subseteq \bigcup_{i=1}^n B(x_i, r_i)$, then there exists $i \in \{1, \ldots, n\}$ such that $r_i \geq 1 + \|x_i\|$. In particular, $W \subseteq B_X \subseteq B(x_i, r_i)$.*
(3) *For every nonempty b_X open subset O of B_X and for every nonempty relatively weakly open subset W of B_X, it follows that $W \cap O \neq \emptyset$.*

Proof. (1) ⇒ (2). Pick a nonempty relatively weakly open subset W of B_X, and assume that $W \subseteq \bigcup_{i=1}^n B(x_i, r_i)$ for certain $x_1, \ldots, x_n \in X$ and $r_1, \ldots, r_n \in \mathbb{R}^+$. Let us prove that there exists $i \in \{1, \ldots, n\}$ such that $r_i \geq 1 + \|x_i\|$. Since X has the Daugavet property, we conclude, using a similar argument to the one given in [22, Lemma 2.8] for weakly open sets, the existence of $y \in W$ such that

$$\|x_i - y\| > 1 + \|x_i\| - \varepsilon$$

holds for every $i \in \{1, \ldots, n\}$. As $y \in W$, then there exists $i \in \{1, \ldots, n\}$ such that $y \in B(x_i, r)$ and thus $r_i \geq 1 + \|x_i\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it is not difficult to get (2).

(2) ⇒ (3). Consider O to be a nonempty b_X open subset of B_X. Up to considering a smaller open set, we can assume that O has the form

$$O := B_X \setminus \bigcup_{i=1}^n B(x_i, r_i)$$

for certain $x_1, \ldots, x_n \in X$ and $r_1, \ldots, r_n \in \mathbb{R}^+$. Consider W to be a nonempty relatively weakly open subset of B_X, and assume by contradiction that $O \cap W = \emptyset$. Then $W \subseteq \bigcup_{i=1}^n B(x_i, r_i)$. By (2) we get that $B_X \subseteq \bigcup_{i=1}^n B(x_i, r_i)$ and, consequently, $O = \emptyset$, which is a contradiction. So (3) follows.

(3) ⇒ (1). Pick $x \in S_X$, $\varepsilon > 0$, and a slice S of B_X. Define $O := B_X \setminus B(x, 2-\varepsilon)$, which is clearly a nonempty b_X open subset of B_X. By (3) there exists $y \in S \cap O$; that is, there exists $y \in S$ such that $\|y - x\| > 2 - \varepsilon$. Consequently, X has the Daugavet property, so we are done. □

As with the octahedrality condition, the previous lemma allows us to strengthen the Daugavet property under separability assumptions.

Theorem 3.2. *Let X be a separable Banach space. The following assertions are equivalent.*

(1) *X has the Daugavet property; that is, for every $x \in S_X$, every nonempty relatively weakly open subset of B_X, and every $\varepsilon > 0$ there exists $y \in W$ such that $\|x + y\| > 2 - \varepsilon$.*
(2) *For every nonempty relatively weakly-star open subset W of $B_{X^{**}}$ there exists $u \in S_{X^{**}} \cap W$ such that*

$$\|x + u\| = 1 + \|x\|$$

holds for every $x \in X$.

Proof. (2) ⇒ (1). Pick $x \in S_X$, $\varepsilon > 0$, and consider a nonempty relatively weakly open subset W of B_X. Define W^* to be the relatively weakly-star open subset of $B_{X^{**}}$ defined by W (i.e., satisfying that $W^* \cap B_X = W$), and consider $u \in W^* \cap S_{X^{**}}$ as in (2). Pick a net $\{x_s\}$ in B_X which is weakly-star convergent to u in $B_{X^{**}}$. On the one hand, because of the weakly-star convergence condition, we can find s_0 such that $s \geq s_0$ implies $x_s \in W^*$, and hence $x_s \in W^* \cap B_X = W$. On the other hand, by the weak-star lower semicontinuity of the norm of X^{**}, we get

$$2 = \|x + u\| \leq \liminf_s \|x_s + x\|,$$

so we can find $s \geq s_0$ such that $\|x_s + x\| > 2 - \varepsilon$, and (1) follows.

(1) ⇒ (2). Since X is separable, the b_X topology has a countable basis (see, e.g., [13, Introduction]). Consequently, consider $\{O_n : n \in \mathbb{N}\}$ to be a basis for the topology b_X of B_X. Since X has the Daugavet property, X has an octahedral norm and, consequently, every pair of nonempty b_X open subsets of B_X has a nonempty intersection (see [13, Lemma 9.1]). Thus, $\bigcap_{k=1}^n O_k$ is a nonempty b_X open subset of B_X for every $n \in \mathbb{N}$. Pick W to be a nonempty relatively weakly-star open subset of $B_{X^{**}}$, and pick U to be another nonempty relatively weakly-star open subset of $B_{X^{**}}$ such that $\overline{U}^{w^*} \subseteq W$. By Lemma 3.1(3), we conclude the existence of $x_n \in (U \cap B_X) \cap \bigcap_{k=1}^n O_k$ for every $n \in \mathbb{N}$. Since $x_n \in \bigcap_{k=1}^n O_k$ for every $n \in \mathbb{N}$, we deduce, following the proof of [13, Lemma 9.l] verbatim, the existence of a w^*-cluster point u of the sequence $\{x_n\}$ in $B_{X^{**}}$ such that

$$\|x - x^{**}\| = 1 + \|x\|$$

holds for every $x \in X$. Moreover, since $\{x_n\}$ is contained in U and u is a weak-star cluster point of $\{x_n\}$, we deduce that $u \in \overline{U}^{w^*} \subseteq W$. Consequently, (2) follows and the theorem is proved. □

Remark 3.3. (1) Let X be a separable Banach space. By [13, Lemma 9.1], it follows that X has an octahedral norm if and only if there exists $u \in S_{X^{**}}$ such that $\|x+u\| = 1+\|x\|$ holds for every $x \in X$. Theorem 3.2 can be read as follows: X has the Daugavet property if and only if the set of such $u \in S_{X^{**}}$ is weak-star dense in $S_{X^{**}}$.

