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ON STAR, SHARP, CORE, AND MINUS PARTIAL ORDERS IN RICKART RINGS

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ABSTRACT. Let $\mathcal{A}$ be a Rickart $*$-ring and let $\leq^*, \leq^\sharp, \leq^\oplus$, and $\leq_\oplus$ denote the star, the sharp, the core, and the dual core partial orders in $\mathcal{A}$, respectively. The sets of all $b \in \mathcal{A}$ such that $a \leq b$, along with the sets of all $b \in \mathcal{A}$ such that $b \leq a$, are characterized, where $a \in \mathcal{A}$ is given and where $\leq$ is one of the partial orders: $\leq^*$, or $\leq^\sharp$, or $\leq^\oplus$, or $\leq_\oplus$. Such sets of elements that are above or below a given element under the minus partial order $\leq^-$ in a Rickart ring $\mathcal{A}$ are also studied. Some recent results of Cvetković-Ilić et al. on partial orders in $\mathcal{B}(H)$, the algebra of all bounded linear operators on a Hilbert space H, are thus generalized.

1. Introduction

Let H be a Hilbert space and let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. For an operator $A \in \mathcal{B}(H)$, the symbols $\operatorname{Ker} A$ and $\operatorname{Im} A$ will denote the kernel and the image of A, respectively, while A^* will denote the adjoint operator of A. Let M be a closed subspace of H. We will denote by P_M the self-adjoint idempotent operator in $\mathcal{B}(H)$ such that $\operatorname{Im} P = M$. Many partial orders may be defined on $\mathcal{B}(H)$. One of such orders is the star partial order $\leq^*$, which is defined as follows. For $A, B \in \mathcal{B}(H)$, we write

$$A \leq^* B \quad \text{if } A^*A = A^*B \text{ and } AA^* = BA^*. \tag{1.1}$$

Note that Drazin [6] defined this order in a more general setting of proper $*$-semigroups. In a very recent paper [3], the set of all $B \in \mathcal{B}(H)$ such that

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$A \leq^* B$, and the set of all $B \in \mathcal{B}(H)$ such that $B \leq^* A$, where $A \in \mathcal{B}(H)$ is given, were characterized; the authors first observed that the description of the set of all operators above a given operator $A \in \mathcal{B}(H)$ under the star partial order follows directly from an equivalent definition of the star partial order provided in [5]. Namely, for a given $A \in B(H)$, we have $A \leq^* B$ if and only if $B = A + P_{\mathrm{Ker}\, A^*} X P_{\mathrm{Ker}\, A}$ for some $X \in \mathcal{B}(H)$. Next, the authors stressed that any operator $B \in \mathcal{B}(H)$ has the following matrix representation with respect to the orthogonal direct sum $H = \overline{\mathrm{Im}\, B} \oplus \mathrm{Ker}\, B^*$:

$$B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathrm{Im}\, B} \\ \mathrm{Ker}\, B^* \end{bmatrix} \to \begin{bmatrix} \overline{\mathrm{Im}\, B} \\ \mathrm{Ker}\, B^* \end{bmatrix}. \tag{1.2}$$

The following result from [3] describes the set of all $A \in \mathcal{B}(H)$ such that $A \leq^* B$ and B is given.

Proposition 1.1. *Let $B \in \mathcal{B}(H)$ be given by (1.2). For $A \in \mathcal{B}(H)$, $A \leq^* B$ if and only if*

$$A = \begin{bmatrix} PB_1 & PB_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathrm{Im}\, B} \\ \mathrm{Ker}\, B^* \end{bmatrix} \to \begin{bmatrix} \overline{\mathrm{Im}\, B} \\ \mathrm{Ker}\, B^* \end{bmatrix}$$

for some self-adjoint idempotent $P \in \mathcal{B}(\overline{\mathrm{Im}\, B})$ which commutes with $D = B_1 B_1^* + B_2 B_2^* \in \mathcal{B}(\overline{\mathrm{Im}\, B})$.

Some other well-known partial orders—the sharp, the core, and the minus partial orders—were studied in [3], where results similar to Proposition 1.1 were obtained. It is the aim of this paper to generalize these results from $\mathcal{B}(H)$ to Rickart rings or to Rickart $*$-rings.

Let $\mathcal{A}$ be a ring. A ring admitting an involution * will be called an *involutory ring*. The set of all idempotent elements in a ring $\mathcal{A}$ will be denoted by $E(\mathcal{A})$. For $a \in \mathcal{A}$, we will denote by a° the right annihilator of a (i.e., the set $a^\circ = \{x \in \mathcal{A} : ax = 0\}$). Similarly, we denote the left annihilator ${}^\circ a$ of a (i.e., the set ${}^\circ a = \{x \in \mathcal{A} : xa = 0\}$). A ring $\mathcal{A}$ is called a *Rickart ring* if for every $a \in \mathcal{A}$ there exist idempotent elements $p, q \in \mathcal{A}$ such that $a^\circ = p \cdot \mathcal{A}$ and ${}^\circ a = \mathcal{A} \cdot q$. Note that every Rickart ring $\mathcal{A}$ has the (multiplicative) identity (see [2] or [10]). An involutory ring $\mathcal{A}$ is called a *Rickart $*$-ring* if the left annihilator ${}^\circ a$ of any element $a \in \mathcal{A}$ is generated by a self-adjoint idempotent $e \in \mathcal{A}$; that is, ${}^\circ a = \mathcal{A} \cdot e$ where $e = e^* = e^2$ (observe that such a self-adjoint idempotent is unique; see [2]). For an involutory ring $\mathcal{A}$, denote the sets $\{x \in \mathcal{A} : ax^* = 0\} = (a^\circ)^*$ and $\{x \in \mathcal{A} : x^* a = 0\} = ({}^\circ a)^*$. Note that $(a^\circ)^* = {}^\circ(a^*)$ and $({}^\circ a)^* = (a^*)^\circ$ for every $a \in \mathcal{A}$. Let a be an element in a Rickart $*$-ring $\mathcal{A}$. There exists by definition a self-adjoint idempotent $f \in \mathcal{A}$ such that ${}^\circ(a^*) = \mathcal{A} \cdot f$. Since f is a self-adjoint element, this is equivalent to $({}^\circ(a^*))^* = f \cdot \mathcal{A}$ (i.e., $a^\circ = f \cdot \mathcal{A}$). So, the analogous property for right annihilators is fulfilled in the case when $\mathcal{A}$ is a Rickart $*$-ring. Note also that $\mathcal{B}(H)$ is a Rickart $*$-ring and hence a Rickart ring. In fact, the class of Rickart rings includes von Neumann algebras and rings with no proper zero divisors (see [2]).

