Banach J. Math. Anal. 11 (2017), no. 3, 698–712
http://dx.doi.org/10.1215/17358787-2017-0012
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

ON THE UNIVERSAL AND STRONG (L^1, L^∞)-PROPERTY RELATED TO FOURIER–WALSH SERIES

MARTIN G. GRIGORYAN

Communicated by S. Astashkin

ABSTRACT. In this article, we construct a function $U \in L^1[0,1)$ with strictly decreasing Fourier–Walsh coefficients $\{c_k(U)\} \searrow$, and having a universal and strong (L^1, L^∞)-property with respect to the Walsh system.

1. INTRODUCTION AND PRELIMINARIES

Let $L^p[0,1)$, $p > 0$ be the class of all measurable functions f on [0,1) satisfying the condition

$$\int_0^1 \big|f(x)\big|^p \, dx < \infty. \tag{1.1}$$

By $L^\infty[0,1)$, we denote the space of all bounded measurable functions on $[0,1)$ with the norm

$$\|\cdot\|_\infty = \sup_{x\in[0,1)} \big\{|\cdot|\big\}. \tag{1.2}$$

Let $\Phi = \{\varphi_k(x)\}$ be the Walsh system, and let $f \in L^p[0,1)$, $p \geq 1$. We denote by $c_k(f)$ the Fourier–Walsh coefficients of f, that is,

$$c_k(f) = \int_0^1 f(x)\varphi_k(x)\, dx, \tag{1.3}$$

Received Jul. 25, 2016; Accepted Oct. 12, 2016.
First published online May 16, 2017.
2010 *Mathematics Subject Classification.* Primary 42A65; Secondary 42A20.
Keywords. universal function, Fourier–Walsh coefficients.

and we denote by $S_n(x,f)$ the nth partial sum of the Fourier–Walsh series of functions f, that is,

$$S_n(x,f)=\sum_{k=0}^{n} c_k(f)\varphi_k(x). \tag{1.4}$$

The spectrum of $f(x)$ (denoted by $\operatorname{spec}(f)$) is the support of $c_k(f)$, that is, the set of integers where $c_k(f)$ is nonzero, that is,

$$\operatorname{spec}(f)=\big\{k\in N, c_k(f)\neq 0\big\}. \tag{1.5}$$

Let $|E|$ be the Lebesgue measure of a measurable set $E\subset[0,1)$, and let $\chi_E(x)$ be its characteristic function, that is,

$$\chi_E(x)=\begin{cases}1, & \text{if } x\in E,\\ 0, & \text{if } x\notin E.\end{cases}$$

Definition 1.1. We say that a function $U\in L^1[0,1)$ has a *universal and strong (L^1, L^∞)-property with respect to the Walsh system* if for each number $\varepsilon\in(0,1)$ and for every almost everywhere finite and measurable function f on $[0,1]$ one can find a function $g\in L^\infty[0,1)$, $|\{x\in[0,1); g(x)\neq f(x)\}|<\epsilon$, such that $|c_k(g)|=c_k(U)$, $\forall k\in\operatorname{spec}(g)$ and the Fourier–Walsh series of $g(x)$ converges uniformly on $[0,1)$.

The above-mentioned definition, while not presented in its most general form, is given at the level of generality in which it will be applied in this article.

In the present article, we construct a function $U\in L^1[0,1)$ with strictly decreasing Fourier–Walsh coefficients $\{c_k(U)\}\searrow$, and having a universal and strong (L^1, L^∞)-property with respect to the Walsh system. This is a continuation of the author's previous work in [7], [6], and [8] (with Galoyan) on the convergence of Fourier series and on the behavior of Fourier coefficients in classical systems. Underlying the author's studies in this area is the so-called Luzin's *C-property* of measurable functions, which reads as follows: for every measurable, almost everywhere finite function f on $[0,1]$ and every $\epsilon>0$, there exists a measurable set $E\subset[0,1]$ with $|E|>1-\epsilon$ and a continuous function g which coincides with $f(x)$ on E. This famous result of Luzin [13, Theoerm 1] dates back to 1912. Luzin's idea of a modification of a function improving its properties was substantially developed later on.

In 1939 Menchoff [14] proved the following fundamental theorem.

Theorem (Menchoff's C-strong property)**.** *For every measurable, almost everywhere finite function f on $[0,2\pi]$ and every $\epsilon>0$, there is a continuous function f_ϵ such that $|\{x\in[0,2\pi]: f_\epsilon(x)\neq f(x)\}|<\epsilon$ and the Fourier series of the function f_ϵ converges uniformly in $[0,2\pi]$.*

In 1988 we were able to show that the trigonometric system possesses the L-strong property for integrable functions; that is, for each $\epsilon>0$ there exists a (measurable) set $E\subset[0,2\pi]$ of measure $|E|>2\pi-\epsilon$ such that for each function $f\in L^1[0,2\pi]$ there exists a function $g\in L^1[0,2\pi]$ equal to $f(x)$ on E and with Fourier series with respect to the trigonometric system convergent to $g(x)$ in the $L^1[0,2\pi]$-norm (see [5]). After Menchoff's proof of the C-strong property, many "correction" type theorems were proved for different systems (see [1], [2], [8], [9], [15], [17], [18], [20]; we refrain from providing a complete survey of all research done in this area). A number of papers (see [7], [5], [16]) have been devoted to correction theorems in which the absolute values of nonzero Fourier coefficients (by the Haar and Walsh systems) of the corrected function are monotonically decreasing.

Here we present results having a direct bearing on the present article. In [6, Theorem 2] we proved the following.

Theorem 1.2. *For any $0 < \epsilon < 1$ and each function $f \in L^\infty[0,1]$ one can find a function $g \in L^\infty[0,1)$, $|\{x \in [0,1); g(x) \neq f(x)\}| < \epsilon$, such that a sequence $\{|c_k(g)|, k \in \operatorname{spec}(g)\}$ is monotonically decreasing.*

In the present article, we prove the following theorem.