(2) Given a separable Banach space X such that X^* additionally has the Daugavet property, Theorem 3.2 can be proved following the argument of [22, Lemma 2.12] for weak-star open subsets instead of w^*-slices.

As an application, we will give some sufficient conditions for a projective tensor product space to enjoy the Daugavet property. For this, we begin with a characterization of the Daugavet property in separable L-embedded Banach spaces.

Theorem 3.4. *Let X be a separable L-embedded Banach space. Assume that $X^{**} = X \oplus_1 Z$. Then, the following are equivalent.*

(1) *X^* has the Daugavet property.*
(2) *X has the Daugavet property.*
(3) *B_Z is weak-star dense in $B_{X^{**}}$.*

Proof. (1) $\Rightarrow$ (2). This implication is obvious.

(2) $\Rightarrow$ (3). Let W be a nonempty relatively weakly-star open subset of $B_{X^{**}}$, and let us prove that $B_Z \cap W \neq \emptyset$. By Theorem 3.2, we can find $u \in W \cap S_{X^{**}}$ such that

$$\|x+u\| = 1 + \|x\|$$

holds for every $x \in X$. Since $u \in X^{**}$, we can find $x \in X$ and $z \in Z$ such that $u = x + z$. Now

$$1 \geq \|z\| = \big\|-x + (x+z)\big\| = 1 + \|x\|.$$

This implies that $x = 0$ and, consequently, $u \in B_Z$. So $W \cap B_Z \neq \emptyset$, as desired.

(3) $\Rightarrow$ (1). This follows from [6, Theorem 2.2]. □

This result generalizes [6, Theorem 3.2] under separability assumptions, where the authors proved that a real or complex JBW*-triple X has the Daugavet property if and only if its predual X_* (which is an L-embedded Banach space) has the Daugavet property.

Now we will apply Theorem 3.2 to study when the projective tensor product of an L-embedded Banach space with the Daugavet property enjoys the Daugavet property. For this we begin with the following abstract lemma. Let us fix a bit of notation related to the tensor product theory before the statement. Recall (see [26, p. 24]) that $(X \widehat{\otimes}_\pi Y)^*$ can be identified with the space $L(X, Y^*)$, where an operator $T \in L(X, Y^*)$ acts on a basic tensor $x \otimes y \in X \widehat{\otimes}_\pi Y$ as $\langle T, x \otimes y\rangle := T(x)(y)$. When we consider an element $T \in L(X, Y^*)$ as a linear and continuous functional on $X \widehat{\otimes}_\pi Y$, we will write its action on $z \in X \widehat{\otimes}_\pi Y$ as $\langle T, z\rangle$. When the same $T \in L(X, Y^*)$ is considered as an operator, its action on $x \in X$ will be denoted as $T(x)$.

Lemma 3.5. *Let X be a separable Banach space with the Daugavet property, and let Y be a nonzero Banach space. Then, for every slice $S := S(B_{X \widehat{\otimes}_\pi Y}, G, \alpha)$ of $B_{X \widehat{\otimes}_\pi Y}$ there exists $u \in S_{X^{**}}$ and $y \in S_Y$ such that $(y \circ G)(u) > 1 - \alpha$ and*

$$\|z + u \otimes y\|_{(X \oplus \mathbb{R}u) \widehat{\otimes}_\pi Y} = 1 + \|z\|$$

holds for every $z \in X \widehat{\otimes}_\pi Y$. Moreover, if $X \oplus \mathbb{R}u$ is an ideal in X^{**} and either X^{**} or Y has the metric approximation property, then $X \widehat{\otimes}_\pi Y$ has the Daugavet property.

Proof. Pick $z \in X \widehat{\otimes}_\pi Y$ and a slice $S := S(B_{X \widehat{\otimes}_\pi Y}, G, \alpha)$. Consider $x \otimes y \in S \cap S_{X \widehat{\otimes}_\pi Y}$ such that $\|x\| = \|y\| = 1$. Note that

$$x \otimes y \in S \Leftrightarrow G(x)(y) > 1 - \alpha.$$

By Theorem 3.2 there exists $u \in S_{X^{**}}$ such that $u(y \circ G) > 1 - \alpha$ and

$$\|w + \lambda u\| = \|w\| + |\lambda|$$

holds for every $w \in X$ and every $\lambda \in \mathbb{R}$. Denote by $X_u := X \oplus \mathbb{R}u$. Now consider $T \in S_{L(X,Y^*)}$ such that $\langle T, z\rangle = \|z\|$, $y^* \in S_{Y^*}$ such that $y^*(y) = 1$, and define $\bar{T} : X_u \longrightarrow Y^*$ by the equation

$$\bar{T}(w + \lambda u) := T(w) + \lambda y^*$$

for all $w \in X$ and all $\lambda \in \mathbb{R}$. Since X_u is isometrically isomorphic to $X \oplus_1 \mathbb{R}$, it is obvious that $\|\bar{T}\| \leq 1$. Consequently,

$$\|z + u \otimes y\|_{X_u \widehat{\otimes}_\pi Y} \geq \langle \bar{T}, z + u \otimes y \rangle = \|z\| + y^*(y) = \|z\| + 1.$$

If X_u is an ideal in X^{**}, then $X_u \widehat{\otimes}_\pi Y$ is an isometric subspace of $X^{**} \widehat{\otimes}_\pi Y$. Moreover, if either X^{**} or Y has the metric approximation property, then $X^{**} \widehat{\otimes}_\pi Y$ is an isometric subspace of $(X \widehat{\otimes}_\pi Y)^{**}$ (see [23, Proposition 2.3] and [25, Theorem 1]). Consequently,

$$\|z + u \otimes y\|_{(X \widehat{\otimes}_\pi Y)^{**}} = 1 + \|z\|_{X \widehat{\otimes}_\pi Y}.$$

