Banach J. Math. Anal. 11 (2017), no. 2, 239–265
http://dx.doi.org/10.1215/17358787-3796878
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

ERGODIC BEHAVIORS OF THE REGULAR OPERATOR MEANS

LAURIAN SUCIU

Communicated by C. Badea

Abstract. This article deals with some ergodic properties for general sequences in the closed convex hull of the orbit of some (not necessarily power-bounded) operators in Banach spaces. A regularity condition more general than that of ergodicity is used to obtain some versions of the Esterle–Katznelson–Tzafriri theorem. Also, the ergodicity of the backward iterates of a sequence is proved under appropriate assumptions as, for example, its peripheral boundedness on the unit circle. The applications concern uniformly Kreiss-bounded operators, and other ergodic results are obtained for the binomial means and some operator means related to the Cesàro means.

1. Introduction and preliminaries

In this paper, $\mathcal{X}$ stands for a complex Banach space, and $\mathcal{B}(\mathcal{X})$ is the Banach algebra of all bounded linear operators on $\mathcal{X}$, I being the identity operator in $\mathcal{B}(\mathcal{X})$. The range, the null-subspace, and the spectrum of $T \in \mathcal{B}(\mathcal{X})$ are denoted by $\mathcal{R}(T)$, $\mathcal{N}(T)$, and $\sigma(T)$, respectively. Also, $\overline{\mathcal{M}}$ denotes the norm closure of a subspace $\mathcal{M}$ of $\mathcal{X}$.

The *ergodic projection* of $T \in \mathcal{B}(\mathcal{X})$ is defined as the idempotent $P_T \in \mathcal{B}(\mathcal{X})$ having the range $\mathcal{N}(T-I)$ and the kernel $\overline{\mathcal{R}(T-I)}$ when the following decomposition (as a direct sum) holds:

$$\mathcal{X} = \overline{\mathcal{R}(T-I)} \oplus \mathcal{N}(T-I). \tag{1.1}$$

As usual, $\mathcal{X}^*$ stands for the dual (or conjugate) space of $\mathcal{X}$, while T^* stands for the adjoint operator of $T \in \mathcal{B}(\mathcal{X})$.

Received Jan. 19, 2016; Accepted Apr. 1, 2016.
2010 *Mathematics Subject Classification.* Primary 47A35; Secondary 47A10, 47A16.
Keywords. Cesàro mean, binomial mean, ergodicity, uniform Kreiss-bounded operator.

If $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are two sequences of positive real numbers, then we use the notation $\alpha_n = O(\beta_n)$ as $n\to\infty$ and $\alpha_n = o(\beta_n)$ as $n\to\infty$ to indicate that the sequence $\{\frac{\alpha_n}{\beta_n}\}$ is bounded, respectively, so that this converges to zero. In particular, $\alpha_n = O(1)$ as $n\to\infty$ or $\alpha_n = o(1)$ as $n\to\infty$ means that $\{\alpha_n\}$ is bounded or it converges to zero, respectively.

We consider some operator means associated to an operator $T\in\mathcal{B}(\mathcal{X})$. More precisely, given a sequence $\{t_n\}_{n\in\mathbb{N}}$ of probabilities on $\mathbb{N}$, where each term t_n is a sequence $t_n = \{t_{nj}\}_{j\in\mathbb{N}}$ satisfying $\sum_{j\geq 0} t_{nj} = 1$ for any $n\in\mathbb{N}$, we consider the sequence $\{T_n\}_{n\in\mathbb{N}}$ in the norm closed convex hull of the powers of T of the form

$$T_n = \sum_{j\geq 0} t_{nj}T^j. \tag{1.2}$$

We denote by $\kappa(T)$ the set of such (possibly infinite) convex combinations of powers of T. According to the terminology of [8], we think of the sequence $\{T_n\}$ as subordinated to $\{T^n\}$ via the probabilities t_n, $n\in\mathbb{N}$.

Many sequences in $\kappa(T)$ can be inferred from matrix summability methods by applying such matrices (for example, the Zweier matrix; see [5]) to the sequence of powers of T. (We discussed such examples in [1], and other particular sequences are mentioned in this paper, especially in the last section.)

Given a sequence $\{T_n\}\subset\kappa(T)$ as in (1.2) with the conditions

$$t_{n0}\neq 1, \qquad \alpha_n := \sum_{j\geq 1} jt_{nj} < \infty \tag{1.3}$$

for all $n\in\mathbb{N}$, we define as in [1] its backward iterate $\{T_n^{(-1)}\}_{n\in\mathbb{N}}$ by

$$T_n^{(-1)} = \sum_{k\geq 0} s_{nk}T^k \quad \text{where } s_{nk} = \frac{1}{\alpha_n}\sum_{j\geq k+1} t_{nj}. \tag{1.4}$$

Clearly, $T_n^{(-1)}\in\kappa(T)$, and the following identity holds:

$$T_n^{(-1)}(T-I) = \frac{1}{\alpha_n}(T_n - I) \quad (n\in\mathbb{N}). \tag{1.5}$$

The definition of the above backward iterate of $\{T_n\}$ was inspired (see [1]) by the Cesàro means of order $p\in\mathbb{N}$ of T. These are the operators $M_n^{(p)}(T)$ given for $n\in\mathbb{N}$ by $M_0^{(p)}(T) = I$, $M_n^{(0)}(T) = T^n$, and for $n,p\geq 1$ by

$$\begin{aligned} M_n^{(p)}(T) &= \frac{p}{(n+1)\cdots(n+p)}\sum_{j=0}^{n}\frac{(j+p-1)!}{j!}M_j^{(p-1)}(T) \\ &= \frac{p}{n+p}\sum_{j=0}^{n}\prod_{k=1}^{p-1}\Big(1-\frac{j}{n+k}\Big)T^j. \end{aligned} \tag{1.6}$$

They satisfy the following recurrence relations valid for $n, p \geq 1$, namely,

$$M_n^{(p)}(T)(T-I) = \frac{p}{n+1}\big(M_{n+1}^{(p-1)}(T) - I\big), \tag{1.7}$$

$$TM_n^{(p)}(T) = \frac{n+p+1}{n+1}M_{n+1}^{(p)}(T) - \frac{p}{n+1}I, \tag{1.8}$$

and it is easy to see that, for $T_n = M_n^{(p)}(T)$, we have $T_n^{(-1)} = M_{n-1}^{(p+1)}(T)$, $n > 1$.

As usual, we write $M_n(T) := M_n^{(1)}(T)$, which is called the *Cesàro mean* of T. When $\{M_n(T)\}$ is bounded (or it converges strongly or uniformly) in $\mathcal{B}(\mathcal{X})$, we say that T is *Cesàro-bounded (Cesàro-ergodic* or *uniformly Cesàro-ergodic)*. According to [22], we say also that T is *uniformly Kreiss-bounded* if $\{M_n(T)\}$ is uniformly peripheral bounded on the unit circle $\mathbb{T}$; that is, $\sup_{\substack{n\in\mathbb{N}\\ \lambda\in\mathbb{T}}} \|M_n(\lambda T)\| < \infty$.

The higher-order Cesàro means were extensively studied in the literature and in some works such as [7], [10], [16]; more recently, [8], [13], [24], and [26] contain ergodic results concerning certain sequences in $\kappa(T)$ in the case when T is a power-bounded operator; that is, $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. We are now interested in the general context where T is not necessarily power-bounded. (We continue such a study which was begun in [1].)

In this article, we have in view two directions: the first is to obtain nonstandard versions of the well-known Esterle–Katznelson–Tzafriri theorem (see [11], [14]) for sequences in $\kappa(T)$, while the second purpose is to investigate some conditions of convergence for $\{T_n\}$ by the uniformly peripheral boundedness of the backward iterate $\{T_n^{(-1)}\}$ (similar to the uniform Kreiss boundedness). In both directions, the essential assumption is a regularity condition on $\{T_n\}$ (or its adjoint), which means the strong convergence to zero of the sequence $\{TT_n - T_{n+n_0}\}$ for some nonnegative integer n_0. The particular cases $n = 0$ and $n = 1$ are intimately related to the strong convergence of $\{T_n\}$ as was recently proved in [17].

The structure of the paper is the following. In Section 2 we describe a method which is used in the general setting of $\kappa(T)$. It is based on the intertwining of T with another operator V acting on a quotient space of $\mathcal{X}$. Under some conditions, the operator V still has good ergodic properties, which then can be transferred to the sequence $\{T_n\}$. For example, the condition of regularity of $\{T_n\}$ introduced here assures that V is an isometry and even unitary when $\sigma(T) \cap \mathbb{T} = \{1\}$. This method was applied in studies concerning the asymptotic behavior of different averages as in [2], [15], [31], [29], and it was partially described in [1]. In this section we characterize the regularity condition of the arithmetic means of a sequence $\{T_n\}$ by some convergence conditions of this sequence. Such conditions are satisfied, for example, when $T_n = M_n(T)$ and when T is a uniformly Kreiss-bounded operator on a Hilbert space as we see in Section 4.

In Section 3 we present two results containing versions of the Esterle–Katznelson–Tzafriri theorem. Both results involve the spectral condition $\sigma(T) \subset \mathbb{D} \cup \{1\}$, $\mathbb{D}$ being the open unit disc in $\mathbb{C}$. The former gives the convergence of $\{T_n(T-I)\}$ (see Theorem 3.1), and the second concerns itself with the powers T^n and the convergence of $\{T_n^*\}$ (see Theorem 3.4) under the condition of boundedness of a sequence $\{T_n\} \subset \kappa(T)$. Both results assume the regularity condition, and their applications concern the higher-order Cesàro means and the binomial means.

In Section 4 we study the ergodicity of the backward iterate $\{T_n^{(-1)}\}$. Here we assume uniform boundedness of $\{T_{n\lambda}^{(-1)}\} \subset \kappa(\lambda T)$ for λ in the unit circle $\mathbb{T}$ as well as an intertwining of T with an isometry by an operator with a closed range (see Theorem 4.1). We prove also the almost-strong convergence (in the sense defined in [17]) of some weighted Cesàro means in Hilbert spaces, assuming only their uniform boundedness on $\mathbb{T}$ (see Theorem 4.5). This last result is based on Kerchy's method for the study of almost convergence of operator sequences involving Banach limits and the almost convergence in Lorentz's sense (see [15], [21]). We apply these results to uniform Kreiss-bounded operators and also to *supercyclic* operators $T \in \mathcal{B}(\mathcal{X})$ (i.e., those for which the set $\{\lambda T^n : n \in \mathbb{N},\ \lambda \in \mathbb{C}\}$ is dense in $\mathcal{X}$; see [4]). Let us note that some ergodic properties of the supercyclic operators satisfying the regularity condition were obtained in [1]. As a remarkable fact, we obtain in this section that the arithmetic means of $\{M_n(T)\}$ converge strongly on a Hilbert space $\mathcal{X}$ when T is uniformly Kreiss-bounded on $\mathcal{X}$, which gives a partial answer to a question in [28].

In Section 5 we obtain the ergodicity of the backward iterates of the binomial means (see Theorem 5.1) and a uniform mean ergodic theorem for these means (see Theorem 5.3). We see here that some ergodic results can be transferred from Cesàro means to binomial means, but not conversely. Also, different relationships between the regularity condition and the ergodicity of some operator means related to Cesàro means are discussed. More precisely, the cases $n = 0$ and $n = 1$ of the regularity condition are considered for some operator means and their backward iterates, which derive from the Cesàro means $M_n(T)$ and $M_n^{(2)}(T)$. We establish nice connections between these conditions and the strong convergence, which lead to some ergodic results obtained by such operator means.

2. Conditions of boundedness and regularity

In the rest of the present article, we investigate the condition of regularity for sequences in $\kappa(T)$ in order to get some convergence results. We use the same technique as in [1], based on the intertwining of T with an operator on another Banach space, canonically induced by the corresponding sequence in $\kappa(T)$.

For a continuous seminorm γ on $\mathcal{X}$, we denote by $\mathcal{N}(\gamma)$ its kernel, and we let $\mathcal{X}_\gamma$ be the completion of the quotient space $\mathcal{X}/_{\mathcal{N}(\gamma)}$ with respect to the norm

$$\|\widetilde{x}\| = \gamma(x) \quad \big(\widetilde{x} = x + \mathcal{N}(\gamma), x \in \mathcal{X}\big).$$

Then the quotient map $Q_\gamma : \mathcal{X} \to \mathcal{X}_\gamma$ has the range dense in $\mathcal{X}_\gamma$, and $\mathcal{N}(Q_\gamma) = \mathcal{N}(\gamma)$.