Theorem 1.3. *There exists a function $U \in L^1[0,1)$ with strictly decreasing Fourier–Walsh coefficients $\{c_k(U)\} \searrow$ such that, for every almost everywhere finite measurable function f on $[0,1]$, one can find a function $g \in L^\infty[0,1)$ with $|\{x \in [0,1) : g(x) \neq f(x)\}| < \epsilon$ such that $|c_k(g)| = c_k(U)$, $\forall k \in \operatorname{spec}(g)$.*

Theorems 1.2 and 1.3 follow from the more general Theorem 1.4.

Theorem 1.4. *There exists a function $U \in L^1[0,1)$ with strictly decreasing Fourier–Walsh coefficients $\{c_k(U)\} \searrow$ such that, for every almost everywhere finite measurable function f on $[0,1]$, one can find a function $g \in L^\infty[0,1)$ with $|\{x \in [0,1) : g(x) \neq f(x)\}| < \delta$ such that $|c_k(g)| = c_k(U)$, $\forall k \in \operatorname{spec}(g)$ and the Fourier–Walsh series of g converges uniformly on $[0,1)$.*

The following corollary is derived from this theorem.

Corollary 1.5. *There exist a function $U \in L^1[0,1)$ and numbers (signs) $\delta_k = \pm 1$, $k \in \mathbb{N}$, such that, for every almost everywhere finite measurable function f on $[0,1]$, one can find a function $g \in L^\infty[0,1)$ with $|\{x \in [0,1); g(x) \neq f(x)\}| < \epsilon$ such that the greedy algorithm of function g with respect to the Walsh system converges uniformly on $[0,1)$ and for some $\{N_m\}_{m=1}^\infty \nearrow$*

$$G_m(x,g) = S_{N_m}(x,g), \quad x \in [0,1), m = 0,1,2,\dots,$$
$$c_k(g) = \delta_k c_k(U), \quad \forall k \in \operatorname{spec}(g).$$

Recall that $G_m(x,g)$ is the mth greedy approximant of f with regard to the Walsh system, that is,

$$G_m(f) := \sum_{n=1}^{m} c_{\sigma(n)}(f)\varphi_{\sigma(n)}(x), \quad m = 1,2,\dots,$$

where $\{\sigma(n)\}_{n=1}^\infty-$ is a decreasing permutation of nonnegative integers such that

$$\left|c_{\sigma(n)}(f)\right| \geq \left|c_{\sigma(n+1)}(f)\right|, \quad n = 1,2,\dots.$$

This nonlinear method of approximation is known as a *greedy algorithm* (see [11]). The above-mentioned definitions, while not presented in their most general form, are given at the level of generality in which they will be applied in this article. Greedy algorithms in Banach spaces with respect to normalized bases have been considered in [3], [10]–[12], and [22].

The following problems remain open.

Question 1. Is it possible to construct the (universal) function U in Theorem 1.4 such that $U \in L^p[0,1)$ for some $p > 1$?

Question 2. Is it possible to choose a modified function g in Theorem 1.4 such that $g \in C[0,1)$?

Question 3. Is Theorem 1.4 (or Theorems 1.2 and 1.3) true for the trigonometric system?

2. Proofs of main lemmas

The Walsh system (see [21]), an extension of the Rademacher system, may be obtained in the following manner. Let r be the periodic function, of least period 1, defined on $[0,1)$ by

$$r = \chi_{[0,1/2)} - \chi_{[1/2,1)}.$$

The Rademacher system, $R = r_n : n = 0, 1, \dots,$ is defined by the conditions

$$r_n(x) = r(2^n x), \quad \forall x \in R, n = 0, 1, \dots,$$

and, in the ordering employed by Paley (see [4], [19], [21]), the nth element of the Walsh system $\{\varphi_n\}$ is given by

$$\varphi_n = \prod_{k=0}^{\infty} r_k^{n_k}, \tag{2.1}$$

where $\sum_{k=0}^{\infty} n_k 2^k$ is the unique binary expansion of n, with each n_k either 0 or 1. It is known that for the Walsh system $\{\varphi_n\}$ (see [19]) and for each natural number m, the following equation is true:

$$\int_0^1 \Big| \sum_{k=0}^{2^m-1} \varphi_k(x) \Big| \, dx = 1. \tag{2.2}$$

By $\Delta = \Delta_m^{(k)}$, we denote the dyadic intervals of the form

$$\Delta_m^{(k)} = \Big[\frac{k-1}{2^m}, \frac{k}{2^m}\Big), \quad k \in [1, 2^m].$$

We use the following lemma, previously proved in [6, Lemma 1].

Lemma 2.1. *Let dyadic interval $\Delta = \Delta_m^{(k)}$, $k \in [1, 2^m]$ and numbers $N_0 \in \mathbb{N}$, $\gamma \neq 0$, $\varepsilon \in (0,1)$ be given. Then there exist a measurable set $E \subset \Delta$ and a polynomial $Q(x)$ with respect to the Walsh system $\{\varphi_k(x)\}$ of the following form*

$$Q = \sum_{k=N_0}^{N} a_k \varphi_k,$$

which satisfy the following conditions:

(1) *the coefficients* $\{a_k\}_{k=N_0}^{N}$ *are* 0 *or* $\pm\gamma|\Delta|$,
(2) $|E| > (1-\varepsilon)|\Delta|$,
(3) $Q(x) = \begin{cases} \gamma, & \text{if } x \in E, \\ 0, & \text{if } x \notin \Delta, \end{cases}$
(4) $\max_{N_0 \le M \le N} \| \sum_{k=N_0}^{M} a_k \varphi_k \|_\infty \le 3|\gamma|\varepsilon^{-1}$.

The main building block in the proof of Theorem 1.4 is Lemma 2.3, which will be proved with the help of Lemma 2.2.