Since $u(y \circ G) = (u \otimes y)(G) > 1 - \alpha$ and $z \in X \widehat{\otimes}_\pi Y$ was arbitrary, we conclude that $X \widehat{\otimes}_\pi Y$ satisfies (2) in Theorem 3.2. Thus, $X \widehat{\otimes}_\pi Y$ enjoys the Daugavet property, which finishes the proof. □

Remark 3.6. The assumption of Lemma 3.5 of X_u being an ideal in X^{**} does not hold in general. Indeed, consider a projective tensor product $L_\infty \widehat{\otimes}_\pi Y$ failing the Daugavet property and where Y has the metric approximation property (see, e.g., [23] for $Y = \ell_3^3$). Then there exist $z := \sum_{i=1}^n f_i \otimes y_i \in L_\infty \widehat{\otimes}_\pi Y$, $\varepsilon_0 > 0$, and a slice $S := S(B_{L_\infty \widehat{\otimes}_\pi Y}, T, \alpha)$ such that, for every $v \in S$, it follows that

$$\|z + v\| \leq \|z\| + \|v\| - \varepsilon_0.$$

Now consider $f \otimes y \in S$, and define $E := \operatorname{span}\{f_1, \ldots, f_n, f\}$, which is a finite-dimensional subspace of L_∞. By [1, Theorem 1.5], there exists a separable ai-ideal W in X containing E. Since W is an ai-ideal in X, then W inherits the Daugavet property (see [3, Proposition 3.8]). Moreover, note that $R := S(B_{W \widehat{\otimes}_\pi Y}, T_{|W}, \alpha)$ contains $f \otimes y$. Furthermore, since W is an ai-ideal in X, then $\|z\|_{X \widehat{\otimes}_\pi Y} = \|z\|_{W \widehat{\otimes}_\pi Y}$. Consequently, by the conditions on z and S, we deduce that

$$\|z + v\|_{W \widehat{\otimes}_\pi Y} \leq \|v\| + \|z\|_{W \widehat{\otimes}_\pi Y} - \varepsilon_0$$

holds for every $v \in R$. This implies that W is a separable Banach space with the Daugavet property and that Y is a Banach space with the metric approximation property such that $W \widehat{\otimes}_\pi Y$ fails the Daugavet property. Then the conclusion follows.

In spite of the previous remark, we will show a class of Banach spaces for which Theorem 3.5 applies.

Theorem 3.7. *Let X be a separable L-embedded Banach space with the Daugavet property, and let Y be a nonzero Banach space. If either X^{**} or Y has the metric approximation property, then $X \widehat{\otimes}_\pi Y$ has the Daugavet property.*

Proof. In this case, $X^{**} = X \oplus_1 Z$ for some subspace $Z \subseteq X^{**}$ for which B_Z is w^*-dense in $B_{X^{**}}$ because of Theorem 3.4. This implies that the element u of the proof of Lemma 3.5 can be taken in S_Z. Pick $u^* \in S_{Z^*}$ such that $u^*(u) = 1$. Observe that, if we define $X_u := X \oplus \mathbb{R}u$, then $X_u^* = X^* \oplus_\infty \mathbb{R}u^*$, so the natural inclusion map $\varphi : X_u^* \longrightarrow X^{***} = X^* \oplus_\infty Z^*$ satisfies that

$$\varphi(x^* + \lambda u^*)(x + \lambda u) = (x^* + \lambda u^*)(x + \lambda u)$$

for every $x+\lambda u \in X_u$, which proves that φ is a Hahn–Banach extension operator. This implies that X_u is an ideal in X^{**} by Theorem 2.1, so Lemma 3.5 applies. □

Note that the key to proving Theorem 3.7 is that B_Z is w^*-dense in $B_{X^{**}}$. Let us show a class of L-embedded Banach spaces for which the previous assumption holds, and for which we will have to introduce a bit of notation. We recall that a complex JB*-triple is a complex Banach space X with a continuous triple product $\{\ldots\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables, conjugate-linear in the middle variable, and satisfies the following.

(1) For all x in X, the mapping $y \to \{xxy\}$ from X to X is a Hermitian operator on X and has nonnegative spectrum.
(2) The main identity

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X.
(3) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Concerning the condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is generally regarded to be Hermitian if $\|\exp(irT)\| = 1$ for every r in $\mathbb{R}$. Examples of complex JB*-triples are all C^*-algebras under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x).$$

Following [17], we define real JB*-triples as norm-closed real subtriples of complex JB*-triples. Here, by a *subtriple* we mean a subspace which is closed under triple products of its elements. Real JBW*-triples were first introduced as those real JB*-triples which are dual Banach spaces in such a way that the triple product becomes separately w^*-continuous (see [17, Definition 4.1 and Theorem 4.4]). Later, Martínez and Peralta [24] showed that the requirement of separate w^*-continuity of the triple product is superabundant. The bidual of every real (resp., complex) JB*-triple X is a JBW*-triple under a suitable triple product which extends the one of X (see [17, Lemma 4.2]) (resp., [11]).

Now we can establish the announced result.