Let $\mathcal{A}$ be a ring with identity 1. Suppose that p and q are idempotent elements in $\mathcal{A}$. Then any $x \in \mathcal{A}$ can be represented in the following form:

$$x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q) = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}_{p\times q}.$$

Here $x_{1,1} = pxq$, $x_{1,2} = px(1-q)$, $x_{2,1} = (1-p)xq$, $x_{2,2} = (1-p)x(1-q)$. If $x = (x_{i,j})_{p\times q}$ and $y = (y_{i,j})_{p\times q}$, then $x+y = (x_{i,j}+y_{i,j})_{p\times q}$. Moreover, if $r \in \mathcal{A}$ is an idempotent, and if $x = (x_{i,j})_{p\times q}$, and $y = (y_{i,j})_{q\times r}$, then $xy = (\sum_{k=1}^{2} x_{i,k}y_{k,j})_{p\times r}$. Thus, if we have idempotents in $\mathcal{A}$, then the usual algebraic operations in $\mathcal{A}$ can be interpreted as simple operations between appropriate 2×2 matrices over $\mathcal{A}$. Furthermore, if $\mathcal{A}$ admits an involution * and $x \in \mathcal{A}$, then we have

$$x^* = \begin{bmatrix} x_{1,1}^* & x_{2,1}^* \\ x_{1,2}^* & x_{2,2}^* \end{bmatrix}_{q^*\times p^*}.$$

Let us denote

$$\mathrm{LP}(a) = \{p \in \mathcal{A} : p = p^2, {}^\circ a = \mathcal{A}\cdot(1-p)\},$$
$$\mathrm{RP}(a) = \{q \in \mathcal{A} : q = q^2, a^\circ = (1-q)\cdot\mathcal{A}\}.$$

Note that the sets $\mathrm{LP}(a)$ and $\mathrm{RP}(a)$ are nonempty in the case when $\mathcal{A}$ is a Rickart ring. If $\mathcal{A}$ is a Rickart $*$-ring, then there exists the unique self-adjoint idempotent in $\mathrm{LP}(a)$, which we will denote by $\mathrm{lp}(a)$. Similarly, let $\mathrm{rp}(a)$ denote the unique self-adjoint idempotent in $\mathrm{RP}(a)$.

The following proposition holds true for any idempotent p in a ring with identity (see [4, Lemma 2.1]).

Proposition 1.2. *Let $\mathcal{A}$ be a ring with identity 1, and let $p \in E(\mathcal{A})$. Then $\mathcal{A}\cdot(1-p) = {}^\circ p$ and $(1-p)\cdot\mathcal{A} = p^\circ$.*

Let $\mathcal{A}$ be a Rickart ring and let $b \in \mathcal{A}$. By Proposition 1.2, there exists an idempotent $s \in \mathcal{A}$ such that ${}^\circ s = {}^\circ b$ (i.e., $s \in \mathrm{LP}(b)$). Let $b = \left[\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right]_{s\times s}$, where $s \in \mathrm{LP}(b)$. Since $b_3 = (1-s)bs$, $b_4 = (1-s)b(1-s)$, and $(1-s) \in {}^\circ s = {}^\circ b$, we may conclude that $b_3 = b_4 = 0$. So, every $b \in \mathcal{A}$ may be represented in the following form:

$$b = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s}, \quad s \in \mathrm{LP}(b). \tag{1.3}$$

Note that in the case when $\mathcal{A}$ is a Rickart $*$-ring, every $b \in \mathcal{A}$ may be represented in the following form:

$$b = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s}, \quad s = \mathrm{lp}(b). \tag{1.4}$$

Throughout the present article, we will use the following notation: $d = b_1b_1^* + b_2b_2^*$.

Remark 1.3. Let $B \in \mathcal{B}(H)$, and denote by I the identity operator in $\mathcal{B}(H)$. Since $\mathcal{B}(H)$ is a Rickart $*$-ring, there exists the unique self-adjoint idempotent operator

$S \in \mathcal{B}(H)$ such that ${}^\circ S = {}^\circ B$ (i.e., $S = \mathrm{lp}(B)$). So, $B = SBS + SB(I - S)$; that is,

$$B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}_{S\times S},$$

where $B_1 = SBS$ and $B_2 = SB(I-S)$. For $C, D \in \mathcal{B}(H)$, Lemma 2.1 in [13] yields ${}^\circ C = {}^\circ D$ if and only if $\overline{\operatorname{Im} C} = \overline{\operatorname{Im} D}$. It follows that $\operatorname{Im} S = \overline{\operatorname{Im} B}$ and thus that $\operatorname{Im}(I - S) = \operatorname{Ker} S = \operatorname{Ker} B^*$. Therefore, we may regard (1.4) as a generalization of matrix representation (1.2) from $\mathcal{B}(H)$ to an arbitrary Rickart $*$-ring $\mathcal{A}$ (for more details, see [15]).

2. The star order

Let $\mathcal{A}$ be an involutory ring with identity 1. A new order was introduced in [13, Definition 4] on involutory rings with identity. The definition is as follows. For $a, b \in \mathcal{A}$, we write $a \le^* b$ if there exist self-adjoint idempotent elements $p, q \in \mathcal{A}$ such that

$${}^\circ a = \mathcal{A}(1-p), \qquad a^\circ = (1-q)\mathcal{A}, \qquad pa = pb, \qquad \text{and} \qquad aq = bq.$$

Authors called this new order the *star order* and showed that when $\mathcal{A}$ is a Rickart $*$-ring, the new order is the same as the order introduced by Drazin (1.1). Namely (see [13, Theorem 1]), for elements a, b in a Rickart $*$-ring $\mathcal{A}$, we have

$$a \le^* b \quad \text{if and only if } a^*a = a^*b \text{ and } aa^* = ba^*.$$

There are many equivalent definitions of the star partial order in Rickart $*$-rings (see [13]); the following one will be used in the rest of this section:

$$a \le^* b \quad \text{if and only if } a = \mathrm{lp}(a)b = b\,\mathrm{rp}(a). \tag{2.1}$$

Another equivalent definition is as follows. For a Rickart $*$-ring $\mathcal{A}$ and for $a, b \in \mathcal{A}$, we have $a \le^* b$ if and only if

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{\mathrm{lp}(a)\times \mathrm{rp}(a)} \qquad \text{and} \qquad b = \begin{bmatrix} a & 0 \\ 0 & b_1 \end{bmatrix}_{\mathrm{lp}(a)\times \mathrm{rp}(a)}, \tag{2.2}$$

where $b_1 \in (1 - \mathrm{lp}(a))\mathcal{A}(1 - \mathrm{rp}(a))$.

Let $\mathcal{A}$ be a Rickart $*$-ring and let $a \in \mathcal{A}$. By (2.2), we may conclude that $a \le^* b$ if and only if

$$b = a + \big(1 - \mathrm{lp}(a)\big)x\big(1 - \mathrm{rp}(a)\big)$$

for some $x \in \mathcal{A}$. With this observation we described the set of all elements $b \in \mathcal{A}$ such that $a \le^* b$ where a is a given element in a Rickart $*$-ring $\mathcal{A}$. With the next theorem we will describe the set of all $a \in \mathcal{A}$ such that $a \le^* b$ where $b \in \mathcal{A}$ is given. We will thus generalize Proposition 1.1.

