Banach J. Math. Anal. 11 (2017), no. 2, 398–415
http://dx.doi.org/10.1215/17358787-0000010X
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

LATTICE PROPERTIES OF THE CORE-PARTIAL ORDER

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Communicated by D. Yang

Abstract. We show that in an arbitrary Hilbert space, the set of group-invertible operators with respect to the core-partial order has the complete lower semilattice structure, meaning that an arbitrary family of operators possesses the core-infimum. We also give a necessary and sufficient condition for the existence of the core-supremum of an arbitrary family, and we study the properties of these lattice operations on pairs of operators.

1. Introduction and motivation

The core inverse and the core-partial order are notions that were recently introduced by Baksalary and Trenkler in [4]. Although the study of these notions originated and was conducted in the space of square matrices (see [13], [14]), they were also extended to some more general structures. Thus, in [17] one can find a generalization of these notions to arbitrary Hilbert spaces, while [18] contains even further generalizations. The existing literature offers an extensive set of properties of the core-partial order regarding its characterizations, generalizations, and the connections with other partial orders, such as the minus-partial order, the star-partial order, and so on. However, to our knowledge, lattice properties of this partial order have not been studied thus far.

Hence, the purpose of this article is to explore the lattice structure of the core-partial order in an arbitrary Hilbert space. To that end, this article has the following structure. In Section 2, we briefly introduce our main notions: the core generalized inverse and the core-partial order. We also give a few basic statements in the form of lemmas. It seems that the statement of Lemma 2.4 from

Received Mar. 12, 2016; Accepted Jun. 29, 2016.
2010 *Mathematics Subject Classification.* Primary 47A05; Secondary 46C05, 15A09.
Keywords. core-partial order, core-infimum, core-supremum, core-parallel.

this section has not been underlined before, but it gives us a way to derive all the given results analogously in terms of the dual core-partial order. In Section 3, we give our main results regarding the existence of the core-supremum and core-infimum. We study arbitrary families of operators, as well as pairs of operators, as a prominent special case. In Section 4, we study further properties of these lattice operations. We characterize operator pairs for which the core-infimum attains, in a sense, a maximal possible value. Also, we establish a relationship between the core-infimum and the parallel sum of operators (for background on the subject of parallel sum, see [3]), and we discuss in which case the core-supremum and core-infimum of two operators A and B belong to the double commutant $\{A, B\}''$ of these operators.

Our intention is also to highlight the computational aspects of given results. For example, a necessary and sufficient condition for the existence of the core-supremum includes solvability of a system of operator equations (see Theorem 3.8). However, in the case of matrices, we give an elegant way to determine whether they have the core-supremum, which includes only matrix multiplication and a calculation of ranks (see Corollary 3.15). Throughout the article, examples are given to demonstrate the extent of the present results.

At the end of this section, we introduce some notation which we use henceforth, and we recall some notions, such as the Moore–Penrose inverse and the group inverse of an operator. (For a thorough study of these and other generalized inverses, the reader is referred to [6].)

If $\mathcal{H}$ is an arbitrary Hilbert space (real or complex), then the algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$, and if $A \in \mathcal{B}(\mathcal{H})$, then $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the range and null space of A, respectively, while A^* denotes its adjoint. Given $A, B \in \mathcal{B}(\mathcal{H})$, the following relations hold and will be frequently used in the subsequent sections without referencing:

$$\big(\mathcal{N}(A) \cap \mathcal{N}(B)\big)^{\perp} = \overline{\mathcal{R}(A^*) + \mathcal{R}(B^*)},$$

and

$$\big(\mathcal{N}(A) + \mathcal{N}(B)\big)^{\perp} = \overline{\mathcal{R}(A^*)} \cap \overline{\mathcal{R}(B^*)}.$$

We will also use the relations obtained by interchanging A and B with A^* and B^* in the preceding equalities. If $A \in \mathcal{B}(\mathcal{H})$ has a closed range, then its adjoint A^* has a closed range too, and a reduction of the operator A defined as $A|_{\mathcal{R}(A^*),\mathcal{R}(A)} : \mathcal{R}(A^*) \to \mathcal{R}(A)$ is an isomorphism. The (bounded) operator defined as $(A|_{\mathcal{R}(A^*),\mathcal{R}(A)})^{-1} \oplus 0$ with respect to the decomposition $\mathcal{R}(A) \oplus \mathcal{N}(A^*) = \mathcal{H}$ is called the *Moore–Penrose generalized inverse* of A and it is denoted by $A^{\dagger}$. We will denote by $\mathcal{B}^1(\mathcal{H})$ the subset of the algebra of bounded operators $\mathcal{B}(\mathcal{H})$, consisting of all operators with the index at most 1, that is, $\mathcal{B}^1(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(A) \oplus \mathcal{N}(A) = \mathcal{H}\}$. This definition implicitly contains the fact that all operators from $\mathcal{B}^1(\mathcal{H})$ have closed ranges. Hence, if $A \in \mathcal{B}^1(\mathcal{H})$, then the subspace $\mathcal{R}(A)$ reduces A to an isomorphism $A|_{\mathcal{R}(A)} : \mathcal{R}(A) \to \mathcal{R}(A)$. The operator defined as $(A|_{\mathcal{R}(A)})^{-1} \oplus 0$ with respect to the decomposition $\mathcal{R}(A) \oplus \mathcal{N}(A) = \mathcal{H}$ is called the *group inverse* of A and is denoted by $A^{\sharp}$. The operator $A \in \mathcal{B}(\mathcal{H})$ with a closed range is considered an EP operator if $\mathcal{R}(A) = \mathcal{R}(A^*)$. As we can see, in this case $A^\sharp = A^\dagger$.

2. The core inverse and the core-partial order

For the sake of completeness, we briefly recall the definition of the core inverse of an operator on an infinite-dimensional Hilbert space $\mathcal{H}$, which was introduced in [17], as well as the definition of the core-partial order. An additional reason for this summary is to emphasize aspects of these notions which are important to us.

If $A \in \mathcal{B}^1(\mathcal{H})$ and if, as before, $A|_{\mathcal{R}(A)}$ stands for the reduction of A: $A|_{\mathcal{R}(A)} : \mathcal{R}(A) \to \mathcal{R}(A)$, then the core inverse of the operator A, denoted by $A^{\textcircled{\#}}$, is defined as $A^{\textcircled{\#}} = (A|_{\mathcal{R}(A)})^{-1} \oplus 0$, but with respect to the decomposition $\mathcal{R}(A) \oplus \mathcal{N}(A^*) = \mathcal{H}$. For equivalent definitions of this generalized inverse, and the discussion concerning Penrose-like equations, the reader is referred to [17].

As we can see, on $\mathcal{R}(A)$ we have the equality $A^{\textcircled{\#}} = A^\sharp$, while on $\mathcal{N}(A^*)$ we have $A^{\textcircled{\#}} = A^\dagger$. In general, as long as the operator A is not EP, these three generalized inverses do not coincide (see [17, Theorem 3.10]). If we denote by $P_{\mathcal{M},\mathcal{N}}$ the projection onto a subspace $\mathcal{M}$ parallel with $\mathcal{N}$, while $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$, then we have $AA^{\textcircled{\#}} = P_{\mathcal{R}(A)}$ and $A^{\textcircled{\#}}A = P_{\mathcal{R}(A),\mathcal{N}(A)}$, since $\mathcal{R}(A^{\textcircled{\#}}) = \mathcal{R}(A)$ and $\mathcal{N}(A^{\textcircled{\#}}) = \mathcal{N}(A^*)$.

Given the "asymmetric" definition of the core inverse, we see that it does not obey some classic duality rule, like the group or Moore–Penrose inverse; for example, in general $(A^{\textcircled{\#}})^* \neq (A^*)^{\textcircled{\#}}$, $(A^{\textcircled{\#}})^{\textcircled{\#}} \neq A$, and so on.

From the computational point of view, a convenient way to express the core-inverse is $A^{\textcircled{\#}} = (A^2A^\dagger)^\dagger$ (see [17, p. 301]), since it only requires the computing of the Moore–Penrose inverses, although with an obvious downside: we need to compute two nested Moore–Penrose inverses. This equality served as a motivation to define the so-called *core generalized inverse* (see [5]), since formally it does not require $A \in \mathcal{B}^1(\mathcal{H})$.

The *core-partial order*, which we denote by $\leq^{\textcircled{\#}}$, is a relation defined in the following way:

$$A \leq^{\textcircled{\#}} B \quad \Leftrightarrow \quad A^{\textcircled{\#}}A = A^{\textcircled{\#}}B \quad \text{and} \quad AA^{\textcircled{\#}} = BA^{\textcircled{\#}}.$$

Obviously, a pair (A, B) belongs to this relation only if $A \in \mathcal{B}^1(\mathcal{H})$, while B does not have to be from $\mathcal{B}^1(\mathcal{H})$. However, since we wish for $\leq^{\textcircled{\#}}$ to be a partial order relation, that is, to be reflexive, we can accomplish this by adjoining to $\leq^{\textcircled{\#}}$ all the pairs (B, B) for every $B \in \mathcal{B}(\mathcal{H})$. More naturally, we can restrict our considerations only to $\mathcal{B}^1(\mathcal{H})$, where this relation is a partial order without any additional conditions. We do the latter, like most of the existing literature on this subject.