In particular, if $\{T_n\} \subset \mathcal{B}(\mathcal{X})$ is a bounded sequence on $\mathcal{R}((T-I)^m)$ for some fixed $m \in \mathbb{N}$, then one can consider the seminorm γ_m given by

$$\gamma_m(x) = \limsup_{n\to\infty} \big\| T_n (T-I)^m x \big\| \quad (x \in \mathcal{X}). \tag{2.1}$$

In this case we write $\widetilde{\mathcal{X}}_m = \mathcal{X}_{\gamma_m}$ and $Q_m = Q_{\gamma_m}$ for short, and also we put

$$\mathcal{X}_m = \big\{x \in \mathcal{X} : \big\|T_n(T-I)^m x\big\| \to 0, n \to \infty\big\}. \tag{2.2}$$

Since γ_m is continuous on $\mathcal{X}$, in fact $\|Q_m T x\| \leq \|T\|\|Q_m x\|$ for all $x \in \mathcal{X}$, there exists a unique operator $V_m \in \mathcal{B}(\widetilde{\mathcal{X}}_m)$ satisfying the relation

$$V_m Q_m = Q_m T. \tag{2.3}$$

In addition, one has $\sigma(V_m) \subset \sigma(T)$. In order to obtain some ergodic properties of T from V_m, we are interested to see when V_m can be chosen as an isometry.

We mention some facts concerning the (null-) spaces $\mathcal{X}_0$ and $\mathcal{X}_1$ from (2.2) in addition to similar facts from [1]. We omit the proofs, which easily follow as in [1].

Proposition 2.1. *Let $T \in \mathcal{B}(\mathcal{X})$ and $\{T_n\} \subset \kappa(T)$ be a bounded sequence on $\overline{\mathcal{R}(T-I)}$. Then the subspaces $\mathcal{X}_0$ and $\mathcal{X}_1$ are closed and*

$$\mathcal{X}_0 = \overline{(T-I)^p \mathcal{X}_1} \subset \overline{\mathcal{X}_1 \cap (T-I)^p \mathcal{X}} \quad (p \geq 1). \tag{2.4}$$

We have $\mathcal{X}_0 = \overline{(T-I)^{p+1}\mathcal{X}}$ for some integer $p \geq 1$ if and only if $(V_1 - I)^p = 0$, or, equivalently, $\overline{(T-I)^p\mathcal{X}} \subset \mathcal{X}_1$. If $\{T_n\}$ is bounded on $\mathcal{X}$, then $\mathcal{X}_0 = \overline{(T-I)\mathcal{X}}$ if and only if $V_0 = I$, or, equivalently, $Q_1 = 0$.

Remark 2.2. From Proposition 2.1 we infer that, if $(T-I)^p\mathcal{X} \subset \mathcal{X}_0$ for some integer $p \geq 1$, then $\mathcal{X}_0 = \overline{(T-I)^p\mathcal{X}}$.

In general, it is possible to have $\mathcal{X}_0 = \overline{(T-I)^2\mathcal{X}} \neq \overline{(T-I)\mathcal{X}} \subset \mathcal{N}(T-I) = \mathcal{X}_1$, or $V_1 = I$ and $(V_0 - I)^2 = 0$. For example, let $T \in \mathcal{B}(\mathcal{X})$ with $(T-I)^2 = 0$, $T \neq I$, and let $T_n = M_n(T)$, $n \in \mathbb{N}$. Then $\mathcal{X}_0 = \{0\}$ by the previous remark, and $T_n(T-I) = T - I$. Therefore, $\{T_n\}$ is bounded on $\mathcal{R}(T-I)$, and we have

$$\mathcal{R}(T-I) \subset \mathcal{N}(T-I) = \mathcal{X}_1 \neq \mathcal{X}.$$

Let us note that in this case, $\{T_n\}$ is unbounded on $\mathcal{X}$ because, for $n \geq 2$, one can express T_n by $M^{(3)}_{n-2}(T)$ to conclude that $T_n = \frac{n}{2}(T-I) + I$.

This example shows that the boundedness on $\overline{\mathcal{R}(T-I)}$ alone cannot ensure valuable ergodic properties of $\{T_n\}$, such as the convergence to zero on this range or the condition $\overline{\mathcal{R}(T-I)} \cap \mathcal{N}(T-I) = \{0\}$. But the convergence on $\mathcal{R}(T-I)$ can be regarded as a special case of the following condition.

According to [1], we say that a sequence $\{T_n\} \subset \kappa(T)$ is n_0-*regular* on $\mathcal{X}$ for some $n_0 \in \mathbb{N}$ if there exists the limit

$$\lim_{n\to\infty} \big\|(TT_n - T_{n+n_0})x\big\| = 0 \quad (x \in \mathcal{X}). \tag{2.5}$$

In the case that (2.5) holds with $n_0 = 0$, we say that the sequence $\{T_n\}$ is *ergodic* (following [16]). But when $n_0 = 1$ in (2.5), we simply say that $\{T_n\}$ is *regular*.

As we have seen in [1], this property can be used to obtain some growth conditions on the powers of operator T or even concerning the convergence of $\{T_n\}$.

Notice that the condition (2.5) ensures that $\lim_{n\to\infty} T_n x \in \mathcal{N}(T-I)$ when the limit exists. In fact, if $\{T_n\}$ converges strongly on $\mathcal{X}$, then the condition (2.5) is satisfied for some (any) $n_0 \in \mathbb{N}$ and every $x \in \mathcal{X}$ if and only if the strong limit of $\{T_n\}$ is P_T (the ergodic projection of T). Since such a type of convergence is considered in the operator-ergodic context (see [16]), it is important to investigate sequences in $\kappa(T)$ satisfying this condition of regularity and, in particular, that of ergodicity.

Proposition 2.3. *Let $T \in \mathcal{B}(\mathcal{X})$ and $\{T_n\} \subset \kappa(T)$ be n_0-regular on $\mathcal{X}$ and bounded on $\mathcal{R}(T-I)$. Then the corresponding operator V_1 in (2.3) is an isometry, and*

$$\mathcal{X}_0 \oplus \mathcal{N}(T-I) = \big\{x \in \mathcal{X} : \{T_n x\} \text{ converges in } \mathcal{X}\big\}. \tag{2.6}$$

Moreover, one has $(T-I)^p\mathcal{X} \subset \mathcal{X}_0$ for some integer $p \geq 1$ if and only if $V_1 = I$ in the case $p \geq 2$, respectively, $V_1 = 0$ when $p = 1$. If $\{T_n\}$ is bounded on $\mathcal{X}$, then the operator V_0 in (2.3) is an isometry, and $V_0 = I$ if and only if $V_1 = 0$.

Remark 2.4. The extreme case $V_0 = I$ just means that $\{T_n\}$ is ergodic, and in this case it is easy to see that $\mathcal{N}(T-I) = \mathcal{N}((T-I)^2)$.

For the Cesàro means $M_n^{(p)}(T)$, the condition of regularity means $\{\frac{1}{n}M_n^{(p)}(T)\}$ converges strongly to 0 on $\mathcal{X}$, and it is equivalent to the ergodicity of $\{M_n^{(p+1)}(T)\}$ (by (1.7), (1.8)). Also, the boundedness of a Cesàro mean gives $\overline{\mathcal{R}(T-I)} = \overline{(T-I)^2\mathcal{X}}$. These facts lead to the following corollary.

Corollary 2.5. *If $T \in \mathcal{B}(\mathcal{X})$ is such that $\{\frac{1}{n^2}M_n^{(p-1)}(T)\}$ converges strongly to zero on $\mathcal{X}$ and $\{M_n^{(p+1)}(T)\}$ is bounded, then $\{M_n^{(p)}(T)\}$ is regular.*

Note that if the peripheral spectrum $\sigma(T) \cap \mathbb{T}$ of T contains at most the singleton $\{1\}$, then only the boundedness of $\{M_n^{(p+1)}(T)\}$ assures the two convergences in this corollary (see [29, Theorem 2.2] or Theorem 3.1(ii) below). The condition (2.5) can be transferred from a sequence $\widehat{T} = \{T_k\} \subset \kappa(T)$ to the arithmetic means of $\widehat{T}$ given by $M_n(\widehat{T}) = \frac{1}{n+1}\sum_{k=0}^{n} T_k, n \in \mathbb{N}$. Furthermore, the property (2.5) for $M_n(\widehat{T})$ can be described in the terms of $\widehat{T}$ as follows.

Proposition 2.6. *Let $T \in \mathcal{B}(\mathcal{X})$ and $\widehat{T} = \{T_n\} \subset \kappa(T)$ be such that $\|M_n(\widehat{T})x\| = o(n)$ as $n \to \infty$ for any $x \in \mathcal{X}$. The following statements hold:*

(i) *$\{M_n(\widehat{T})\}$ satisfies (2.5) with $n_0 \geq 1$ and $x \in \mathcal{X}$ if and only if*

$$\Big\|\sum_{k=0}^{n}(TT_k - T_{k+n_0})x\Big\| = o(n) \quad (n \to \infty),$$

(ii) *$\{M_n(\widehat{T})\}$ is ergodic if and only if for some integer $n_0 \geq 1$ and every $x \in \mathcal{X}$ we have*

$$\Big\|\sum_{k=0}^{n}(TT_k - T_{k+n_0})x + \sum_{k=1}^{n_0} T_{n+k}x\Big\| = o(n) \quad (n \to \infty),$$

(iii) $\|(M_n(\widehat{T}) - M_{n+n_0}(\widehat{T}))x\| = o(1)$ *as* $n \to \infty$ *for some* $n_0 \geq 1$ *and* $x \in \mathcal{X}$ *if and only if*

$$\Big\|\sum_{k=1}^{n_0} T_{n+k}x\Big\| = o(n) \quad (n \to \infty).$$

Proof. Let $T, \widehat{T}$ be as above with $n > n_0 \geq 1$. We have by a simple computation

$$TM_n(\widehat{T}) - M_{n+n_0}(\widehat{T}) = \frac{1}{n+1}\sum_{k=0}^{n}(TT_k - T_{k+n_0}) + \frac{n_0}{n+1}\big[M_{n+n_0}(\widehat{T}) - M_{n_0-1}(\widehat{T})\big],$$

and so the equivalence of (i) follows by using the hypothesis on $M_n(\widehat{T})$. Also, the equivalences of (ii) and (iii) are obtained from the relations

$$M_n(\widehat{T})(T-I) = \frac{1}{n+1}\Big(\sum_{k=0}^{n}(TT_k - T_{k+n_0}) + \sum_{k=1}^{n_0} T_{n+k} - n_0 M_{n_0-1}(\widehat{T})\Big)$$

and, respectively,

$$M_n(\widehat{T}) - M_{n+n_0}(\widehat{T}) = \frac{n_0}{n+1}M_{n+n_0}(\widehat{T}) - \frac{1}{n+1}\sum_{k=1}^{n_0} T_{n+k} \quad (n \geq n_0 \geq 1).$$

□

Remark 2.7. The assertion (ii) holds without the convergence stipulated in the hypothesis. In addition, $\{M_n(\widehat{T})\}$ is ergodic if $\{T_n\}$ is either ergodic or bounded (by (ii) and (iii)). When $\{T_n\}$ is n_0-regular with $n_0 \geq 1$, $\{M_n(\widehat{T})\}$ is ergodic if and only if $\{\frac{1}{n}\sum_{k=0}^{n_0} T_{n+k}\}$ converges strongly to zero on $\mathcal{X}$. In the case when $\{T_n\}$ is bounded, the sequence $\{M_n(\widehat{T})\}$ is ergodic if and only if it is n_0-regular for $n_0 \geq 1$.

The strong convergence of $\{M_n(\widehat{T})\}$ is related to that of $\widehat{T} = \{T_n\}$. More precisely, it is known from [17] that $\{T_n\}$ converges strongly to P_T if and only if it is bounded, ergodic, regular, and $\{T_n\}$ *almost converges strongly* to P_T, which means that

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\big\|M_k(\widehat{T}_n)x - P_T x\big\| = 0 \quad (x \in \mathcal{X}), \tag{2.7}$$

where $\widehat{T}_n = \{T_{n+j}\}_{j\in\mathbb{N}}$. This shows that ergodicity and regularity are necessary but not sufficient conditions for strong convergence of a sequence.

In general, the limit in (2.7) exists even if $\{T_n\}$ is neither regular nor ergodic, as we will see in the example below. This example shows also that the hypothesis on $\widehat{T}$ in the following corollary is essential.

Corollary 2.8. *Let* $T \in \mathcal{B}(\mathcal{X})$, *and let* $\widehat{T} = \{T_n\} \subset \kappa(T)$ *be a bounded and regular sequence. Then* $\{M_n(\widehat{T})\}$ *converges strongly on* $\mathcal{X}$ *if and only if it almost converges strongly on* $\mathcal{X}$.

In Section 4 we illustrate this corollary by proving that the arithmetic means of $\{M_n(T)\}$ converges strongly when T is a uniformly Kreiss-bounded operator on a Hilbert space.

Example 2.9. Consider $\mathcal{X} = L^p[0,1]$ with $1 < p < \infty$, $p \neq 2$, and let V be the Volterra operator on $\mathcal{X}$ given by

$$Vx(t) = \int_0^t x(s)\,\mathrm{d}s \quad \text{for } x \in \mathcal{X}, t \in [0,1].$$

Put $S = V - I$, $T = 2V - 3I$, and let $T_k = S^k = \frac{1}{2^k}(T+I)^k$, $k \in \mathbb{N}$. Then $\widehat{T} = \{T_k\}$ is unbounded, but $\{M_n(\widehat{T})\}$ is bounded (see [22]). By symmetry $\{M_n(-S)\}$ is bounded too, and since $\sigma(-S) = \{1\}$, one has $\frac{1}{n}\|S^n\| \to 0$ (as we remarked after Corollary 2.5). Since $1 \notin \sigma(S)$, we have $\mathcal{R}(S-I) = \mathcal{X}$, while by the uniform ergodic theorem (see [19, Main Theorem]) we have $\|M_n(\widehat{T})\| = \|M_n(S)\| \to 0$. This implies that $\{T_k\}$ almost converges strongly on $\mathcal{X}$.