Theorem 2.1. *Let $\mathcal{A}$ be a Rickart $*$-ring and let $b \in \mathcal{A}$ be given by (1.4). For $a \in \mathcal{A}$, we have $a \le^* b$ if and only if $a = \left[\begin{smallmatrix} pb_1 & pb_2 \\ 0 & 0 \end{smallmatrix}\right]_{s\times s}$, where $p \in s\mathcal{A}s$, $p = p^* = p^2$, and $pd = dp$.*

Proof. Let $a, b \in \mathcal{A}$ and let $b = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s}$, where $s = \operatorname{lp}(b)$. Suppose first that $a \leq^* b$ and let $a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{s\times s}$. By (2.1), $a = \operatorname{lp}(a)b = b\operatorname{rp}(a)$. It follows that ${}^\circ b \subseteq {}^\circ a$, which implies that $(1-s)a = 0$ and hence that $a_3 = 0$ and $a_4 = 0$. Since ${}^\circ(\operatorname{lp}(a)) = {}^\circ a$, we obtain $(1-s)\operatorname{lp}(a) = 0$, and hence

$$\operatorname{lp}(a) = \begin{bmatrix} r_1 & r_2 \\ 0 & 0 \end{bmatrix}_{s\times s}.$$

From $\operatorname{lp}(a) = (\operatorname{lp}(a))^*$ we have

$$\operatorname{lp}(a) = \begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix}_{s\times s}.$$

Note that $r_1 = r_1^* = r_1^2$ and denote $r_1 = p$. Since $a = \operatorname{lp}(a)b$,

$$a = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s}.$$

Recall that $d = b_1b_1^* + b_2b_2^*$. Since $a \leq^* b$, we have $aa^* = ba^*$. So,

$$\begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} b_1^*p & 0 \\ b_2^*p & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} b_1^*p & 0 \\ b_2^*p & 0 \end{bmatrix}_{s\times s},$$

and hence $pdp = dp$. Since p and d are self-adjoint, we have $pdp = pd$; thus $pd = dp$.

Conversely, let $a = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s}$, where $s = \operatorname{lp}(b)$, $p \in s\mathcal{A}s$, $p = p^* = p^2$, and $pd = dp$. Recall that $b = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s}$. It follows that

$$ba^* = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} b_1^*p & 0 \\ b_2^*p & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} dp & 0 \\ 0 & 0 \end{bmatrix}_{s\times s}$$

and that

$$aa^* = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} b_1^*p & 0 \\ b_2^*p & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} pdp & 0 \\ 0 & 0 \end{bmatrix}_{s\times s}.$$

Since $pd = dp$, we may conclude that $aa^* = ba^*$. We may similarly show that $a^*a = a^*b$. So, $a \leq^* b$. □

3. The sharp order

Other characterizations, similar to the one described with Proposition 1.1, of sets of operators from $\mathcal{B}(H)$ that are above or below a given operator under a certain (sharp, or core, or minus) partial order were given in [3]. In this section, we will generalize results from [3] concerning the sharp partial order.

Let S be a semigroup and let $a \in S$. Any solution $x = a^-$ to the equation $axa = a$ is called an *inner generalized inverse* of a. If such a^- exists, then a is called *regular*. We say that a has the group inverse $a^\sharp$ in S if $x = a^\sharp$ satisfies the following equations: $axa = a$, $xax = x$, and $ax = xa$. The group inverse, if it exists, is unique (see [9]). For a ring $\mathcal{A}$ with identity, denote by $\mathcal{G}(\mathcal{A})$ the subset of elements in $\mathcal{A}$ which have the group inverse. Mitra introduced in [14] (see also [7]) the sharp partial order on the set of all $n \times n$ complex matrices M_n which have the group inverse. Namely, for $A, B \in \mathcal{G}(M_n)$, we write $A \leq^\sharp B$

if $A^\sharp A = A^\sharp B$ and $AA^\sharp = BA^\sharp$. This order was further generalized in [12] and [16]. The definition from [12, pp. 1715–1722] is as follows. Let $\mathcal{A}$ be a ring with identity 1. For $a \in \mathcal{G}(\mathcal{A})$ and $b \in \mathcal{A}$, we write

$$a \leq^\sharp b \quad \text{if } a^\sharp a = a^\sharp b \text{ and } aa^\sharp = ba^\sharp.$$

The order $\leq^\sharp$ is called the *sharp partial order* on $\mathcal{G}(\mathcal{A})$. It was shown in [12] that $\leq^\sharp$ is indeed a partial order on $\mathcal{G}(\mathcal{A})$. Note that the assumption in [12] for the above definition was that element b also has the group inverse (i.e., $a, b \in \mathcal{G}(\mathcal{A})$); however, it is easy to check that the results from [12] which follow are valid also when we assume that $a \in \mathcal{G}(\mathcal{A})$ and $b \in \mathcal{A}$. Namely, some equivalent definitions of the sharp partial order were presented in [12]. For example, for $a \in \mathcal{G}(\mathcal{A})$ and $b \in \mathcal{A}$, we have $a \leq^\sharp b$ if and only if there exists $p \in E(\mathcal{A})$ such that

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p \times p} \qquad \text{and} \qquad b = \begin{bmatrix} a & 0 \\ 0 & b_1 \end{bmatrix}_{p \times p},$$

where $b_1 \in (1-p)\mathcal{A}(1-p)$. So, for a given $a \in \mathcal{G}(\mathcal{A})$, we have $a \leq^\sharp b$ if and only if

$$b = a + (1-p)x(1-p),$$

where $p \in E(\mathcal{A})$ such that ${}^\circ p \subseteq {}^\circ a$ and $p^\circ \subseteq a^\circ$, and x is an element in $\mathcal{A}$. This observation describes the set of all elements $b \in \mathcal{A}$ such that $a \leq^\sharp b$ where $a \in \mathcal{G}(\mathcal{A})$ is given (see also [16, Theorem 3.5]). Let us now prove the following auxiliary result, which is analogous to Theorem 3.5 from [3].