Again, an important fact from the computational point of view is that, in order to check whether $A \leq^{\textcircled{\#}} B$, we do not need to compute any generalized inverses, since [17, (26)] gives

$$A \leq^{\textcircled{\#}} B \quad \Leftrightarrow \quad A^*A = A^*B \quad \text{and} \quad A^2 = BA. \tag{2.1}$$

It is convenient to state the following properties of the $\textcircled{\#}$-partial order in the form of lemmas, for later reference. We only include the proof of Lemma 2.3, which seems to be scattered throughout existing literature. The proofs of the other two lemmas are easily derived from the definition.

Lemma 2.1. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$. Then $A \leq^{\textcircled{\#}} B$ if and only if A and B coincide on $\mathcal{R}(A)$ and $B(\mathcal{N}(A)) \subseteq \mathcal{N}(A^*)$. Moreover, if $A \leq^{\textcircled{\#}} B$, then A^* and B^* coincide on $\mathcal{R}(A)$.*

Lemma 2.2. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$ be such that $A \leq^{\textcircled{\#}} B$. Then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(A) \supseteq \mathcal{N}(B)$. Moreover, $A = B$ if and only if $\mathcal{R}(A) = \mathcal{R}(B)$ if and only if $\mathcal{N}(A) = \mathcal{N}(B)$.*

Lemma 2.3. *If $B \in \mathcal{B}(\mathcal{H})$ is a projection and $A \in \mathcal{B}^1(\mathcal{H})$ is such that $A \leq^{\textcircled{\#}} B$, then A is a projection. Moreover, if B is an orthogonal projection, then so is A.*

Proof. Since the $\textcircled{\#}$-partial order induces the minus-partial order, the first statement is contained in [3, Corollary 4.14]. For the second statement, it is enough to show that $A \leq^{\textcircled{\#}} I$ if and only if A is an orthogonal projection. This can be directly obtained from the definition. □

We mention that, together with the $\textcircled{\#}$-inverse and $\textcircled{\#}$-partial order, one could consider the dual $\textcircled{\#}$-inverse and dual $\textcircled{\#}$-partial order (see [17]). Namely, if $A \in \mathcal{B}^1(\mathcal{H})$, and if the operator $A|_{\mathcal{R}(A^*),\mathcal{R}(A)}$ is defined as before, then the operator $A_{\textcircled{\#}}$ defined as $(A|_{\mathcal{R}(A^*),\mathcal{R}(A)})^{-1} \oplus 0$ with respect to the decomposition $\mathcal{R}(A) \oplus \mathcal{N}(A)$ is called the *dual $\textcircled{\#}$-inverse* of A. So on $\mathcal{R}(A)$ we have $A_{\textcircled{\#}} = A^\dagger$, while on $\mathcal{N}(A)$ we have $A_{\textcircled{\#}} = A$. The dual $\textcircled{\#}$-partial order is defined as

$$A \leq_{\textcircled{\#}} B \quad \Leftrightarrow \quad A_{\textcircled{\#}}A = A_{\textcircled{\#}}B \quad \text{and} \quad AA_{\textcircled{\#}} = BA_{\textcircled{\#}}.$$

In similar fashion to (2.1), we can obtain (see also [4, p. 693])

$$A \leq_{\textcircled{\#}} B \quad \Leftrightarrow \quad AA^* = BA^* \quad \text{and} \quad A^2 = AB. \tag{2.2}$$

Lemma 2.4. *If $A, B \in \mathcal{B}^1(\mathcal{H})$, then*

$$A \leq^{\textcircled{\#}} B \quad \Leftrightarrow \quad A^* \leq_{\textcircled{\#}} B^*.$$

Proof. This is derived directly from (2.1) and (2.2). □

The preceding lemma could also be derived from the fact that if $A \in \mathcal{B}^1(\mathcal{H})$, then the dual $\textcircled{\#}$-inverse of A is $A_{\textcircled{\#}} = ((A^*)^{\textcircled{\#}})^*$ (see [17, Theorems 3.4 and 6.1]). Therefore, we focus our study only on a "regular" $\textcircled{\#}$-partial order.

3. Infimum and supremum in the $\textcircled{\#}$-partial order

In this section, we will prove that the set $\mathcal{B}^1(\mathcal{H})$ with respect to the $\textcircled{\#}$-partial order is in fact a complete lower semilattice, meaning that an arbitrary subset of $\mathcal{B}^1(\mathcal{H})$ has the $\textcircled{\#}$-infimum. This will follow from the fact proved in Theorem 3.3 stating that $\mathcal{B}^1(\mathcal{H})$ has the so-called *upper bound property*: for any subset $\{A_j \mid j \in J\} \subseteq \mathcal{B}^1(\mathcal{H})$, the existence of the $\textcircled{\#}$-supremum is equivalent with the existence of one common $\textcircled{\#}$-upper bound. However, it is easy to see that not all $A, B \in \mathcal{B}^1(\mathcal{H})$ have a common $\textcircled{\#}$-upper bound (e.g., take $A \neq B$ to be invertible).

We will also give some necessary and sufficient conditions for the existence of the ⓓ-supremum of two operators. Henceforth, we denote the lattice operations in this partial order with $\wedge^{ⓓ}$ and $\vee^{ⓓ}$.

In the following statements, let $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$ denote a family of operators with a common ⓓ-upper bound $A \in \mathcal{B}^1(\mathcal{H})$. Denote by $\mathcal{R}_1$ the vector space spanned by the set of vectors $\bigcup_{i\in I} \mathcal{R}(A_i)$, that is, $\mathcal{R}_1 = \{x_{i_1} + \cdots + x_{i_n} \mid x_{i_1} \in \mathcal{R}(A_{i_1}), \ldots, x_{i_n} \in \mathcal{R}(A_{i_n}), i_1, \ldots, i_n \in I, n \in \mathbb{N}\}$, and put $\mathcal{R} = \overline{\mathcal{R}_1}$. Let $\mathcal{N}$ denote $\bigcap_{i\in I} \mathcal{N}(A_i)$, and let $\mathcal{N}^*$ denote $\mathcal{R}^\perp = \bigcap_{i\in I} \mathcal{N}(A_i^*)$.

Lemma 3.1. *It holds that $\mathcal{R} \subseteq \mathcal{R}(A)$, that the reduction $A : \mathcal{R} \to \mathcal{R}$ is well defined, and that it is a bijection. Moreover, the reduction $A' : \mathcal{R} \to \mathcal{R}$ is the same for any common ⓓ-upper bound $A' \in \mathcal{B}^1(\mathcal{H})$ of the family $\{A_i \mid i \in I\}$.*

Proof. On every subspace $\mathcal{R}(A_i)$, the operators A and A_i coincide, and so $A(\mathcal{R}(A_i)) = \mathcal{R}(A_i)$. Thus $A(\mathcal{R}_1) = \mathcal{R}_1$, which gives $A(\mathcal{R}) \subseteq \mathcal{R}$, showing that this reduction is well defined. Also, from $A(\mathcal{R}_1) = \mathcal{R}_1$, we conclude that $\mathcal{R} \subseteq \mathcal{R}(A)$, showing that this reduction is injective.

Let $y \in \mathcal{R}$ be arbitrary. Then there is some $x \in \mathcal{R}(A)$ such that $Ax = y$, and let us prove that $x \in \mathcal{R}$. Since $y \in \mathcal{R}$, there is a sequence $(y_n) \subseteq \mathcal{R}_1$ such that $y_n \to y$. For every $y_n \in \mathcal{R}_1$, there is a finite sequence of indices $i_{n,1}, i_{n,2}, \ldots, i_{n,k_n}$ and vectors $b_{i_{n,1}}, b_{i_{n,2}}, \ldots, b_{i_{n,k_n}}$ such that $y_n = b_{i_{n,1}} + b_{i_{n,2}} + \cdots + b_{i_{n,k_n}}$, where $b_{i_{n,1}} \in \mathcal{R}(A_{i_{n,1}})$, $b_{i_{n,2}} \in \mathcal{R}(A_{i_{n,2}}), \ldots, b_{i_{n,k_n}} \in \mathcal{R}(A_{i_{n,k_n}})$. The operators A_i are of index at most 1, so there are $a_{i_{n,1}} \in \mathcal{R}(A_{i_{n,1}})$, $a_{i_{n,2}} \in \mathcal{R}(A_{i_{n,2}}), \ldots, a_{i_{n,k_n}} \in \mathcal{R}(A_{i_{n,k_n}})$ such that $A_{i_{n,1}} a_{i_{n,1}} = b_{i_{n,1}}$, $A_{i_{n,2}} a_{i_{n,2}} = b_{i_{n,2}}, \ldots, A_{i_{n,k_n}} a_{i_{n,k_n}} = b_{i_{n,k_n}}$. Denote by $x_n = a_{i_{n,1}} + a_{i_{n,2}} + \cdots + a_{i_{n,k_n}}$. Then $x_n \in \mathcal{R}_1$, and since A coincides with A_i on $\mathcal{R}(A_i)$, we have that $Ax_n = y_n$.

We now have $A(x_n - x) = y_n - Ax \to y - y = 0$. Since $x_n - x \in \mathcal{R}(A) = \overline{\mathcal{R}(A)}$, we conclude that $x_n - x \to 0$ (i.e., $x \in \mathcal{R}$). Thus, the reduction is also surjective.

