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EMBEDDED CONSTANT MEAN CURVATURE HYPERSURFACES ON SPHERES*

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Abstract. Let $m \geq 2$ and $n \geq 2$ be any pair of integers. In this paper we prove that if H lies between $\cot(\frac{\pi}{m})$ and $b_{m,n} = \frac{(m^2-2)\sqrt{n-1}}{n\sqrt{m^2-1}}$, there exists a non isoparametric, compact embedded hypersurface in S^{n+1} with constant mean curvature H that admits $O(n) \times Z_m$ in its group of isometries. These hypersurfaces therefore have exactly 2 principal curvatures. When $m = 2$ and H is close to the boundary value $0 = \cot(\frac{\pi}{2})$, such a hypersurface looks like two very close n-dimensional spheres with two catenoid necks attached, similar to constructions made by Kapouleas. When $m > 2$ and H is close to $\cot(\frac{\pi}{m})$, it looks like a necklace made out of m spheres with $m+1$ catenoid necks attached, similar to constructions made by Butscher and Pacard. In general, when H is close to $b_{m,n}$ the hypersurface is close to an isoparametric hypersurface with the same mean curvature. For hyperbolic spaces we prove that every $H \geq 0$ can be realized as the mean curvature of an embedded CMC hypersurface in H^{n+1}. Moreover we prove that when $H > 1$ this hypersurface admits $O(n) \times Z$ in its group of isometries. As a corollary of the properties we prove for these hypersurfaces, we construct, for any $n \geq 6$, non-isoparametric compact minimal hypersurfaces in S^{n+1} whose cones in $\mathbf{R}^{n+2}$ are stable. Also, we prove that the stability index of every non-isoparametric minimal hypersurface with two principal curvatures in S^{n+1} exceeds $n+3$.

Key words. Constant mean curvature, embedded, principal curvatures.

AMS subject classifications. 53C42, 53A10

1. Introduction. Minimal hypersurfaces of spheres that have exactly two principal curvatures at each point were initially studied by Otsuki in [14]. He reduced the problem of classifying them, to that of solving an ODE, and the problem of deciding about their compactness, to the problem of studying an integral that relates periods of two functions involved in the immersions that he found. For surfaces in $\mathbf{R}^3$, Delaunay in 1841 [6] showed that if one rolls a conic section on a line in a plane and then rotates about that line the trace of a focus, one obtains a CMC surface of revolution. CMC stands for constant mean curvature. This rolling construction was generalized for the case of CMC hypersurfaces in $\mathbf{R}^{n+1}$ by Hsiang and Yu in the early eighties [10], [11] and for CMC hypersurfaces in the hyperbolic space and the sphere by Sterling in 1987 [21].

After Oksuki's paper in 1970, several properties for a CMC hypersurface $M \subset S^{n+1}$ with exactly two principal curvatures were proved in [7], [4], [17], [22], [9], [1], [2], [23], [24], [25], and [13] among others.

For the case $n = 2$, we give explicit trigonometric formulas for immersions of CMC hypersurfaces in S^3. A gallery of pictures of the stereographic projection of some of these surfaces, made by Schmitt, can be found in the GANG (Geometry Analysis Numerics Graphics, University of Massachusetts) web page. These surfaces are called unduloidal tori in S^3 with m-lobes because all of them have Z_m, for some m, in their group of symmetries. In this paper we will prove that this symmetry property holds in every dimension and we will also prove that for every positive integer m, if H lies between

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$$a_{n,m} = \cot \frac{\pi}{m} \quad \text{and} \quad b_{n,m} = \frac{(m^2-2)\sqrt{n-1}}{n\sqrt{m^2-1}}$$

then, there exists an embedded non-isoparametric unduloidal n-dimensional hypersurface in S^{n+1} with m-lobes and constant mean curvature H. Some previous results on the problem of determining which values of H can be realized as mean curvatures of CMC embedded hypersurfaces on n dimensional spheres were found by Otsuki [15] and Furuya [8]. They showed that $H = 0$ can not be the mean curvature of a non isoparametric minimal embedded hypersurface in S^{n+1} with two principal curvatures. Later on, Leite and Brito in [3], showed that small positive values of H can be realized as non isoparametric values for CMC hypersurfaces in S^{n+1} with two principal curvatures. This result by Leiti and Brito can be considered as the first step toward the solution of the problem considered in our paper: Given $m \geq 2$, say exactly which values of H allow to *embed* a compact hypersurface with mean curvature H, and $O(n-2) \times Z_m$ symmetry into S^n. I would like to point out that a big part of our paper is the understanding of a formula given by an integral. In the particular case when $H = 0$, this integral was studied by Furuya in 1971 and by Otsuki in 1972 and, in the general case, this integral was studied by Brito and Leite in 1990.

Lemma (4.1) and its corollary (4.2) play an important role in the main result and they are responsible for the explicit bounds $a_{n,m}$ and $b_{n,m}$ for H given above. The Lemma and techniques developed in this paper can be used to obtain similar results for hypersurfaces with two principal curvatures with generalized mean curvature H_k constant were H_k is the H^{th} symmetric function of the principal curvatures. A few weeks after the results of this paper were posted on the ArXiv, Cheng, Li and Wei obtained similar result for hypersurfaces with constant fourth mean curvature H_4 on spheres [5]. Similar results were also obtained by the author for hypersurfaces on space forms [18] and [19].

Since the formulas obtained for the CMC immersions of the sphere are very explicit, it is not difficult to generalize them to obtain similar results in Euclidean spaces and hyperbolic spaces. See sections (6.2) and (6.1).