Lemma 2.2. *Let dyadic interval* Δ *and numbers* $m_0 \in \mathbb{N}$, $\gamma \neq 0$, $\delta \in (0,1)$, $0 < \theta < \frac{|\gamma|}{\delta}$ *be given. Then there exist a function* $g(x)$, *a measurable set* $E \subset \Delta$, *and polynomials* $H(x)$, $Q(x)$ *with respect to the Walsh system* $\{\varphi_k\}$ *of the following form*

$$H(x) = \sum_{k=2^{m_0-1}}^{2^m} b_k \varphi_k(x),$$

$$Q(x) = \sum_{k=2^{m_0}}^{2^m-1} \varepsilon_k b_k \varphi_k(x),$$

which satisfy the following conditions:

(1) $0 < b_{k+1} < b_k < \theta$, $\forall k \in [2^{m_0}, 2^m)$,
(2) $\varepsilon_k = 0, \pm 1$, $\forall k \in [2^{m_0}, 2^m)$,
(3) $\int_0^1 |H(x)|\, dx < \theta$,
(4) $|E| > (1-\delta)|\Delta|$,
(5) $g(x) = \begin{cases} \gamma, & \text{if } x \in E, \\ 0, & \text{if } x \notin \Delta, \end{cases}$
(6) $\|g(x) - Q(x)\|_\infty < \theta$,
(7) $\max_{2^{m_0-1} \le n < 2^m} \| \sum_{k=2^{m_0}}^{n} \varepsilon_k b_k \varphi_k(x) \|_\infty < \frac{7|\gamma|}{\delta}$.

Proof. We choose a natural number $\nu_0 > 1$ such that

$$2^{-\nu_0}|\gamma| < \frac{\theta}{4}, \tag{2.3}$$

and we present the interval Δ in the form of the union of dyadic intervals

$$\Delta = \bigcup_{\nu=1}^{2^{\nu_0}} \Delta_\nu, \tag{2.4}$$

with

$$|\Delta_\nu| = 2^{-\nu_0}|\Delta|. \tag{2.5}$$

By successively applying Lemma 1, we determine some sets $E_\nu \subset \Delta_\nu$, $\nu = 1, 2, \dots, 2^{\nu_0}$ and polynomials

$$Q_\nu(x) = \sum_{j=2^{m_{\nu-1}}}^{2^{m_\nu}-1} a_j \varphi_j(x), \quad a_j = 0 \text{ or } \pm\gamma|\Delta_j|, \text{ if } j \in [2^{m_{\nu-1}}, 2^{m_\nu}), \tag{2.6}$$

which satisfy the following conditions:

$$|E_\nu| > (1-\epsilon)|\Delta_\nu|, \tag{2.7}$$

$$Q_\nu(x) = \begin{cases} \gamma, & \text{if } x \in E_\nu, \\ 0, & \text{if } x \notin \Delta_\nu, \end{cases} \tag{2.8}$$

$$\max_{2^{m_\nu - 1} \le m < 2^{m_\nu}} \Big\| \sum_{j=2^{m_{\nu-1}}}^{m} a_j \varphi_j(x) \Big\|_\infty < \frac{3|\gamma|}{\delta}. \tag{2.9}$$

We define

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu, \tag{2.10}$$

$$b_k = \frac{|\gamma|}{2^{\nu_0}}|\Delta| + \frac{\theta}{2^{2k}}, \quad k \in [2^{m_0}, 2^m), m = m_{\nu_0} - 1, \tag{2.11}$$

$$H(x) = \sum_{k=2^{m_0}}^{2^m - 1} b_k \varphi_k(x), \quad m = m_{\nu_0} - 1, \tag{2.12}$$

$$g(x) = \sum_{\nu=1}^{\nu_0} Q_\nu(x) = \sum_{\nu=1}^{\nu_0} \sum_{j=2^{m_{\nu-1}}}^{2^{m_\nu}-1} a_j \varphi_j(x) = \sum_{j=2^{m_0}}^{2^m - 1} a_j \varphi_j(x), \tag{2.13}$$

$$Q(x) = \sum_{k=2^{m_0}}^{2^m - 1} \varepsilon_k b_k \varphi_k(x), \tag{2.14}$$

where

$$\varepsilon_k = \operatorname{sign}(a_k), \quad \forall k \in [2^{m_0}, 2^m). \tag{2.15}$$

It immediately follows from (2.4), (2.6), (2.7), (2.10), (2.11), and (2.13)–(2.15) that

$$|E| > (1-\delta)|\Delta|,$$

$$0 < b_{k+1} < b_k < \theta, \quad \varepsilon_k = \pm 1, 0, \forall k \in [2^{m_0}, 2^m),$$

$$g(x) = \begin{cases} \gamma, & \text{if } x \in E, \\ 0, & \text{if } x \notin \Delta, \end{cases}$$

$$\big\| g(x) - Q(x) \big\|_\infty < \theta.$$

By (2.11) and (2.12), we obtain

$$H(x) = \frac{|\gamma|}{2^{\nu_0}}|\Delta| \Big(\sum_{j=0}^{2^m - 1} \varphi_j(x) - \sum_{j=0}^{2^{m_0}-1} \varphi_j(x) \Big) + \theta \sum_{j=2^{m_0}}^{2^m - 1} \frac{1}{2^{2j}} \varphi_j(x).$$

From this and (2.1) and (2.2), we have

$$\int_0^1 \big| H(x) \big| \, dx \le \frac{|\gamma|}{2^{\nu_0}}|\Delta| + \frac{\theta}{2} \le \theta.$$