To prove the last part of the statement, note that A and A' coincide on every $\mathcal{R}(A_i)$, and so on $\mathcal{R}_1$, but due to continuity, they also coincide on $\mathcal{R}$. ☐

Theorem 3.2. *It holds that $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$.*

Proof. Suppose that $x \in \mathcal{R} \cap \mathcal{N}$. From Lemma 3.1, we have that $Ax \in \mathcal{R}$. On the other hand, since $x \in \mathcal{N}$, from Lemma 2.1 we have that $Ax \in \mathcal{N}^* = \mathcal{R}^\perp$. This yields $Ax = 0$, but $x \in \mathcal{R} \subseteq \mathcal{R}(A)$. Thus $x = 0$, showing that $\mathcal{R} \cap \mathcal{N} = \{0\}$.

It remains to prove that $\mathcal{R} + \mathcal{N} = \mathcal{H}$. Let us first prove that the (well-defined) reduction $A : \mathcal{R}(A) \cap \mathcal{N} \to \mathcal{R}(A) \cap \mathcal{N}^*$ is a bijection. This reduction is injective, since A is injective on $\mathcal{R}(A)$. To show that it is surjective, pick any $y \in \mathcal{R}(A) \cap \mathcal{N}^*$. There is $x \in \mathcal{R}(A)$ such that $Ax = y$. For every $i \in I$, we have $A_i^{ⓓ} Ax = A_i^{ⓓ} y = 0$, and since $A_i \leq^{ⓓ} A$, we deduce that $0 = A_i^{ⓓ} Ax = A_i^{ⓓ} A_i x$; that is, $x \in \mathcal{N}(A_i)$. Thus $x \in \mathcal{R}(A) \cap \mathcal{N}$, and so this reduction is also surjective.

Denote by $S = \mathcal{R}(A) \cap \mathcal{N}$. Since $\mathcal{N}(A)$ is a part of $\mathcal{N}$ and $\mathcal{R}(A) \oplus \mathcal{N}(A) = \mathcal{H}$, we can easily conclude that $S \oplus \mathcal{N}(A) = \mathcal{N}$. We have that $\mathcal{R} \cap S = \{0\}$, so if we prove that $\mathcal{R} \oplus S = \mathcal{R}(A)$, we will have

$$\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A) = \mathcal{R} \oplus S \oplus \mathcal{N}(A) = \mathcal{R} \oplus \mathcal{N}.$$

Denote by $S_1 = \mathcal{R}(A) \cap \mathcal{N}^*$. Since $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}^*$, we can easily conclude (in the same way as before) that $\mathcal{R}(A) = \mathcal{R} \oplus S_1$.

Now take any $x \in \mathcal{R}(A)$, and let $y = Ax$. Then $y = r + s_1$, where $r \in \mathcal{R}$ and $s_1 \in S_1$. From Lemma 3.1 it follows that there is $\rho \in \mathcal{R}$ such that $A\rho = r$, and since $A : S \to S_1$ is a bijection, it follows that there is $\sigma \in S$ such that $A\sigma = s_1$. So $Ax = y = A(\rho + \sigma)$, while $x, \rho + \sigma \in \mathcal{R}(A)$. So $x = \rho + \sigma \in \mathcal{R} \oplus S$. Thus $\mathcal{R}(A) = \mathcal{R} \oplus S$, and the theorem is proved. $\square$

Note that the sum in Theorem 3.2 is not orthogonal in general. It would be orthogonal, for example, if all A_i are EP operators.

In the following theorem, we prove the upper-bound property of the structure $(\mathcal{B}^1(\mathcal{H}), \leq^{\textcircled{\#}})$.

Theorem 3.3. *If $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$, then the following statements are equivalent.*

(i) *There exists $A \in \mathcal{B}^1(\mathcal{H})$ such that $A_i \leq^{\textcircled{\#}} A$ for every $i \in I$.*

(ii) *There exists $\bigvee_{i\in I}^{\textcircled{\#}} A_i$.*

Proof. Since (ii) $\Rightarrow$ (i) is clear, we prove (i) $\Rightarrow$ (ii).

Denote by $\mathcal{R}$ and $\mathcal{N}$ the subsets defined by the family $\{A_i \mid i \in I\}$ as before, and let $P = P_{\mathcal{R},\mathcal{N}}$, which exists by Theorem 3.2. We will prove that $B = AP$ is the $\textcircled{\#}$-supremum of this family. From Theorem 3.2 and Lemma 3.1, it follows that $B \in \mathcal{B}^1(\mathcal{H})$ with $\mathcal{R}(B) = \mathcal{R}$ and $\mathcal{N}(B) = \mathcal{N}$. Using $A_iP = A_i$ and $PA_i^{\textcircled{\#}} = A_i^{\textcircled{\#}}$ for every $i \in I$ (the first equality follows from $\mathcal{N}(P) \subseteq \mathcal{N}(A_i)$ and the second one from $\mathcal{R}(A_i^{\textcircled{\#}}) \subseteq \mathcal{R}(P)$), from $A_iA_i^{\textcircled{\#}} = AA_i^{\textcircled{\#}}$ and $A_i^{\textcircled{\#}}A_i = A_i^{\textcircled{\#}}A$, respectively, we get $A_iA_i^{\textcircled{\#}} = BA_i^{\textcircled{\#}}$ and $A_i^{\textcircled{\#}}A_i = A_i^{\textcircled{\#}}B$. Thus, B is indeed one $\textcircled{\#}$-common upper bound for $\{A_i \mid i \in I\}$. Suppose that B_1 is another one, and let us prove that $B \leq^{\textcircled{\#}} B_1$.

From Lemma 3.1, we know that B and B_1 are the same on $\mathcal{R} = \mathcal{R}(B)$. Hence, we have $BB^{\textcircled{\#}} = B_1B^{\textcircled{\#}}$. We already know that the operators $B^{\textcircled{\#}}B$ and $B^{\textcircled{\#}}B_1$ are the same on $\mathcal{R}$, while on $\mathcal{N}$ they are both equal to the null operator: the first one because $\mathcal{N}(B) = \mathcal{N}$, and the second one since $B_1(\mathcal{N}) \subseteq \mathcal{N}^*$ (see Lemma 2.1), while $\mathcal{N}^* = \mathcal{R}^\perp = \mathcal{R}(B)^\perp = \mathcal{N}(B^{\textcircled{\#}})$. So from $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$ (see Theorem 3.2), we get that $B^{\textcircled{\#}}B = B^{\textcircled{\#}}B_1$. This completes the proof. $\square$

Previous considerations can be summarized in the next corollary.

Corollary 3.4. *If a family $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$ has some common $\textcircled{\#}$-upper bound $A \in \mathcal{B}^1(\mathcal{H})$, then $\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$, the operator $AP_{\mathcal{R},\mathcal{N}}$ does not depend on the choice of A and it is the $\textcircled{\#}$-supremum of the family $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$. Moreover, $\mathcal{R}(\bigvee_{i\in I}^{\textcircled{\#}} A_i) = \mathcal{R}$ and $\mathcal{N}(\bigvee_{i\in I}^{\textcircled{\#}} A_i) = \mathcal{N}$.*

Theorem 3.5. *If $\{A_j \mid j \in J\} \subseteq \mathcal{B}^1(\mathcal{H})$ is an arbitrary family, then $\bigwedge_{j\in J}^{\textcircled{\#}} A_j$ exists.*

Proof. Since the set of all common $\textcircled{\#}$-lower bounds of $\{A_j \mid j \in J\}$ is nonempty (it contains the null operator), and has at least one common $\textcircled{\#}$-upper bound (any A_j will suffice), from Theorem 3.3 we conclude that it has the $\textcircled{\#}$-supremum. Now, by a simple order-theoretic argument, it follows that this $\textcircled{\sharp}$-supremum is in fact the $\textcircled{\sharp}$-infimum for $\{A_j \mid j \in J\}$. $\square$

Theorem 3.2 gives one necessary condition for the existence of a common $\textcircled{\sharp}$-upper bound of a family of operators. We will derive a necessary and sufficient condition for an arbitrary family $\{A_i \mid i \in I\}$ to have at least one common $\textcircled{\sharp}$-upper bound—that is, to have the $\textcircled{\sharp}$-supremum. Special attention will be given to the families $\{A_1, A_2\}$, where, under some restrictions, these conditions are simplified. For example, if A_1 and A_2 are square matrices, we only need to check these simplified conditions.

Lemma 3.6. *If a family $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$ has a common $\textcircled{\sharp}$-upper bound $A \in \mathcal{B}^1(\mathcal{H})$, then for every $i, j \in I$ we have $A_i^{\textcircled{\sharp}} A_j A_j^{\textcircled{\sharp}} = A_i^{\textcircled{\sharp}} A_i A_j^{\textcircled{\sharp}}$.*

Proof. This follows directly from equalities which define the relations $A_i \leq^{\textcircled{\sharp}} A$ and $A_j \leq^{\textcircled{\sharp}} A$. $\square$

Lemma 3.7. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$. The following statements are equivalent:*

(i) $A^{\textcircled{\sharp}} B B^{\textcircled{\sharp}} = A^{\textcircled{\sharp}} A B^{\textcircled{\sharp}}$,
(ii) $A A^{\textcircled{\sharp}} B = A B^{\textcircled{\sharp}} B$,
(iii) $A^* A B = A^* B B$.