As a consequence of the symmetries proven for all compact constant mean curvatures in S^{n+1} with two principal curvatures everywhere, we proved that all such examples with $H = 0$ have stability index greater than $n+3$. There is a conjecture stating that the only minimal hypersurfaces in S^{n+1} with stability index $n+3$ are the isoparametric ones with two principal curvatures. Some partial results for this conjecture were proven in [16]. Also, since it is not difficult to prove that the square of the norm of the second fundamental form of these examples can be chosen to be as close as we want from those of the isoparametric examples, we point out that some of Otsuki's minimal hypersurfaces produce examples of non isoparametric compact stable minimal truncated cones in $\mathbf{R}^{n+2}$ for $n \geq 6$. Recall that these examples are not embedded. Stable embedded minimal cones in $\mathbf{R}^{n+2}$ for some values n were constructed by Hsiang and Sterling in [12].

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2. Preliminaries. Let M be an n-dimensional hypersurface of the $(n+1)$-dimensional unit sphere $S^{n+1} \subset \mathbf{R}^{n+2}$. Let $\nu : M \to S^{n+1}$ be a Gauss map and

$A_p : T_pM \to T_pM$ the shape operator. Notice that

$$A_p(v) = -\bar{\nabla}_v \nu \quad \text{for all} \quad v \in T_pM$$

where $\bar{\nabla}$ is the Euclidean connection in $\mathbf{R}^{n+2}$. We will denote by $||A||^2$ the square of the norm of the shape operator.

If X, Y and Z are vector fields on M, $\nabla_X Y$ represents the Levi-Civita connection on M with respect to the metric induced by S^{n+1} and $[X,Y] = \nabla_X Y - \nabla_Y X$ represents the Lie bracket, then the curvature tensor on M is defined by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \tag{2.1}$$

and the covariant derivative of A is defined by

$$DA(X,Y,Z) = Z\langle A(X), Y\rangle - \langle A(\nabla_Z X), Y\rangle - \langle A(X), \nabla_Z Y\rangle \tag{2.2}$$

the Gauss equation is given by

$$R(X,Y)Z = \langle X,Z\rangle Y - \langle Y,Z\rangle X + \langle A(X),Z\rangle A(Y) - \langle A(Y),Z\rangle A(X) \tag{2.3}$$

and the Codazzi equations are given by

$$DA(X,Y,Z) = DA(Z,Y,X). \tag{2.4}$$

Let us denote by $\kappa_1, \ldots, \kappa_n$ the principal curvatures of M and, by $H = \dfrac{\kappa_1 + \cdots + \kappa_n}{n}$ the mean curvature of M. We will assume that M has exactly two principal curvatures everywhere and that H is a constant function on M. Since it is known that M has to be isoparametric in the case that the multiplicities of both principal curvatures are greater than 1, [14], we will assume that

$$\kappa_1 = \cdots = \kappa_{n-1} = \lambda, \quad \kappa_n = \mu \quad \text{and} \quad (n-1)\lambda + \mu = nH.$$

By changing ν by $-\nu$ if necessary we can assume without loss of generality that $\lambda - \mu > 0$. Recall that this hypersurface does not have umbilical points because we are assuming it has exactly two principal curvatures everywhere. Let $\{e_1, \ldots, e_n\}$ denote a locally defined orthonormal frame such that

$$A(e_i) = \lambda e_i \;\text{ for }\; i = 1, \ldots, n-1 \;\text{ and }\; A(e_n) = \mu e_n. \tag{2.5}$$

The next Theorem is well known [14]. For completeness sake and partly to prepare for the deduction of other formulas, we give a proof here.

Theorem 2.1. *If $M \subset S^{n+1}$ is a CMC hypersurface with two principal curvatures and dimension greater than 2, and $\{e_1, \ldots, e_n\}$ is a locally defined orthonormal frame such that (2.5) holds, then*

$$
\begin{aligned}
v(\lambda) &= 0 \quad \textit{for any} \quad v \in \textit{Span}\{e_1, \dots, e_{n-1}\} \\
\nabla_v e_n &= \frac{e_n(\lambda)}{\mu - \lambda} v \quad \textit{for any} \quad v \in \textit{Span}\{e_1, \dots, e_{n-1}\} \\
\nabla_{e_n} e_n &= 0 \\
1 + \lambda\mu &= e_n\Big(\frac{e_n(\lambda)}{\lambda - \mu}\Big) - \Big(\frac{e_n(\lambda)}{\lambda - \mu}\Big)^2
\end{aligned}
$$

$$
[e_i, e_j] \in \textit{Span}\{e_1, \dots, e_{n-1}\} \quad \textit{for any} \quad i, j \in \{1, \dots, n-1\}.
$$

Proof. For any $i, j \in \{1, \dots, n-1\}$ with $i \neq j$ (here we are using the fact that the dimension of M is greater than 2) and any $k \in \{1, \dots, n\}$, we have that,

$$
\begin{aligned}
DA(e_i, e_j, e_k) &= e_k\langle\, A(e_i), e_j \,\rangle - \langle\, A(\nabla_{e_k} e_i), e_j \,\rangle - \langle\, A(e_i), \nabla_{e_k} e_j \,\rangle \\
&= e_k(\lambda\langle\, e_i, e_j \,\rangle) - \langle\, \nabla_{e_k} e_i, A(e_j) \,\rangle - \lambda\langle\, e_i, \nabla_{e_k} e_j \,\rangle \\
&= e_k(0) - \lambda\langle\, \nabla_{e_k} e_i, e_j \,\rangle - \lambda\langle\, e_i, \nabla_{e_k} e_j \,\rangle \\
&= 0 - \lambda e_k(\langle\, e_i, e_j \,\rangle) \\
&= 0.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
DA(e_i, e_i, e_j) &= e_j\langle\, A(e_i), e_i \,\rangle - \langle\, A(\nabla_{e_j} e_i), e_i \,\rangle - \langle\, A(e_i), \nabla_{e_j} e_i \,\rangle \\
&= e_j(\lambda) - \lambda e_j(\langle\, e_i, e_i \,\rangle) \\
&= e_j(\lambda).
\end{aligned}
$$