Lemma 3.1. *Let $\mathcal{A}$ be a ring with identity 1, let $a \in \mathcal{G}(\mathcal{A})$, and let $b \in \mathcal{A}$. Suppose that $b = \left[\begin{smallmatrix} b_1 & b_2 \\ 0 & 0 \end{smallmatrix}\right]_{s \times s}$, where $s \in E(\mathcal{A})$. If $a \leq^\sharp b$, then $a = \left[\begin{smallmatrix} pb_1 & pb_2 \\ 0 & 0 \end{smallmatrix}\right]_{s \times s}$, where $p \in s\mathcal{A}\,s \cap E(\mathcal{A})$, $pb_1^2 = b_1pb_1$, and $pb_1b_2 = b_1pb_2$.*

Proof. Suppose that $b = \left[\begin{smallmatrix} b_1 & b_2 \\ 0 & 0 \end{smallmatrix}\right]_{s \times s}$ where $s \in E(\mathcal{A})$, and let $a = \left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right]_{s \times s}$. Since $a \leq^\sharp b$, we may conclude (see Theorem 9 and concluding remarks in [12]) that there exists an idempotent $r \in \mathcal{A}$ such that ${}^\circ r = {}^\circ a$, $r^\circ = a^\circ$, $ra = rb$, and $ar = br$. So, $a(1-r) = 0$, and hence $a = br$. This implies that ${}^\circ b \subseteq {}^\circ a$, and thus that $a_3 = a_4 = 0$ and ${}^\circ b \subseteq {}^\circ r$. It follows that $(1-s)r = 0$, and hence

$$r = \begin{bmatrix} r_1 & r_2 \\ 0 & 0 \end{bmatrix}_{s \times s}.$$

Since $r^2 = r$, we have $r_1^2 = r_1$. Let $r_1 = p$. From ${}^\circ r = {}^\circ a$, we have $a = ra = rb$; thus

$$a = \begin{bmatrix} p & r_2 \\ 0 & 0 \end{bmatrix}_{s \times s} \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s \times s} = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s \times s}.$$

Note that $p \in s\mathcal{A}\,s \cap E(\mathcal{A})$. Since $a \leq^\sharp b$, we also have (see [12]) $ab = ba = a^2$. It follows that

$$\begin{bmatrix} pb_1^2 & pb_1b_2 \\ 0 & 0 \end{bmatrix}_{s \times s} = ab = ba = \begin{bmatrix} b_1pb_1 & b_1pb_2 \\ 0 & 0 \end{bmatrix}_{s \times s},$$

and thus that $pb_1^2 = b_1pb_1$ and $pb_1b_2 = b_1pb_2$. $\square$

We say that an involutory ring $\mathcal{A}$ is *proper* if, for every $x \in \mathcal{A}$ where $xx^* = 0$, we have $x = 0$. Recall that every Rickart $*$-ring is a proper involutory ring (see, e.g., [2]). The proof of the following proposition is short and may be found in [2, p. 10].

Proposition 3.2. *Let $\mathcal{A}$ be a proper involutory ring. Then ${}^\circ x = {}^\circ(xx^*)$ for every $x \in \mathcal{A}$.*

For a Rickart $*$-ring $\mathcal{A}$, we will now characterize the set of all elements $a \in \mathcal{G}(\mathcal{A})$ such that $a \leq^\sharp b$, where $b \in \mathcal{A}$ is given.

Theorem 3.3. *Let $\mathcal{A}$ be a Rickart $*$-ring and let $b \in \mathcal{A}$ be given by (1.4). For $a \in \mathcal{G}(\mathcal{A})$, we have $a \leq^\sharp b$ if and only if $a = \left[\begin{smallmatrix} pb_1 & pb_2 \\ 0 & 0 \end{smallmatrix}\right]_{s\times s}$, where $p \in s\mathcal{A}\,s \cap E(\mathcal{A})$ and $pb_1 = b_1p$.*

Proof. Suppose that $a \in \mathcal{G}(\mathcal{A})$, $b \in \mathcal{A}$, and $b = \left[\begin{smallmatrix} b_1 & b_2 \\ 0 & 0 \end{smallmatrix}\right]_{s\times s}$, where $s = \mathrm{lp}(b)$. Let first $a \leq^\sharp b$. By Lemma 3.1, $a = \left[\begin{smallmatrix} pb_1 & pb_2 \\ 0 & 0 \end{smallmatrix}\right]_{s\times s}$ where $p \in s\mathcal{A}\,s \cap E(\mathcal{A})$, $pb_1^2 = b_1pb_1$, and $pb_1b_2 = b_1pb_2$. Let us now show that $pb_1 = b_1p$. We have $pb_1b_1b_1^* = b_1pb_1b_1^*$ and $pb_1b_2b_2^* = b_1pb_2b_2^*$. Adding these equations, we obtain $pb_1d = b_1pd$ where $d = b_1b_1^* + b_2b_2^*$. So, $(pb_1 - b_1p) \in {}^\circ d$. Note that

$$bb^* = \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix}_{s\times s}$$

(i.e., $d = bb^*$), and recall that $\mathcal{A}$ is a proper involutory ring. Since ${}^\circ b = {}^\circ(\mathrm{lp}(b)) = {}^\circ s$, we may conclude by Proposition 3.2 that ${}^\circ(bb^*) = {}^\circ s$. So, ${}^\circ d = {}^\circ s$, and hence $(pb_1 - b_1p)s = 0$. Recall that $b_1 = sbs$ and that $p \in s\mathcal{A}\,s$. It follows that $(pb_1 - b_1p) \in s\mathcal{A}\,s$, and thus that $0 = (pb_1 - b_1p)s = pb_1 - b_1p$ (i.e., $pb_1 = b_1p$).

Conversely, let $a = \left[\begin{smallmatrix} pb_1 & pb_2 \\ 0 & 0 \end{smallmatrix}\right]_{s\times s}$, where $p \in s\mathcal{A}\,s \cap E(\mathcal{A})$ and $pb_1 = b_1p$. By Theorem 13 in [12], $a \leq^\sharp b$ if and only if $(b-a) \in {}^\circ a \cap a^\circ$. We have

$$\begin{aligned}(b-a)a &= \begin{bmatrix} b_1 - pb_1 & b_2 - pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} \\ &= \begin{bmatrix} (b_1 - pb_1)pb_1 & (b_1 - pb_1)pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s}.\end{aligned}$$

Since $pb_1 = b_1p$, it follows that $(b_1 - pb_1)pb_1 = pb_1^2 - pb_1^2 = 0$ and $(b_1 - pb_1)pb_2 = pb_1b_2 - pb_1b_2 = 0$. This yields $(b-a)a = 0$ (i.e., $(b-a) \in {}^\circ a$). Again, since $pb_1 = b_1p$, we obtain

$$a(b-a) = \begin{bmatrix} pb_1(b_1 - pb_1) & pb_1(b_2 - pb_2) \\ 0 & 0 \end{bmatrix}_{s\times s} = 0,$$

and thus $(b-a) \in a^\circ$. So, $(b-a) \in {}^\circ a \cap a^\circ$, which yields $a \leq^\sharp b$. $\square$

With Theorem 3.3 we generalized a result from [3, Corollary 3.1] where the authors described the set of all operators $A \in \mathcal{G}(\mathcal{B}(H))$ that are under the sharp partial order below a given (relatively) regular operator $B \in \mathcal{B}(H)$. Note that in Theorem 3.3 we omitted the regularity condition, namely, $b \in \mathcal{A}$ was an arbitrary given (not necessarily regular) element in a Rickart $*$-ring $\mathcal{A}$.