Proposition 3.8. *Let X be a real or complex* JBW**-triple, let X_* be its predual, and consider a nonzero Banach space Y. If X_* has the Daugavet property and either Y or X^* has the metric approximation property, then $X_* \widehat{\otimes}_\pi Y$ has the Daugavet property.*

Proof. In this case, X_* is an L-embedded Banach space with the Daugavet property. Hence, it follows that $X^* = X_* \oplus_1 Z$ for some subspace Z of X^*. Since X_* has the Daugavet property, then X_* does not have any extreme point (see [6, Theorem 3.2]). Consequently, B_Z is w^*-dense in B_{X^*}, and the proof of Theorem 3.7 applies. □

4. A Daugavet index of thickness

Lemma 3.1 together with the definition of the index $T(X)$ motivates the definition of the index

$$\mathcal{T}(X) := \inf\left\{ r > 0 : \begin{array}{c} \exists n \in \mathbb{N}, x_1, \ldots, x_n \in S_X \\ \exists \emptyset \neq W \subseteq B_X \text{ weakly open } W \subseteq \bigcup_{i=1}^n B(x_i, r) \end{array} \right\}. \quad (4.1)$$

Moreover, in dual Banach spaces, it makes sense considering the index

$$\begin{aligned} &\mathcal{T}_{w^*}(X) \\ &:= \inf\left\{ r > 0 : \begin{array}{c} \exists n \in \mathbb{N}, x_1, \ldots, x_n \in S_X \\ \exists \emptyset \neq W \subseteq B_X \text{ weak-star open } W \subseteq \bigcup_{i=1}^n B(x_i, r) \end{array} \right\}. \end{aligned} \quad (4.2)$$

It is obvious from Lemma 3.1 that a Banach space X has the Daugavet property if and only if $\mathcal{T}(X) = 2$, which in turn is equivalent to the fact that $\mathcal{T}_{w^*}(X^*) = 2$. It is also clear, from the definition of $\mathcal{T}(X)$, that $\mathcal{T}(X) \leq T(X)$, but the inequality may be strict. Indeed, given a nonempty relatively weakly open subset W of B_X and $x \in W$, it is clear that $W \subseteq B(x, \operatorname{diam}(W))$. Consequently, the following proposition is clear.

Proposition 4.1. *Let X be a Banach space whose unit ball contains nonempty relatively weakly open subsets of B_X whose diameter is smaller than ε. Then $\mathcal{T}(X) \leq \varepsilon$. In particular, if X has a dentable unit ball (i.e., the unit ball contains slices of arbitrarily small diameter), then $\mathcal{T}(X) = 0$.*

Now we will introduce the following lemma, which allows us to consider $n = 1$ in the definition of the indices $\mathcal{T}$ and $\mathcal{T}_{w^*}$. This fact will be used throughout the section without any explicit reference.

Lemma 4.2. *Let X be a Banach space, and let $r > 0$. Assume that for every nonempty relatively weakly open subset W of B_X and every $x \in S_X$ there exists $w \in W$ such that $\|x - w\| > r$. Then, for every $n \in \mathbb{N}$, every $x_1, \ldots, x_n \in S_X$, and every nonempty relatively weakly open subset W of B_X, there exists $w \in W$ such that $\|x_i - w\| > r$ holds for every $i \in \{1, \ldots, n\}$.*

Note that a similar statement can be established for weakly-star open sets.

Proof. We will use induction on n. The case $n = 1$ is just the hypothesis of the lemma. So assume that the thesis holds for n, and let us prove it for $n+1$. Consider a nonempty relatively weakly open subset W of B_X and $x_1, \ldots, x_{n+1} \in S_X$. By assumption, there exists $y \in W$ such that $\|x_{n+1} - y\| > r$. This means that $V := W \setminus (x_{n+1} + rB_X)$ is nonempty. Moreover, V is a relatively weakly open subset of B_X since B_X is weakly closed and W is relatively weakly open. If we apply the induction hypothesis, we can find $w \in V \subseteq W$ such that $\|x_i - w\| > r$ holds for every $i \in \{1, \ldots, n\}$. The condition $\|x_{n+1} - w\| > r$ holds because of the definition of V. This finishes the proof. □

Let us now analyze the index $\mathcal{T}(X)$ for some classical Banach spaces.

Example 4.3.

(1) It is known that $T(\ell_1) = 2$ but $\mathcal{T}(\ell_1) = 0$. This shows that the inequality $\mathcal{T}(X) \leq T(X)$ can be strict.

(2) $\mathcal{T}(c_0) = 1$. Indeed, the inequality $\mathcal{T}(c_0) \geq 1$ follows immediately from the known fact that every nonempty relatively weakly open subset of B_{c_0} has diameter 2 (see, e.g., [2]). On the other hand, $S(B_{c_0}, e_1^*, \alpha) \subseteq B(e_1, 1)$, so $\mathcal{T}(c_0) = 1$ as desired. This proves that the converse of Proposition 4.1 does not hold.

(3) $\mathcal{T}_{w^*}(\ell_\infty) = 1$. Indeed, $\mathcal{T}_{w^*}(\ell_\infty) \leq 1$ as in the previous example. Moreover, the inequality $\mathcal{T}_{w^*}(\ell_\infty) \geq 1$ follows from the fact that every nonempty relatively weakly-star open subset of B_{ℓ_∞} has diameter 2 (see, e.g., [2]).

(4) $\mathcal{T}(\mathcal{C}([0,1])) = 2$ since $\mathcal{C}([0,1])$ has the Daugavet property. However, the unit ball of $\mathcal{C}([0,1])^*$ has denting points and, consequently, $\mathcal{T}(\mathcal{C}([0,1])^*) = 0$. Thus $\mathcal{T}_{w^*}(\mathcal{C}([0,1])^{**}) < 2$.

This index still has a relation with the Daugavet equation even when $\mathcal{T}(X) < 2$. The proof of the following result follows the ideas of [22, Theorem 2.3], but we include the proof for the sake of completeness.