Let $\forall n \in [2^{m_0}, 2^m)$. Then for some $\nu \in [1, 2^{-\nu_0}]$, we have $n \in [2^{m_{\nu-1}}, 2^{m_\nu})$

$$\sum_{j=2^{m_0}}^{n} \varepsilon_j b_j \varphi_j(x) = \sum_{\nu=1}^{\nu-1} \sum_{j=2^{m_{k-1}}}^{2^{m_k}-1} \varepsilon_j b_j \varphi_j(x) + \sum_{j=2^{m_{\nu-1}}}^{n} \varepsilon_j b_j \varphi_j(x)$$
$$= \sum_{k=1}^{\nu-1} Q_k(x) + \sum_{j=2^{m_{\nu-1}}}^{n} a_j \varphi_j(x) + \theta \sum_{j=2^{m_0}}^{n} \frac{1}{2^{2j}} \varepsilon_j \varphi_j(x).$$

Thus, from (2.1), (2.6), (2.9), and (2.15), it follows that

$$\Big| \sum_{k=2^{m_0}}^{n} \varepsilon_k b_k \varphi_k(x) \Big| \le \frac{6|\gamma|}{\delta} + \theta < \frac{7|\gamma|}{\delta}.$$

Thus Lemma 2.2 is proved. □

Lemma 2.3. *Let number $m_0 > 1$, $\theta, \delta \in (0,1)$ and polynomial $f(x)$ with respect to the Walsh system be given. Then there exist a function $g(x)$, a measurable set $E \subset \Delta$, and polynomials $H(x)$, $Q(x)$ with respect to the Walsh system $\{\varphi_k\}$ of the following form*

$$H(x) = \sum_{k=2^{m_0}}^{2^m-1} b_k \varphi_k(x),$$
$$Q(x) = \sum_{k=2^{m_0}}^{2^m-1} \varepsilon_k b_k \varphi_k(x),$$

which satisfy the following conditions:

(1) $0 < b_{k+1} < b_k < \theta$, $\forall k \in [2^{m_0}, 2^m)$,
(2) $\varepsilon_k = 0, \pm 1$, $\forall k \in [2^{m_0}, 2^m)$,
(3) $|E| > (1-\delta)|\Delta|$,
(4) $\int_0^1 |H(x)|\, dx < \theta$,
(5) $g(x) = f(x)$, *for all* $x \in E$,
(6) $\|g(x)\|_\infty < \frac{16\|f\|_\infty}{\delta}$,
(7) $\|g(x) - Q(x)\|_\infty < \theta$,
(8) $\max_{2^{m_0-1} \le n < 2^m} \| \sum_{k=2^{m_0}}^{n} \varepsilon_k b_k \varphi_k(x)\|_\infty < \frac{15\|f\|_\infty}{\delta}$.

Proof. By presenting the function $f(x)$ in the form

$$f(x) = \sum_{\nu=1}^{\nu_0} \gamma_\nu \chi_{\Delta_\nu}(x), \tag{2.16}$$

where $\gamma_\nu \neq 0$, $1 \le \nu \le \nu_0$ and $\{\Delta_\nu\}_{\nu=1}^{\nu_0}$ are disjoint dyadic subintervals of the section $[0,1)$, and by successively applying Lemma 2.1 for each subinterval Δ_ν, $1 \le \nu \le \nu_0$, we can find some sets $E_\nu \subset \Delta_\nu$, functions $g_\nu(x)$, and polynomials

$$H_\nu(x) = \sum_{k=2^{m_{\nu-1}}}^{2^{m_\nu}-1} b_k\varphi_k(x), \quad 1 \le \nu \le \nu_0, \tag{2.17}$$

$$Q_\nu(x) = \sum_{k=2^{m_{\nu-1}}}^{2^{m_\nu}-1} \varepsilon_k b_k\varphi_k(x), \quad 1 \le \nu \le \nu_0, \tag{2.18}$$

which for all $\nu \in [1, \nu_0]$ satisfy the following conditions:

$$0 < b_{k+1} < b_k < \theta, \quad \forall k \in [2^{m_{\nu-1}}, 2^{m_\nu}), \tag{2.19}$$

$$\varepsilon_k = 0, \pm 1, \quad \forall k \in [2^{m_{\nu-1}}, 2^{m_\nu}), \tag{2.20}$$

$$\int_0^1 \big|H_\nu(x)\big|\, dx < \frac{\theta}{2^\nu}, \tag{2.21}$$

$$|E_\nu| > (1-\delta)|\Delta_\nu|, \tag{2.22}$$

$$g_\nu(x) = \begin{cases} \gamma_\nu, & \text{if } x \in E_\nu, \\ 0, & \text{if } x \notin \Delta_\nu, \end{cases} \tag{2.23}$$

$$\big\|g_\nu(x) - Q_\nu(x)\big\|_\infty < \frac{\min\{\theta, \|f\|_\infty\}}{2^{\nu-1}}, \tag{2.24}$$

$$\max_{2^{m_{\nu-1}} \le n < 2^{m_\nu}} \Big\| \sum_{k=2^{m_{\nu-1}}}^{n} \varepsilon_k b_k\varphi_k(x)\Big\|_\infty < \frac{7|\gamma_\nu|}{\delta}. \tag{2.25}$$

We define

$$g(x) = \sum_{\nu=1}^{\nu_0} g_\nu(x), \tag{2.26}$$

$$H(x) = \sum_{\nu=1}^{\nu_0} H_\nu(x) = \sum_{\nu=1}^{\nu_0} \sum_{k=2^{m_{\nu-1}}}^{2^{m_\nu}-1} b_k\varphi_k(x) = \sum_{k=2^{m_0}}^{2^m-1} b_k\varphi_k(x), \quad m = m_{\nu_0}, \tag{2.27}$$

$$Q(x) = \sum_{\nu=1}^{\nu_0} Q_\nu(x) = \sum_{\nu=1}^{\nu_0} \sum_{k=2^{m_{\nu-1}}}^{2^{m_\nu}-1} \varepsilon_k b_k\varphi_k(x) = \sum_{k=2^{m_0}}^{2^m-1} \varepsilon_k b_k\varphi_k(x), \tag{2.28}$$