On the other hand, it is easy to see that, for any integer $n_0 \geq 1$,

$$TT_k - T_{k+n_0} = \Big(T_k - \frac{1}{2}\sum_{j=0}^{n_0-1} T_{k+j}\Big)(T-I) = 2S^k\Big(I - \frac{n_0}{2}M_{n_0}(S)\Big)(S-I).$$

As $\sigma(S) = \{-1\}$ one has $1 \notin \sigma(\frac{n_0}{2}M_{n_0}(S))$; that is, $I - \frac{n_0}{2}M_{n_0}(S)$ is invertible. Since $\{S^k\}$ is unbounded, we infer that $\{TT_k - T_{k+n_0}\}$ does not converge strongly to zero. Then $\{T_k\}$ does not satisfy (2.5) and consequently is not ergodic because $T_k(T-I) = 2(TT_k - T_{k+1})$.

3. Versions of the Esterle–Katznelson–Tzafriri theorem

The spectral condition mentioned above, namely,

$$\sigma(T) \cap \mathbb{T} \subset \{1\}, \tag{3.1}$$

is frequently used in the operator-ergodic context, for instance, this appears in the Esterle–Katznelson–Tzafriri theorem (see [11], [14]) concerning power-bounded operators. But for some sequences in $\kappa(T)$, this condition can lead to the extreme cases quoted in Proposition 2.3. Thus an extension of the Esterle–Katznelson–Tzafriri theorem for some sequences in $\kappa(T)$ can be now obtained.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{X})$ be an operator satisfying the spectral condition (3.1), and let $\{T_n\} \subset \kappa(T)$ be a bounded sequence on $\overline{\mathcal{R}(T-I)^m}$ for some integer m. The following statements hold.*

(i) *If $\{T_n\}$ satisfies (2.5) on $\mathcal{R}(T-I)^m$, then either $\mathcal{X}_0 = \overline{\mathcal{R}(T-I)^m}$ or $\mathcal{X}_0 = \overline{\mathcal{R}(T-I)^{m+1}}$.*

(ii) *If $\{T_n\}$ satisfies the stronger regularity assumption*

$$\lim_{n\to\infty} \big\|(TT_n - T_{n+n_0})|\overline{\mathcal{R}(T-I)^m}\big\| = 0 \tag{3.2}$$

for some $n \in \mathbb{N}$, then $\{T_n(T-I)|_{\overline{\mathcal{R}(T-I)^m}}\}$ converges in norm to zero.

Proof. (i) Let γ_m be given by (2.1). If $\mathcal{X}_{\gamma_m} = \{0\}$, then $\mathcal{X}_0 = \overline{\mathcal{R}(T-I)^m}$. If $\mathcal{X}_{\gamma_m} \neq \{0\}$, then by using the condition (2.5) on $\mathcal{R}(T-I)^m$ one can easily see that the corresponding operator V_m in (2.3) is an isometry with $\sigma(V_m) \subset \sigma(T)$. But by (3.1), we have $\sigma(V_m) \neq \overline{\mathbb{D}}$; hence V_m is unitary. In fact, $V_m = I$ by

Gelfand's theorem [12, Satz]. This means that $T_n(T-I)^{m+1} \to 0$ strongly, and the conclusion follows by Remark 2.2.

(ii) Let $\mathcal{X}_m = \overline{\mathcal{R}(T-I)^m}$. Consider the operator $\widehat{T} : \mathcal{B}(\mathcal{X}_m) \to \mathcal{B}(\mathcal{X}_m)$ defined by

$$\widehat{T}S = T|_{\mathcal{X}_m} S \quad \big(S \in \mathcal{B}(\mathcal{X}_m)\big).$$

Clearly, $\widehat{T}$ satisfies the spectral condition (3.1), and by (3.2) the sequence $\{\widehat{T}_n\}$ corresponding to $\widehat{T}$ satisfies also the condition (2.5) with $n_0 \geq 0$ on $\mathcal{B}(\mathcal{X}_m)$. Then by (i) we have that $\{\widehat{T}_n(\widehat{T}-I)\}$ converges strongly to zero in $\mathcal{B}(\mathcal{X}_m)$. By applying this sequence to the identity operator in $\mathcal{B}(\mathcal{X}_m)$, we infer that $\{T_n(T-I)|_{\mathcal{X}_m}\}$ converges to zero in the norm of $\mathcal{B}(\mathcal{X}_m)$. □

In particular, assertion (i) shows that the spectral condition together with the boundedness and the regularity of $\{T_n\}$ on $\mathcal{X}$ ensures the ergodicity of this sequence. But assertion (ii) just gives a version of Esterle–Katznelson–Tzafriri's theorem for general sequences $\{T_n\} \subset \kappa(T)$ in the following form.

If $\{T_n\}$ is bounded and the conditions (3.1) and (3.2) with $m = 0$ and $n_0 \geq 1$ are satisfied, then $\|T_n(T-I)\| \to 0$.

Clearly, the Esterle–Katznelson–Tzafriri theorem can be obtained from this result for $T_n = T^n$, the condition (3.2) being satisfied with $m = 0$ and $n_0 = 1$. For Cesàro means, we have as a direct application of (1.7) that if $\|M_n^{(p+1)}(T)\| = O(n^\alpha)$, then $\|M_n^{(p)}(T)\| = O(n^{\alpha+1})$, $n \to \infty$, while Theorem 3.1 implies even $\|M_n^{(p)}(T)\| = o(n^{\alpha+1})$, $n \to \infty$, where α is a positive scalar.

Concerning the above results, it turns out that such a "comparative growth of means" remains true in a very general context. Using the backward iterates, as a direct application of Theorem 3.1 we obtain the following.

Corollary 3.2. *Let $T \in \mathcal{B}(\mathcal{X})$ be an operator satisfying (3.1), and let $\{T_n\}$ be a sequence in $\kappa(T)$ of the form $T_n = \sum_{j\geq 0} t_{nj}T^j$, which is bounded on $\overline{\mathcal{R}(T-I)^m}$ with $\alpha_n := \lim_{n\to\infty} \sum_{j\geq 1} jt_{nj} = \infty$, and $t_{n_0} \neq 1$ for $n \in \mathbb{N}$. The following statements hold.*

(i) *If the sequence $\{T_n^{(-1)}\}$ satisfies (2.5) on $\mathcal{R}(T-I)^m$ for some $m, n_0 \geq 0$, then*

$$\|T_n x\| = o(\alpha_n) \quad \big(x \in \overline{\mathcal{R}(T-I)^m}\big).$$

(ii) *If the sequence $\{T_n^{(-1)}\}$ satisfies (3.2) for some $m, n_0 \geq 0$, then*

$$\big\|T_n|\overline{\mathcal{R}(T-I)^m}\big\| = o(\alpha_n).$$

In the particular case of the Cesàro means, we are led to the following corollary. The proof is a straightforward application of the above ideas and will be omitted.

Corollary 3.3. *Let $T \in \mathcal{B}(\mathcal{X})$ be an operator satisfying (3.1) such that $\{M_n^{(p)}(T)|\overline{\mathcal{R}(T-I)^r}\}$ is bounded for some integers p, r with $0 \leq r < p$. The following statements hold.*

(i) *For all integers $q < p - r$ and all $x \in \mathcal{X}$, we have $\|M_n^{(q)}(T)x\| = o(n^{p-q})$, $n \to \infty$.*

(ii) *If $r > 0$, and $q \geq p - r$, then $\|M_n^{(q)}(T)\| = O(n^r)$, $n \to \infty$. Furthermore, one has $\|M_n^{(q)}(T)x\| = o(n^r)$, $n \to \infty$, for any $x \in \mathcal{X}$ if and only if $\overline{\mathcal{R}(T-I)^r} = \overline{\mathcal{R}(T-I)^{r+1}}$.*

(iii) *If $r = 0$, then $\|M_n^{(q)}(T)\| = o(n^{p-q})$ as $n \to \infty$ when $q < p$.*

Another version of Esterle–Katznelson–Tzafriri's theorem is the following.

Theorem 3.4. *Let $T \in \mathcal{B}(\mathcal{X})$ be such that $\overline{\mathcal{R}(T^* - I)}$ is weak*-closed in $\mathcal{X}^*$. Let $\{T_n\} \subset \kappa(T)$ be a bounded sequence which has backward iterates such that the sequence $\{T_n^*\}$ satisfies condition (2.5). The following statements are equivalent:*

(i) *T satisfies (3.1), and $\{T^n(T-I)\}$ is bounded;*

(ii) *$\{T^n(T-I)^2\}$ converges uniformly to zero, and $\{T^n(T-I)x\}$ converges weakly to zero for all $x \in \mathcal{X}$;*

(iii) *$\{T^n(T-I)^2\}$ is as in (ii), and $\{T^{*n}(T^* - I)\}$ converges strongly to zero on $\mathcal{X}^*$.*

Moreover, if T satisfies (3.1), then $\{T_n^*\}$ converges strongly on $\mathcal{X}^*$. In addition, $\{T_n\}$ converges strongly to P_T on $\mathcal{X}$ if and only if $P_{T^*} = S^*$ for some operator $S \in \mathcal{B}(\mathcal{X})$ and $\{T_n\}$ satisfies (2.5). In this case, the sequence $\{T^n(T-I)\}$ converges strongly to zero on $\mathcal{X}$ if (one of) the above equivalent statements hold.

Proof. Suppose first that (3.1) is satisfied (under the other assumptions from the hypothesis). Since $\{T_n^*\}$ satisfies (2.5), the corresponding isometry V_0 for $\{T_n^*\}$ given by Proposition 2.3 is either $V_0 = 0$ or $V_0 = I$. In both cases we have $T_n^*(T^* - I)f \to 0$ for $f \in \mathcal{X}^*$.

Consider the (algebraic) direct sum $\mathcal{X}^* = \overline{\mathcal{R}(T^* - I)} \oplus \mathcal{G}$ for some subspace $\mathcal{G}$ of $\mathcal{X}^*$. Then as $\{T_n^*\}$ is bounded, for $g \in \mathcal{G}$ there exists $g^* \in \mathcal{X}^*$ and a subnet $\{n_\nu\} \subset \mathbb{N}$ such that $T_{n_\nu}^* g \to g^*$ in the *weak**-topology of $\mathcal{X}^*$. But by the previous conclusion, we obtain $(T^* - I)g^* = w^* - \lim_\nu T_{n_\nu}^*(T^* - I)g = 0$; that is, $g^* \in \mathcal{N}(T^* - I)$. By hypothesis we can consider the backward iterates $T_n^{*(-1)}$ of T_n^*, and by (1.4) and the previous convergence we have

$$g^* - g = w^* - \lim_\nu (T_{n_\nu}^* - I)g = w^* - \lim_\nu \alpha_{n_\nu} T_{n_\nu}^{*(-1)}(T^* - I)g,$$

where α_n is defined by (1.3). Therefore, $g^* - g \in \overline{\mathcal{R}(T^* - I)}$ because this range is *weak**-closed. Then $g \in \overline{\mathcal{R}(T^* - I)} \oplus \mathcal{N}(T^* - I)$ for $g \in \mathcal{G}$, and with the choice of $\mathcal{G}$ we obtain

$$\mathcal{X}^* = \overline{\mathcal{R}(T^* - I)} \oplus \mathcal{N}(T^* - I).$$

Now, by this decomposition, we conclude that the sequence $\{T_n^*\}$ converges strongly on $\mathcal{X}^*$ to P_{T^*}, as we already remarked that it converges to zero on $\mathcal{R}(T^* - I)$.

Clearly, if the sequence $\{T_n\}$ also converges strongly to P_T on $\mathcal{X}$, then $P_{T^*} = P_T^*$, and $\{T_n\}$ satisfies (2.5). Conversely, let us assume that $P_{T^*} = S^*$ for some $S \in \mathcal{B}(\mathcal{X})$. Then, for every $f \in \mathcal{X}^*$ and every $x \in \mathcal{X}$, one has $(T_n^* f)x \to (S^* f)x$, which means that $T_n x \to Sx$ weakly in $\mathcal{X}$. This gives the decomposition

$$\mathcal{X} = \overline{\mathcal{R}(T - I)} \oplus \mathcal{N}(T - I).$$

If $\{T_n\}$ also satisfies (2.5), then by (3.1) and Proposition 2.3 we have $T_n(T-I)x \to 0$ for $x \in \mathcal{X}$. Hence $\{T_n\}$ converges strongly to P_T on $\mathcal{X}$ by the above decomposition of $\mathcal{X}$.

Assume now (i), that is, the spectral condition (3.1) and that $\{T^n(T-I)\}$ is bounded. Then the isometry V_1 in (2.3) corresponding to the sequence $\{T^n\}$ by Proposition 2.3 is either $V_1 = 0$ or $V_1 = I$. Hence $T^n(T-I)^2x \to 0$ for $x \in \mathcal{X}$.