Proposition 4.4. *Let X be a Banach space. Then, for every norm 1 and weakly compact operator $T : X \longrightarrow X$, it follows that*

$$\|T + I\| \geq \mathcal{T}(X).$$

Similarly, it follows that

$$\|T + I\| \geq \mathcal{T}_{w^*}(X^*).$$

Proof. Pick a weakly compact operator $T : X \longrightarrow X$ such that $\|T\| = 1$ and $\varepsilon > 0$. Then $K = \overline{T(B_X)}$ is weakly compact and, consequently, we can find a denting point y_0 of K such that $\|y_0\| > 1 - \varepsilon$. For $0 < \delta < \varepsilon$ we can find a slice $S := \{y \in K : y^*(y) > 1 - \delta\}$ containing y_0 and having diameter smaller than ε (see, e.g., [8, Theorem 3.6.1]). For $x^* = T^*(y^*)$, we have $\|x^*\| = 1$ and

$$T\big(S(B_X, x^*, \delta)\big) \subseteq S.$$

Now we can find $x \in S(B_X, x^*, \delta)$ such that $\|x + \frac{y_0}{\|y_0\|}\| > \mathcal{T}(X) - \varepsilon$, so $\|x + y_0\| > \mathcal{T}(X) - 2\varepsilon$. Moreover, $T(x) \in S$ and thus $\|T(x) - y_0\| < \varepsilon$. Consequently,

$$\|T + I\| \geq \big\|T(x) + x\big\| \geq \|x + y_0\| - \big\|T(x) - y_0\big\| > \mathcal{T}(X) - 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude the desired result.

The second part of the proof follows from the fact that T^* is also weakly compact and then $T^*(B_{X^*})$ has the Radon–Nikodym property, so $T(B_{X^*})$ is w^*-dentable (see [8, Theorem 4.2.13(f)]). □

Now we turn to analyze the index $\mathcal{T}$ with respect to ℓ_p-sums for $1 \leq p \leq \infty$.

Proposition 4.5. *Let X and Y be Banach spaces. Then we have the following.*

(1) $\mathcal{T}(X \oplus_\infty Y) \geq \min\{\mathcal{T}(X), \mathcal{T}(Y)\}$. *Moreover, if $\mathcal{T}(X \oplus_\infty Y) > 1$, then the equality holds.*

(2) $\mathcal{T}(X \oplus_1 Y) \leq \min\{\mathcal{T}(X), \mathcal{T}(Y)\}$.

(3) $\mathcal{T}(X \oplus_p Y) \leq (\frac{(2^{\frac{1}{p}}+1)^p+1}{2})^{\frac{1}{p}}$ *for every* $1 \leq p \leq \infty$.

Proof. (1). Consider a nonempty relatively weakly open subset W of $B_{X\oplus_\infty Y}$, $\varepsilon > 0$, and $(x,y) \in S_{X\oplus_\infty Y}$. By [2, Theorem 4.5] we can find nonempty weakly open sets U of B_X and V of B_Y such that $U \times V \subseteq W$. Because of the definition of the norm of $X \oplus_\infty Y$, we get that $\|x\| = 1$ or $\|y\| = 1$. We will assume without loss of generality that $\|x\| = 1$. In that case, we can find $u \in U$ such that $\|x - u\| > \mathcal{T}(X) - \varepsilon$. Clearly, given $v \in V$, it follows that $(u,v) \in U \times V \subseteq W$. Moreover, it follows that

$$\big\|(x,y) - (u,y)\big\| = \max\big\{\|x-u\|, \|y-v\|\big\} \geq \mathcal{T}(X) - \varepsilon.$$

Consequently, we get $\mathcal{T}(X \oplus_\infty Y) \geq \min\{\mathcal{T}(X), \mathcal{T}(Y)\}$. For the converse inequality, assume that $\min\{\mathcal{T}(X), \mathcal{T}(Y)\} = \mathcal{T}(X)$ and that $\mathcal{T}(X \oplus_\infty Y) > 1$, and pick $\varepsilon > 0$ such that $\mathcal{T}(X \oplus_\infty Y) - \varepsilon > 1$. Pick $x \in S_X$, and consider a nonempty relatively weakly open subset W of B_X. Since $W \times B_Y$ is weakly open in $B_{X\oplus_\infty Y}$, there exists $(w,y) \in W \times B_Y$ such that

$$1 < \mathcal{T}(X \oplus_\infty Y) - \varepsilon \leq \big\|(x,0) - (w,y)\big\| = \max\big\{\|x-w\|, \|y\|\big\}.$$

Since $\|y\| \leq 1$, it follows that $\|x - w\| > \mathcal{T}(X \oplus_\infty Y) - \varepsilon$ and, as $x \in W$, we conclude that $\mathcal{T}(X) \geq \mathcal{T}(X \oplus_\infty Y) - \varepsilon$. Since $0 < \varepsilon < \mathcal{T}(X \oplus_\infty Y) - 1$ was arbitrary, we conclude that $\mathcal{T}(X \oplus_\infty Y) = \min\{\mathcal{T}(X), \mathcal{T}(Y)\}$, so (1) is proved.

(2). Consider $Z := X \oplus_1 Y$, assume without loss of generality that $\min\{\mathcal{T}(X), \mathcal{T}(Y)\} = \mathcal{T}(X)$, and pick $\varepsilon > 0$. Then there exists a basic nonempty relatively weakly open subset $W = \bigcap_{i=1}^m S(B_X, x_i^*, \alpha)$ of B_X, and there is $x \in S_X$ such that

$$W \subseteq B\big(x, \mathcal{T}(X) + \varepsilon\big).$$

Now, by the proof of [4, Proposition 3.1], taking $0 < \eta < \alpha$ we conclude that $S(B_Z, (x_i^*, 0), \eta) \subseteq S(B_X, x_i^*, \alpha) + \eta B_Y$ holds for every $i \in \{1, \ldots, m\}$. Consequently,

$$\bigcap_{i=1}^m S\big(B_Z, (x_i^*, 0), \eta\big) \subseteq B\big(x, \mathcal{T}(X) + \varepsilon\big) \times \eta B_Y \subseteq B\big((x,0), \mathcal{T}(X) + \eta + \varepsilon\big).$$

Since α can be chosen to be arbitrarily small (see [18, Lemma 2.1]), we get that $\mathcal{T}(Z) \leq \mathcal{T}(X)$.