Proof. The assertion follows directly from the convenient multiplications from the left and right, and the fact that (i) is equivalent with $(A-B)\mathcal{R}(B) \subseteq \mathcal{N}(A^*)$. $\square$

The condition from Lemma 3.7 will appear in necessary and sufficient conditions for the existence of the $\textcircled{\sharp}$-supremum. Observe the computational advantage that statement (iii) has over the other two (equivalent) statements: for example, if A and B are two square matrices, condition (iii) is checked readily, and there is no need to compute the generalized inverses.

We briefly recall the notion of the *coherent pairs* introduced in [10]. Let $\{A_j \mid j \in J\} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{M}_j \subseteq \mathcal{H}$ be closed subspaces for every $j \in J$. We say that pairs $\{(A_j, \mathcal{M}_j) \mid j \in J\}$ are *coherent* if there is some $B \in \mathcal{B}(\mathcal{H})$ which coincides with A_j on $\mathcal{M}_j$ for every $j \in J$. In other words, pairs $\{(A_j, \mathcal{M}_j) \mid j \in J\}$ are coherent if there exists some $B \in \mathcal{B}(\mathcal{H})$ which satisfies $A_j P_{\mathcal{M}_j} = B P_{\mathcal{M}_j}$ for every $j \in J$.

Theorem 3.8. *Let $\{A_i \mid i \in I\} \subseteq \mathcal{B}^1(\mathcal{H})$. Then $\bigvee_{i \in I}^{\textcircled{\sharp}} A_i$ exists if and only if the following conditions are satisfied:*

(1) *$\{(A_i, \mathcal{R}(A_i)) \mid i \in I\}$ are coherent pairs and $\{(A_i^{\textcircled{\sharp}}, \mathcal{R}(A_i)) \mid i \in I\}$ are coherent pairs;*
(2) *for every $i, j \in I$, it holds that $A_i^{\textcircled{\sharp}} A_j A_j^{\textcircled{\sharp}} = A_i^{\textcircled{\sharp}} A_i A_j^{\textcircled{\sharp}}$;*
(3) *$\mathcal{H} = \mathcal{R} \oplus \mathcal{N}$, where $\mathcal{R}$ is the closure of the subspace spanned by the set $\bigcup_{i \in I} \mathcal{R}(A_i)$, while $\mathcal{N} = \bigcap_{i \in I} \mathcal{N}(A_i)$.*

Proof. If $A = \bigvee_{i \in I}^{\textcircled{\sharp}} A_i$ exists, then A coincides with A_i on $\mathcal{R}(A_i)$, for every $i \in I$, and $A^{\textcircled{\sharp}}$ coincides with $A_i^{\textcircled{\sharp}}$ on $\mathcal{R}(A_i)$, for every $i \in I$, so condition (1) is satisfied. Conditions (2) and (3) follow from Lemma 3.6 and Theorem 3.2.

Now suppose that (1), (2), and (3) are fulfilled. Denote by $A_1 \in \mathcal{B}(\mathcal{H})$ (resp., $B_1 \in \mathcal{B}(\mathcal{H})$) the operator that coincides with A_i on $\mathcal{R}(A_i)$ for every $i \in I$ (resp., with $A_i^{\textcircled{\#}}$ on $\mathcal{R}(A_i)$ for every $i \in I$). Let $P = P_{\mathcal{R},\mathcal{N}}$, $A = A_1 P$, $B = B_1 P$, and as before, let $\mathcal{R}_1$ be the subspace spanned by the set $\bigcup_{i \in I} \mathcal{R}(A_i)$. In that case, we have $A(\mathcal{R}_1) = \mathcal{R}_1$ and $\mathcal{R}(A) = \mathcal{R}(AP) = A(\mathcal{R}) \subseteq \overline{A(\mathcal{R}_1)} = \mathcal{R}$. Similarly, $\mathcal{R}(B) \subseteq \mathcal{R}$, since $B(\mathcal{R}(A_i^{\textcircled{\#}})) = \mathcal{R}(A_i)$ for every $i \in I$. Thus we can take reductions $\tilde{A}$ and $\tilde{B}$ of A and B on $\mathcal{R}$. Operator $\tilde{B}\tilde{A}$ is equal to identity on every $\mathcal{R}(A_i)$, thus on $\mathcal{R}_1$. Since it is bounded, it is equal to identity on the whole of $\mathcal{R}$. Similarly, $\tilde{A}\tilde{B} = I$. This means that $\tilde{A}$ and $\tilde{B}$ are both injective and surjective, which leads us to the conclusion that $\mathcal{N}(A) = \mathcal{N}$; then $A \in \mathcal{B}^1(\mathcal{H})$ and $B = A^\sharp$.

We will complete the proof by showing that A is one $\textcircled{\#}$-common upper bound for $\{A_i \mid i \in I\}$. Operators A and A_i coincide on $\mathcal{R}(A_i)$, and so $A_i A_i^{\textcircled{\#}} = A A_i^{\textcircled{\#}}$, for every $i \in I$. The equality $A_i^{\textcircled{\#}} A_i = A_i^{\textcircled{\#}} A$ obviously holds on $\mathcal{N}$, but also on every $\mathcal{R}(A_j)$, $j \in I$; if we take any $y \in \mathcal{R}(A_j)$, then $Ay = A_j y$ and there is some x such that $y = A_j^{\textcircled{\#}} x$, so $A_i^{\textcircled{\#}}(A - A_i)y = A_i^{\textcircled{\#}}(A_j - A_i)y = A_i^{\textcircled{\#}}(A_j - A_i)A_j^{\textcircled{\#}} x = 0$, by (2). By continuity and (3), we have that $A_i^{\textcircled{\#}} A_i = A_i^{\textcircled{\#}} A$. Therefore, A is indeed one $\textcircled{\#}$-common upper bound for $\{A_i \mid i \in I\}$, and by Theorem 3.3, $\bigvee_{i \in I}^{\textcircled{\#}} A_i$ exists. □

In what follows, we deal with the case $\{A_i \mid i \in I\} = \{A, B\}$.

Lemma 3.9. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$ be such that $A^{\textcircled{\#}} B B^{\textcircled{\#}} = A^{\textcircled{\#}} A B^{\textcircled{\#}}$ and $B^{\textcircled{\#}} A A^{\textcircled{\#}} = B^{\textcircled{\#}} B A^{\textcircled{\#}}$. Then*

(a) *A and B coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$;*
(b) *if $A^{\textcircled{\#}}$ and $B^{\textcircled{\#}}$ coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$, then $(\mathcal{R}(A) + \mathcal{R}(B)) \cap [\mathcal{N}(A) \cap \mathcal{N}(B)] = \{0\}$;*
(c) *if $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional, then $A^{\textcircled{\#}}$ and $B^{\textcircled{\#}}$ coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$.*

Proof. (a) From $A^{\textcircled{\#}} B B^{\textcircled{\#}} = A^{\textcircled{\#}} A B^{\textcircled{\#}}$, we get $(A - B)(\mathcal{R}(B)) \subseteq \mathcal{N}(A^*)$ (see Lemma 3.7). Similarly, from $B^{\textcircled{\#}} A A^{\textcircled{\#}} = B^{\textcircled{\#}} B A^{\textcircled{\#}}$, we get $(A - B)(\mathcal{R}(A)) \subseteq \mathcal{N}(B^*)$. Hence, $(A - B)(\mathcal{R}(A) \cap \mathcal{R}(B)) \subseteq \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$, but on the other hand, $(A - B)(\mathcal{R}(A) \cap \mathcal{R}(B)) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$. So $(A - B)(\mathcal{R}(A) \cap \mathcal{R}(B)) = \{0\}$, that is, A and B coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$.

(b) Let $n \in (\mathcal{R}(A) + \mathcal{R}(B)) \cap [\mathcal{N}(A) \cap \mathcal{N}(B)]$ be arbitrary. There are $r_A \in \mathcal{R}(A)$ and $r_B \in \mathcal{R}(B)$ such that $n = r_A - r_B$. Since $n \in \mathcal{N}(A) \cap \mathcal{N}(B)$, we get $Ar_A = Ar_B$ and $Br_A = Br_B$. Let us prove that $Ar_A = Br_B$. From $A^{\textcircled{\#}} B B^{\textcircled{\#}} = A^{\textcircled{\#}} A B^{\textcircled{\#}}$, in the same way as before, we get that $(A - B)r_B \in \mathcal{N}(A^*)$. Since $Ar_B = Ar_A$, this means that $Ar_A - Br_B \in \mathcal{N}(A^*)$. Similarly, we have $Ar_A - Br_B \in \mathcal{N}(B^*)$, and so $Ar_A - Br_B \in \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$. On the other hand, $Ar_A - Br_B \in \mathcal{R}(A) + \mathcal{R}(B)$, and so $Ar_A - Br_B = 0$, that is, $Ar_A = Br_B \in \mathcal{R}(A) \cap \mathcal{R}(B)$. Since $A^{\textcircled{\#}}$ and $B^{\textcircled{\#}}$ coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$: $r_A = A^{\textcircled{\#}} A r_A = B^{\textcircled{\#}} B r_B = r_B$, and $n = 0$.