By the Codazzi equation (2.4), we now get $e_j(\lambda) = 0$, for all $j \in \{1, \dots, n-1\}$, and therefore $v(\lambda) = 0$ for any $v \in \mathrm{Span}\{e_1, \dots, e_{n-1}\}$. Now,

$$
\begin{aligned}
DA(e_i, e_n, e_j) &= e_j\langle\, A(e_i), e_n \,\rangle - \langle\, A(\nabla_{e_j} e_i), e_n \,\rangle - \langle\, A(e_i), \nabla_{e_j} e_n \,\rangle \\
&= e_j(\lambda\langle\, e_i, e_n \,\rangle) - \langle\, \nabla_{e_j} e_i, A(e_n) \,\rangle - \lambda\langle\, e_i, \nabla_{e_j} e_n \,\rangle \\
&= e_j(0) - \mu\langle\, \nabla_{e_j} e_i, e_n \,\rangle - \lambda\langle\, e_i, \nabla_{e_j} e_n \,\rangle + (\lambda\langle\, \nabla_{e_j} e_i, e_n \,\rangle - \lambda\langle\, \nabla_{e_j} e_i, e_n \,\rangle) \\
&= (\lambda - \mu)\langle\, \nabla_{e_j} e_i, e_n \,\rangle - \lambda\ e_j(\langle\, e_i, e_n \,\rangle) \\
&= (\mu - \lambda)\langle\, e_i, \nabla_{e_j} e_n \,\rangle.
\end{aligned}
$$

Since $\mu - \lambda > 0$, using the Codazzi equations we get

$$
\langle\, e_i, \nabla_{e_j} e_n \,\rangle = 0 \quad \text{for any} \quad i, j \in \{1, \dots, n-1\} \quad \text{with} \quad i \neq j. \tag{2.6}
$$

Now, for any $i \in \{1, \dots, n-1\}$, using computations like those above we can prove

$$
DA(e_i, e_i, e_n) = e_n(\lambda) \;=\; DA(e_i, e_n, e_i) = (\mu - \lambda)\langle\, e_i, \nabla_{e_i} e_n \,\rangle
$$

and

$$DA(e_n,e_n,e_i)=e_i(\mu)\;=\;0\;=\;DA(e_i,e_n,e_n)=(\mu-\lambda)\langle\, e_i,\nabla_{e_n}e_n\,\rangle.$$

Therefore,

$$\langle\, e_i,\nabla_{e_i}e_n\,\rangle=\frac{e_n(\lambda)}{\mu-\lambda}\quad\text{and}\quad\langle\, e_i,\nabla_{e_n}e_n\,\rangle=0\quad\text{for any}\quad i\in\{1,\dots,n-1\}. \tag{2.7}$$

Since e_n is a unit vector field, we have that $\langle\,\nabla_{e_k}e_n,e_n\,\rangle=0$ for any k. From the equations (2.6) and (2.7) we conclude that

$$\nabla_v e_n=\frac{e_n(\lambda)}{\mu-\lambda}\,v\quad\text{for any}\quad v\in\operatorname{Span}\{e_1,\dots,e_{n-1}\}\quad\text{and}\quad\nabla_{e_n}e_n=0.$$

Noticing that for any $i,j\in\{1,\dots,n-1\}$ with $i\neq j$, using equation (2.6), we see that

$$\langle\,[e_i,e_j],e_n\,\rangle=\langle\,\nabla_{e_i}e_j-\nabla_{e_j}e_i,e_n\,\rangle=\langle\, e_i,\nabla_{e_j}e_n\,\rangle-\langle\, e_j,\nabla_{e_i}e_n\,\rangle=0.$$

Therefore $[e_i,e_j]\in\operatorname{Span}\{e_1,\dots,e_{n-1}\}$. Finally we will use Gauss equation to prove the differential equation on λ. First we point out that, using equation (2.7), we can prove $\langle\,[e_n,e_1],e_n\,\rangle=0$ and therefore $[e_n,e_1]\in\operatorname{Span}\{e_1,\dots,e_{n-1}\}$. By the Gauss equation we then get,

$$\begin{aligned}
1+\lambda\mu&=\langle\, R(e_n,e_1)e_n,e_1\,\rangle\\
&=\langle\,\nabla_{e_1}\nabla_{e_n}e_n-\nabla_{e_n}\nabla_{e_1}e_n+\nabla_{[e_n,e_1]}e_n,e_1\,\rangle\\
&=\langle\,0-\nabla_{e_n}(\frac{e_n(\lambda)}{\mu-\lambda}\,e_1)+\frac{e_n(\lambda)}{\mu-\lambda}\,[e_n,e_1]\,,e_1\rangle\\
&=-e_n(\frac{e_n(\lambda)}{\mu-\lambda})+\frac{e_n(\lambda)}{\mu-\lambda}\langle\,\nabla_{e_n}e_1-\nabla_{e_1}e_n,e_1\,\rangle\\
&=-e_n(\frac{e_n(\lambda)}{\mu-\lambda})-(\frac{e_n(\lambda)}{\mu-\lambda})^2\\
&=e_n(\frac{e_n(\lambda)}{\lambda-\mu})-(\frac{e_n(\lambda)}{\lambda-\mu})^2.
\end{aligned}$$

□

3. Construction of the examples. Maintaining the notation of the previous section, we now prove a series of identities and results that make it easier to state and prove the theorem that defines the examples at the end of this section.