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu. \tag{2.29}$$

By (2.19)–(2.24), we obtain

$$g(x) = f(x) \quad \text{for } x \in E,$$
$$|E| > 1 - \delta,$$
$$0 < b_{k+1} < b_k < \theta, \quad \varepsilon_k = 0, \pm 1, \forall k \in [2^{m_0}, 2^m).$$

From (2.23)–(2.24), for all $\nu \in [1, \nu_0]$, it follows that

$$\big|Q_\nu(x)\big| \le \frac{\|f\|_\infty}{2^{\nu-1}}, \quad x \notin \Delta_\nu. \tag{2.30}$$

By (2.24)–(2.27), we obtain

$$\begin{gathered}\int_0^1 \big|H(x)\big|\,dx \le \sum_{\nu=1}^{\nu_0} \int_0^1 \big|H_\nu(x)\big|\,dx < \sum_{\nu=1}^{\nu_0} \frac{\theta}{2^\nu} \le \theta, \\ \big\|g(x) - Q(x)\big\|_\infty < \sum_{\nu=1}^{\nu_0} \big\|g_\nu(x) - Q_\nu(x)\big\|_\infty \le \min\{\theta, \|f\|_\infty\}.\end{gathered} \tag{2.31}$$

Let $n \in [2^{m_0}, 2^m)$. and let $x \in [0,1)$. Then for some $\nu \in [1, \nu_0]$ we have $n \in [2^{m_{\nu-1}}, 2^{m_\nu})$, and for some $\nu' \in [1, \nu_0]$ we have $x \in \Delta_{\nu'}$. Then taking into account (2.18) and (2.28), we get

$$\sum_{k=2^{m_0}}^{n} \varepsilon_k b_k \varphi_k(x) = \sum_{j=1}^{\nu-1} Q_j(x) + \sum_{j=2^{m_{\nu-1}}}^{n} \varepsilon_k b_k \varphi_k(x)$$

and

$$\begin{aligned}\Big|\sum_{k=2^{m_0}}^{n} \varepsilon_k b_k \varphi_k(x)\Big| &\le \sum_{j=1}^{\nu_0} \big|Q_j(x)\big| \chi_{\Delta_{\nu'}}(x) + \sum_{j=1}^{\nu_0} \big|Q_j(x)\big| \chi_{[0,1)\setminus\Delta_{\nu'}}(x) \\ &\quad + \Big|\sum_{k=2^{m_{\nu-1}}}^{n} \varepsilon_k b_k \varphi_k(x)\Big| \\ &< \frac{7}{\delta}\|f\|_\infty + \sum_{\nu=1}^{\nu_0} \frac{\|f\|_\infty}{2^{\nu-1}} + \frac{7}{\delta}\|f\|_\infty \le \frac{15}{\delta}\|f\|_\infty.\end{aligned}$$

From this and (2.28) and (2.31), it follows that

$$\big\|g(x)\big\|_\infty < \frac{16\|f\|_\infty}{\delta}.$$

Thus Lemma 2.3 is proved. □

3. Proof of Theorem 1.4

By numbering polynomials with respect to the Walsh system having rational coefficients, we can present them as a sequence

$$\{f_n(x)\}_{n=1}^{\infty}. \tag{3.1}$$

By consecutively applying Lemma 3, one can find a sequence of functions $\{g_n^{(j)}(x)\}_{j=1}^n$; $n \ge 1$, sets $\{E_n^{(j)}\}_{j=1}^n$; $n \ge 1$, and polynomials of the form

$$H_n^{(j)}(x) = \sum_{k=M_n^{(j-1)}}^{M_n^{(j)}-1} b_k^{(n,j)} W_k(x), \quad 1 \le j \le n, n \in [1,\infty), b_k^{(n,j)} \searrow 0, \tag{3.2}$$

$$Q_n^{(j)}(x) = \sum_{k=M_n^{(j-1)}}^{M_n^{(j)}-1} \varepsilon_k^{(n,j)} b_k^{(n,j)} W_k(x), \quad 1 \le j \le n, n \in [1,\infty), \tag{3.3}$$

where

$$\begin{aligned} &\varepsilon_k^{(n,j)} = \pm 1, 0, \quad k \in [M_n^{(j-1)}, M_n^{(j)}), 1 \le j \le n, n = 1, 2, \dots, \\ &M_n^{(j)} = 2^{m_n^{(j)}}, \\ &\quad 0 \le M_1^{(0)} < M_1^{(1)} = M_2^{(0)} < M_2^{(1)} < M_2^{(2)} \\ &\qquad < M_{n-1}^{(n-1)} = M_n^{(0)} < M_n^{(1)} < \cdots < M_n^{(n)} = M_{n+1}^{(0)} < M_{n+1}^{(1)} \dots, \end{aligned} \tag{3.4}$$

which satisfy the following conditions:

$$g_n^{(j)}(x) = f_n(x), \quad x \in E_n^{(j)}, \tag{3.5}$$

$$|E_n^{(j)}| = 1 - 2^{-j}, \tag{3.6}$$

$$\left\| g_n^{(j)}(x) - Q_n^{(j)}(x) \right\|_\infty < 2^{-4n}, \quad 1 \le j \le n, \tag{3.7}$$

$$\left\| g_n^{(j)}(x) \right\|_\infty < 16 \cdot 2^j \|f_n\|_\infty, \quad 1 \le j \le n, \tag{3.8}$$

$$\max_{M_n^{(j-1)} \le l < M_n^{(j)}} \Big\| \sum_{k=M_n^{(j-1)}}^{l} \varepsilon_k^{(n,j)} b_k^{(n,j)} W_k(x) \Big\|_\infty < 15 \cdot 2^j \|f_n\|_\infty, \quad 1 \le j \le n, \tag{3.9}$$

$$\Big(\int_0^1 \big| H_n^{(j)}(x) \big| \, dx \Big) < 4^{-(n+j)}, \quad 1 \le j \le n. \tag{3.10}$$