Next, we consider the operator $\widetilde{T}$ on $\mathcal{B}(\mathcal{X})$ given by

$$\widetilde{T}S = TS \quad \big(S \in \mathcal{B}(\mathcal{X})\big).$$

Then $\{\widetilde{T}^n(\widetilde{T}-I)\}$ is bounded, and $\sigma(\widetilde{T}) \cap \mathbb{T} \subset \{1\}$, and so one has $\widetilde{T}^n(\widetilde{T}-I)^2S \to 0$ for $S \in \mathcal{B}(\mathcal{X})$. Hence it follows that $\|T^n(T-I)^2\| \to 0$. This also implies that $\|T^{*n}(T^*-I)f\| \to 0$ for $f \in \overline{\mathcal{R}(T^*-I)}$ because $\{T^{*n}(T^*-I)\}$ is bounded, while by the above decomposition of $\mathcal{X}^*$ we infer that this convergence holds for all $f \in \mathcal{X}^*$. Then the implication (i) $\Rightarrow$ (iii) is proved, while the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial, taking into account that the spectral condition follows from the convergence to zero of $\{T^n(T-I)^2\}$ by the spectral mapping theorem. In addition, if one of the statements (i)–(iii) holds and if $\{T_n\}$ converges strongly on $\mathcal{X}$, then (ii) gives $T^n(T-I)x \to 0$ for $x \in \overline{\mathcal{R}(T-I)}$; hence this convergence also happens for each $x \in \mathcal{X}$ by the above decomposition of $\mathcal{X}$. This ends the proof. □

Corollary 3.5. *Let $T \in \mathcal{B}(\mathcal{X})$ be an operator satisfying (3.1) such that $T-I$ has a right or a left inverse in $\mathcal{B}(\mathcal{X})$. If $\{T_n\} \subset \kappa(T)$ is as in Theorem 3.4, then $\{T^n\}$ converges uniformly to zero and $\{T_n^*\}$ converges strongly to zero on $\mathcal{X}^*$. In addition, $\{T_n\}$ converges strongly to zero on $\mathcal{X}$ if and only if it satisfies (2.5).*

Proof. Assume first that $S(T-I) = I$ for some $S \in \mathcal{B}(\mathcal{X})$. Then $\mathcal{R}(T^*-I) = \mathcal{X}^*$ and $T_n^* \to 0$ strongly on $\mathcal{X}^*$ by Theorem 3.4, $\mathcal{N}(T^*-I) = \{0\}$. Hence $T-I$ is invertible; that is, $1 \notin \sigma(T)$. By (3.1) this means that $\sigma(T) \subset \mathbb{D}$ or $\|T^n\| \to 0$.

If $(T-I)S = I$, then one has $\mathcal{R}(T-I) = \mathcal{X}$ and $\mathcal{N}(T^*-I) = \{0\}$. Since $T_n^*(T^*-I)f \to 0$ for $f \in \mathcal{X}^*$ (by proof of the theorem), we obtain $T_n^*f = S^*T_n^*(T^*-I)f \to 0$, $f \in \mathcal{X}^*$. This also implies that $T_nx \to 0$ weakly in $\mathcal{X}$ for $x \in \mathcal{X}$; hence $\mathcal{N}(T-I) = \{0\}$, and so $1 \notin \sigma(T)$.

Now, if $\{T_n\}$ satisfies (2.5), then the spectral condition and Proposition 2.3 ensure (as for T_n^*) that $T_nx \to 0$ for $x \in \mathcal{R}(T-I) = \mathcal{X}$. Clearly, the regularity condition is just necessary for this convergence. □

If in this corollary the assumption of regularity of $\{T_n^*\}$ is changed to the uniform regularity of $\{T_n\}$ in (2.5) (that is, (3.2) with $m = 0$), then one can conclude that $\|T_n\| \to 0$.

In reflexive spaces, the assumption on $\overline{\mathcal{R}(T^*-I)}$ to be *weak**-closed is superfluous. Thus, one obtains the following corollary.

Corollary 3.6. *Let $\mathcal{X}$ be a reflexive space, and let $T \in \mathcal{B}(\mathcal{X})$ be an operator satisfying the condition (3.1). If $\{T_n\} \subset \kappa(T)$ is bounded and satisfies (2.5), then $\{T_n\}$ converges strongly on $\mathcal{X}$. Moreover, the sequence $\{T^n(T-I)\}$ converges strongly to zero if and only if it is bounded.*

Remark 3.7. The above results can easily be applied to Cesàro and the *discretized Abel sums* $A_n(T)$, where

$$A_n(T) = \frac{1}{n}\sum_{j=0}^{\infty}\Big(1-\frac{1}{n}\Big)^j T^j = A_n^{(-1)}(T) \quad (n \in \mathbb{N}). \tag{3.3}$$

Notice that $\{A_n(T)\}$ is simultaneously ergodic and regular, which means that $\frac{1}{n}A_n(T) \to 0$ strongly.

Another application refers to the *binomial means* $B_n(T) \in \kappa(T)$ given by

$$B_n(T) = \frac{1}{2^n}\sum_{k=0}^{n} C_n^k T^k. \tag{3.4}$$

Proposition 3.8. *Let $T \in \mathcal{B}(\mathcal{X})$ with $\sigma(T) \subset \overline{\mathbb{D}}$ be such that the sequence $\{M_n(B)\}$ is bounded, where $B = \{B_n(T)\}$. Then $\|B_n(T)\| = o(n)$ as $n \to \infty$. In addition, if $\overline{\mathcal{R}(T^* - I)}$ is weak*-closed, then $\{M_n(B^*)\}$ converges strongly on $\mathcal{X}^*$.*

Proof. Considering the operator $S = \frac{1}{2}(T+I)$, we have that $B_n(T) = S^n$, $T_n = M_n(S)$, and $\mathcal{R}(T^*-I) = \mathcal{R}(S^*-I)$. Also, $\sigma(S)\cap\mathbb{T} \subset \{1\}$ because $\sigma(T) \subset \overline{\mathbb{D}}$. The spectral condition on S and the boundedness of $\{T_n\}$ assures by [29, Theorem 2.2] that $\frac{1}{n}\|B_n(T)\| = \frac{1}{n}\|S^n\| \to 0$, which is the first assertion. The second assertion follows from Theorem 3.4 because $\{T_n^*\}$ is bounded and hence regular. $\square$

Remark 3.9. When $\mathcal{X}$ is reflexive, the proposition gives that $\{M_n(\{B_k(T)\})\}$ converges strongly on $\mathcal{X}$ if it is bounded. In the case that $\{B_n(T)\}$ is bounded, then it follows directly from the Esterle–Katznelson–Tzafriri's theorem that $\|B_n(T)(T-I)\| \to 0$, and if $\mathcal{X}$ is reflexive, then $\{B_n(T)\}$ converges strongly on $\mathcal{X}$. A direct proof of the last convergence for T a Hilbert space contraction was given in [9].

Let us remark that the boundedness and the regularity of $\{T_n^*\}$ are essential conditions in Theorem 3.4, as we see in the following example.

Example 3.10. Let $T \in \mathcal{B}(\mathcal{X}\oplus\mathcal{X})$ be given by the matrix representation

$$T = \begin{bmatrix} I & J \\ 0 & I \end{bmatrix},$$

where I is the identity operator on $\mathcal{X}$ and where $J(0\oplus y) = y\oplus 0$, $y \in \mathcal{X}$. Clearly, $\sigma(T) = \{1\}$, and for $n \geq 1$ one has $T^n(T-I) = T-I$. Then the sequence $\{T^n(T-I)\}$ is bounded, but $\{T^n(T-I)y\}$ does not converge weakly to zero for $0 \neq y \in \{0\}\oplus\mathcal{X}$.

In addition, one has $\mathcal{R}(T-I) = \mathcal{N}(T-I) = \mathcal{X}\oplus\{0\}$. Thus, by Proposition 2.3, it follows that every sequence $\{T_n\} \subset \kappa(T)$ satisfying the condition (1.3) is either unbounded or is not regular. In particular, $T_n = M_n^{(p)}(T)$ for $p \geq 1$ is unbounded and not regular. Observe also that the assertions (ii) and (iii) are not true even if (i) holds in this case.

Notice that the weak convergence of $\{T^n(T-I)\}$ in statement (ii) of Theorem 3.4 is an optimal requirement. In reflexive spaces this convergence becomes the strong convergence by the other condition of (ii). But we cannot obtain the norm convergence to zero of this sequence in general, as we see in the following example.

Example 3.11. Consider the Volterra operator V on $L^2[0,1]$, and let $T = I - V$. Then T is power-bounded with $\sigma(T) = \{1\}$, and $M_n(T) \to 0$ strongly. Define the operator $\mathcal{T}$ on $\mathcal{L} = L^2[0,1] \oplus L^2[0,1]$ by

$$\mathcal{T} = \begin{bmatrix} T & T - I \\ 0 & T \end{bmatrix}.$$

Then $\sigma(\mathcal{T}) = \{1\}$, $\mathcal{T}$ is Cesàro-bounded, and $\frac{1}{n}\mathcal{T}^n \to 0$ strongly; hence $M_n(\mathcal{T}) \to 0$ strongly ($\mathcal{T}$ acts on a reflexive space) because $\mathcal{L} = \overline{\mathcal{R}(\mathcal{T} - I)} = \overline{\mathcal{R}(T - I)} \oplus \overline{\mathcal{R}(T - I)}$. Thus, by Theorem 3.4, the sequence $\{\mathcal{T}^n(\mathcal{T} - I)\}$ converges strongly to zero if it is bounded. But its boundedness follows easily from the matrix of $\mathcal{T}^n(\mathcal{T} - I)$ because $nT^{n-1}(T - I)^2 \to 0$ strongly, as was proved in [18]. Also, by [30, Lemma 2.1], we have $\|\mathcal{T}^n(\mathcal{T} - I)\| \nrightarrow 0$.

4. Uniform peripheral boundedness and ergodicity

For a sequence $\{T_n\} \subset \kappa(T)$ and $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$, we define $T_{n\lambda} \in \kappa(\lambda T)$ by the same convex combination of T_n; that is, if $T_n = \sum_{j\ge 0} t_{nj} T^j$ as in (1.2), then

$$T_{n\lambda} = \sum_{j\ge 0} t_{nj} \lambda^j T^j.$$

In the case where $\alpha_n := \sum_{j\ge 1} j t_{nj} < \infty$, one can also define the backward iterates $T_n^{(-1)} \in \kappa(T)$ as in (1.4). We investigate the ergodicity of $\{T_n^{(-1)}\}$ by considering the condition (2.5) and the corresponding operators Q_1 and V_1 from Proposition 2.3 for this sequence. Equivalently, by (1.5), we are interested in the strong convergence of $\{\frac{1}{\alpha_n} T_n\}$.

Theorem 4.1. *Let $T \in \mathcal{B}(\mathcal{X})$ and let $\{T_n\} \subset \kappa(T)$ be a sequence which has backward iterate $\{T_n^{(-1)}\}$ satisfying the conditions (2.5) and*

$$\sup_{\substack{n\in\mathbb{N}\\ \lambda\in\mathbb{T}}} \|T_{n\lambda}^{(-1)}\| < \infty. \tag{4.1}$$

Suppose that the operator Q_1 has closed range and that the sequence $\{\alpha_n\}$ is unbounded and satisfies the condition

$$\sup_{n,q\in\mathbb{N}} \frac{\alpha_n}{\alpha_{n+q}} < \infty. \tag{4.2}$$

Then, for any $x \in \mathcal{X}$, we have

$$\|T_n x\| = o(\alpha_n) \quad (n \to \infty). \tag{4.3}$$

Proof. The condition (4.1) implies by (1.5) that the sequence $\{\frac{1}{\alpha_n} T_n\}$ is bounded. Hence one can consider the continuous seminorm γ on $\mathcal{X}$ given for $x \in \mathcal{X}$ by

$$\gamma(x) = \limsup_{n\to\infty} \frac{1}{\alpha_n} \|T_n x\| = \limsup_{n\to\infty} \left\|T_n^{(-1)}(T - I)x\right\|.$$