3.1. The function w and its solution along a line of curvature. Since $(n-1)\lambda+\mu=nH$, we have

$$\lambda-\mu=\lambda-(nH-(n-1)\lambda)=n(\lambda-H)=nw^{-n}\quad\text{where}\quad w=(\lambda-H)^{-\frac{1}{n}}. \tag{3.1}$$

Recall that we are assuming that $\lambda-\mu$ is always positive, so w is a smooth differentiable function. By the definition of w in (3.1) we have

$$e_n(w) = -\frac{1}{n}(\lambda - H)^{-\frac{n+1}{n}} e_n(\lambda) = -\frac{1}{n} w^{n+1}\, e_n(\lambda) = -w\frac{e_n(\lambda)}{\lambda - \mu}. \tag{3.2}$$

Using w, the second order differential equation in Theorem (2.1) can be written as

$$e_n\big(\,\frac{e_n(w)}{w}\,\big) + \big(\,\frac{e_n(w)}{w}\,\big)^2 + 1 + \lambda\mu = 0 \tag{3.3}$$

and if we write λ and μ in terms of w we get

$$e_n\big(\,\frac{e_n(w)}{w}\,\big) + \big(\,\frac{e_n(w)}{w}\,\big)^2 - \frac{(n-1)}{w^{2n}} - \frac{(n-2)H}{w^n} + H^2 + 1 = 0. \tag{3.4}$$

Deriving the previous equation, we have used the following identities,

$$\lambda = w^{-n} + H \quad \text{and} \quad \mu = H - (n-1)w^{-n}. \tag{3.5}$$

From Equation (3.2) we now get

$$e_n(\lambda) = -(\lambda - \mu)\,\frac{e_n(w)}{w}. \tag{3.6}$$

This allows us to write one of the equations in Theorem (2.1) as

$$\bar\nabla_v e_n = \frac{e_n(w)}{w}\, v \quad \text{for any} \quad v \in \mathrm{Span}\{e_1, \dots, e_{n-1}\}. \tag{3.7}$$

Notice that equation (3.4) reduces to

$$\frac{e_n(e_n(w))}{w} - \frac{(n-1)}{w^{2n}} - \frac{(n-2)H}{w^n} + H^2 + 1 = 0 \tag{3.8}$$

and therefore multiplying by $2we_n(w)$ we see that there exists a constant C such that,

$$(e_n(w))^2 + w^{2-2n} + (1+H^2)w^2 + 2Hw^{2-n} = C. \tag{3.9}$$

The equation above plays an important role in the constructions of immersions with CMC in S^n and it was also proven by Wei in [24]. Let $x : M \to \mathbf{R}^{n+2}$ denote the position vector, viewed as a map, and by $\bar\nabla$ the Euclidean connection on $\mathbf{R}^{n+1}$. Using the equations in Theorem (2.1) and the fact that $\bar\nabla_v x = v$, $\langle\, x, \nu(x)\,\rangle = 0$ and $\langle\, \nu(x), \nu(x)\,\rangle = 1$, we get that

$$\bar\nabla_{e_n} e_n = -x + \mu\nu \tag{3.10}$$
$$\bar\nabla_{e_n} \nu = -\mu e_n \tag{3.11}$$
$$\bar\nabla_{e_n} x = e_n. \tag{3.12}$$

Fix a point $p_0 \in M$, and let us denote by $\gamma(u)$ the only geodesic in M such that $\gamma(0) = p_0$ and $\gamma'(0) = e_n(p_0)$. Since $\nabla_{e_n} e_n$ vanishes, then $\gamma(u) = e_n(\gamma(u))$. Notice that $\gamma(u)$ is also a line of curvature. Let $g(u) = w(\gamma(u))$. Equation (3.9) implies that

$$(g')^2 + g^{2-2n} + (1+H^2)g^2 + 2Hg^{2-n} = C \tag{3.13}$$

or equivalently,

$$\frac{g^{n-1}\, g'}{\sqrt{Cg^{2n-2} - 1 - (1+H^2)g^{2n} - 2Hg^n}} = \pm 1. \tag{3.14}$$

It is clear that the constant C must be positive and moreover, in order to solve this equation we need to consider a constant C such that the polynomial

$$\xi(s) = Cs^{2n-2} - 1 - (1+H^2)s^{2n} - 2Hs^n \tag{3.15}$$

is positive on a interval (t_1, t_2) with $0 < t_1 < t_2$ and $\xi(t_1) = 0 = \xi(t_2)$. Notice that for every H we may pick a C such that ξ is positive on an interval because ξ is a polynomial of even degree with negative leading coefficient, $\xi(0) = -1$, and if C is big enough, this polynomial takes positive values for positive values of s. Let us assume that t_1 and t_2 are as above and also that $\xi'(t_1)$ and $\xi'(t_2)$ are not zero, so that the following formula for G is well defined on $[t_1, t_2]$:

$$G(s) = \int_{t_1}^{s} \frac{t^{n-1}}{\sqrt{Ct^{2n-2} - 1 - (1+H^2)t^{2n} - 2Ht^n}} dt \quad \text{for} \quad t_1 \leq s \leq t_2.$$

Let $T = 2G(t_2)$. Since $G'(s) > 0$ for $s \in (t_1, t_2)$, G has an inverse. Denoting it by $F : [0, \frac{T}{2}] \to [t_1, t_2]$, a direct verification shows that the T-periodic function given by

$$g(u) = F(u) \quad \text{for} \quad 0 \leq u \leq \frac{T}{2} \quad \text{and} \quad g(u) = F(T-u) \quad \text{for} \quad \frac{T}{2} \leq u \leq T$$

solves equation (3.14).