We define a function U in the following way:

$$\begin{aligned} U(x) &= \sum_{n=1}^\infty \sum_{j=1}^n H_n^{(j)}(x) = \sum_{n=1}^\infty \sum_{j=1}^n \Big(\sum_{k=M_n^{(j-1)}}^{M_n^{(j)}-1} b_k^{(n,j)} W_k(x) \Big) \\ &= \sum_{k=0}^\infty b_k W_k(x), \\ &\quad \text{where } b_k := b_k^{(n,j)}, k \in [M_n^{(j-1)}, M_n^{(j)}), 1 \le j \le n, n = 1, 2, \dots. \end{aligned} \tag{3.11}$$

It is clear that

$$\begin{aligned} \int_0^1 \big| U(x) \big| \, dx &\le \sum_{n=1}^\infty \sum_{j=1}^n \Big(\int_0^1 \big| H_n^{(j)}(x) \big| \, dx \Big) \\ &< \sum_{n=1}^\infty \sum_{j=1}^n 4^{-(n+j)} < 1, \quad b_k \searrow 0. \end{aligned} \tag{3.12}$$

From this and (3.10) and (3.11), we have

$$\int_0^1 \Big| \sum_{k=0}^{M_n^{(n)}} b_k W_k(x) - U(x) \Big| \, dx \le 2^{-n} \to 0;$$

therefore,

$$b_k = c_k(U), \quad k = 0, 1, 2, \dots. \tag{3.13}$$

Let δ be an arbitrary number $\delta \in (0,1)$, and let f be an almost everywhere finite measurable function on $[0,1)$. Taking into account Luzin's theorem (see

[13]), one may assume without loss of generality that $f \in C[0,1)$. It is easy to see that one can choose a sequence $\{f_{k_n}\}_{n=1}^{\infty}$ from the sequence (3.1) such that

$$\lim_{N\to\infty}\Big\|\sum_{n=1}^{N} f_{k_n}(x) - f(x)\Big\|_\infty = 0, \tag{3.14}$$

$$\big\|f_{k_n}(x)\big\|_\infty \le 4^{-2n}, \quad n \ge 2, \tag{3.15}$$

where

$$k_1 > j_0 = [\log_{\frac{1}{2}} \delta] + 1. \tag{3.16}$$

(We have that $[a]$- is the integer part of the real number a.)

We put

$$Q_1(x) = Q_{k_1}^{(j_0+1)}(x), \qquad E_1 = E_{k_1}^{(j_0+1)}, \qquad g_1 = g_{k_1}^{(j_0+1)}.$$

Suppose that the natural numbers $k_1 = \nu_1 < \cdots < \nu_{q-1}$, the functions $f_{\nu_n}(x)$, $g_n(x)$, $1 \le n \le q-1$, the sets E_n, $1 \le n \le q-1$, and the polynomials

$$Q_n(x) = Q_{\nu_n}^{(n+j_0)}(x) = \sum_{k=M_{\nu_n}^{(n+j_0-1)}}^{M_{\nu_n}^{(n+j_0)}-1} \varepsilon_k^{(\nu_n, n+j_0)} b_k W_k(x)$$

are already defined and which for all $1 \le n \le q-1$ satisfy the following conditions:

$$\begin{gathered} g_n(x) = f_{k_n}(x), \quad x \in E_n, \\ \Big\|\sum_{k=1}^{n}\big[Q_k(x) - g_k(x)\big]\Big\|_\infty < 4^{-(n-1)}, \\ |E_n| > 1 - \delta 2^{-n}, \\ \big\|g_n(x)\big\|_\infty < 5\delta^{-1}2^{-(n-8)}, \\ \max_{M_{\nu_n}^{(n+j_0-1)} \le l < M_{\nu_n}^{(n+j_0)}} \Big\|\sum_{k=M_{\nu_n}^{(n+j_0-1)}}^{l} \varepsilon_k^{(\nu_n, n+j_0)} b_k W_k(x)\Big\|_\infty < 2^{-n}. \end{gathered} \tag{3.17}$$

It is easy to see that one can choose a function $f_{\nu_q}(x)$ $(\nu_q > \nu_{q-1})$ from the sequence (3.1) such that

$$\Big\|f_{\nu_q}(x) - \Big(f_{k_q}(x) - \sum_{i=1}^{q-1}\big[Q_i(x) - g_i(x)\big]\Big)\Big\|_\infty < \delta 4^{-2q}. \tag{3.18}$$

By virtue of (3.15), (3.17), and (3.18), we have

$$\begin{aligned} \|f_{\nu_q}\|_\infty \le{}& \Big\|f_{\nu_q}(x) - \Big(f_{k_q}(x) - \sum_{i=1}^{q-1}\big[Q_i(x) - g_i(x)\big]\Big)\Big\|_\infty \\ &+ \|f_{k_q}\|_\infty + \Big\|\sum_{i=1}^{q-1}\big[Q_i(x) - g_i(x)\big]\Big\|_\infty \\ <{}& \delta 4^{-(q-3)}. \end{aligned} \tag{3.19}$$