We show first that the sequence $\{\gamma(\alpha_n T_{n\lambda}^{(-1)}x)\}$ is uniformly bounded for $n \in \mathbb{N}$, $\lambda \in \mathbb{T}$, and $\|x\| = 1$. Indeed, for any $q \in \mathbb{N}$, we have by direct computation

$$
\begin{aligned}
T_q(\alpha_n T_{n\lambda}^{(-1)}) &= \sum_{m\geq 0} t_{qm} \sum_{k\geq 0}\Big(\sum_{j\geq k+1} t_{nj}\Big)\lambda^k T^{k+m} \\
&= \alpha_n t_{q0} T_{n\lambda}^{(-1)} + \sum_{m\geq 1} t_{qm}\overline{\lambda}^m \sum_{i\geq m}\Big(\sum_{j=i+1-m}^{i} + \sum_{j\geq i+1}\Big) t_{nj}(\lambda T)^i \\
&= \alpha_n t_{q0} T_{n\lambda}^{(-1)} \\
&\quad + \sum_{m\geq 1} t_{qm}\overline{\lambda}^m \Big[\sum_{i\geq m}\Big(\sum_{j=i+1-m}^{i} t_{nj}\Big) + \Big(\sum_{i\geq 0} - \sum_{i=0}^{m-1}\Big)\Big(\sum_{j\geq i+1} t_{nj}\Big)\Big](\lambda T)^i \\
&= \alpha_n \sum_{m\geq 0} t_{qm}\overline{\lambda}^m T_{n\lambda}^{(-1)} \\
&\quad + \sum_{m\geq 2} t_{qm}\overline{\lambda}^m \Big[\sum_{i=1}^{m-1} i t_{ni} + m\sum_{i\geq m} t_{ni} - \sum_{i=0}^{m-1}\Big(\sum_{j\geq i+1} t_{nj}\Big)\Big](\lambda T)^i \\
&= \sum_{m\geq 0} \alpha_n t_{qm}\overline{\lambda}^m T_{n\lambda}^{(-1)} \\
&\quad + \sum_{m\geq 2} t_{qm}\overline{\lambda}^m \Big[\sum_{i=1}^{m-1}\Big(i t_{ni} - \sum_{j\geq i+1} t_{nj}\Big) + \sum_{i\geq m} m t_{ni} - \sum_{j\geq 1} t_{nj}\Big](\lambda T)^i \\
&= \alpha_n \sum_{m\geq 0} t_{qm}\overline{\lambda}^m T_{n\lambda}^{(-1)}.
\end{aligned}
$$

For the last equality we see that the expression in brackets is zero (which is easy to verify). We infer that

$$
\begin{aligned}
\gamma(\alpha_n T_{n\lambda}^{(-1)}x) &= \limsup_{q\to\infty} \frac{\alpha_n}{\alpha_{n+q}} \|T_{n+q} T_{n\lambda}^{(-1)} x\| \\
&= \limsup_{q\to\infty} \frac{\alpha_n}{\alpha_{n+q}} \Big\|\sum_{m\geq 0} t_{n+q,m}\overline{\lambda}^m T_{n\lambda}^{(-1)} x\Big\| \\
&\leq \sup_{\substack{n,q\in\mathbb{N}\\ \lambda\in\mathbb{T}}} \frac{\alpha_n}{\alpha_{n+q}} \Big(\sum_{m\geq 0} t_{n+q,m}\Big)\|T_{n\lambda}^{(-1)}x\| < \infty,
\end{aligned}
$$

whence the claim follows by (4.1), (4.2), and using the fact that $T_{n+q} \in \kappa(T)$.

Now let Q_1 and V_1 be the operators from (2.3) corresponding to the bounded sequence $\{T_n^{(-1)}\}$. In fact, as this sequence satisfies (2.5), V_1 will be an isometry (by Proposition 2.3). Then, by the previous conclusion and using the equality

$$\gamma(\alpha_n T_{n\lambda}^{(-1)}x) = \big\|Q_1(\alpha_n T_{n\lambda}^{(-1)}x)\big\|,$$

one obtains the condition

$$\beta := \sup_{\substack{n\in\mathbb{N}\\ \lambda\in\mathbb{T}}} \Big\|\sum_{k\geq 0}\Big(\sum_{j\geq k+1} t_{nj}\Big)\lambda^k V_1^k Q_1\Big\| < \infty. \tag{4.4}$$

We show that this implies even $Q_1 = 0$ under the assumption that $\mathcal{R}(Q_1)$ is closed in the corresponding Banach space $\mathcal{X}_\gamma$; in fact, $\mathcal{R}(Q_1) = \mathcal{X}_\gamma$ by the density of this range.

Assume to the contrary that $Q_1 \neq 0$. Let $\mu \in \mathbb{T}$ be an approximate eigenvalue for V_1. Then there exists a sequence $\{\widetilde{x}_q\} \subset \mathcal{X}_\gamma$ with $\|\widetilde{x}_q\| = 1$ for any q, and $(V_1 - \mu I)\widetilde{x}_q \to 0$. Since Q_1 maps $\mathcal{X}$ onto $\mathcal{X}_\gamma$, there exists $r > 0$ such that $\widetilde{x}_q = Q_1 x_q$ for some $x_q \in \mathcal{X}$ with $\|x_q\| \leq r$.

Now let $f_q \in \mathcal{X}_\gamma^*$ with $\|f_q\| = 1$ and $|f_q(\widetilde{x}_q)| > \frac{1}{2}$ for any q. Then from (4.4) we have, for $n, q \geq 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{k \geq 0} \Big(\sum_{j \geq k+1} t_{nj} \Big) f_q(V_1^k \widetilde{x}_q) e^{ikt} \Big|^2 \, \mathrm{d}t \leq \beta^2 r^2,$$

while by Parseval's formula we obtain

$$\sum_{k \geq 0} \Big| \Big(\sum_{j \geq k+1} t_{nj} \Big) f_q(V_1^k \widetilde{x}_q) \Big|^2 \leq \beta^2 r^2.$$

But with the choice of μ we have, for every $k \in \mathbb{N}$,

$$\lim_{q \to \infty} f_q\big((V_1^k - \mu^k)\widetilde{x}_q \big) = 0;$$

hence from the preceding inequality, we infer for each integer $N \geq 1$ that

$$\limsup_{q \to \infty} \sum_{k=0}^{N} \Big(\sum_{j \geq k+1} t_{nj} \Big) \big| f_q(\widetilde{x}_q) \big|^2 \leq \beta^2 r^2.$$

Since $|f_q(\widetilde{x}_q)| > \frac{1}{2}$ for any q, and letting $N \to \infty$, we get finally $\alpha_n \leq (2\beta r)^2$, a contradiction because $\{\alpha_n\}$ is assumed unbounded by the hypothesis. Hence $Q_1 = 0$, which even means that the sequence $\{\frac{1}{\alpha_n} T_n\}$ converges strongly to zero on $\mathcal{X}$. $\square$

Corollary 4.2. *Let $T \in \mathcal{B}(\mathcal{X})$ with $\sigma(T) \subset \overline{\mathbb{D}}$, and let $\{T_n\} \subset \kappa(T)$ be such that condition (4.1) is satisfied. If the operator Q_1 has the closed range while $\{\alpha_n\}$ is increasing to infinity and $\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$, then the convergence (4.3) holds true.*

Proof. Clearly, the conditions (4.2) and consequently (4.4) are satisfied, and putting $V = V_1$ and $\widetilde{V}_n$ in the corresponding sequence in $\kappa(V_1)$ obtained by the relation $Q_1 T_n = \widetilde{V}_n Q_1$, by (4.4) we then have

$$\widetilde{V}_{n\lambda}^{(-1)} \widetilde{x} = \frac{1}{\alpha_n} \sum_{k \geq 0} \Big(\sum_{j \geq k+1} t_{nj} \Big) \lambda^k V^k \widetilde{x} \to 0$$

for all $\widetilde{x} \in \mathcal{X}_\gamma = \mathcal{R}(Q_1)$ and every $\lambda \in \mathbb{T}$. Hence $\mathcal{X}_\gamma = \overline{\mathcal{R}(V - \lambda I)}$ for $\lambda \in \mathbb{T}$ (by the previous convergence), and as $\sigma(V) \subset \sigma(T) \subset \overline{\mathbb{D}}$, two cases arise: either $\sigma(V) \cap \mathbb{T}$ is nonempty or $\sigma(V) \subset \mathbb{D}$. In the former case $V \neq 0$ has an approximate eigenvalue on $\mathbb{T}$, while this fact together with the assumption that $\mathcal{R}(Q_1) = \mathcal{X}_\gamma$ leads (as in the previous proof) to the conclusion that $V = 0$, a contradiction.

Thus the second case occurs; that is, $\|V^n\| \to 0$. In this last case, for $0 < \varepsilon < 1$, there exists $n_0 \in \mathbb{N}$ with $\|V^{n_0}\| < \varepsilon$ so that, for $x \in \mathcal{X}$, we have

$$\begin{aligned}\|Q_1 x\| &= \limsup_{n\to\infty} \frac{1}{\alpha_{n+n_0}} \|T^{n+n_0} x\| = \lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+n_0}} \|Q_1 T^{n_0} x\|, \\ &= \|V^{n_0} Q_1 x\| \le \varepsilon \|Q_1 x\|,\end{aligned}$$

which yields $Q_1 = 0$. Therefore, $\frac{1}{\alpha_n} T_n \to 0$ strongly on $\mathcal{X}$, which finishes the proof. □

The assumption on $\mathcal{R}(Q_1)$ or the regularity assumption can also be suppressed in Theorem 4.1 in some cases. For instance, if the spectral condition (3.1) is satisfied for $T \in \mathcal{B}(\mathcal{X})$ while $\{T_n\} \subset \kappa(T)$ has a bounded and n_0-regular backward iterate $\{T_n^{(-1)}\}$ for an integer $n_0 \ge 1$, then $\{T_n^{(-1)}\}$ is ergodic (by Theorem 3.1(i)); that is, the convergence (4.3) holds. In this case the conclusion of Theorem 4.1 occurs under a weaker assumption than the uniform boundedness (4.1).

Let us comment on a special case where the condition on $\mathcal{R}(Q_1)$ in Theorem 4.1 is satisfied. Namely, if T is a supercyclic operator, then under the conditions (4.1) and (2.3) for $\{T_n\}$, one has $Q_1 T = V_1 Q_1$, and it follows that V_1 is also supercyclic on $\mathcal{X}_\gamma$. Thus, if V_1 is an isometry (for example, when $\{T_n^{(-1)}\}$ satisfies (2.5)), then we have $\dim \widetilde{\mathcal{X}}_0 \le 1$ (as it is known from [3]). Therefore, $\mathcal{R}(Q_1)$ is closed, and in fact $\mathcal{R}(Q_1) = \mathcal{X}_\gamma = \{0\}$. Thus, from Theorem 4.1 and [1, Corollary 3.7], we infer the following result which extends in the general context of the sequences in $\kappa(T)$ the well-known result of Ansari–Bourdon [3], which says that $\{T^n\}$ converges strongly to zero if T is supercyclic and power-bounded. In particular, our results apply to uniformly Kreiss-bounded operators.

Corollary 4.3. *Let $T \in \mathcal{B}(\mathcal{X})$ be supercyclic, and let $\{T_n\} \subset \kappa(T)$ be a sequence which has backward iterates satisfying conditions (4.1), (4.2), and (2.5). Then $\{T_n^{(-1)}\}$ is ergodic. Moreover, it converges strongly on $\mathcal{X}$ if and only if $1 \notin \sigma_p(T^*) \setminus \sigma_p(T)$. In this case, the limit is nonzero if and only if $1 \in \sigma_p(T)$.*

In particular, if T is supercyclic and uniformly Kreiss-bounded, then $\{M_n(T)\}$ is ergodic in reflexive spaces by the mean ergodic theorem (see [16]) and this corollary, while the means $M_n(T)$ always converge strongly, and the limit is zero if and only if $1 \notin \sigma_p(T)$. Such an operator was recently constructed in [1].

The above results can be applied to many useful operator means such as for Cesàro means or binomial means. For the Abel mean $A_n(T)$, the boundedness condition (4.1) is equivalent to a similar condition for $M_n^{(p)}(T)$, which simultaneously holds for all $p \ge 2$ (see [1], [27], [28]). This is the *Kreiss boundedness condition* on the resolvent function of T, namely,

$$\left\|(T - \lambda I)^{-1}\right\| \le \frac{C}{|\lambda| - 1} \quad \left(|\lambda| > 1\right)$$

for some constant $C > 0$. Moreover, it is known from [27] that the above Kreiss condition implies $\|M_n(T)\| = O(\log n)$, $n \to \infty$. Hence, if the condition (4.1) is satisfied for $T_n = M_n^{(p)}(T)$ with some (any) integer $p \ge 2$, then $\|M_n(T)\| = o(n)$ as $n \to \infty$. Hence the conclusion of Theorem 4.1 is interesting for Cesàro means of order 1 and for other sequences in $\kappa(T)$. When T is power-bounded, each sequence $\{T_n\} \subset \kappa(T)$ is bounded, and the convergence (4.3) always occurs. But Theorem 4.1 gives the same conclusion under other assumptions concerning the backward iterates of $\{T^n\}$.

Corollary 4.4. *Let $T \in \mathcal{B}(\mathcal{X})$ be uniformly Kreiss-bounded. If $\mathcal{R}(Q_1)$ is closed, where the operator Q_1 corresponds to sequence $\{M_n(T)\}$ by Proposition 2.3, then $\|T^n x\| = o(n)$ as $n \to \infty$ for each $x \in \mathcal{X}$.*

Turning to the general context of the sequences in $\kappa(T)$, let us note that the method based on the seminorm from (2.1) does not give more precise information in the case of Hilbert spaces because the corresponding complete quotient space $\widetilde{\mathcal{X}}$ is not always a Hilbert space. But if the seminorms induced by Banach limits are used, then it is possible to make $\widetilde{\mathcal{X}}$ a Hilbert space when $\mathcal{X}$ is such (see [15]). In this case, the method is related only with the almost convergence of bounded sequences of scalars. By using this general argument, we give now a Hilbertian version of Theorem 4.1, where almost-strong convergence (in the sense of (2.7)) of some weighted Cesàro means is obtained.