3.2. The vector field η **.** Now define the following vector field along M

$$\eta = -\frac{e_n(w)}{w}\, e_n + \lambda\, \nu - x.$$

It has the following properties

1. $\langle \eta, \eta \rangle = (\frac{e_n(w)}{w})^2 + \lambda^2 + 1 = \frac{C}{w^2}$, which follows from Equation (3.9) and the definition of λ in terms of w, (3.5).
2. $\bar{\nabla}_{e_n} \eta = -\frac{e_n(w)}{w} \eta$. This crucial fact makes all the constructions work in this section. The equation follows from Equations (3.10), (3.11) and (3.12) and the first and second differential equations for the function w, especially, Equation (3.3) and Equation (3.6).
3. For any $i \in \{1, \ldots, n-1\}$, $\bar{\nabla}_{e_i}(x + \frac{w^2}{C}\, \eta)$ vanishes. The proof of this identity is similar, and additionally, uses the Equation (3.7).
4. $\langle x + \frac{w^2}{C}\, \eta, x + \frac{w^2}{C}\, \eta \rangle = 1 - \frac{w^2}{C}$.

3.3. Vector fields that lie on a plane . Now that we have computed $g(u) = w(\gamma(u))$, we can better understand the geodesic γ. The equations (3.10), (3.11) and (3.12) imply that

$$X(u) = e_n(\gamma(u)), \quad Y(u) = \nu(\gamma(u)) \quad \text{and} \quad Z(u) = \gamma(u)$$

satisfy an ordinary linear differential equation in the variable u with periodic coefficients (notice that $\mu(\gamma(u))$ is a function of $g(u)$). By the existence and uniqueness theorem of ordinary differential equations, the solutions $X(u)$, $Y(u)$ and $Z(u)$ must lie in the three dimensional space

$$\Gamma_{p_0} = \text{Span}\{e_n(p_0), \nu(p_0), p_0\}. \tag{3.16}$$

For the sake of simplicity, we will consider the T-periodic function $r : \mathbf{R} \to \mathbf{R}$ defined by

$$r(u) = \frac{g(u)}{\sqrt{C}}.$$

It is not difficult to check that r satisfies the equations

$$\frac{r''}{r} + 1 + \lambda\mu = 0, \quad (r')^2 + r^2\,(1+\lambda^2) = 1, \quad \lambda' = -(\lambda-\mu)\frac{r'}{r}. \tag{3.17}$$

In the previous equations we are abusing notation with the name of the functions λ and μ. Here and whenever the context dictates it, they will also denote the functions $\lambda(\gamma(u))$ and $\mu(\gamma(u))$ respectively. To construct our examples, the function $1-r^2$ needs to be positive. We can achieve this by assuming $H \geq 0$ because that will imply $\lambda > 0$, and therefore $r < 1$.

Define the following vector fields along γ

$$\begin{aligned}
B_1(u) &= \eta(\gamma(u)) = -\frac{r'}{r}\,X + \lambda\,Y - Z \\
B_2(u) &= -\frac{rr'}{\sqrt{1-r^2}}\,X + \frac{r^2\lambda}{\sqrt{1-r^2}}\,Y + \sqrt{1-r^2}Z \\
B_3(u) &= \frac{r\lambda}{\sqrt{1-r^2}}\,X + \frac{r'}{\sqrt{1-r^2}}\,Y.
\end{aligned}$$

Using the equations in section (3.2), Equations (3.10), (3.11) and (3.12) giving the derivative of the vector fields X, Y and Z, and Equation (3.17), we can check the following properties.

1. $B_1(u)$, $B_2(u)$ and $B_3(u)$ lie on the three dimensional subspace Γ_{p_0}.

2. $$B_1' = -\frac{r'}{r}B_1.$$

3. $$\langle B_1, B_2\rangle = 0, \quad \langle B_1, B_3\rangle = 0 \quad \text{and} \quad \langle B_2, B_3\rangle = 0.$$

4.
$$\langle B_2, B_2\rangle = 1, \quad \langle B_3, B_3\rangle = 1 \quad \text{and} \quad \langle B_1, B_1\rangle = \frac{1}{r^2}.$$

5. From the previous items we get that
$$B_2' = hB_3 \quad \text{and} \quad B_3' = -hB_2 \quad \text{for some function} \quad h : \mathbf{R} \to \mathbf{R}.$$
These equations hold because
$$\langle B_2', B_1\rangle = -\langle B_1', B_2\rangle = \frac{r'}{r}\langle B_1, B_2\rangle = 0 \quad \text{likewise} \quad \langle B_3', B_1\rangle = 0.$$

6. From the previous item we get that the vectors B_2 and B_3 lie in a two dimensional subspace.

7. We have
$$\langle B_3', Z\rangle = -\frac{r\lambda}{\sqrt{1-r^2}} \quad \text{and} \quad \langle B_2, Z\rangle = \sqrt{1-r^2}.$$
Therefore the function h in the previous item is given by $\frac{r\lambda}{1-r^2}$. It follows that,
$$B_2' = \frac{r\lambda}{1-r^2}B_3 \quad \text{and} \quad B_3' = -\frac{r\lambda}{1-r^2}B_2.$$
The fact that h does not change sign when $\lambda > 0$, in particular when $H \geq 0$, will help us prove that for some choices of C the hypersurface M is embedded.