4. The core order

Baksalary and Trenkler introduced in [1] a new order $\leq^{\oplus}$, called the *core partial order*, on the set of all $n \times n$ complex matrices M_n which have the group inverse. One of the equivalent definitions is the following (see [11, p. 1638]): For A, $B \in \mathcal{G}(M_n)$, we write

$$A \leq^{\oplus} B \quad \text{if } A^*A = A^*B \text{ and } AA^{\sharp} = BA^{\sharp}. \tag{4.1}$$

We may observe that the core partial order is "between" the star partial order and the sharp partial order. To present an extension of this order from the set $\mathcal{G}(M_n)$ to $\mathcal{G}(\mathcal{B}(H))$ or even to some wider set, let us first recall the following lemma (see [13]) and definition (see [16, p. 5]).

Lemma 4.1 ([13, Lemma 2.4])**.** *Let $\mathcal{A}$ be a Rickart $*$-ring and let $a, b \in \mathcal{A}$. The following are then equivalent:*

(i) $a = \mathrm{lp}(a)b$,
(ii) $a = pb$ *for some self-adjoint element* $p \in \mathcal{A}$,
(iii) $a^*a = a^*b$.

Let $\mathcal{A}$ be an arbitrary ring with identity 1. Then we define

$$\mathcal{I}_{\mathcal{A}} = \{a \in \mathcal{A} : {}^{\circ}a = {}^{\circ}p \text{ and } a^{\circ} = p^{\circ} \text{ for some idempotent } p \in \mathcal{A}\}.$$

For $a \in \mathcal{I}_{\mathcal{A}}$, the idempotent p is unique (see [16]), so we may write $p = p_a$.

Let X be a Banach space and let $\mathcal{B}(X)$ be the set of all bounded linear operators on X. In a special case when $\mathcal{A} = \mathcal{B}(X)$, it turns out (see [16, Theorem 2.6]) that $\mathcal{I}_{\mathcal{B}(X)} = \{A \in \mathcal{B}(X) : \overline{\operatorname{Im} A} \oplus \operatorname{Ker} A = X\}$. Note that if $\dim X < \infty$, then the set $\{A \in \mathcal{B}(X) : \operatorname{Im} A \oplus \operatorname{Ker} A = X\}$ coincides with the set of all group invertible operators in $\mathcal{B}(X)$.

Let $a \in \mathcal{I}_{\mathcal{A}}$ and let $b \in \mathcal{A}$, where $\mathcal{A}$ is a ring with identity. It is not hard to prove (see the proof of Lemma 3.3 in [16]) that then

$$a = b\,p_a \quad \text{if and only if } a^2 = ba. \tag{4.2}$$

Suppose now that $a \in \mathcal{A}$ has the group inverse $a^{\sharp}$, and let $b \in \mathcal{A}$. It is easy to check that $a^2 = ba$ if and only if $aa^{\sharp} = ba^{\sharp}$. By these observations (compare them with (4.1) and Lemma 4.1), the core partial order was extended in [16] from the set $\mathcal{G}(M_n)$ to $\mathcal{I}_{\mathcal{A}}$ where $\mathcal{A}$ is a Rickart $*$-ring. The definition is as follows.

Definition 4.2 ([16, Definition 4.5]). Let $\mathcal{A}$ be a Rickart $*$-ring. For $a, b \in \mathcal{A}$, we write $a \leq^{\oplus} b$ if $a \in \mathcal{I}_{\mathcal{A}}$ and

$$a = \mathrm{lp}(a)b = b\,p_a.$$

It was proved in [16] that the relation $\leq^{\oplus}$ is indeed a partial order on $\mathcal{I}_{\mathcal{A}}$ where $\mathcal{A}$ is a Rickart $*$-ring. Very recently, another generalization of the core partial order has been independently introduced in [3], where the authors generalized this order from $\mathcal{G}(M_n)$ to $\mathcal{B}(H)$—where H is a Hilbert space—in the following way.

Definition 4.3. For $A, B \in \mathcal{B}(H)$, we write $A \leq^{\oplus} B$ if there exists a self-adjoint idempotent operator $P \in \mathcal{B}(H)$ and an idempotent operator $Q \in \mathcal{B}(H)$ such that

(i) $\operatorname{Im} P = \overline{\operatorname{Im} A}$,
(ii) $\operatorname{Im} Q = \overline{\operatorname{Im} A}$, $\operatorname{Ker} Q = \operatorname{Ker} A$,
(iii) $PA = PB$,
(iv) $AQ = BQ$.

Clearly, Definition 4.3 makes sense only for operators belonging to the set $\{A \in \mathcal{B}(H) : \overline{\operatorname{Im} A} \oplus \operatorname{Ker} A = H\}$. Recall that $\{A \in \mathcal{B}(H) : \overline{\operatorname{Im} A} \oplus \operatorname{Ker} A = H\} = \mathcal{I}_{\mathcal{B}(H)}$. The proof of the following lemma is easy and may follow a known technique, and therefore we leave it to the reader (see [13, Lemma 2.1]).

Lemma 4.4. *Definitions 4.2 and 4.3 are equivalent on $\mathcal{I}_{\mathcal{B}(H)}$.*

In [16], the set of all elements $b \in \mathcal{A}$ that are above a given element $a \in \mathcal{I}_{\mathcal{A}}$ under the core partial order where $\mathcal{A}$ is a Rickart $*$-ring was characterized. Namely,

$$a \leq^{\oplus} b \quad \text{iff } b = \begin{bmatrix} a & 0 \\ 0 & b_1 \end{bmatrix}_{\operatorname{lp}(a) \times p_a} \quad \text{iff } b = a + \big(1 - \operatorname{lp}(a)\big)x(1 - p_a)$$

for some $b_1 \in (1 - \operatorname{lp}(a))\mathcal{A}(1 - p_a)$ and $x \in \mathcal{A}$. We will characterize with the next theorem the set of all elements $a \in \mathcal{I}_{\mathcal{A}}$ that are below a given element $b \in \mathcal{A}$. Thus, we will generalize Theorem 3.7 from [3], which describes the set of all operators $A \in \mathcal{G}(\mathcal{B}(H))$ that are below a given operator $B \in \mathcal{G}(\mathcal{B}(H))$ under the core partial order.