(3) This follows because $\mathcal{T}(X \oplus_p Y) \leq T(X \oplus_p Y) \leq (\frac{(2^{\frac{1}{p}}+1)^p+1}{2})^{\frac{1}{p}}$, where the last inequality was proved in [15, Proposition 2.7]. □

Example 4.6. Let $X := c_0$, $Y := \mathbb{R}$, and $Z := X \oplus_\infty Y$. Then Z is isometrically isomorphic to c_0 and thus $\mathcal{T}(Z) = 1 > \min\{\mathcal{T}(X), \mathcal{T}(Y)\} = \mathcal{T}(\mathbb{R}) = 0$. This proves that the inequality in (1) may be strict if we remove the assumption on $\mathcal{T}(X \oplus_\infty Y)$.

Let us now show some results related to the index $\mathcal{T}$ with respect to subspaces.

Proposition 4.7. *Let X be a Banach space, and let Y be an almost isometric ideal in X. Then $\mathcal{T}(X) \leq \mathcal{T}(Y)$.*

Proof. Pick a positive $\varepsilon > 0$, a basic nonempty relatively weakly open subset $W = \bigcap_{j=1}^m S(B_Y, y_j^*, \alpha_j)$ of B_Y, and $y \in S_Y$ such that

$$W \subseteq B\big(y, \mathcal{T}(Y) + \varepsilon\big).$$

Consider by [3, Theorem 1.4] a Hahn–Banach extension operator $\varphi : Y^* \longrightarrow X^*$ such that, for all finite-dimensional subspaces $E \subseteq X$ and $F \subseteq X^*$, there exists a linear and bounded operator $T : E \longrightarrow Y$ satisfying

(1) $T(e) = e$ for all $e \in E \cap Y$,
(2) $(1+\varepsilon)^{-1}\|e\| \leq \|T(e)\| \leq (1+\varepsilon)\|T(e)\|$ holds for all $e \in E$,
(3) $\varphi(y^*)(e) = y^*(T(e))$ for all $e \in E, y^* \in F$.

Let us prove that $U := \bigcap_{j=1}^m S(B_X, \varphi(y_j^*), \alpha_j) \subseteq B(y, (1+\varepsilon)(\mathcal{T}(Y) + \varepsilon))$ (note that, since W is nonempty, so is U). To this aim pick $x \in U$, define $E := \operatorname{span}\{y, x\} \subseteq X$ and $F := \operatorname{span}\{y_1^*, \ldots, y_m^*\} \subseteq Y^*$, and consider the associated operator $T : E \longrightarrow Y$ satisfying (1), (2), and (3). Now, given $j \in \{1, \ldots, m\}$, we have

$$1 - \alpha_j < \varphi(y_j^*)(x) = y_j^*\big(T(x)\big),$$

so $T(x) \in S$. Consequently, $\|T(x) - y\| \leq \mathcal{T}(Y) + \varepsilon$ holds true. Hence

$$\|x - y\| \leq (1+\varepsilon)\|T(x-y)\| = (1+\varepsilon)\|T(x) - y\| \leq (1+\varepsilon)\big(\mathcal{T}(Y) + \varepsilon\big),$$

which proves the desired inclusion and finishes the proof. □

Remark 4.8. Since every Banach space is an ai-ideal in its bidual, Example 4.3(4) shows that the inequality in the previous proposition may be strict.

We will finish the section with another result related to the inheritance to subspaces inspired by [5, Theorem 2.2].

Proposition 4.9. *Let X be a Banach space, and let Y be a finite-codimensional subspace of X. Then $\mathcal{T}(Y) \geq \mathcal{T}(X)$.*

Proof. Pick a weakly open set $W := \{y \in Y : |y_i^*(y - y_0)| < \varepsilon \text{ for all } i \in \{1, \ldots, n\}\}$, where $n \in \mathbb{N}$, $y_1^*, \ldots, y_n^* \in S_{Y^*}$, $y_0 \in S_Y$ and $\varepsilon > 0$ satisfies that

$$W \cap B_Y \neq \emptyset,$$

and pick $y \in S_Y$ and $0 < \delta < \varepsilon$. Let us find $z \in W \cap B_Y$ such that $\|y - z\| \geq \mathcal{T}(X) - \delta$ holds. To this aim, assume, up to an application of the Hahn–Banach theorem, that $y_i^* \in S_{X^*}$ holds for all $i \in \{1, \ldots, n\}$. Define

$$U := \Big\{x \in X : \big|y_i^*(x - y_0)\big| < \varepsilon - \frac{\delta}{4} \text{ for all } i \in \{1, \ldots, n\}\Big\}.$$

Consider $p : X \longrightarrow X/Y$ to be the quotient map. Now $p(U)$ is a weakly open set of X/Y which contains 0. Since X/Y is finite-dimensional, there exists a weakly open neighborhood of 0 V such that $V \subseteq p(U)$ and that

$$\operatorname{diam}(V) < \frac{\delta}{16}.$$

Consider $B := p^{-1}(V) \cap U \cap B_X$, which is a nonempty relatively weakly open subset of B_X. Since $y \in S_Y \subseteq S_X$, we can find $x \in B$ such that

$$\|y - x\| > \mathcal{T}(X) - \frac{\delta}{16}.$$

As $p(x) \in V$ and $\operatorname{diam}(V) < \frac{\delta}{16}$, we can find $u \in Y$ such that $\|x - u\| < \frac{\delta}{16}$. Define $z := \frac{u}{\|u\|} \in S_Y$, and note that $\|x - z\| < \frac{\delta}{4}$. Moreover, given $j \in \{1, \dots, n\}$, we get

$$\left|y_j^*(z - y_0)\right| \leq \left|y_j^*(x - y_0)\right| + \frac{\delta}{4} < \varepsilon,$$

so $z \in W$. Finally, it follows that

$$\|y - z\| \geq \|y - x\| - \|x - z\| > \mathcal{T}(X) - \frac{\delta}{2} > \mathcal{T}(X) - \delta.$$

Since $\delta > 0$ was arbitrary, we conclude that $\mathcal{T}(Y) \geq \mathcal{T}(X)$, so we are done. $\square$

Remark 4.10. The inequality in the previous proposition may be strict. Indeed, consider $Y := L_1([0,1])$ and $X := Y \oplus_1 \mathbb{R}$. From Proposition 4.5 we get that $\mathcal{T}(X) \leq \mathcal{T}(\mathbb{R}) = 0$, while $\mathcal{T}(Y) = 2$.