8. If we assume without loss of generality that
$$\frac{1}{|B_1(0)|}B_1(0) = (0,\dots,1,0,0), \quad B_2(0) = (0,\dots,0,1,0)$$
$$\text{and} \quad B_3(0) = (0,\dots,0,0,1)$$
then,
$$\begin{aligned} B_1(u) &= \frac{1}{r}(0,\dots 0,1,0,0)\\ B_2(u) &= \sin(\theta(u))(0,\dots 0,0,1) + \cos(\theta(u))(0,\dots,0,1,0)\\ B_3(u) &= \cos(\theta(u))(0,\dots 0,0,1) - \sin(\theta(u))(0,\dots,0,1,0)\end{aligned}$$
where $\theta : \mathbf{R} \to \mathbf{R}$ is given by
$$\theta(u) = \int_0^u \frac{r(s)\lambda(s)}{1-r^2(s)}ds.$$

9. If
$$K = K(H,n,C) = \theta(T) = \int_0^T \frac{r(s)\lambda(s)}{1-r^2(s)}ds = 2\int_0^{\frac{T}{2}} \frac{r(s)\lambda(s)}{1-r^2(s)}ds$$
then, for any positive integer m and any $u \in [mT,(m+1)T]$ we have
$$\theta(u) = mK + \theta(u-mT).$$
This property is a consequence of the existence and uniqueness theorem for differential equation and will be used to prove the invariance of M under some rotations.

10. If $q(u)=\gamma(u)+r^2(u)\eta(\gamma(u))$, then

$$\langle q,q\rangle = 1-r^2 \quad \text{and} \quad B_2=\frac{q}{|q|} \quad \text{i.e} \quad q=\sqrt{1-r^2}\,B_2.$$

3.4. A classification of constant mean curvature hypersurfaces in spheres with two principal curvatures . We are ready to define the examples of constant mean curvature hypersurfaces on S^{n+1} when $n\geq 2$. Here is the theorem:

THEOREM 3.1. *Let n be an integer greater than 1 and let H be a non-negative real number.*

1. *Let $g_C:\mathbf{R}\to\mathbf{R}$ be a T-periodic solution of the equation (3.13) associated with this H and a positive constant C. If $\lambda, r, \theta:\mathbf{R}\to\mathbf{R}$ are defined by*

$$r=\frac{g_C}{\sqrt{C}}, \quad \lambda = H+g_C^{-n} \quad \text{and} \quad \theta(u)=\int_0^u \frac{r(s)\lambda(s)}{1-r^2(s)}ds$$

then, the map $\phi: S^{n-1}\times\mathbf{R}\to S^{n+1}$ given by

$$\phi(y,u)=(\,r(u)\,y, \sqrt{1-r(u)^2}\,\cos(\theta(u)), \sqrt{1-r(u)^2}\,\sin(\theta(u))\,) \tag{3.18}$$

is an immersion with constant mean curvature H.

2. *If $K(H,n,C)=2\int_0^{\frac{T}{2}}\frac{r(u)\lambda(u)}{1-r^2(u)}du=\frac{2\pi}{k}$ for some positive integer k, then, the image of the immersion ϕ is an embedded compact hypersurface in S^{n+1}. More generally, if $K(H,n,C)=\frac{2k\pi}{m}$ for a pair (k,m) of integers, then, the image of ϕ is a compact hypersurface in S^{n+1}.*
3. *Let n be an integer greater than 2, and let $M\subset S^{n+1}$ be a connected compact hypersurface with two principal curvatures λ with multiplicity $n-1$, and μ with multiplicity 1. If $\lambda-\mu$ is positive and the mean curvature $H=(n-1)\lambda+\mu$ is a non-negative constant, then, up to a rigid motion of the sphere, M can be written as an immersion of the form (3.18). Moreover, M contains $O(n)\times Z_m$ in its isometry group, where m is the positive integer such that $K(H,n,C)=\frac{2k\pi}{m}$, with k and m relatively prime.*

Proof. Defining B_1 and B_2 as before we have that

$$\phi(y,u)=r(u)(y,0,0)+\sqrt{1-r(u)^2}B_2(u).$$

A direct verification shows

$$\frac{\partial\phi}{\partial u}=r'\,(y,0,0)-\frac{r\,r'}{\sqrt{1-r^2}}\,B_2+\frac{\lambda\,r}{\sqrt{1-r^2}}\,B_3.$$

We have $\langle\frac{\partial\phi}{\partial u},\frac{\partial\phi}{\partial u}\rangle=1$ and that the tangent space of the immersion at (y,u) is given by

$$T_{\phi(y,u)}=\{(v,0,0)+s\,\frac{\partial\phi}{\partial u}:\langle v,y\rangle=0 \quad \text{and} \quad s\in\mathbf{R}\}.$$