Theorem 4.5. *Let $\mathcal{A}$ be a Rickart $*$-ring and let $b \in \mathcal{A}$ be given by (1.4). For $a \in \mathcal{I}_{\mathcal{A}}$, $a \leq^{\oplus} b$ if and only if $a = \left[\begin{smallmatrix} pb_1 & pb_2 \\ 0 & 0 \end{smallmatrix}\right]_{s \times s}$, where $p \in s\mathcal{A}s$, $p = p^* = p^2$, and $pb_1p = b_1p$.*

Proof. Let $b \in \mathcal{A}$ be given as in (1.4). For $a \in \mathcal{I}_{\mathcal{A}}$, suppose that $a \leq^{\oplus} b$. By Definition 4.2, we have $a = \operatorname{lp}(a)b = b\,p_a$, and hence ${}^{\circ}b \subseteq {}^{\circ}a$. It follows that $(1 - s)a = 0$, and hence that $a = \left[\begin{smallmatrix} a_1 & a_2 \\ 0 & 0 \end{smallmatrix}\right]_{s \times s}$. Since $(1 - s) \in {}^{\circ}a = {}^{\circ}(\operatorname{lp}(a))$, we may also conclude that $\operatorname{lp}(a) = \left[\begin{smallmatrix} p_1 & p_2 \\ 0 & 0 \end{smallmatrix}\right]_{s \times s}$. Recall that $\operatorname{lp}(a)$ is a self-adjoint idempotent, so $p_2 = 0$, and hence $\operatorname{lp}(a) = s\operatorname{lp}(a)s$. Let $p = \operatorname{lp}(a)$. Since $a = \operatorname{lp}(a)b$, we have

$$a = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{s \times s} \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s \times s} = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s \times s}.$$

By (4.2), $a = b\,p_a$ implies that $a^2 = ba$. So,

$$\begin{bmatrix} pb_1pb_1 & pb_1pb_2 \\ 0 & 0 \end{bmatrix}_{s \times s} = \begin{bmatrix} b_1pb_1 & b_1pb_2 \\ 0 & 0 \end{bmatrix}_{s \times s},$$

and hence $b_1pb_1 = pb_1pb_1$ and $b_1pb_2 = pb_1pb_2$. It follows that $b_1pb_1b_1^* = pb_1pb_1b_1^*$ and that $b_1pb_2b_2^* = pb_1pb_2b_2^*$, which yields

$$(b_1p - pb_1p)d = 0,$$

where $d = b_1b_1^* + b_2b_2^* = bb^*$. Since $\mathcal{A}$ is a proper involutory ring, ${}^{\circ}b = {}^{\circ}(bb^*)$ by Proposition 3.2. Recall that ${}^{\circ}b = {}^{\circ}s$, and hence that ${}^{\circ}d = {}^{\circ}s$. Therefore, since $(b_1p - pb_1p) \in^{\circ} d$, we have $(b_1p - pb_1p)s = 0$. From $p = sps$, we may conclude that $b_1p - pb_1p = 0$ (i.e., that $b_1p = pb_1p$).

Conversely, for $a \in \mathcal{I}_{\mathcal{A}}$, let $a = \left[\begin{smallmatrix} pb_1 & pb_2 \\ 0 & 0 \end{smallmatrix}\right]_{s\times s}$, where $p \in s\mathcal{A}s$, $p = p^* = p^2$, and $pb_1p = b_1p$. Then

$$a^*a = \begin{bmatrix} b_1^*p & 0 \\ b_2^*p & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} b_1^*pb_1 & b_1^*pb_2 \\ b_2^*pb_1 & b_2^*pb_2 \end{bmatrix}_{s\times s}$$

and

$$a^*b = \begin{bmatrix} b_1^*p & 0 \\ b_2^*p & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} b_1^*pb_1 & b_1^*pb_2 \\ b_2^*pb_1 & b_2^*pb_2 \end{bmatrix}_{s\times s},$$

which yields by Lemma 4.1, $a = \mathrm{lp}(a)b$. Also,

$$a^2 = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} pb_1pb_1 & pb_1pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s}$$

and

$$ba = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} b_1pb_1 & b_1pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s}.$$

Since $pb_1p = b_1p$, we have $pb_1pb_1 = b_1pb_1$ and $pb_1pb_2 = b_1pb_2$. It follows that $a^2 = ba$, and hence by (4.2), that $a = b\,p_a$. So $a = \mathrm{lp}(a)b = b\,p_a$, and thus by Definition 4.2, $a \leq^{\oplus} b$. ☐

Let us conclude this section with another order, the dual core partial order, which was introduced in [1] on the set of all matrices in M_n which have the group inverse and is generalized in [16] to the set $\mathcal{I}_{\mathcal{A}}$, where $\mathcal{A}$ is an arbitrary Rickart $*$-ring. The definition is as follows.

Definition 4.6. Let $\mathcal{A}$ be a Rickart $*$-ring. For $a, b \in \mathcal{A}$, we write $a \leq_{\oplus} b$ if $a \in \mathcal{I}_{\mathcal{A}}$ and

$$a = p_a b = b\,\mathrm{rp}(a).$$

In the same way as this was done for the core partial order, we may describe the set of all elements $b \in \mathcal{A}$ that are above a given element $a \in \mathcal{I}_{\mathcal{A}}$ under the dual core partial order where $\mathcal{A}$ is a Rickart $*$-ring. Namely,

$$a \leq_{\oplus} b \quad \text{iff } b = \begin{bmatrix} a & 0 \\ 0 & b_1 \end{bmatrix}_{p_a \times \mathrm{rp}(a)} \quad \text{iff } b = a + (1-p_a)x\big(1-\mathrm{rp}(a)\big)$$

for some $b_1 \in (1-p_a)\mathcal{A}(1-\mathrm{rp}(a))$ and $x \in \mathcal{A}$.

Note now that in the case when $\mathcal{A}$ is a Rickart $*$-ring, any $b \in \mathcal{A}$ may be represented in the following form:

$$b = \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \end{bmatrix}_{z\times z}, \quad z = \mathrm{rp}(b). \tag{4.3}$$

The next corollary describes the set of all elements $a \in \mathcal{I}_{\mathcal{A}}$ below a given element $b \in \mathcal{A}$ under the dual core partial order $\leq_{\oplus}$. Its proof is similar to that of Theorem 4.5 above; however, it is more elegant to consider a new Rickart $*$-ring $(\mathcal{A}, +, \cdot_L, *)$, where

$$a \cdot_L b := ba \tag{4.4}$$

for $a, b \in \mathcal{A}$. It is then easy to see (see Definitions 4.2 and 4.6) that for every $a, b \in \mathcal{A}$ we have

$$a \leq_{\oplus} b \quad \text{if and only if } a \leq_L^{\oplus} b,$$

where $\leq_L^{\oplus}$ is the core partial order in $(\mathcal{A}, +, \cdot_L, *)$. Corollary 4.7 follows then immediately by (4.4) and Theorem 4.5.