We put

$$g_q(x) = f_{k_q}(x) + \left[g_{\nu_q}^{(q+j_0)}(x) - f_{\nu_q}(x)\right], \tag{3.20}$$

$$Q_q(x) = Q_{\nu_q}^{(q+j_0)}(x) = \sum_{k=M_{\nu_q}^{(q+j_0-1)}}^{M_{\nu_q}^{(q+j_0)}-1} \varepsilon_k^{(\nu_q,n+j_0)} b_k W_k(x), \tag{3.21}$$

$$E_q(x) = E_{\nu_q}^{(q+j_0)}. \tag{3.22}$$

Taking into account (3.5) and (3.20), we get

$$g_q(x) = f_{k_q}(x), \quad x \in E_q, |E_q| > 1 - \delta 2^{-q}. \tag{3.23}$$

By virtue of (3.6) and (3.19)–(3.21), we obtain

$$\begin{aligned} \left\|\sum_{j=1}^{q}\left[Q_j(x) - g_j(x)\right]\right\|_\infty &= \left\|\sum_{j=1}^{q-1}\left[Q_j(x) - g_j(x)\right] + Q_q(x) - g_q(x)\right\|_\infty \\ &\le \left\|f_{\nu_q}(x) - \left(f_{k_q}(x) - \sum_{i=1}^{q-1}\left[Q_i(x) - g_i(x)\right]\right)\right\|_\infty \\ &\quad + \|g_{\nu_q}^{(q+j_0)} - Q_{\nu_q}^{(q+j_0)}\|_\infty \\ &< 4^{-(q-1)}. \end{aligned} \tag{3.24}$$

Obviously (see (3.9), (3.16), (3.19), (3.21)),

$$\max_{M_{\nu_q}^{(q+j_0-1)} \le l < M_{\nu_q}^{(q+j_0)}} \left\|\sum_{k=M_{\nu_q}^{(q+j_0-1)}}^{l} \varepsilon_k^{(\nu_q,q+j_0)} b_k W_k(x)\right\|_\infty < 2^{-q}. \tag{3.25}$$

From (3.8) and (3.17)–(3.19), it follows that

$$\begin{aligned} \left\|g_q(x)\right\|_\infty &\le \left\|f_{\nu_q}(x) - \left(f_{k_q}(x) - \sum_{i=1}^{q-1}\left[Q_i(x) - g_i(x)\right]\right)\right\|_\infty \\ &\quad + \left\|\sum_{j=1}^{q-1}\left[Q_j(x) - g_j(x)\right]\right\|_\infty + \|g_{\nu_q}^{(q+j_0)}\|_\infty \\ &< 4^{-2q} + 4^{-q+2} + 2^{q+j_0}\left\|f_{\nu_q}(x)\right\|_\infty < 2^{-q+8}. \end{aligned} \tag{3.26}$$

It is clear that by using an induction, one can determine a sequence of functions $\{g_q(x)\}_{q=1}^\infty$, sets $\{E_q\}_{q=1}^\infty$, and polynomials $\{Q_q(x)\}$ which satisfy the conditions (3.23)–(3.26) for all $q \ge 1$.

We put

$$E = \bigcap_{q=1}^{\infty} E_q. \tag{3.27}$$

From (3.23) and (3.27), it follows that

$$|E| > 1 - \delta.$$

Furthermore, according to (3.26), we have

$$\Big\|\sum_{q=1}^{\infty} g_q(x)\Big\|_\infty \le \sum_{q=1}^{\infty}\big\|g_q(x)\big\|_\infty < \infty. \tag{3.28}$$

We define the function $\tilde f(x)$ and the sequence of numbers $\{\varepsilon_k\}$ in the following way:

$$\tilde f(x) = \sum_{q=1}^{\infty} g_q(x), \tag{3.29}$$

$$\varepsilon_k = \begin{cases} \varepsilon_k^{(\nu_q, q+j_0)}, & k \in [M_{\nu_q}^{(q+j_0-1)}, M_{\nu_q}^{(q+j_0)}), q = 1, 2, \dots, \\ 0, & k \notin \bigcup_{q=1}^{\infty} [M_{\nu_q}^{(q+j_0-1)}, M_{\nu_q}^{(q+j_0)}). \end{cases} \tag{3.30}$$

From (3.14), (3.23), (3.27), and (3.29), it follows that

$$\tilde f(x) \in L^\infty[0,1], \qquad \tilde f(x) = f(x), \quad x \in E.$$

It is easy to note (see (3.4), (3.21), (3.25)) that

$$\begin{aligned} &\max_{M_{\nu_q}^{(0)} \le l < M_{\nu_{q+1}}^{(0)}} \Big\| \sum_{k=M_{\nu_q}^{(q+j_0-1)}}^{l} \varepsilon_k^{(\nu_q, q+j_0)} b_k W_k(x) \Big\|_\infty \\ &= \max_{M_{\nu_q}^{(q+j_0-1)} \le l < M_{\nu_q}^{(q+j_0)}} \Big\| \sum_{k=M_{\nu_q}^{(q+j_0-1)}}^{l} \varepsilon_k^{(\nu_q, q+j_0)} b_k W_k(x) \Big\|_\infty < 2^{-q}. \end{aligned} \tag{3.31}$$

Taking into consideration (3.26), (3.29), and (3.30), we get

$$\begin{aligned} &\Big\| \sum_{k=0}^{M_{\nu_q}^{(q+j_0-1)}-1} \varepsilon_k b_k \varphi_k(x) - \tilde f(x) \Big\| \\ &\quad = \Big\| \sum_{n=1}^{q-1} \Big(\sum_{k=M_{\nu_n}^{(n+j_0-1)}}^{M_{\nu_n}^{(n+j_0)}-1} \varepsilon_k^{(\nu_n, n+j_0)} b_k \varphi_k(x) \Big) - \tilde f(x) \Big\|_\infty \\ &\quad = \Big\| \sum_{n=1}^{q-1} Q_n(x) - \tilde f(x) \Big\|_\infty \\ &\quad \le \Big\| \sum_{n=1}^{q-1} \big(Q_n(x) - g_n(x)\big) \Big\|_\infty + \sum_{n=q}^{\infty} \|g_n(x)\|_\infty \le 5 \cdot 2^{-(q-10)}. \end{aligned}$$

From this and from (3.31), it follows that the series $\sum_{k=0}^{\infty} \varepsilon_k b_k \varphi_k(x)$ converges to the function $\tilde f(x)$ uniformly on $[0,1)$, and therefore (see (3.13))

$$c_k(\tilde f) = \int_0^1 \tilde f(x) \varphi_k(x)\, dx = \varepsilon_k b_k = \varepsilon_k c_k(U), \quad k = 0, 1, 2, \dots .$$