Recall (see [15]) that a *Banach limit* is a positive linear functional L on $l^\infty = l^\infty(\mathbb{N}, \mathbb{C})$ with $\|L\| = L(\{1\})$ and $L(u\eta) = L(\eta)$ for every $\eta \in l^\infty$, where $\{1\} = (1, 1, \ldots)$ and u stands for the truncated backward shift on l^∞.

Theorem 4.5. *Suppose $\mathcal{X}$ to be a Hilbert space, and let $T \in \mathcal{B}(\mathcal{X})$ satisfying the condition*

$$\sup_{\substack{n \geq 1 \\ \lambda \in \mathbb{T}}} \frac{1}{n^r} \big\| M_n(\lambda T) \big\| < \infty \tag{4.5}$$

for some integer $r \geq 0$. Then the sequence $\{\frac{1}{n^{r+1}} \|T^n x\|\}$ almost converges to zero for every $x \in \mathcal{X}$. Moreover, for any $\lambda \in \mathbb{T}$, the sequence $\{\frac{1}{n^r} M_n(\lambda T) x\}$ almost converges to zero for every $x \in \overline{\mathcal{R}(\lambda T - I)} + \mathcal{N}(\lambda T - I)$ in the case $r > 0$, respectively, to $P_{\lambda T} x$ for every $x \in \mathcal{X}$ when $r = 0$.

Proof. Consider first the bounded sequences $\xi_x = \{\frac{1}{n^{r+1}} \|T^n x\|\}$ for $x \in \mathcal{X}$. By a well-known result of Lorentz [21], the almost convergence to zero of ξ_x is equivalent with the fact that $L(\xi_x) = 0$ for each Banach limit L. The nontrivial case is when $\{\frac{\|T^n\|}{n^{r+1}}\}$ does not almost converge to zero. In this case, by [15, Theorem 2] there exists an isometry V acting on a Hilbert space $\mathcal{H}$ and an operator $Q \in \mathcal{B}(\mathcal{X}, \mathcal{H})$ such that $QT = VQ$ and

$$\mathcal{N}(Q) = \bigcap_L \big\{ x \in \mathcal{X} : L(\xi_x) = 0 \big\}.$$

We wish to prove that $\mathcal{N}(Q) = \mathcal{X}$. Assuming first that $\mathcal{X}$ is separable, we need to prove that

$$\|Qx\| = L\Big(\Big\{ \frac{1}{n^{r+1}} \|T^n x\| \Big\} \Big) = 0$$

for some Banach limit L (see the proof of [15, Theorem 2 and Lemma 3]) and any $x \in \mathcal{X}$.

Assume that $Qx_0 \neq 0$ for some $x_0 \in \mathcal{X}$ and L, as above. It is known from [21] that

$$\|Qx_0\| = L(\xi_{x_0}) \leq \limsup_{n\to\infty} \frac{1}{n^{r+1}} \|T^n x_0\|.$$

By a standard argument we can obtain from (4.5) (as in the proof of [1, Theorem 4.8]) the estimate

$$\beta := \sup_{\substack{n\geq 1\\ |\lambda|=1}} \Big\| \sum_{k=0}^{n} \lambda^k V^k Q x_0 \Big\| < \infty.$$

From this we infer that, for any $n \geq 1$,

$$\int_0^{2\pi} \Big\| \sum_{k=0}^{n} e^{ikt} V^k Q x_0 \Big\|^2 \frac{\mathrm{d}t}{2\pi} \leq \beta^2,$$

and taking into account that V is an isometry, this leads to contradiction $(n+1)\|Qx_0\|^2 \leq \beta^2$. Hence $Q = 0$; that is,

$$\lim_{k\to\infty} \sup_{n\geq 1} \frac{1}{k+1} \sum_{j=0}^{k} \frac{\|T^{n+j}x\|}{(n+j)^{r+1}} = 0,$$

which means that the sequences $\{\frac{\|T^n x\|}{n^{r+1}}\}$ almost converge to zero for any $x \in \mathcal{X}$.

In the case that $\mathcal{X}$ is nonseparable, by the construction from the proof of [15, Theorem 2] one can find an orthogonal decomposition $\mathcal{X} = \bigoplus_{\alpha\in\mathcal{A}} \mathcal{X}_\alpha$ such that $\mathcal{X}_\alpha$ is separable and invariant for T. Then, for each $\alpha \in \mathcal{A}$, there exist as above a Hilbert space $\mathcal{H}_\alpha$, V_α an isometry on $\mathcal{H}_\alpha$, and $Q_\alpha \in \mathcal{B}(\mathcal{X}_\alpha, \mathcal{H}_\alpha)$ such that $Q_\alpha T_\alpha = V_\alpha Q_\alpha$ where $T_\alpha = T|_{\mathcal{X}_\alpha}$. By the conclusion from the separable case we have $Q_\alpha = 0$; that is, $\mathcal{H}_\alpha = \overline{\mathcal{R}(Q_\alpha)} = \{0\}$ for each $\alpha \in \mathcal{A}$, which gives that the sequence ξ_x almost converges for any $x \in \mathcal{X}$. This proves the first statement.

Now, if we denote the sequence from the previous limit by $\{\beta_k\}$, then we have by (1.7) for $n, k \geq 1$, $\lambda \in \mathbb{T}$ and $x \in \mathcal{X}$, the relations

$$\begin{aligned}
\beta_k &\geq \sup_{n\geq 1}\Big(\frac{1}{k+1} \sum_{j=1}^{k} \frac{1}{(n+j)^r} \Big\| M_{n+j-1}(\lambda T)(\lambda T - I)x + \frac{x}{n+j} \Big\| \Big) \\
&= \sup_{n\geq 1} \frac{1}{k+1} \sum_{j=1}^{k} \Big(\frac{n+j-1}{n+j} \Big)^r \frac{1}{(n+j-1)^r} \Big\| M_{n+j-1}(\lambda T)(\lambda T - I)x + \frac{x}{n+j} \Big\| \\
&\geq \sup_{n\geq 1} \Big(\frac{n}{n+1} \Big)^r \frac{1}{k+1} \sum_{j=0}^{k-1} \frac{1}{(n+j)^r} \Big\| M_{n+j}(\lambda T)(\lambda T - I)x + \frac{x}{n+j+1} \Big\| \\
&\geq \frac{1}{2^r} \Big(\frac{1}{k+1} \Big\| \sum_{j=0}^{k-1} \frac{1}{(n+j)^r} M_{n+j}(\lambda T)(\lambda T - I)x \Big\| - \frac{1}{k+1} \sum_{j=0}^{k-1} \frac{\|x\|}{(n+j)^{r+1}} \Big).
\end{aligned}$$

From this we infer that

$$\sup_{n\geq 1}\frac{1}{k}\Big\|\sum_{j=0}^{k-1}\frac{1}{(n+j)^r}M_{n+j}(\lambda T)(\lambda T-I)x\Big\|\leq \frac{k+1}{2^{-r}k}\Big(\beta_k+\sup_{n\geq 1}\frac{1}{k}\sum_{j=0}^{k-1}\frac{\|x\|}{(n+j)^{r+1}}\Big).$$

Since the sequence from the right-hand side converges to 0, we have

$$\lim_{k\to\infty}\sup_{n\geq 1}\frac{1}{k}\Big\|\sum_{j=0}^{k-1}\frac{1}{(n+j)^r}M_{n+j}(\lambda T)x\Big\|=0$$

for every $\lambda\in\mathbb{T}$ and $x\in\mathcal{R}(\lambda T-I)$. But by (4.5) this statement holds also for any $x\in\overline{\mathcal{R}(\lambda T-I)}$, while for $x\in\mathcal{N}(\lambda T-I)$ the limit exists trivially when $r>0$. Therefore, $\{\frac{1}{n^r}M_n(\lambda T)\}$ almost converges strongly to zero on $\overline{\mathcal{R}(\lambda T-I)}+\mathcal{N}(\lambda T-I)$.

In the case $r=0$, we have by (4.5) that $\{M_n^{(2)}(\lambda T)\}$ strongly converges on $\mathcal{X}$, which yields the decomposition $\mathcal{X}=\overline{\mathcal{R}(\lambda T-I)}\oplus\mathcal{N}(\lambda T-I)$. Then when $r=0$, we get

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\Big\|\frac{1}{k}\sum_{j=0}^{k-1}M_{n+j}(\lambda T)x-P_{\lambda T}x\Big\|=0$$

for $x\in\mathcal{X}$, which means by (2.7) that the sequence $\{M_n(\lambda T)\}$ almost converges strongly on $\mathcal{X}$ for every $\lambda\in\mathbb{T}$. This ends the proof. $\square$

The interesting fact reflected by Theorem 4.5 is that, for a uniformly Kreiss-bounded operator T on a Hilbert space, the Cesàro mean $\{M_n(\lambda T)\}$ almost converges strongly on $\mathcal{X}$ for every $\lambda\in\mathbb{T}$. But for a power-bounded operator T its powers T^n do not almost converge strongly in general, for example, when T is the unilateral forward shift on $l^2(\mathbb{N},\mathbb{C})$.

As a consequence of Theorem 4.5 and Corollary 2.8, we obtain the strong convergence of the arithmetic means from Proposition 2.6 when $T_n=M_n(T)$.

Corollary 4.6. *If $T\in\mathcal{B}(\mathcal{X})$ is uniformly Kreiss-bounded on a Hilbert space, then the sequence of arithmetic means of the Cesàro mean of λT strongly converges to $P_{\lambda T}$ for every $\lambda\in\mathbb{T}$.*

5. Applications to operator means derived from Cesàro means

The first application refers to the binomial means $B_n(T)$ and their backward iterates.

Theorem 5.1. *For any operator $T\in\mathcal{B}(\mathcal{X})$, the backward iterate of $B_n(T)$ is given by the relations*

$$B_n^{(-1)}(T)=\frac{1}{n2^{n-1}}\sum_{k=1}^{n}kC_n^kM_{k-1}(T)=M_{n-1}\Big(\frac{I+T}{2}\Big) \tag{5.1}$$

for every integer $n\geq 1$. In addition, the following statements hold.

(i) *If* $\|B_n^{(-1)}(\lambda T)\| = O(1)$ *as* $n \to \infty$ *and* $\lambda \in \mathbb{T}$*, and the corresponding operator* Q_1 *for* $\{B_n^{(-1)}(T)\}$ *satisfying (2.3) has closed range, then* $\|B_n(T)x\| = o(n)$ *as* $n \to \infty$ *for every* $x \in \mathcal{X}$*.*

(ii) *If* $\sigma(T) \subset \overline{\mathbb{D}}$ *and* $\|B_n^{(-1)}(T)\| = O(1)$ *as* $n \to \infty$*, then* $\|B_n(T)\| = o(n)$ *as* $n \to \infty$*.*

(iii) *If the Lebesgue measure of* $\sigma(T) \cap \mathbb{T}$ *is zero and* $\|B_n(\lambda T)\| = O(1)$ *as* $n \to \infty$ *and* $\lambda \in \mathbb{T}$*, then* $\|T^n\| = o(n2^n)$ *as* $n \to \infty$*.*

Proof. To see the first relation in (5.1), we note that $\alpha_n = \frac{n}{2}$ for the sequence $\{B_n(T)\} \subset \kappa(T)$, and that the scalar coefficients of the backward iterates $B_n^{(-1)}(T) := B_n(T)^{(-1)}$ are given by

$$s_{nk} = \frac{2}{n} - \frac{1}{n2^{n-1}} \sum_{j=0}^{k} C_n^j \quad (0 \le k \le n-1), \qquad s_{nk} = t_{n,k+1} = 0 \quad (k \ge n).$$

Then for $n \ge 1$ we have

$$\begin{aligned} B_n^{(-1)}(T) &= \sum_{k=0}^{n-1} \Big(\frac{2}{n} - \frac{1}{n2^{n-1}} \sum_{j=0}^{k} C_n^j \Big) T^k \\ &= 2M_{n-1}(T) - \frac{1}{n2^{n-1}} \Big(n(2^n - 1)M_{n-1}(T) - \sum_{k=1}^{n-1} kC_n^k M_{k-1}(T) \Big) \\ &= \frac{1}{n2^{n-1}} \sum_{k=1}^{n} kC_n^k M_{k-1}(T). \end{aligned}$$

This is the first equality in (5.1), while the second can be easily obtained.

Now the assertion (i) follows directly from Theorem 4.1 because $\{B_n^{(-1)}(T)\}$ is regular by (5.1) and the hypothesis from (i), and $\alpha_n \to \infty$ while $\{\alpha_n\}$ satisfies (4.2) in this case.

Next, the assertion (ii) is a consequence of Corollary 3.3(i) applied to the sequence $T_n = B_n^{(-1)}(T) = M_n(S)$, where $S = \frac{I+T}{2}$. Here $\{T_n\}$ is bounded by the hypothesis of (ii) and $\sigma(S) \cap \mathbb{T} \subset \{1\}$ because $\sigma(T) \subset \overline{\mathbb{D}}$. Hence $\|S^n\| = o(n)$ as $n \to \infty$ (by the quoted corollary), which is just the conclusion in (ii).