A direct verification shows that the map

$$\nu(y,u) = -r(u)\lambda(u)\,(y,0,0) + \frac{r^2(u)\,\lambda(u)}{\sqrt{1-r^2(u)}}\,B_2(u) + \frac{r'(u)}{\sqrt{1-r^2(u)}}\,B_3(u)$$

satisfies $\langle \nu,\nu\rangle = 1$, $\langle \nu, \frac{\partial\phi}{\partial u}\rangle = 0$, and for any $v \in \mathbf{R}^n$ with $\langle v, y\rangle = 0$ we have $\langle \nu, (v,0,0)\rangle = 0$. It then follows that ν is a Gauss map of the immersion ϕ. The fact that ϕ has constant mean curvature H follows because for any unit vector v in $\mathbf{R}^n$ perpendicular to y, we have

$$\beta(t) = (r\cos(t)y + r\sin(t)v, 0, 0) + \sqrt{1-r^2}\,B_2 = \phi(\cos(t)y + r\sin(t)v, u)$$

satisfies that $\beta(0) = \phi(y,u)$, $\beta'(0) = rv$ and

$$\frac{d\nu(\beta(t))}{dt}\Big|_{t=0} = d\nu(rv) = -r\lambda\, v.$$

Therefore, the tangent vectors of the form $(v,0,0)$ are principal directions with principal curvature λ and multiplicity $n-1$. Now, since $\langle \frac{\partial\phi}{\partial u}, (v,0,0)\rangle = 0$, we have that $\frac{\partial\phi}{\partial u}$ defines a principal direction, i.e. we must have that $\frac{\partial\nu}{\partial u}$ is a multiple of $\frac{\partial\phi}{\partial u}$. A direct verification shows that if we define $\mu : \mathbf{R} \to \mathbf{R}$ by $\mu(u) = nH - (n-1)\lambda(u)$, then,

$$\langle \frac{\partial\nu}{\partial u}, y\rangle = -\lambda'\, r - \lambda r' = (\lambda-\mu)r' - \lambda\, r' = -\mu\, r' = -(nH - (n-1)\lambda)r'.$$

We also have that $\langle \frac{\partial\phi}{\partial u}, y\rangle = r'$, therefore,

$$\frac{\partial\nu}{\partial u} = d\nu(\frac{\partial\phi}{\partial u}) = -\mu\,\frac{\partial\phi}{\partial u} = -(nH-(n-1)\lambda)\frac{\partial\phi}{\partial u}.$$

It follows that the other principal curvature is $nH-(n-1)\lambda$. Therefore ϕ defines an immersion with constant mean curvature H, which proves the first item in the Theorem.

In order to prove the second item, we notice that if $K(H,n,C) = \frac{2\pi}{k}$ for some positive k, $\theta(kT) = 2\pi$, which makes the image of ϕ compact. It is also embedded because ϕ is one-to-one for values of u between 0 and kT as we can easily check using the fact that whenever $H \geq 0$, the function θ is strictly increasing. Recall that under these circumstances $\theta(0) = 0$ and $\theta(kT) = 2\pi$. The proof of the other statement in this item is similar.

Let us prove the next item. For $n > 2$, consider a minimal hypersurface M with the properties of the statement. We will use the notation we used in the preliminaries, in particular the function $w : M \to \mathbf{R}$ is defined by the relation $(\lambda - \mu) = nw^n$. We will also assume that $B_1(0)$, $B_2(0)$ and $B_3(0)$ are chosen as before. By Theorem (2.1) we get that the distribution $\mathrm{Span}\{e_1, \ldots, e_{n-1}\}$ is completely integrable. Let us fix a point p_0 in M and define the geodesic $\gamma : \mathbf{R} \to M$, and the functions $r : \mathbf{R} \to \mathbf{R}$ as before and let us denote by $M_u \subset M$ the $(n-1)$-dimensional integral submanifold of M of this distribution that passes through $\gamma(u)$. We define the vector field η on M as before. Recall that $B_1(u) = \eta(\gamma(u))$. Fixing a value u, let us define the maps

$$\rho_u, \zeta_u : M_u \to \mathbf{R}^{n+2} \quad \text{by} \quad \rho_u(x) = x + \frac{w^2(x)}{C}\,\eta \quad \text{and} \quad \zeta_u(x) = \nu(x) + \lambda(x)\,x.$$

Using the equations in section (3.2) we find that the maps ρ_u and ζ_u are constant. Therefore,

$$\rho_u(x) = x + \frac{w^2(x)}{C}\,\eta = \gamma(u) + r^2(u)B_1 = \sqrt{1-r^2}B_2.$$

Notice that for every $x \in M_u$, we have

$$|x - \rho_u(x)|^2 = |Z(u) - \sqrt{1-r^2}B_2(u)|^2 = r^2(u).$$

Therefore M_u is contained in a sphere with center in $\sqrt{1-r^2}B_2$ and radius r. We have that the vectors $e_1, \dots, e_{n-1}$ are perpendicular to the vectors

$$\rho_u(x) = \sqrt{1-r^2(u)}B_2(u) \quad \text{and} \quad \zeta_u(x) = Y(u) + \lambda(u)Z(u).$$

Since $\langle Y(u) + \lambda(u)Z(u), B_1(u)\rangle = 0$, $\langle Y(u) + \lambda(u)Z(u), B_2(u)\rangle = \frac{\lambda r^2}{\sqrt{1-r^2}}$ and $\langle Y(u) + \lambda(u)Z(u), B_3(u)\rangle = \frac{r'}{\sqrt{1-r^2}}$, we get that

$$\zeta_u(x) = \frac{\lambda r^2}{\sqrt{1-r^2}}\,B_2 + \frac{r'}{\sqrt{1-r^2}}\,B_3.$$

It follows that, anytime $r'(u) \neq 0$, all tangent vectors of M_u must lie in the n-dimensional space perpendicular to the two dimensional space spanned by $B_1(u)$ and $B_2(u)$. Since this two dimensional space is independent of u, we conclude that every point $x \in M_u$, satisfies that

$$x - \rho_u(x) = r(u)(y, 0, 0) \quad \text{where} \quad |y|^2 = 1$$

or equivalently,

$$x = r(u)\,(y, 0, 0) + \rho_u(x) = r(u)(y, 0, 0) + \sqrt{1-r(u)^2}B_2(u).$$

Since the set of points where r' is discrete, we conclude that the expression for the points $x \in M_u$ holds true for all u. The theorem then follows because the manifold M is connected.