Corollary 4.7. *Let $\mathcal{A}$ be a Rickart $*$-ring and let $b \in \mathcal{A}$ be given by (4.3). For $a \in \mathcal{I}_{\mathcal{A}}$, $a \leq_{\oplus} b$ if and only if $a = \left[\begin{smallmatrix} b_1 p & 0 \\ b_2 p & 0 \end{smallmatrix}\right]_{z \times z}$, where $p \in z\mathcal{A}z$, $p = p^* = p^2$, and $pb_1 p = pb_1$.*

5. The minus order

A regular semigroup is a semigroup in which every element is regular. The minus partial order was originally introduced by Hartwig in [8] in the following way. For a regular semigroup S and for $a, b \in S$, we write

$$a \leq^- b \quad \text{if } a'a = a'b \text{ and } aa' = ba', \tag{5.1}$$

where a' is a reflexive generalized inverse of a (i.e., $a' \in \{x \in S : a = axa, x = xax\}$). Note here that for a regular element $a \in S$ there exists a reflexive generalized inverse a' of a: $a' = a^- a a^-$, where $aa^- a = a$. Šemrl studied in [17] the minus partial order on $\mathcal{B}(H)$. He did not want to restrict himself only to operators in $\mathcal{B}(H)$ that are (relatively) regular, so he introduced a new order on $\mathcal{B}(H)$, proved that it is indeed a partial order on $\mathcal{B}(H)$ for a general Hilbert space H and showed that the new partial order is the same as Hartwig's minus partial order $\leq^-$ when H is finite-dimensional. In [4], Šemrl's definition of the minus partial order was generalized in the following way.

Definition 5.1 ([4, Definition 2.1]). Let $\mathcal{A}$ be a ring with identity 1 and let $a, b \in \mathcal{A}$. Then we write $a \leq^- b$ if there exist idempotent elements $p, q \in \mathcal{A}$ such that ${}^\circ a = \mathcal{A}(1-p)$, $a^\circ = (1-q)\mathcal{A}$, $pa = pb$, and $aq = bq$.

It was proved in [4] that this order $\leq^-$ is indeed a partial order when $\mathcal{A}$ is a Rickart ring. Suppose now for a moment that $\mathcal{A}$ is a ring with identity 1 in which every element is regular (i.e., $\mathcal{A}$ is a von Neumann regular ring). Theorem 2.2 in [4] states that on $\mathcal{A}$, Hartwig's definition (5.1) and Definition 5.1 are equivalent. It turns out that the proof of this theorem may be used to prove the following observation: Let $\mathcal{A}$ be a ring with identity 1. For a regular element $a \in \mathcal{A}$ and any (not necessarily regular) $b \in \mathcal{A}$, we have $a \leq^- b$ in the sense of Definition 5.1 if and only if $a'a = a'b$ and $aa' = ba'$, where a' is some reflexive generalized inverse of a.

In [12], the present author described the set of all elements $b \in \mathcal{A}$ that are above a given element $a \in \mathcal{A}$ under the minus partial order where $\mathcal{A}$ is a Rickart ring. Namely, for a given $a \in \mathcal{A}$,

$$a \leq^- b \quad \text{if and only if } b = \begin{bmatrix} a & 0 \\ 0 & b_1 \end{bmatrix}_{p \times q}$$

for some $b_1 \in (1-p)\mathcal{A}(1-q)$ and $p, q \in E(\mathcal{A})$. The set of all operators in $\mathcal{B}(H)$ that are below a given operator $B \in \mathcal{B}(H)$ was studied in [3]. Authors managed to characterize the set of all (relatively) regular operators $A \in \mathcal{B}(H)$ that are below a given (relatively) regular operator $B \in \mathcal{B}(H)$. It is natural to ask if the result from [3] is valid also for general (not necessarily (relatively) regular) operators in $\mathcal{B}(H)$ or are there some counterexamples. We tried to answer this question in a more general setting of Rickart rings; however, we managed to obtain the following result, which is again limited to the set of regular elements in a Rickart ring $\mathcal{A}$.

Theorem 5.2. *Let $\mathcal{A}$ be a Rickart ring and let $b \in \mathcal{A}$ be given by (1.3). For a regular element $a \in \mathcal{A}$, we have $a \leq^- b$ if and only if there exists a reflexive generalized inverse a' of a such that $ba'b = a$ and $a = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s}$, where $aa' = \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{s\times s}$; that is, $p = saa's$, $p_1 \in s\mathcal{A}(1-s)$, and $(1-s)aa's = 0 = (1-s)aa'(1-s)$.*

Proof. Let $\mathcal{A}$ be a Rickart ring and let $b \in \mathcal{A}$ be given by (1.3). For a regular element $a \in \mathcal{A}$, suppose first that $a \leq^- b$. So, there exists a reflexive generalized inverse a' of a such that $a'a = a'b$ and $aa' = ba'$. Fix such a' and let $\widetilde{p} = aa'$ and $\widetilde{q} = a'a$. We have, $\widetilde{p}^2 = \widetilde{p}$, $\widetilde{q}^2 = \widetilde{q}$, $\widetilde{p}a = \widetilde{p}b$, $a\widetilde{q} = b\widetilde{q}$. From $\widetilde{p} = aa'$ we obtain $a = \widetilde{p}a$ and from $\widetilde{q} = a'a$ it follows that $a = a\widetilde{q}$. Thus, $a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{\widetilde{p}\times\widetilde{q}}$. Let $b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{\widetilde{p}\times\widetilde{q}}$. We have, $b_1 = \widetilde{p}b\widetilde{q} = \widetilde{p}a\widetilde{q} = a$, $b_2 = \widetilde{p}b(1-\widetilde{q}) = \widetilde{p}a(1-\widetilde{q}) = 0$, and similarly $b_3 = 0$. So,

$$b = \begin{bmatrix} a & 0 \\ 0 & b_4 \end{bmatrix}_{\widetilde{p}\times\widetilde{q}}.$$

From $\widetilde{p} = aa'$, we have $a'\widetilde{p} = a'$, and $\widetilde{q} = a'a$ yields $\widetilde{q}a' = a'$. Thus, $a' = \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix}_{\widetilde{q}\times\widetilde{p}}$. It follows that

$$ba'b = \begin{bmatrix} a & 0 \\ 0 & b_4 \end{bmatrix}_{\widetilde{p}\times\widetilde{q}} \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix}_{\widetilde{q}\times\widetilde{p}} \begin{bmatrix} a & 0 \\ 0 & b_4 \end{bmatrix}_{\widetilde{p}\times\widetilde{q}} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{\widetilde{p}\times\widetilde{q}} = a.$$