To prove (iii), we first remark that the condition $\|B_n(\lambda T)\| \le c$ for some constant $c > 0$ means that $\frac{1}{2^n}(I + \lambda T)^n \le c$, which by (5.1) implies $\|B_n^{(-1)}(\lambda T)\| \le c$ for every $\lambda \in \mathbb{T}$, $n \in \mathbb{N}$. Then the conclusion of (iii) follows from [1, Theorem 4.1] applied to the sequence $\{B_n(T)\}$. For this we only mention that, since the coefficients of $B_n(T)$ are $t_{nj} = \frac{C_n^j}{2^n}$ for $0 \le j \le n$ and $t_{nj} = 0$ for $j > n$, we have $\sum_{j \ge n} t_{nj} = t_{nn} = \frac{1}{2^n}$ and $\alpha_n = \sum_{j=1}^{n} j t_{nj} = \frac{n}{2}$. Then by the above-quoted theorem (taking into account the spectral condition in (iii)), we have

$$\frac{t_{nn}}{\alpha_{n-1}} \|T^{n-1}\| = o(1) \quad \text{as } n \to \infty;$$

that is, $\|T^n\| = o(n2^n)$, $n \to \infty$. This ends the proof. $\square$

Note that if T is uniformly Kreiss-bounded, then $\|B_n^{(-1)}(\lambda T)\| = O(1)$ as $n \to \infty$ and $\lambda \in \mathbb{T}$, but this does not imply that S has the same property as T. On the other hand, if T is Cesàro-bounded, then S has the same property (by (5.1)), and in this case $\sigma(T) \subset \overline{\mathbb{D}}$, and so we infer the following corollary.

Corollary 5.2. *For every Cesàro-bounded operator $T \in \mathcal{B}(\mathcal{X})$, we have*

$$\big\|B_n(T)\big\| = o(n) \quad (n \to \infty).$$

In addition, if $\mathcal{X} = \overline{\mathcal{R}(T-I)} \oplus \mathcal{N}(T-I)$, then $\{B_n^{(-1)}(T)\}$ converges strongly to the ergodic projection P_T.

Proof. The first assertion follows from Theorem 5.1(ii). Now assume that T is Cesàro-bounded and that $\mathcal{X} = \overline{\mathcal{R}(T-I)} \oplus \mathcal{N}(T-I)$. If $S = \frac{1}{2}(I+T)$, then $\mathcal{X} = \overline{\mathcal{R}(S-I)} \oplus \mathcal{N}(S-I)$, and $\frac{1}{n}\|S^n\| \to 0$ by the previous conclusion. But these imply that S is Cesàro-ergodic, which by (5.1) means that $\{B_n^{(-1)}(T)\}$ converges strongly on $\mathcal{X}$. $\square$

The second assertion of the corollary implies also that if T is Cesàro ergodic, then S has the same property. Note that the statement (iii) in Theorem 5.1 can be considered a version for binomial means of O. Nevanlinna's result [23] for the Cesàro means. In reflexive spaces Theorem 5.1 implies under the hypotheses of (i) and (ii) that the sequence $\{B_n^{(-1)}(T)\}$ converges strongly on $\mathcal{X}$. In particular, this happens when T is Cesàro-bounded (by Corollary 5.2).

We mention now another result which holds for binomial means, in fact an improvement of the corresponding result for Cesàro means. More precisely, since $\mathcal{R}(T-I) = \mathcal{R}(S-I)$ and $\sigma(S) \cap \mathbb{T} \subset \{1\}$ if $\sigma(T) \subset \overline{\mathbb{D}}$, we have the following version for binomial means of the uniform mean ergodic theorem of Lin [19, Main Theorem].

Theorem 5.3. *For $T \in \mathcal{B}(\mathcal{X})$ with $\sigma(T) \subset \overline{\mathbb{D}}$, the following statements are equivalent:*

(i) *$\{B_n(T)\}$ converges uniformly to the ergodic projection P_T of T;*
(ii) *$\{B_n^{(-1)}(T)\}$ converges uniformly to P_T;*
(iii) *$\{B_n^{(-1)}(T)\}$ is bounded, and $\mathcal{R}(T-I)$ is closed;*
(iv) *1 is a simple pole of the resolvent function of T, jor $1 \notin \sigma(T)$.*

Proof. Assertion (i) means that $\|S^n - P_T\| \to 0$, which also yields $\|M_n(S) - P_T\| \to 0$; that is, $\|B_n^{(-1)}(T) - P_T\| \to 0$ by (5.1). Then (i) implies (ii), which implies (iii) by (5.1) and by [19, Main Theorem]. Also, (i) follows from (iii) by (5.1) and [20, Theorem 2.7]. Now (i) gives the decomposition $\mathcal{X} = \mathcal{R}(T-I) \oplus \mathcal{N}(T-I)$, which implies immediately the assertions of (iv) (see [6]). Conversely, the assertion (iv) ensures the previous decomposition of $\mathcal{X}$ or, equivalently, $\mathcal{X} = \mathcal{R}(S-I) \oplus \mathcal{N}(S-I)$. Also, one has $\mathcal{R}((S-I)^2) = \mathcal{R}(S-I)$ (see [6]), and putting $S_0 = S|_{\mathcal{R}(S-I)}$, we have $1 \notin \sigma(S_0)$. Hence $\sigma(S_0) = \sigma(S) \setminus \{1\} \subset \mathbb{D}$; that is, $\|S_0^n\| \to 0$, which implies (by the decomposition of $\mathcal{X}$) that $\|S^n - P_S\| \to 0$. Consequently, (iv) implies (i). $\square$

Let us note that the assertion (iv) is ensured by the uniform convergence of a (every) Cesàro mean $\{M_n^{(p)}(T)\}$ or of the Abel mean $\{A_n(T)\}$. In particular, it follows that if T is uniformly Cesàro-ergodic, then S has the same property (that is, all assertions (i)–(iv) are true in this case).

Remark 5.4. If T is power-bounded, then S is power-bounded, too. But the uniform Kreiss boundedness cannot be transferred between T and S. More precisely, the condition $\|M_n(\lambda S)\| = O(n)$ as $n \to \infty$ and $\lambda \in \mathbb{T}$ can equivalently be expressed by the partial sums of the Taylor expansion at infinity of the resolvent function of S (as in [1, Theorem 4.7]), which in terms of T becomes

$$\sup_{\substack{n\in\mathbb{N}\\ |\lambda+1|>2}} |\lambda+1| \Big\| \sum_{k=0}^{n} \lambda^{-k-1} T^k \Big\| < \infty. \tag{5.2}$$

This just means that S is uniformly Kreiss-bounded, but it does not force T to be uniformly Kreiss-bounded in general (see the theorem in [1] quoted before).

To end this discussion, let us remark that the spectral condition in the hypothesis of Theorem 5.3 is essential for the equivalence of (i) to other assertions. For this purpose, let us consider the operator $T = 2V - 3I$, where V is the above Volterra operator on $L^p[0,1]$, $1 \le p \le \infty$. Then the operator $S = \frac{1}{2}(T+I) = V - I$ is uniformly Kreiss-bounded, and we have $\frac{1}{n}\|S^n\| \to 0$ by [22]. As $\mathcal{R}(S-I) = L^p[0,1]$ because $\sigma(S) = \{-1\}$, one obtains that $B_n^{(-1)}(T) = M_n(S) \to 0$ uniformly. Obviously $\sigma(T) = \{-3\}$. In the case $p = 2$, the operators S and $-S$ are power-bounded, and $\sigma(-S) = \{1\}$. In this case, $\|B_n(T)\| = \|S^n\| = \|(-S)^n\| \nrightarrow 0$. However, by the Esterle–Katznelson–Tzafriri theorem one has $\|S^n(S-I)\| \to 0$, which leads in the end to $S^n \to 0$ strongly. We conclude in this case that the assertion (i) in Theorem 5.3 is not true even if the other assertions (ii)–(iv) hold.

The above facts show that different sequences in $\kappa(T)$ can have different asymptotic behaviors.

We see next that some ergodic properties for other important operator means in applications can be expressed in terms of corresponding properties for the Cesàro means. More precisely, we study the connection between ergodicity and regularity (the cases $n = 0$ and $n = 1$ of condition (2.5)) on the one hand and the strong convergence on the other hand for certain operator means and their backward iterates, which derive from Cesàro means.

We first refer to the square of the Cesàro mean, that is, the averages $M_n(T)^2$ (which were also considered in [25]). It is easy to see that these can be expressed by the relation

$$M_n(T)^2 = \frac{1}{n+1}\left[(2n+1)M_{2n}^{(2)}(T) - nM_{n-1}^{(2)}(T)\right] \quad (n \ge 1). \tag{5.3}$$

It is clear that $M_n(T)^2 \in \kappa(T)$, and this relation shows that the boundedness, the ergodicity, or the strong convergence of $M_n(T)^2$ is ensured by those of $M_n^{(2)}(T)$, respectively.

Concerning the regularity (that is, (2.5) with $n_0 = 1$), we have in view the following relation:

$$TM_n(T)^2 - M_{n+1}(T)^2 = \frac{2n+3}{(n+2)^2}\big(TM_n(T)^2 - M_{2(n+1)}(T)\big).$$

Then it is natural to assume that $\{\frac{1}{n}M_n(T)^2\}$ converges strongly to zero on $\mathcal{X}$, and under this assumption we have that $\{M_n(T)^2\}$ is regular if and only if $\{M^{(2)}_{2n-1}(T)\}$ is ergodic. In this case, $\{M^{(2)}_n(T)\}$ is ergodic if and only if $\{M_n(T)^2\}$ is ergodic and regular. This fact can be equivalently expressed by involving the subsequence $\{M_{2n}(T)^2\}$ as below.

First, from (5.3) with $2n$ instead of n, we derive under the above assumption and the regularity of $\{M_n(T)^2\}$ that $\{M_{2n}(T)^2\}$ is ergodic if and only if $\{M^{(2)}_{4n}(T)\}$ is ergodic. Concerning the regularity of $M_{2n}(T)^2$, we obtain by (5.3) and a direct computation that

$$\begin{aligned}
&TM_{2n}(T)^2 - M_{2(n+1)}(T)^2\\
&\quad= \frac{1}{(2n+3)^2}\big[8(n+1)TM_{2n}(T)^2 + 4(n+1)M_{2n+1}(T) - 3(4n+5)M_{4(n+1)}(T)\\
&\qquad + \big((2n+1)M_{2n}(T)(T-I)+I\big)^2(T+2T^2)\big].
\end{aligned}$$

Under the assumption of regularity of $\{M_n(T)^2\}$ and of ergodicity of $\{M_{2n}(T)^2\}$, we infer that all terms on the right-hand side of this relation converge strongly to zero, excepting the term corresponding to $M_{2n+1}(T)$. In this case, the sequence $\{M_{2n}(T)^2\}$ is regular if and only if $\frac{1}{n}M_{2n+1}(T) \to 0$ strongly on $\mathcal{X}$. This together with the regularity of $\{M_n(T)^2\}$ implies that $\frac{1}{n}M_n(T) \to 0$ strongly on $\mathcal{X}$, which means that $\{M^{(2)}_n(T)\}$ is ergodic. We can summarize these facts in the following.

Proposition 5.5. *Let $T \in \mathcal{B}(\mathcal{X})$ be such that the sequence $\{\frac{1}{n}M_n(T)^2\}$ converges strongly to zero. The following statements are equivalent:*

(i) $\{M^{(2)}_n(T)\}$ *is ergodic,*
(ii) $\{M_n(T)^2\}$ *is ergodic and regular,*
(iii) $\{M_n(T)^2\}$ *is regular, and* $\{M_{2n}(T)^2\}$ *is ergodic and regular.*

Note that by (5.3) the condition in the hypothesis of this proposition is satisfied if $\{\frac{1}{n}M^{(2)}_n(T)\}$ converges strongly to zero on $\mathcal{X}$, which means that $\{M^{(3)}_n(T)\}$ is ergodic, which will be next assumed. Then by (1.8) and the above regularity of $\{M_n(T)^2\}$, we obtain the assertion (i) below.

On the other hand, if $\{M_n(T)^2\}$ converges strongly on $\mathcal{X}$, then $\mathcal{X} = \overline{\mathcal{R}(T-I)} \oplus \mathcal{N}(T-I)$, while Proposition 5.5 gives that $\{M^{(2)}_n(T)\}$ is ergodic. In addition, if this sequence is bounded, then it converges strongly on $\mathcal{X}$ by using the decomposition of $\mathcal{X}$. From these facts and (5.3), we infer the following ergodic result.

Theorem 5.6. *The following statements hold for $T \in \mathcal{B}(\mathcal{X})$:*

(i) $\{M^{(2)}_n(T)\}$ *is ergodic if and only if* $\{M^{(3)}_n(T)\}$ *is ergodic and* $\{M_n(T)^2\}$ *is regular,*

(ii) $\{M_n^{(2)}(T)\}$ *converges on* $\mathcal{X}$ *if and only if it is bounded and* $\{M_n(T)^2\}$ *converges strongly on* $\mathcal{X}$.