(c) From (a) we have that A and B map $\mathcal{R}(A) \cap \mathcal{R}(B)$ into itself and that they are injective on this space, so if $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional, they are also bijective. Thus, for every $y \in \mathcal{R}(A) \cap \mathcal{R}(B)$ there is $x \in \mathcal{R}(A) \cap \mathcal{R}(B)$ such that

$Ax = Bx = y$. This means that $A^{\oplus}y$ as well as $B^{\oplus}y$ are exactly equal to x. Thus $A^{\oplus}$ and $B^{\oplus}$ coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$. $\square$

Example 3.10. Let $\mathcal{H}_1$ be an infinite-dimensional separable Hilbert space with an orthonormal basis $\{e_1, e_2, e_3, \dots\}$, and let $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_1 \times \mathcal{H}_1$. If $A, B : \mathcal{H} \to \mathcal{H}$ are maps defined in the following manner,

$$A : \Big(\sum_{i=1}^{\infty} x_i e_i, \sum_{i=1}^{\infty} y_i e_i, \sum_{i=1}^{\infty} z_i e_i\Big) \mapsto \Big(\sum_{i=1}^{\infty} x_{i+1} e_i, x_1 e_1 + \sum_{i=2}^{\infty} y_{i-1} e_i, 0\Big),$$
$$B : \Big(\sum_{i=1}^{\infty} x_i e_i, \sum_{i=1}^{\infty} y_i e_i, \sum_{i=1}^{\infty} z_i e_i\Big) \mapsto \Big(0, z_1 e_1 + \sum_{i=2}^{\infty} y_{i-1} e_i, \sum_{i=1}^{\infty} z_{i+1} e_i\Big),$$

then it is not difficult to see that $A, B \in \mathcal{B}^1(\mathcal{H})$ with $\mathcal{R}(A) = \mathcal{H}_1 \times \mathcal{H}_1 \times \{0\}$ and $\mathcal{R}(B) = \{0\} \times \mathcal{H}_1 \times \mathcal{H}_1$. Moreover, A and B coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$. Furthermore, we have

$$A^{\oplus} : \Big(\sum_{i=1}^{\infty} x_i e_i, \sum_{i=1}^{\infty} y_i e_i, \sum_{i=1}^{\infty} z_i e_i\Big) \mapsto \Big(y_1 e_1 + \sum_{i=2}^{\infty} x_{i-1} e_i, \sum_{i=1}^{\infty} y_{i+1} e_i, 0\Big),$$
$$B^{\oplus} : \Big(\sum_{i=1}^{\infty} x_i e_i, \sum_{i=1}^{\infty} y_i e_i, \sum_{i=1}^{\infty} z_i e_i\Big) \mapsto \Big(0, \sum_{i=1}^{\infty} y_{i+1} e_i, y_1 e_1 + \sum_{i=2}^{\infty} z_{i-1} e_i\Big).$$

Thus, $A^{\oplus}$ and $B^{\oplus}$ do not coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$, since $A^{\oplus}(0, e_1, 0) \neq B^{\oplus}(0, e_1, 0)$.

As we can see, a mere coincidence of A and B on $\mathcal{R}(A) \cap \mathcal{R}(B)$ cannot assure that operators $A^{\oplus}$ and $B^{\oplus}$ also coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$ if the underlying Hilbert space is infinite-dimensional. However, the operators A and B constructed here do not fulfill $A^{\oplus}AB^{\oplus} = A^{\oplus}BB^{\oplus}$ and so on, so perhaps the condition of finite-dimensionality of $\mathcal{R}(A) \cap \mathcal{R}(B)$ in Lemma 3.9 statement (c) (and likewise in further statements) is dispensable.

Lemma 3.11. *If $A, B \in \mathcal{B}^1(\mathcal{H})$ are such that $A^{\oplus}BB^{\oplus} = A^{\oplus}AB^{\oplus}$ and $B^{\oplus}AA^{\oplus} = B^{\oplus}BA^{\oplus}$, and if $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, then $(A, \mathcal{R}(A))$ and $(B, \mathcal{R}(B))$ are coherent pairs. Furthermore, if $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional, then $(A^{\oplus}, \mathcal{R}(A))$ and $(B^{\oplus}, \mathcal{R}(B))$ are also coherent pairs.*

Proof. From Lemma 3.9, we have that A and B coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$. Since $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, it is not difficult to deduce that $(A, \mathcal{R}(A))$ and $(B, \mathcal{R}(B))$ are coherent (see [10, Proposition 2.1]). If $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional, again from Lemma 3.7 we get that $(A^{\oplus}, \mathcal{R}(A))$ and $(B^{\oplus}, \mathcal{R}(B))$ are coherent. $\square$

The following theorem simplifies the conditions of Theorem 3.2 in the case in which $\mathcal{R}(A) + \mathcal{R}(B)$ is closed and $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional.

Theorem 3.12. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$ be such operators that $\mathcal{R}(A) + \mathcal{R}(B)$ is closed, while $\mathcal{R}(A) \cap \mathcal{R}(B)$ is finite-dimensional. Then $A \vee^{\oplus} B$ exists if and only if the following conditions are satisfied:*

(1′) $A^{\oplus}BB^{\oplus} = A^{\oplus}AB^{\oplus}$ *and* $B^{\oplus}AA^{\oplus} = B^{\oplus}BA^{\oplus}$,
(2′) $\mathcal{H} = (\mathcal{R}(A) + \mathcal{R}(B)) \dotplus [\mathcal{N}(A) \cap \mathcal{N}(B)]$.

Proof. We need to prove only that (1′) and (2′) imply conditions (1), (2), and (3) of Theorem 3.8. Clearly, (2) holds. From Lemma 3.11, we have that (1) also holds. To see that (3) holds, we use Lemma 3.9, the fact that $\mathcal{R}(A)+\mathcal{R}(B)$ is closed, and (2′). □

The following example shows that condition (2′) of Theorem 3.12 cannot be omitted.

Example 3.13. Let $A \in \mathcal{B}(\mathcal{H})$ be some (not necessarily orthogonal) projection. In that case, $A^{\oplus} = P_{\mathcal{R}(A)}$, $AA^{\oplus} = A^{\oplus} = P_{\mathcal{R}(A)}$, and $A^{\oplus}A = A$. So if $A, B \in \mathcal{B}(\mathcal{H})$ are projections such that $\mathcal{R}(A) = \mathcal{R}(B)$, we certainly have $A^{\oplus}BB^{\oplus} = A^{\oplus}AB^{\oplus}$ and $B^{\oplus}AA^{\oplus} = B^{\oplus}BA^{\oplus}$.

We can easily choose two projections A and B with the same range and such that $\mathcal{N}(A)\cap\mathcal{N}(B) = \{0\}$, as long as the dimension of $\mathcal{H}$ is greater than 1. Thus, in general, $A^{\oplus}BB^{\oplus} = A^{\oplus}AB^{\oplus}$ and $B^{\oplus}AA^{\oplus} = B^{\oplus}BA^{\oplus}$ do not imply that $\overline{\mathcal{R}(A)+\mathcal{R}(B)} + (\mathcal{N}(A)\cap\mathcal{N}(B)) = \mathcal{H}$.

We now refer to the case in which A and B are two square matrices. Of course, we interpret all matrices as linear operators on a finite-dimensional Hilbert space in the usual manner. If A and B are two square matrices of appropriate size and with index at most 1, instead of checking condition (1′) of Theorem 3.12, we readily check an equivalent condition, namely, condition (iii) of Lemma 3.7. In order to give a more computation-ready character to condition (2′), we present the following proposition. If X and Y are two square $n \times n$ matrices, then with $[X\ \ Y]$ we denote the matrix obtained by adjoining the columns of the matrix Y to the columns of the matrix X.

Proposition 3.14. *Let A and B be two complex $n\times n$ matrices such that $(\mathcal{R}(A)+\mathcal{R}(B)) \cap [\mathcal{N}(A)\cap\mathcal{N}(B)] = \{0\}$. The following statements are equivalent:*

(i) $(\mathcal{R}(A)+\mathcal{R}(B)) \oplus [\mathcal{N}(A)\cap\mathcal{N}(B)] = \mathbb{C}^n$,
(ii) $\operatorname{rank}([A\ \ B]) = \operatorname{rank}([A^*\ \ B^*])$,
(iii) $\operatorname{rank}(AA^*+BB^*) = \operatorname{rank}(A^*A+B^*B)$.

Proof. Since $(\mathcal{R}(A)+\mathcal{R}(B)) \cap [\mathcal{N}(A)\cap\mathcal{N}(B)] = \{0\}$, then $(\mathcal{R}(A)+\mathcal{R}(B)) \oplus [\mathcal{N}(A)\cap\mathcal{N}(B)] = \mathbb{C}^n$ if and only if $\dim(\mathcal{R}(A)+\mathcal{R}(B))+\dim(\mathcal{N}(A)\cap\mathcal{N}(B)) = n$. We already know that $(\mathcal{R}(A^*)+\mathcal{R}(B^*)) \overset{\perp}{\oplus} [\mathcal{N}(A)\cap\mathcal{N}(B)] = \mathbb{C}^n$; thus $\dim(\mathcal{R}(A)+\mathcal{R}(B))+\dim(\mathcal{N}(A)\cap\mathcal{N}(B)) = n$ if and only if $\dim(\mathcal{R}(A)+\mathcal{R}(B)) = \dim(\mathcal{R}(A^*)+\mathcal{R}(B^*))$. In this way, we obtain (i) $\Leftrightarrow$ (ii).