5. Some remarks and open questions

In general, it is false that the property of being an L-embedded Banach space is hereditary (see [16, Chapter IV]) and, to the best of our knowledge, it is not known whether an ideal in an L-embedded Banach space is itself an L-embedded Banach space (see [25, p. 608]). However, for the class of those L-embedded Banach spaces for which every subspace which is an ideal is itself an L-embedded Banach space (e.g., von Neumann algebras (see the proof of [25, Proposition 5])), the conclusion of Theorem 3.7 holds removing the separability assumption.

Proposition 5.1. *Let X be an L-embedded Banach space with the Daugavet property, and let Y be a nonzero Banach space. Assume that every ideal in X is itself an L-embedded Banach space. If either X^{**} or Y has the metric approximation property, then $X \widehat{\otimes}_\pi Y$ has the Daugavet property.*

Proof. Pick $z := \sum_{i=1}^n x_i \otimes y_i \in X \widehat{\otimes}_\pi Y$, and consider a slice $S := S(B_{X \widehat{\otimes}_\pi Y}, G, \alpha)$. Since $\|G\| = 1$, we can find $x \otimes y \in S \cap S_X$ with $\|x\| = \|y\| = 1$. Define

$$E := \operatorname{span}\{x_1, \dots, x_n, x\} \subseteq X.$$

Now E is a finite-dimensional subspace of X. By [1, Theorem 1.5] we can find an ai-ideal in X, say, W, containing E. Now note that $\|G_{|W}\| \geq \|G(x)\| > 1 - \alpha$, so we can consider

$$T := \left\{z \in B_{W \widehat{\otimes}_\pi Y} : G(z) > 1 - \alpha\right\},$$

which is a slice of $B_{W \widehat{\otimes}_\pi Y}$. Moreover, since W is an ai-ideal in X, we get that $z \in W \widehat{\otimes}_\pi Y$ and that $\|z\|_{W \widehat{\otimes}_\pi Y} = \|z\|_{X \widehat{\otimes}_\pi Y}$. Furthermore, note that W^{**} has the metric approximation property whenever X^{**} has the metric approximation property because $W^{\circ\circ}$ is 1-complemented in X^{**}. Since W is an L-embedded Banach space by the assumptions, we conclude from Theorem 3.7 that $W \widehat{\otimes}_\pi Y$

has the Daugavet property and, consequently, there exists $w \in T$ such that $\|z + w\|_{W \widehat{\otimes}_\pi Y} > 1 + \|z\| - \varepsilon$. Since $W \widehat{\otimes}_\pi Y$ is an isometric subspace of $X \widehat{\otimes}_\pi Y$, we conclude that

$$\|z + w\|_{X \widehat{\otimes}_\pi Y} > 1 + \|z\|_{X \widehat{\otimes}_\pi Y} - \varepsilon.$$

Moreover, since $w \in T$, we get that $w \in S$. Hence, $X \widehat{\otimes}_\pi Y$ enjoys the Daugavet property, as desired. □

In view of the previous proposition, it is natural to pose the following question.

Problem 5.2. Let X be an L-embedded space with the Daugavet property, and let Y be a nonzero Banach space. If either X^{**} or Y has the metric approximation property, does $X \widehat{\otimes}_\pi Y$ have the Daugavet property?

It is known that $X \widehat{\otimes}_\pi Y$ has, at least, an octahedral norm under the assumptions of the previous question (see [23, Theorem 4.3]).

With respect to Section 4, in view of the characterizations given in Theorem 2.2 and Proposition 4.4, it is natural to wonder the following.

Problem 5.3. Does the equality

$$\inf\left\{\|T + I\| : T \in S_{L(X,X)} \text{ and } T \text{ is weakly compact}\right\} = \max\left\{\mathcal{T}(X), \mathcal{T}_{w^*}(X^*)\right\}$$

hold for every Banach space X?

Acknowledgments. We want to thank Luis García-Lirola for fruitful conversations and the editor and one of the anonymous referees for careful readings that helped to improve the final version of this article. The author is especially grateful for the valuable remarks about Lemma 4.2 and its proof.

The author's work was supported by research grant Contratos predoctorales FPU del Plan Propio del Vicerrectorado de Investigación y Transferencia de la Universidad de Granada, by MINECO (Spain) grant MTM2015-65020-P, and by Junta de Andalucía grant FQM-0185.

References

1. T. A. Abrahamsen, *Linear extensions, almost isometries, and diameter two*, Extracta Math. **30** (2015), no. 2, 135–151. Zbl 1365.46006. MR3496032. 75, 81
2. T. A. Abrahamsen, V. Lima, and O. Nygaard, *Remarks on diameter 2 properties*, J. Convex Anal. **20** (2013), no. 2, 439–452. Zbl 1274.46027. MR3098474. 78, 79
3. T. A. Abrahamsen, V. Lima, and O. Nygaard, *Almost isometric ideals in Banach spaces*, Glasg. Math. J. **56** (2014), no. 2, 395–407. Zbl 1303.46012. MR3187906. DOI 10.1017/S0017089513000335. 70, 75, 80
4. M. D. Acosta, J. Becerra-Guerrero, and G. López-Pérez, *Stability results of diameter two properties*, J. Convex Anal. **22** (2015), no. 1, 1–17. Zbl 1332.46010. MR3346177. 79
5. J. Becerra-Guerrero, G. López-Pérez, and A. Rueda Zoca, *Subspaces of Banach spaces with big slices*, Banach J. Math. Anal. **10** (2016), no. 4, 771–782. Zbl 1365.46007. MR3548625. DOI 10.1215/17358787-3649392. 80
6. J. Becerra-Guerrero and M. Martín, *The Daugavet property of C^*-algebras, JB^*-triples, and of their isometric preduals*, J. Funct. Anal. **224** (2005), no. 2, 316–337. Zbl 1081.46009. MR2146042. DOI 10.1016/j.jfa.2004.11.004. 74, 76
7. J. Becerra-Guerrero and A. Rodríguez-Palacios, *Banach spaces with the Daugavet property, and the centralizer*, J. Funct. Anal. **254** (2008), no. 8, 2294–2302. Zbl 1149.46012. MR2402110. DOI 10.1016/j.jfa.2007.11.017. 69