Thus Theorem 1.4 is proved. □

References

1. S. A. Episkoposian, *On the existence of universal series by the generalized Walsh system*, Banach J. Math. Anal. **10** (2016), no. 2, 415–429. Zbl 06575518. MR3489647. DOI 10.1215/17358787-3589331. 700
2. S. A. Episkoposian and J. Müller, *Universality properties of Walsh-Fourier series*, Monatsh. Math. **175** (2014), no. 4, 511–518. Zbl 1311.42067. MR3273663. DOI 10.1007/s00605-014-0631-5. 700
3. G. G. Gevorkyan and A. Kamont, *Two remarks on quasi-greedy bases in the space* L_1, J. Contemp. Math. Anal. **40** (2005), no. 1, 2–14. Zbl 1339.46026. MR2265671. 701
4. B. Golubov, A. Efimov, and V. Skvortsov, *Walsh Series and Transforms: Theory and Applications*, Math. Appl. (Soviet Series) **64**, Kluwer Academic, Dordrecht, 1991. Zbl 0785.42010. MR1155844. DOI 10.1007/978-94-011-3288-6. 701
5. M. G. Grigoryan, *On the convergence of Fourier series in the metric of* L^1, Anal. Math. **17** (1991), no. 3, 211–237. Zbl 0754.42006. MR1198962. DOI 10.1007/BF01907344. 700
6. M. G. Grigoryan, *Uniform convergence of the greedy algorithm with respect to the Walsh system*, Studia Math. **198** (2010), no. 2, 197–206. Zbl 1203.42046. MR2640077. DOI 10.4064/sm198-2-6. 699, 700, 701
7. M. G. Grigoryan, *Modifications of functions, Fourier coefficients and nonlinear approximation* (in Russian), Mat. Sb. **203** (2012), no. 3, 49–78; English translation in Sb. Math. **203** (2012), no. 3–4, 351–379. Zbl 1246.42024. MR2961732. 699, 700
8. M. G. Grigoryan and L. N. Galoyan, *On the uniform convergence of negative order Cesaro means of Fourier series*, J. Math. Anal. Appl. **434** (2016), no. 1, 554–567. Zbl 1331.42004. MR3404573. DOI 10.1016/j.jmaa.2015.09.001. 699, 700
9. M. G. Grigoryan and A. A. Sargsyan, *On the universal function for the class* $L^p[0,1]$, $p \in (0,1)$, J. Funct. Anal. **270** (2016), no. 8, 3111–3133. Zbl 1333.42049. MR3470436. DOI 10.1016/j.jfa.2016.02.021. 700
10. A. Kamont, *General Haar systems and greedy approximation*, Studia Math. **145** (2001), no. 2, 165–184. Zbl 0980.41017. MR1828003. DOI 10.4064/sm145-2-5. 701
11. S. V. Konyagin and V. N. Temlyakov, *A remark on greedy approximation in Banach spaces*, East J. Approx. **5** (1999), no. 3, 365–379. Zbl 1084.46509. MR1716087. 700, 701
12. T. W. Körner, *Decreasing rearranged Fourier series*, J. Fourier Anal. Appl. **5** (1999), no. 1, 1–19. Zbl 0978.42004. MR1682270. 701
13. N. N. Luzin, *On the fundamental theorem of the integral calculus* (in Russian), Mat. Sb. **28** (1912), 266–294; English translation in Sb. Math. **28** (1912). 699, 708
14. D. E. Menchoff, *Sur la convergence uniforme des series de Fourier*, Sb. Math. **53** (1942), no. 11, 67–96. Zbl 0060.18502. MR0012685. 699
15. D. E. Menchoff, *On Fourier series of summable functions*, Trans. Moscow Math. Soc. **1** (1952), 5–38. MR0049368. 700
16. K. A. Navasardyan, *On series in the Walsh system with monotone coefficients* (in Russian), Izv. Nats. Akad. Nauk Armenii Mat. **42** (2007), no. 5, 51–64; English translation in J. Contemp. Math. Anal. **42** (2007), no. 5, 258–269. Zbl 1188.42012. MR2416718. DOI 10.3103/S1068362307050032. 700
17. A. M. Olevskii, *Modifications of functions and Fourier series*, Russian Math. Surveys **40** (1985), no. 3, 181–224. Zbl 0609.42004. MR0795189. 700
18. K. I. Oskolkov, *The uniform modulus of continuity of summable functions on sets of positive measure* (in Russian), Dokl. Akad. Nauk SSSR **229** (1976), no. 2, 304–306; English translation in Dokl. Math. **229** (1976), no. 2, 1028–1030. Zbl 0349.26003. MR0470150. 700
19. R. E. A. C. Paley, *A remarkable set of orthogonal functions, I*, Proc. Lond. Math. Soc. **34** (1932), 241–264; *II*, 265–279. Zbl 0005.24806. MR1576148. DOI 10.1112/plms/s2-34.1.241. Zbl 0005.24901. MR1576149. 701
20. J. J. Price, *Walsh series and adjustment of functions on small sets*, Illinois J. Math. **13** (1969), 131–136. Zbl 0164.37103. MR0234199. 700

21. J. L. Walsh, *A closed set of normal orthogonal functions*, Amer. J. Math. **45** (1923), no. 1, 5–24. JFM 49.0293.03. MR1506485. 701
22. P. Wojtaszczyk, *Greedy algorithm for general biorthogonal systems*, J. Approx. Theory **107** (2000), no. 2, 293–314. Zbl 0974.65053. MR1806955. 701

Chair of Higher Mathematics, Faculty of Physics, Yerevan State University, Alex Manoogian str. 1, 0025, Yerevan, Republic of Armenia.
E-mail address: gmarting@ysu.am