To show that (ii) $\Leftrightarrow$ (iii), recall the result of Crimmins (see [12, Theorem 2.2]): $\mathcal{R}(A)+\mathcal{R}(B) = \mathcal{R}((AA^*+BB^*)^{1/2}) = \mathcal{R}(AA^*+BB^*)$, since $\mathbb{C}^n$ is finite-dimensional, and every subspace is closed. Now (ii) $\Leftrightarrow$ (iii) is clear. □

Corollary 3.15. *If $A, B \in \mathbb{C}^{n\times n}$ are two matrices of index at most 1, then $A \vee^{\oplus} B$ exists if and only if the following conditions are satisfied:*

(1″) $A^*AB = A^*B^2$ *and* $B^*BA = B^*A^2$;
(2″) $\operatorname{rank}(AA^*+BB^*) = \operatorname{rank}(A^*A+B^*B)$.

Proof. From Lemmas 3.7 and 3.9, we have that (1″) implies that $(\mathcal{R}(A)+\mathcal{R}(B))\cap(\mathcal{N}(A)\cap\mathcal{N}(B))=\{0\}$. Thus, according to Proposition 3.14, we have that condition (2″) implies (2′). Hence if (1″) and (2″) are fulfilled, then so are (1′) and (2′), showing that $A \vee^{\text{\textcircled{\#}}} B$ exists. The other implication is clear with Theorem 3.2, Lemmas 3.6 and 3.7, and Proposition 3.14 at our disposal. □

4. Properties of the core-infimum and core-supremum

The ⃝♯-supremum can exist for some A and B while it does not exist for any of the pairs: (A^*, B^*), $(A^{\text{\textcircled{\#}}}, B^{\text{\textcircled{\#}}})$, $(A^\sharp, B^\sharp)$, $(A^{\text{\textcircled{\#}}}A, B^{\text{\textcircled{\#}}}B)$, (A^*A, B^*B), (AA^*, BB^*), $(|A|, |B|)$, where $|T|$ stands for the modulus of an operator $|T|=\sqrt{T^*T}$. It is also possible that $(A \wedge^{\text{\textcircled{\#}}} B)^\bullet$ differs from $A^\bullet \wedge^{\text{\textcircled{\#}}} B^\bullet$, where $\bullet$ can stand for $^*, ^{\text{\textcircled{\#}}}, ^\sharp$, and so on. This is due to the fact that the ⃝♯-partial order is not transferable from $A \leq^{\text{\textcircled{\#}}} B$ to $A^\bullet \leq^{\text{\textcircled{\#}}} B^\bullet$. These observations, demonstrated in the following example, are unlike the ones for the star-partial order, where we can expect this kind of duality (for corresponding statements, see [2] and [10]).

Example 4.1. Let $\mathcal{H}=\mathbb{C}^3$, and let A and B be defined as follows:

$$A=\begin{bmatrix}1&0&0\\1&2&3\\0&0&0\end{bmatrix},\qquad B=\begin{bmatrix}3/4&0&\sqrt{3}/4\\1&2&3\\\sqrt{3}/4&0&1/4\end{bmatrix}.$$

Using Corollary 3.15, we readily check that $A \vee^{\text{\textcircled{\#}}} B$ exists. On the other hand, we have

$$A^{\text{\textcircled{\#}}}=\begin{bmatrix}1&0&0\\-1/2&1/2&0\\0&0&0\end{bmatrix},\qquad B^{\text{\textcircled{\#}}}=\begin{bmatrix}3/4&0&\sqrt{3}/4\\(-3-3\sqrt{3})/8&1/2&(-3-\sqrt{3})/8\\\sqrt{3}/4&0&1/4\end{bmatrix},$$

$$A^\sharp=\begin{bmatrix}1&0&0\\-1/2&1/2&3/4\\0&0&0\end{bmatrix},\qquad B^\sharp=\begin{bmatrix}3/4&0&\sqrt{3}/4\\(-5-9\sqrt{3})/16&1/2&(3-3\sqrt{3})/16\\\sqrt{3}/4&0&1/4\end{bmatrix}.$$

So we can see that the ⃝♯-supremum does not exist for any of the abovementioned pairs. Moreover, if $D=A\wedge^{\text{\textcircled{\#}}} B$, then

$$D=\begin{bmatrix}0&0&0\\1&2&3\\0&0&0\end{bmatrix},\qquad D^{\text{\textcircled{\#}}}=\begin{bmatrix}0&0&0\\0&1/2&0\\0&0&0\end{bmatrix},\qquad D^\sharp=\begin{bmatrix}0&0&0\\1/4&1/2&3/4\\0&0&0\end{bmatrix}.$$

Then $D^\bullet \not\leq^{\text{\textcircled{\#}}} A^\bullet$, where $\bullet$ can be any of the following: $^*, ^{\text{\textcircled{\#}}}, ^\sharp$.

If A and B are orthogonal projections, Lemma 2.3 shows that the ⃝♯-supremum and ⃝♯-infimum of A and B coincide with the regular supremum and infimum of A and B in the lattice of all orthogonal projections on $\mathcal{H}$. Namely, $A\wedge^{\text{\textcircled{\#}}} B = A\wedge B = P_{\mathcal{R}(A)\cap\mathcal{R}(B)}$ and $A\vee^{\text{\textcircled{\#}}} B = A\vee B = P_{\overline{\mathcal{R}(A)+\mathcal{R}(B)}}$. However, for oblique projections the ⃝♯-supremum need not exist, which we can see from Examples 3.13 and 4.11.

Observe that from Lemma 2.2, we have the following inclusions: $\mathcal{R}(A \wedge^{\textcircled{\#}} B) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\overline{\mathcal{N}(A)+\mathcal{N}(B)} \subseteq \mathcal{N}(A \wedge^{\textcircled{\#}} B)$. Equality is obtained if, for example, A and B are orthogonal projections. In the following theorem, we describe the pairs of operators for which these inclusions become equalities. The analogous problem for the star-partial order was discussed in [10].

Theorem 4.2. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$, and let $C = A \wedge^{\textcircled{\#}} B$. Then $\mathcal{R}(C) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(C) = \overline{\mathcal{N}(A)+\mathcal{N}(B)}$ if and only if the following conditions are satisfied:*

(1) *A and B coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$,*
(2) *A^* and B^* coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$,*
(3) *$\mathcal{H} = (\mathcal{R}(A) \cap \mathcal{R}(B)) + \overline{\mathcal{N}(A)+\mathcal{N}(B)}$.*

Proof. Suppose first that $\mathcal{R}(C) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and that $\mathcal{N}(C) = \overline{\mathcal{N}(A)+\mathcal{N}(B)}$. Since $C \in \mathcal{B}^1(\mathcal{H})$, we have that condition (3) is satisfied. Conditions (1) and (2) follow from Lemma 2.1, since both A and B are $\textcircled{\#}$-larger than C.

Now suppose that conditions (1), (2), and (3) are satisfied. If $n \in \overline{\mathcal{N}(B)}$ and $y \in \mathcal{R}(A) \cap \mathcal{R}(B)$, then $\langle An, y\rangle = \langle n, A^*y\rangle = \langle n, B^*y\rangle = \langle Bn, y\rangle = 0$, where $\langle \cdot,\cdot\rangle$ stands for the inner product in $\mathcal{H}$. This shows that $A(\overline{\mathcal{N}(A)+\mathcal{N}(B)}) \subseteq (\mathcal{R}(A) \cap \mathcal{R}(B))^\perp$, while from (1) we have $A(\mathcal{R}(A) \cap \mathcal{R}(B)) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B)$. From these conclusions, we get that the sum in (3) is direct and also that $A(\mathcal{R}(A) \cap \mathcal{R}(B)) = \mathcal{R}(A) \cap \mathcal{R}(B)$. The same holds for B. Thus, we have

$$A = \begin{bmatrix} D_{11} & 0 \\ 0 & A_1 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ \overline{\mathcal{N}(A)+\mathcal{N}(B)} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ (\mathcal{R}(A) \cap \mathcal{R}(B))^\perp \end{bmatrix},$$

$$B = \begin{bmatrix} D_{11} & 0 \\ 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ \overline{\mathcal{N}(A)+\mathcal{N}(B)} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ (\mathcal{R}(A) \cap \mathcal{R}(B))^\perp \end{bmatrix},$$

where D_{11} is an isomorphism. Let us define D in the following way:

$$D = \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ \overline{\mathcal{N}(A)+\mathcal{N}(B)} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ (\mathcal{R}(A) \cap \mathcal{R}(B))^\perp \end{bmatrix},$$

in which case we have $D \in \mathcal{B}^1(\mathcal{H})$ and

$$D^{\textcircled{\#}} = \begin{bmatrix} D_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ (\mathcal{R}(A) \cap \mathcal{R}(B))^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \cap \mathcal{R}(B) \\ \overline{\mathcal{N}(A)+\mathcal{N}(B)} \end{bmatrix}.$$

A direct calculation now shows that $D \leq^{\textcircled{\#}} A$ and $D \leq^{\textcircled{\#}} B$, and so $D \leq^{\textcircled{\#}} C$. On the other hand, from Lemma 2.2, since C is $\textcircled{\#}$-smaller than A and B, we have $\mathcal{R}(C) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{R}(D) \subseteq \mathcal{R}(C)$, implying that $D = C$. ☐

For the sake of efficiency, the operators satisfying conditions of Theorem 4.2 will be called *core-parallel* (or $\textcircled{\#}$-parallel).