8. R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon-Nikodým Property*, Lecture Notes in Math. **993**, Springer, Berlin, 1983. Zbl 0512.46017. MR0704815. DOI 10.1007/BFb0069321. 78
9. J. M. F. Castillo, P. L. Papini, and M. Simões, *Thick coverings for the unit ball of a Banach space*, Houston J. Math. **41** (2015), no. 1, 177–186. Zbl 1348.46012. MR3347943. 70
10. I. K. Daugavet, *A property of completely continuous operators in the space C*, Uspekhi Mat. Nauk **18** (1963), no. 5, 157–158. Zbl 0138.38603. MR0157225. 68
11. S. Dineen, "The second dual of a JB*-triple system" in *Complex Analysis, Functional Analysis and Approximation Theory (Campinas, 1984)*, North-Holland Math. Stud. **125**, North-Holland, Amsterdam, 1986, 67–69. Zbl 0653.46053. MR0893410. 76
12. G. Godefroy, *Metric characterization of first Baire class linear forms and octahedral norms*, Studia Math. **95** (1989), no. 1, 1–15. Zbl 0698.46011. MR1024271. 71
13. G. Godefroy and N. J. Kalton, "The ball topology and its applications" in *Banach Space Theory (Iowa City, IA, 1987)*, Contemp. Math. **85**, Amer. Math. Soc., Providence, 1989, 195–237. Zbl 0676.46003. MR0983386. DOI 10.1090/conm/085/983386. 71, 73
14. G. Godefroy, N. J. Kalton, and P. D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. **104** (1993), no. 1, 13–59. Zbl 0814.46012. MR1208038. 70
15. R. Haller and J. Langemets, *Geometry of Banach spaces with an octahedral norm*, Acta Comment. Univ. Tartu. Math. **18** (2014), no. 1, 125–133. Zbl 1314.46013. MR3339882. DOI 10.12697/ACUTM.2014.18.13. 79
16. P. Harmand, D. Werner, and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. **1547**, Springer, Berlin, 1993. Zbl 0789.46011. MR1238713. DOI 10.1007/BFb0084355. 69, 81
17. J. M. Isidro, W. Kaup, and A. Rodríguez-Palacios, *On real forms of JB^*-triples*, Manuscripta Math. **86** (1995), no. 3, 311–335. Zbl 0834.17047. MR1323795. DOI 10.1007/BF02567997. 76
18. Y. Ivakhno and V. Kadets, *Unconditional sums of spaces with bad projections*, preprint, `http://www.research.net/publication/235745598` (accessed 29 August 2017). 79
19. Y. Ivakhno, V. Kadets, and D. Werner, *The Daugavet property for spaces of Lipschitz functions*, Math. Scand. **101** (2007), no. 2, 261–279. Zbl 1177.46010. MR2379289. DOI 10.7146/math.scand.a-15044. 68
20. V. Kadets, N. Kalton, and D. Werner, *Remarks on rich subspaces of Banach spaces*, Studia Math. **159** (2003), no. 2, 195–206. Zbl 1055.46006. MR2052218. DOI 10.4064/sm159-2-3. 69
21. V. Kadets, V. Shepelska, and D. Werner, *Thickness of the unit sphere, ℓ_1-types, and the almost Daugavet property*, Houston J. Math. **37** (2011), no. 3, 867–878. Zbl 1235.46014. MR2844454. 70
22. V. Kadets, R. V. Shvydkoy, G. G. Sirotkin, and D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc. **352**, no. 2 (2000), 855–873. Zbl 0938.46016. MR1621757. DOI 10.1090/S0002-9947-99-02377-6. 68, 71, 72, 73, 78
23. J. Langemets, V. Lima, and A. Rueda Zoca, *Octahedral norms in tensor products of Banach spaces*, preprint, arXiv:1609.02062v3 [math.FA]. 69, 70, 75, 82
24. J. Martínez and A. M. Peralta, *Separate weak*-continuity of the triple product in dual real* JB**-triples*, Math. Z. **234** (2000), no. 4, 635–646. Zbl 0977.17032. MR1778403. DOI 10.1007/s002090050002. 76
25. T. S. S. R. K. Rao, *On ideals in Banach spaces*, Rocky Mountain J. Math. **31** (2001), no. 2, 595–609. Zbl 0988.46012. MR1840956. DOI 10.1216/rmjm/1020171577. 70, 75, 81
26. R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monogr. Math., Springer, London, 2002. Zbl 1090.46001. MR1888309. DOI 10.1007/978-1-4471-3903-4. 70, 74
27. R. V. Shvydkoy, *Geometric aspects of the Daugavet property*, J. Funct. Anal. **176** (2000), no. 2, 198–212. Zbl 0964.46006. MR1784413. DOI 10.1006/jfan.2000.3626. 68, 71

28. D. Werner, *Recent progress on the Daugavet property*, Irish Math. Soc. Bull. **46** (2001), 77–97. Zbl 0991.46009. MR1856978. 68, 69
29. R. Whitley, *The size of the unit sphere*, Canad. J. Math. **20** (1968), 450–455. Zbl 0153.44203. MR0228997. DOI 10.4153/CJM-1968-041-1. 69, 70

Universidad de Granada, Facultad de Ciencias. Departamento de Análisis Matemático and Instituto de Matemáticas de la Universidad de Granada (IEMath-GR), 18071-Granada (Spain).
E-mail address: abrahamrueda@ugr.es