If T is Kreiss-bounded, then as we already noted, $\{M_n^{(2)}(T)\}$ is ergodic, or, equivalently, $\{M_n(T)^2\}$ is regular. Moreover, the two sequences simultaneously converge strongly on $\mathcal{X}$. On the other hand, on reflexive spaces, the regularity of $\{M_n(T)^2\}$ is equivalent with the strong convergence of this sequence when ergodicity of $\{M_n^{(3)}(T)\}$ is assumed.

Similar properties can be obtained for the backward iterates of $T_n = M_n(T)^2$, which will be denoted by $M_n^{(-2)}(T)$ for short. We write $T_n = \sum_{j=0}^{2n} t_{nj}T^j$ using (5.3). Then $\alpha_n = \sum_{j\geq 1} jt_{nj} = n$, and for $n \geq 2$ we have

$$M_n^{(-2)}(T) = \frac{1}{3(n+1)}\big[2(2n+1)M_{2n-1}^{(3)}(T) - (n-1)M_{n-2}^{(3)}(T)\big]. \tag{5.4}$$

We investigate the regularity under the assumption that $\{M_n^{(4)}(T)\}$ is ergodic; that is, that $\frac{1}{n}M_n^{(3)}(T) \to 0$ strongly on $\mathcal{X}$. We have, for $n \geq 1$,

$$\begin{aligned}
&TM_n^{(-2)}(T) - M_{n+1}^{(-2)}(T) \\
&\quad= \frac{1}{3(n+1)}\big[2(2n+1)\big(TM_{2n-1}^{(3)}(T) - M_{2n}^{(3)}(T)\big) \\
&\qquad - (n-1)\big(TM_{n-2}^{(3)}(T) - M_{n-1}^{(3)}(T)\big)\big] \\
&\qquad + \frac{2}{3}\Big[\Big(2 - \frac{1}{n+1}\Big)M_{2n}^{(3)}(T) - \Big(2 - \frac{1}{n+2}\Big)M_{2n+1}^{(3)}(T)\Big] \\
&\qquad + \frac{2}{3(n+1)(n+2)}M_{n-1}^{(3)}(T).
\end{aligned}$$

Now, by our assumption and the relation (1.7), the first and the last terms on the right-hand side converge strongly to zero. Hence $\{M_n^{(-2)}(T)\}$ is regular if and only if $M_{2n}^{(3)}(T) - M_{2n+1}^{(3)}(T) \to 0$ strongly on $\mathcal{X}$. But by (1.8), this implies that $\frac{1}{2n+1}M_{2n+1}^{(2)}(T) \to 0$ strongly, which means that $\{M_{2n}^{(3)}(T)\}$ is ergodic. We infer that $\{M_n^{(3)}(T)\}$ is ergodic when $\{M_n^{(-2)}(T)\}$ is regular. Conversely, if $\{M_n^{(3)}(T)\}$ is ergodic, then $\{M_n^{(4)}(T)\}$ is such, while by (1.6) and (1.8) we have that $\{M_n^{(-2)}(T)\}$ is regular (having in view the above characterization of this property).

A relationship between the strong convergence of $\{M_n^{(-2)}(T)\}$ and $\{M_n^{(3)}(T)\}$ can also be established as above for $\{M_n(T)^2\}$. These facts lead to the following theorem.

Theorem 5.7. *The following statements hold for* $T \in \mathcal{B}(\mathcal{X})$*:*

(i) $\{M_n^{(3)}(T)\}$ *is ergodic if and only if* $\{M_n^{(4)}(T)\}$ *is ergodic and* $\{M_n^{(-2)}(T)\}$ *is regular,*

(ii) $\{M_n^{(3)}(T)\}$ *converges on* $\mathcal{X}$ *if and only if it is bounded and* $\{M_n^{(-2)}(T)\}$ *converges strongly on* $\mathcal{X}$.

Remark 5.8. If T is Kreiss-bounded, then $\{M_n(T)^2\}$ and $\{M_n^{(-2)}(T)\}$ are ergodic and regular, while in reflexive spaces both sequences converge strongly.

Another sequence $\{T_n\} \subset \kappa(T)$ related to the Cesàro means is given by

$$T_n = \frac{2}{n(n+1)} \sum_{j=1}^{n} jT^j = \frac{2(n+1)}{n} M_n(T) - \frac{n+2}{n} M_n^{(2)}(T). \tag{5.5}$$

We have the following result derived from (5.5) and Theorem 5.6.

Corollary 5.9. *Let $T \in \mathcal{B}(\mathcal{X})$ be such that $\{M_n^{(2)}(T)\}$ is bounded. The following statements are equivalent:*

(i) *$\{M_n(T)\}$ converges strongly on $\mathcal{X}$,*
(ii) *$\{T_n\}$ converges strongly on $\mathcal{X}$ and $\{M_n(T)^2\}$ is regular,*
(iii) *$\{M_n(T)^2\}$ converges strongly on $\mathcal{X}$ and $\{T_n\}$ is ergodic.*

Remark 5.10. If T is Kreiss-bounded, then by (5.5) the sequences $\{T_n\}$ and $\{M_n(T)\}$ are simultaneously ergodic (resp., regular). But if T is uniformly Kreiss-bounded on a Hilbert space $\mathcal{H}$, then by Corollary 4.6 and (5.5) the sequence $\{M_n(\{T_k\})\}$ converges strongly on $\mathcal{H}$.

The above results show that in $\kappa(T)$ it is possible to transfer some ergodic properties between Cesàro means and other sequences related to these in both senses, which can be important for applications. Also, the results obtained on binomial means in the preceding sections show that some ergodic properties can be true for some operator means related to Cesàro means even if such properties do not hold for the latter.

Acknowledgment. We are very grateful to the referees for appropriate and constructive suggestions which improved the exposition of this article.

References

1. A. Aleman and L. Suciu, *On ergodic operator means in Banach spaces*, Integral Equations Operator Theory **85** (2016), no. 2, 259–287. Zbl 06599317. MR3511367. DOI 10.1007/s00020-016-2298-x. 240, 241, 242, 243, 254, 256, 258, 260
2. G. R. Allan, "Power-bounded elements and radical Banach algebras" in *Linear Operators (Warsaw, 1994)*, Banach Center Publ. **38**, Polish Academy of Sciences–Inst. of Mathematics, Warsaw, 1997, 9–16. Zbl 0884.47003. MR1456997. 241
3. S. I. Ansari and P. S. Bourdon, *Some properties of cyclic operators*, Acta Sci. Math. (Szeged) **63** (1997), no. 1–2, 195–207. Zbl 0892.47004. MR1459787. 254
4. F. Bayart and E. Matheron, *Dynamics of Linear Operators*, Cambridge Tracts in Math. **179**, Cambridge Univ. Press, Cambridge, 2009. Zbl 1187.47001. MR2533318. DOI 10.1017/CBO9780511581113. 242
5. J. Boos, *Classical and Modern Methods in Summability*, Oxford Math. Monogr., Oxford Univ. Press, New York, 2000. Zbl 0954.40001. MR1817226. 240
6. L. Burlando, *A generalization of the uniform ergodic theorem to poles of arbitrary order*, Studia Math. **122** (1997), no. 1, 75–98. Zbl 0869.47007. MR1425876. 259
7. L. W. Cohen, *On the mean ergodic theorem*, Ann. of Math. (2) **41** (1940), no. 3, 505–509. Zbl 0024.21401. MR0002027. 241
8. N. Dungey, *Subordinated discrete semigroups of operators*, Trans. Amer. Math. Soc. **363** (2011), no. 4, 1721–1741. Zbl 1228.47012. MR2746662. DOI 10.1090/S0002-9947-2010-05094-9. 240, 241

9. K. Dykema and H. Schultz, *Brown measure and iterates of the Aluthge transform for some operators arising from measurable actions*, Trans. Amer. Math. Soc. **361** (2009), no. 12, 6583–6593. Zbl 1181.47006. MR2538606. DOI 10.1090/S0002-9947-09-04762-X. 250
10. W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67** (1949), no. 1, 217–240. Zbl 0034.06404. MR0036455. 241
11. J. Esterle, "Quasimultipliers, representations of H^∞, and the closed ideal problem for commutative Banach algebras" in *Radical Banach Algebras and Automatic Continuity (Long Beach, 1981)*, Lecture Notes in Math. **975**, Springer, Berlin, 1983, 66–162. Zbl 0536.46041. MR0697579. DOI 10.1007/BFb0064548. 241, 246
12. I. Gelfand, *Zur Theorie der Charaktere der Abelschen topologischen Gruppen*, Mat. Sbornik **9** (1941), 49–50. Zbl 0024.32301. MR0004635. 247
13. J. Glück, *On the peripheral spectrum of positive operators*, Positivity **20** (2016), no. 2, 307–336. Zbl 06591946. MR3505354. DOI 10.1007/s11117-015-0357-1. 241
14. Y. Katznelson and L. Tzafriri, *On power-bounded operators*, J. Funct. Anal. **68** (1986), 313–328. Zbl 0611.47005. MR0859138. DOI 10.1016/0022-1236(86)90101-1. 241, 246
15. L. Kérchy, *Operators with regular norm-sequences*, Acta Sci. Math. (Szeged) **63** (1997), no. 3–4, 571–605. Zbl 0893.47006. MR1480500. 241, 242, 255, 256
16. U. Krengel, *Ergodic Theorems*, De Gruyter Stud. Math. **6**, de Gruyter, Berlin, 1985. Zbl 0575.28009. MR0797411. DOI 10.1515/9783110844641. 241, 243, 244, 254
17. M. K. Kuo, *Tauberian conditions for almost convergence*, Positivity **13** (2009), no. 4, 611–619. Zbl 1186.40007. MR2538509. DOI 10.1007/s11117-008-2282-z. 241, 242, 245
18. Z. Léka, *A note on the powers of Cesàro bounded operators*, Czechoslovak Math. J. **60** (2010), no. 4, 1091–1100. Zbl 1220.47014. MR2738971. DOI 10.1007/s10587-010-0074-7. 251
19. M. Lin, *On the uniform ergodic theorem*, Proc. Amer. Math. Soc. **43** (1974), no. 2, 337–340. Zbl 0252.47004. MR0417821. 246, 259
20. M. Lin, D. Shoikhet, and L. Suciu, *Remarks on uniform ergodic theorems*, Acta Sci. Math. (Szeged) **81** (2015), 251–283. Zbl 06656645. MR3381884. DOI 10.14232/actasm-012-307-4. 259
21. G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), no. 1, 167–190. Zbl 0031.29501. MR0027868. 242, 255, 256
22. A. Montes-Rodríguez, J. Sánchez-Álvarez, and J. Zemánek, *Uniform Abel–Kreiss boundedness and the extremal behaviour of the Volterra operator*, Proc. Lond. Math. Soc. (3) **91** (2005), no. 3, 761–788. Zbl 1088.47040. MR2180462. DOI 10.1112/S002461150501539X. 241, 246, 260
23. O. Nevanlinna, *Resolvent conditions and powers of operators*, Studia Math. **145** (2001), no. 2, 113–134. Zbl 0981.47002. MR1828000. DOI 10.4064/sm145-2-2. 259
24. A. V. Romanov, *Weak convergence of operator means*, Izv. Math. **75** (2011), no. 6, 1165–1183. Zbl 1248.47013. MR2918894. DOI 10.1070/IM2011v075n06ABEH002568. 241
25. H. C. Rönnefarth, *On properties of the powers of a bounded linear operator and their characterization by its spectrum and resolvent*, Ph.D. dissertation, Technische Universität Berlin, Berlin, Germany, 1996. 260
26. M. Schreiber, *Uniform families of ergodic operator nets*, Semigroup Forum **86** (2013), no. 2, 321–336. Zbl 1273.43009. MR3034777. DOI 10.1007/s00233-012-9444-9. 241
27. J. C. Strikwerda and B. A. Wade, *A survey of the Kreiss matrix theorem for power bounded families of matrices and its extensions*, Banach Center Publ. **38** (1994), 339–360. Zbl 0877.15029. MR1457017. 254
28. L. Suciu, *Estimations of the operator resolvent by higher order Cesàro means*, Results Math. **69** (2016), no. 3–4, 457–475. Zbl 06585580. MR3499574. DOI 10.1007/s00025-016-0533-z. 242, 254
29. L. Suciu and J. Zemánek, *Growth conditions and Cesàro means of higher order*, Acta Sci. Math. (Szeged) **79** (2013), no. 3–4, 545–581. Zbl 1313.47017. MR3134504. 241, 244, 250

30. Y. Tomilov and J. Zemánek, *A new way of constructing examples in operator ergodic theory*, Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 1, 209–225. Zbl 1073.47503. MR2075049. DOI 10.1017/S0305004103007436. 251
31. Q. P. Vũ, *A short proof of the Y. Katznelson's and L. Tzafriri's theorem*, Proc. Amer. Math. Soc. **115** (1992), no. 4, 1023-1024. Zbl 0781.47003. MR1087468. DOI 10.2307/2159349. 241

Department of Mathematics and Informatics, "Lucian Blaga" University of Sibiu, Dr. Ion Raţiu 5-7, Sibiu, 550012, Romania.
E-mail address: laurians2002@yahoo.com