Example 4.3. We should note that $\mathcal{R}(A \wedge^{\textcircled{\#}} B) = \mathcal{R}(A) \cap \mathcal{R}(B)$ is not equivalent with $\mathcal{N}(A \wedge^{\textcircled{\#}} B) = \overline{\mathcal{N}(A)+\mathcal{N}(B)}$.

We can take two non-null operators $A, B \in \mathcal{B}^1(\mathcal{H})$ with $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ and $\mathcal{N}(A) = \mathcal{N}(B)$, as long as $\dim \mathcal{H} \geq 2$. Then from $\mathcal{R}(A \wedge^{\textcircled{\#}} B) \subseteq \mathcal{R}(A) \cap \mathcal{R}(B)$, we get $\mathcal{R}(A \wedge^{\textcircled{\#}} B) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{N}(A \wedge^{\textcircled{\#}} B) = \mathcal{H} \neq \overline{\mathcal{N}(A)+\mathcal{N}(B)}$.

On the other hand, we can also take two rank 1 operators $A, B \in \mathcal{B}^1(\mathcal{H})$ with $\mathcal{R}(A) = \mathcal{R}(B)$, $\mathcal{N}(A) \neq \mathcal{N}(B)$, and such that A and B do not coincide on $\mathcal{R}(A) \cap \mathcal{R}(B)$. Since condition (1) from Theorem 4.2 is not satisfied, we have $\mathcal{R}(A \wedge^{\sharp} B) \subsetneq \mathcal{R}(A) \cap \mathcal{R}(B)$, which together with $\dim \mathcal{R}(A) \cap \mathcal{R}(B) = 1$ gives $\mathcal{R}(A \wedge^{\sharp} B) = \{0\}$; that is, $A \wedge^{\sharp} B = 0$. Now we have $\mathcal{N}(A \wedge^{\sharp} B) = \overline{\mathcal{N}(A) + \mathcal{N}(B)} = \mathcal{H}$, but $\mathcal{R}(A \wedge^{\sharp} B) \neq \mathcal{R}(A) \cap \mathcal{R}(B)$.

Example 4.4. Let us demonstrate that none of the conditions (1), (2), and (3) in Theorem 4.2 can be omitted.

The pair of operators A and B described in Example 4.3 with $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ shows that conditions (1) and (2) can hold, while condition (3) does not hold.

If $C = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $D = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then the pair $(A, B) = (C, D)$ satisfies conditions (1) and (3) (in fact, the sum in (3) is direct), but it does not satisfy (2), while the pair $(A, B) = (C^*, D)$ satisfies (2) and (3) (again, the sum is direct) and does not satisfy (1).

One "computational version" of Theorem 4.2 is contained in the following proposition.

Proposition 4.5. *Let $A, B \in \mathbb{C}^{n\times n}$ be two matrices of index at most 1, and let $C = 2I - AA^\dagger - BB^\dagger$. Then A and B are $\textcircled{\sharp}$-parallel if and only if the following conditions are satisfied:*

(1′) $\operatorname{rank}([A^* - B^* \;\; C]) = \operatorname{rank}(C)$,
(2′) $\operatorname{rank}([A - B \;\; C]) = \operatorname{rank}(C)$,
(3′) $\operatorname{rank}(AA^* + BB^*) = \operatorname{rank}(A^*A + B^*B)$.

Proof. Recall that if P and Q are two orthogonal projections such that $\mathcal{R}(P+Q)$ is closed, then $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{R}(P+Q)$ (see, e.g., [1]).

Condition (1) of Theorem 4.2 is equivalent with $\mathcal{R}(A) \cap \mathcal{R}(B) \subseteq \mathcal{N}(A - B)$. Since $\mathcal{R}(A) \cap \mathcal{R}(B) = (\mathcal{R}(I - AA^\dagger) + \mathcal{R}(I - BB^\dagger))^\perp = \mathcal{R}(2I - AA^\dagger - BB^\dagger)^\perp$, we have that (1) is equivalent with $\mathcal{R}(C)^\perp \subseteq \mathcal{N}(A - B)$, that is, with $\mathcal{R}(A^* - B^*) \subseteq \mathcal{R}(C)$. This is exactly (1′).

Similarly, (2) is equivalent with (2′).

Observe that implicit in the proof of Theorem 4.2 was the fact that (1) and (2) imply $(\mathcal{R}(A) \cap \mathcal{R}(B)) \cap (\mathcal{N}(A) + \mathcal{N}(B)) = \{0\}$. Under the condition $(\mathcal{R}(A) \cap \mathcal{R}(B)) \cap (\mathcal{N}(A) + \mathcal{N}(B)) = \{0\}$, the equality $\mathcal{H} = (\mathcal{R}(A) \cap \mathcal{R}(B)) \oplus (\mathcal{N}(A) + \mathcal{N}(B))$ holds if and only if $\dim(\mathcal{R}(A) \cap \mathcal{R}(B)) = \dim(\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$. From the relation $\dim(\mathcal{R}(A) + \mathcal{R}(B)) = \dim \mathcal{R}(A) + \dim \mathcal{R}(B) - \dim(\mathcal{R}(A) \cap \mathcal{R}(B))$, and likewise for A^* and B^*, and the fact that $\dim \mathcal{R}(T) = \dim \mathcal{R}(T^*)$, we see that the equality $\dim(\mathcal{R}(A) \cap \mathcal{R}(B)) = \dim(\mathcal{R}(A^*) \cap \mathcal{R}(B^*))$ is equivalent with $\dim(\mathcal{R}(A) + \mathcal{R}(B)) = \dim(\mathcal{R}(A^*) + \mathcal{R}(B^*))$, which is, like Proposition 3.14, equivalent with (3′). Thus, under (1) and (2) of Theorem 4.2, (3) is equivalent with (3′). Since we already proved that (1) is equivalent with (1′) and (2) is equivalent with (2′), we see that conditions (1), (2), and (3) of Theorem 4.2 are simultaneously satisfied if and only if conditions (1′), (2′), and (3′) are simultaneously satisfied, which proves the assertion of the theorem. $\square$

For orthogonal projections P and Q such that $\mathcal{R}(P+Q)$ is closed, with $2P(P+Q)^\dagger Q$ we obtain an operator (in fact, the orthogonal projection) with the range

$\mathcal{R}(P) \cap \mathcal{R}(Q)$ (see [1]). Thus, condition (1′) of Proposition 4.5 can be replaced with $(A-B)AA^\dagger(AA^\dagger + BB^\dagger)^\dagger BB^\dagger = 0$, and similarly for condition (2′).

Although any two orthogonal projections are ⓣ-parallel (conditions (1) and (2) are obviously satisfied when A and B are orthogonal projections, and (3) would follow from $\mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{R}(A^*) \cap \mathcal{R}(B^*) = \overline{\mathcal{N}(A) + \mathcal{N}(B)}^{\perp}$), if one of the projections is not orthogonal, it is fairly obvious that, in general, they are not ⓣ-parallel. This can also be seen from Example 4.11 below. In the following proposition, we give another example of ⓣ-parallel operators.

Proposition 4.6. *Let $P, Q \in \mathcal{B}(\mathcal{H})$ be orthogonal projections. Then $\mathcal{R}(PQ)$ is closed if and only if $\mathcal{R}(QP)$ is closed, in which case PQ and QP are from $\mathcal{B}^1(\mathcal{H})$ and they are ⓣ-parallel.*

Proof. Since $(PQ)^* = QP$, the first statement is clear. The second statement follows from [17, Theorem 3.11]. The final statement follows from $\mathcal{R}(PQ) = \mathcal{R}(P) \cap (\mathcal{R}(Q) + \mathcal{N}(P))$ and $\mathcal{N}(PQ) = \mathcal{N}(Q) \oplus (\mathcal{R}(Q) \cap \mathcal{N}(P))$, and similarly for QP (note that $\mathcal{R}(Q)+\mathcal{N}(P)$ and $\mathcal{N}(Q)+\mathcal{R}(P)$ are closed; see [8, Theorem 22]). In fact, $\mathcal{R}(PQ) \cap \mathcal{R}(QP) = \mathcal{R}(P) \cap \mathcal{R}(Q)$ and $\overline{\mathcal{N}(PQ) + \mathcal{N}(QP)} = \overline{\mathcal{N}(P) + \mathcal{N}(Q)}$ (thus $PQ \wedge^{\textcircled{\sharp}} QP = P \wedge Q$). □

Note that from [7, Theorem 4.1], we have that if $\mathcal{R}(PQ)$ is closed, then $QP \in \mathcal{B}^1(\mathcal{H})$, so it offers an alternative way to deduce the second statement of Proposition 4.6.

There is an interesting correlation between the parallel sum of two (not necessarily positive) operators A and B and their ⓣ-infimum. Such properties for different matrix partial orders were first demonstrated by Mitra (see, e.g., [15] and [16]) and in [9] they were studied for the star-partial order. Our goal here is to prove that for the ⓣ-parallel operators, their ⓣ-infimum is exactly twice their parallel sum. Moreover, this equality characterizes the ⓣ-parallel operators.

We briefly outline the general definition of the parallel sum of operators introduced in [3]. We also state some basic facts regarding this notion. For the historical background of this notion and also for the proofs of these facts, the reader is referred to [3].