Let $\widetilde{p} = \begin{bmatrix} p & p_1 \\ p_2 & p_3 \end{bmatrix}_{s\times s}$. Recall that b is given by (1.3), and note that $\widetilde{p} = aa'$ and $a = \widetilde{p}a$ yield ${}^\circ a = {}^\circ\widetilde{p}$, and that $a = b\widetilde{q}$ yields ${}^\circ b \subseteq {}^\circ a$. So, since $(1-s) \in {}^\circ b$, it follows that $p_2 = p_3 = 0$, and therefore $a = \widetilde{p}b$ implies that

$$a = \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s}.$$

Conversely, suppose that there exists a reflexive generalized inverse a' of a such that $ba'b = a$ and $a = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s}$, where $aa' = \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{s\times s}$. Denote $\widetilde{p} = aa'$ and $\widetilde{q} = a'a$. As before, we have $\widetilde{p}a = a$ and $a\widetilde{q} = a$, and hence ${}^\circ a = {}^\circ\widetilde{p}$ and $\widetilde{q}^\circ = a^\circ$. Also, $\widetilde{p}^2 = \widetilde{p}$, $\widetilde{q}^2 = \widetilde{q}$, and

$$\widetilde{p}a = \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{s\times s} \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} = \widetilde{p}b.$$

Let $a' = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{s\times s}$. Since $a' = a'aa'$, we have

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{s\times s} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{s\times s} \begin{bmatrix} p & p_1 \\ 0 & 0 \end{bmatrix}_{s\times s},$$

and thus $c_1 = c_1 p$ and $c_3 = c_3 p$. It follows that

$$\widetilde{q} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{s\times s} \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s} = \begin{bmatrix} c_1pb_1 & c_1pb_2 \\ c_3pb_1 & c_3pb_2 \end{bmatrix}_{s\times s}$$
$$= \begin{bmatrix} c_1b_1 & c_1b_2 \\ c_3b_1 & c_3b_2 \end{bmatrix}_{s\times s} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{s\times s} \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_{s\times s} = a'b.$$

We have $b\widetilde{q} = ba'b = a = a\widetilde{q}$. By Definition 5.1, together with Proposition 1.2, we may conclude that $a \leq^- b$. □

The following observation is valid for general Rickart rings.

Proposition 5.3. *Let $\mathcal{A}$ be a Rickart ring and let $b \in \mathcal{A}$ be given by (1.3). If $a \leq^- b$, $a \in \mathcal{A}$, then there exists an idempotent $p \in s\mathcal{A}s$ such that*

$$a = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s}.$$

We will omit the proof of this proposition since it is similar to (parts of) proofs of some results from this paper (for example, see the first part of the proof of Theorem 2.1). Let us conclude the paper with the following conjecture (cf. Theorem 3.2 in [3]).

Open question. For a Rickart ring $\mathcal{A}$, let $b \in \mathcal{A}$ be given by (1.3). For $a \in \mathcal{A}$, we have $a \leq^- b$ if and only if there exists an idempotent $p \in s\mathcal{A}s$ such that

$$a = \begin{bmatrix} pb_1 & pb_2 \\ 0 & 0 \end{bmatrix}_{s\times s}.$$

We leave the reader with the question whether this conjecture holds (at least on the set of regular elements in $\mathcal{A}$), or perhaps there are some counterexamples.

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References

1. O. M. Baksalary and G. Trenkler, *Core inverse of matrices*, Linear Multilinear Algebra **58** (2010), no. 6, 681–697. Zbl 1202.15009. MR2722752. DOI 10.1080/03081080902778222. 502, 504
2. S. K. Berberian, *Baer *-rings*, Grundlehren Math. Wiss. **195**, Springer, New York, 1972. MR0429975. 496, 501
3. D. S. Cvetković-Ilić, D. Mosić, and Y. Wei, *Partial orders on $\mathcal{B}(H)$*, Linear Algebra Appl. **481** (2015), 115–130. MR3349648. DOI 10.1016/j.laa.2015.04.025. 495, 496, 499, 500, 501, 502, 503, 505, 506, 507
4. D. S. Djordjević, D. S. Rakić, and J. Marovt, *Minus partial order in Rickart rings*, Publ. Math. Debrecen **87** (2015), no. 3–4, 291–305. MR3411994. DOI 10.5486/PMD.2015.7161. 497, 505
5. G. Dolinar and J. Marovt, *Star partial order on $\mathcal{B}(H)$*, Linear Algebra Appl. **434** (2011), no. 1, 319–326. Zbl 1214.47006. MR2737252. DOI 10.1016/j.laa.2010.08.023. 496
6. M. P. Drazin, *Natural structures on semigroups with involution*, Bull. Amer. Math. Soc. **84** (1978), no. 1, 139–141. Zbl 0395.20044. MR0486234. 495
7. J. Groß, *Remarks on the sharp partial order and the ordering of squares of matrices*, Linear Algebra Appl. **417** (2006), no. 1, 87–93. MR2238597. DOI 10.1016/j.laa.2005.10.036. 499

8. R. E. Hartwig, *How to partially order regular elements*, Sci. Math. Jpn. **25** (1980), no. 1, 1–13. Zbl 0442.06006. MR0571255. 505
9. R. E. Hartwig and J. Luh, *A note on the group structure on unit regular ring elements*, Pacific J. Math. **71** (1977), no. 2, 449–461. Zbl 0355.16005. MR0442018. 499
10. I. Kaplansky, *Rings of Operators*, Benjamin, New York, 1968. MR0244778. 496
11. S. B. Malik, L. Rueda, and N. Thome, *Further properties on the core partial order and other matrix partial orders*, Linear Multilinear Algebra **62** (2014), no. 12, 1629–1648. Zbl 1306.15030. MR3265626. DOI 10.1080/03081087.2013.839676. 502
12. J. Marovt, *On partial orders in Rickart rings*, Linear Multilinear Algebra **63** (2015), no. 9, 1707–1723. Zbl 1326.06017. MR3305004. DOI 10.1080/03081087.2014.972314. 500, 501, 505
13. J. Marovt, D. S. Rakić, and D. S. Djordjević, *Star, left-star, and right-star partial orders in Rickart $*$-rings*, Linear Multilinear Algebra **63** (2015), no. 2, 343–365. MR3273759. DOI 10.1080/03081087.2013.866670. 498, 502, 503
14. S. K. Mitra, *On group inverses and the sharp order*, Linear Algebra Appl. **92** (1987), 17–37. Zbl 0619.15006. MR0894635. DOI 10.1016/0024-3795(87)90248-5. 499
15. D. S. Rakić, *Decomposition of a ring induced by minus partial order*, Electron. J. Linear Algebra **23** (2012), 1040–1059. MR3007204. DOI 10.13001/1081-3810.1573. 498
16. D. S. Rakić, *Generalization of sharp and core partial order using annihilators*, Banach J. Math. Anal. **9** (2015), no. 3, 228–242. MR3296136. DOI 10.15352/bjma/09-3-16. 500, 502, 503, 504
17. P. Šemrl, *Automorphisms of $\mathcal{B}(H)$ with respect to minus partial order*, J. Math. Anal. Appl. **369** (2010), no. 1, 205–213. Zbl 1195.47027. MR2643859. DOI 10.1016/j.jmaa.2010.02.059. 505

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