Recall the famous theorem of Douglas [11, Theorem 1], which states that a Hilbert space operator equation $B = AX$ has a (bounded) solution if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. In that case, exactly one solution X has the property $\mathcal{R}(X) \subseteq \mathcal{N}(A)^\perp$, and such a solution is called the *reduced solution*.

Operators $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are said to be *weakly parallel summable* if the following range inclusions hold:

$$\mathcal{R}(A) \subseteq \mathcal{R}\big(\sqrt{|A^* + B^*|}\big), \qquad \mathcal{R}(B) \subseteq \mathcal{R}\big(\sqrt{|A^* + B^*|}\big),$$
$$\mathcal{R}(A^*) \subseteq \mathcal{R}\big(\sqrt{|A + B|}\big), \qquad \mathcal{R}(B^*) \subseteq \mathcal{R}\big(\sqrt{|A + B|}\big).$$

If U denotes the partial isometry of the polar decomposition of $A+B$, that is, $A + B = U|A+B|$, $\mathcal{N}(U) = \mathcal{N}(A+B)$, and E_A and F_A denote the reduced solutions of the equations, respectively,

$$A = \sqrt{|A^* + B^*|}UX, \qquad A^* = \sqrt{|A + B|}X,$$

the parallel sum $A : B$ of A and B is defined as $A : B = A - F_A^* E_A$.

If we have

$$\mathcal{R}(A) \subseteq \mathcal{R}(A + B), \qquad \mathcal{R}(A^*) \subseteq \mathcal{R}(A^* + B^*),$$

then we say that operators A and B are *parallel summable*. The condition $\mathcal{R}(A) \subseteq \mathcal{R}(A + B)$ is, as easily noted, equivalent with range additivity $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A + B)$.

If the range of $A + B$ is closed, then A and B are weakly parallel summable if and only if they are parallel summable, and in that case

$$A : B = A(A + B)^\dagger B = B(A + B)^\dagger A.$$

We gather a few basic facts about the parallel sum of operators in the following proposition.

Proposition 4.7. *If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are weakly parallel summable operators, then*

(a) *$A : B = B : A$, A^* and B^* are weakly parallel summable and $(A : B)^* = A^* : B^*$;*
(b) *if $x, y \in \mathcal{H}$ are such that $Ax = By \in \mathcal{R}(A) \cap \mathcal{R}(B)$, then $Ax = By = (A : B)(x + y)$;*
(c) *$\mathcal{R}(A) \cap \mathcal{R}(B) \subseteq \mathcal{R}(A : B) \subseteq \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$.*

Proposition 4.8. *If $A, B \in \mathcal{B}^1(\mathcal{H})$ are weakly parallel summable operators such that $A : B \in \mathcal{B}^1(\mathcal{H})$, then $A \wedge^{\textcircled{\sharp}} B \leq^{\textcircled{\sharp}} 2(A : B)$.*

Proof. In order to show that $(A \wedge^{\textcircled{\sharp}} B)(A \wedge^{\textcircled{\sharp}} B)^{\textcircled{\sharp}} = 2(A : B)(A \wedge^{\textcircled{\sharp}} B)^{\textcircled{\sharp}}$, we proceed in the same way as in the proof of [9, Lemma 3.1]. Since $(A \wedge^{\textcircled{\sharp}} B)^{\textcircled{\sharp}}(A \wedge^{\textcircled{\sharp}} B) = (A \wedge^{\textcircled{\sharp}} B)^{\textcircled{\sharp}} \cdot 2(A : B)$ is equivalent with $(A \wedge^{\textcircled{\sharp}} B)^*(A \wedge^{\textcircled{\sharp}} B) = (A \wedge^{\textcircled{\sharp}} B)^* \cdot 2(A : B)$, that is, with $(A \wedge^{\textcircled{\sharp}} B)^*(A \wedge^{\textcircled{\sharp}} B) = 2(A : B)^*(A \wedge^{\textcircled{\sharp}} B)$, the proof of this equality follows in the same manner, given that A^* and B^* coincide with $(A \wedge^{\textcircled{\sharp}} B)^*$ on $\mathcal{R}(A \wedge^{\textcircled{\sharp}} B)$ (see Lemma 2.1). $\square$

Theorem 4.9. *If $A, B \in \mathcal{B}^1(\mathcal{H})$ are weakly parallel summable operators, then the following statements are equivalent:*

(i) *$A : B \in \mathcal{B}^1(\mathcal{H})$ and $2(A : B) = A \wedge^{\textcircled{\sharp}} B$,*
(ii) *A and B are $\textcircled{\sharp}$-parallel.*

Proof. (i) $\Rightarrow$ (ii) This follows from Proposition 4.7 and the fact that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed.

(ii) $\Rightarrow$ (i) From Proposition 4.7 we first note that $A : B \in \mathcal{B}^1(\mathcal{H})$. From Proposition 4.8 we have that $A \wedge^{\textcircled{\sharp}} B \leq^{\textcircled{\sharp}} 2(A : B)$, which together with $\mathcal{R}(A \wedge^{\textcircled{\sharp}} B) = \mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{R}(2(A : B))$ and Lemma 2.2 gives $A \wedge^{\textcircled{\sharp}} B = 2(A : B)$. $\square$

Lastly, we prove certain commutativity properties of the $\textcircled{\sharp}$-supremum and $\textcircled{\sharp}$-infimum. Recall that if $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ with $\mathcal{M}'$, then we denote the set $\{T \in \mathcal{B}(\mathcal{H}) : TM = MT, \text{for all } M \in \mathcal{M}\}$, called the *commutant* of $\mathcal{M}$. The *double commutant* of $\mathcal{M}$ is $(\mathcal{M}')' = \mathcal{M}''$. We prove that $A \vee^{\textcircled{\sharp}} B \in \{A, B\}''$ when this supremum exists, and that $A \wedge^{\textcircled{\#}} B \in \{A, B\}''$ when A and B are $\textcircled{\#}$-parallel. We also show that if A and B are not $\textcircled{\#}$-parallel, then in general $A \wedge^{\textcircled{\#}} B \notin \{A, B\}''$.

Theorem 4.10. *Let $A, B \in \mathcal{B}^1(\mathcal{H})$. Then the following hold.*

(a) *If $A \vee^{\textcircled{\#}} B$ exists, then $A \vee^{\textcircled{\#}} B \in \{A, B\}''$.*
(b) *If A and B are $\textcircled{\#}$-parallel, then $A \wedge^{\textcircled{\#}} B \in \{A, B\}''$.*

Proof. Let $T \in \{A, B\}'$ be arbitrary. In that case, $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$, $T(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$, and similarly for B.

(a) If $A \vee^{\textcircled{\#}} B$ exists, we easily obtain that both of the operators $(A \vee^{\textcircled{\#}} B)T$ and $T(A \vee^{\textcircled{\#}} B)$ are the null operator on $\mathcal{N}(A) \cap \mathcal{N}(B)$. If $x \in \mathcal{R}(A)$, then $Tx \in \mathcal{R}(A)$, and so $(A \vee^{\textcircled{\#}} B)Tx = ATx = TAx = T(A \vee^{\textcircled{\#}} B)x$, since A and $A \vee^{\textcircled{\#}} B$ coincide on $\mathcal{R}(A)$. Similarly, $(A \vee^{\textcircled{\#}} B)T$ and $T(A \vee^{\textcircled{\#}} B)$ coincide on $\mathcal{R}(B)$, which gives $(A \vee^{\textcircled{\#}} B)T = T(A \vee^{\textcircled{\#}} B)$ (see Corollary 3.4). Thus $A \vee^{\textcircled{\#}} B \in \{A, B\}''$.

(b) If A and B are $\textcircled{\#}$-parallel, the proof is similar, with only one difference: we prove that operators $(A \wedge^{\textcircled{\#}} B)T$ and $T(A \wedge^{\textcircled{\#}} B)$ coincide on $\mathcal{N}(A), \mathcal{N}(B)$ and $\mathcal{R}(A) \cap \mathcal{R}(B)$. Of course, we have in mind Theorem 4.2. $\square$

Example 4.11. Let $\mathcal{H} = \mathbb{C}^4$, and let A and B be defined as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In that case,

$$A \wedge^{\textcircled{\#}} B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and operators A and B are not $\textcircled{\#}$-parallel. If we take

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we can see that $T \in \{A, B\}'$, but $(A \wedge^{\textcircled{\#}} B)T \neq T(A \wedge^{\textcircled{\#}} B)$. Hence, if we just remove the condition that A and B are $\textcircled{\#}$-parallel from part (b) of Theorem 4.10, then the statement would not hold. On the other hand, we can easily find two operators A and B which are not $\textcircled{\#}$-parallel and $A \wedge^{\textcircled{\#}} B = 0$, so trivially we would have $A \wedge^{\textcircled{\#}} B \in \{A, B\}''$. Thus A and B being $\textcircled{\#}$-parallel is not a necessary condition for $A \wedge^{\textcircled{\#}} B \in \{A, B\}''$.

From Theorem 4.10, we can see that the $\textcircled{\#}$-supremum of two operators, as well as their $\textcircled{\#}$-infimum, if they are $\textcircled{\#}$-parallel, belong to the von Neumann algebra generated by these two operators.

Acknowledgments. The author is grateful to the referees for valuable comments and suggestions which significantly improved the presentation of this article. The author's work was partially supported by Ministry of Education, Science and Technological Development (Republic of Serbia) grant 174007.

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