The property on the group of isometries of the manifold follows because we can write M as the image of the map

$$\phi(y,u) = (\,r(u)\,y, \sqrt{1-r(u)^2}\,\cos(\theta(u)), \sqrt{1-r(u)^2}\,\sin(\theta(u))\,). \tag{3.19}$$

The group $O(n)$ acts isometrically on M because any isometry in $\mathbf{R}^{n+2}$ that fixes the origin and the last two entries of $\mathbf{R}^{n+2}$ leaves our manifold M invariant. The group Z_m includes in the isometry group because the closed curve given by the last two entries is built by joining m pieces of the the curve

$$\alpha(u) = (\sqrt{1-r(u)^2}\,\cos(\theta(u)), \sqrt{1-r(u)^2}\,\sin(\theta(u))) \quad 0 \leq u \leq K(H,n,C) = \frac{2k\pi}{m}.$$

This last statement is true by the the following observation already pointed out in the previous section.

For any positive integer j and $u \in [jT, (j+1)T]$ we have that $\theta(u) = jK + \theta(u - jT)$. □

Corollary 3.2. *If M is one of the compact examples in the previous theorem with $H = 0$, then, the stability index, i.e, the number of negative eigenvalues of the operator $J(f) = -\Delta f - nf - ||A||^2\, f$ is greater than $n+3$.*

Proof. Theorem (3.1.1) in [16] states that if $M \subset S^{n+1}$ is a compact minimal hypersurface different from a Clifford torus with the property that for any non-zero vector $v \in \mathbf{R}^{n+2}$ there exists an $(n+2)\times(n+2)$ orthogonal matrix B such that $B(M) = M$ and $B(v) \neq v$, then the stability index of M is greater than $n+3$. Let M be one of the examples from the previous theorem. Since M is compact, M is left invariant by the orthogonal matrices in the group $O(n) \times Z_m$ where m satisfies that $K(0,n,C) = \frac{2k\pi}{m}$, with k and m relatively prime. Independently, Otsuki in [15] and Furuya in [8] showed that M can not be embedded by showing that $\pi < K(0,n,C) < 2\pi$, these inequalities also implies that $m \geq 2$. Since for any non-zero vector $v \in \mathbf{R}^{n+2}$, there exists a matrix $B \in O(n) \times Z_m$ such that $B(v) \neq v$, then, the stability index of M must be greater than $n+3$. □

4. Embedded hypersurface with CMC in S^{n+1}. In this section we will study the existence of compact examples in S^{n+1} by studying the values $K(H,n,C)$. The key for this is the following.

Lemma 4.1. *Let $f : (-\delta,\delta) \to \mathbf{R}$ be a smooth function such that $f(0) = f'(0) = 0$ and $f''(0) = -2a < 0$. For positive values of c close to 0, let $t(c)$ be the first positive root of the function $f(t) + c$. Then*

$$\lim_{c\to 0^+} \int_0^{t(c)} \frac{dt}{\sqrt{f(t)+c}} = \frac{\pi}{2\sqrt{a}}.$$

Proof. For any $b > a$ let us define the function $h(t) = f'(t) + 2bt$. Since $h'(0) = 2(b-a) > 0$ there exists a positive ϵ such that $h'(t) > 0$ for all $t \in [0,\epsilon]$. Now for any c such that $t(c) < \epsilon$, the function

$$g(t) = f(t) + c - (bt(c)^2 - bt^2)$$

satisfies that $g(t(c)) = 0$ and $g'(t) = h(t) > 0$. Therefore, $g(t) < 0$ for any $t \in [0, t(c)]$. By the definition of $g(t)$ we get that

$$0 < f(t) + c < bt(c)^2 - bt^2 \quad \text{for all} \quad t \in [0, t(c))$$

and therefore

$$\frac{\pi}{2\sqrt{b}} = \int_0^{t(c)} \frac{dt}{\sqrt{bt(c)^2 - bt^2}} < \int_0^{t(c)} \frac{dt}{\sqrt{f(t)+c}}.$$

Likewise, for any $b < a$, the same argument shows

$$\int_0^{t(c)} \frac{dt}{\sqrt{f(t)+c}} < \int_0^{t(c)} \frac{dt}{\sqrt{bt(c)^2 - bt^2}} = \frac{\pi}{2\sqrt{b}}.$$

Since $b \neq a$ can be chosen arbitrarily close to a, we obtain the lemma. □

COROLLARY 4.2. *Let ϵ and δ be positive real numbers and let $f : (t_0-\epsilon, t_0+\epsilon) \to \mathbf{R}$ and $g : (-\delta,\delta)\times(t_0-\epsilon, t_0+\epsilon) \to \mathbf{R}$ be smooth functions such that $f(t_0) = f'(t_0) = 0$ and $f''(t_0) = -2a < 0$. If for any small $c > 0$, $t_1(c) < t_0 < t_2(c)$ are such that $f(t_1(c)) + c = 0 = f(t_2(c))) + c$, then*

$$\lim_{c\to 0^+} \int_{t_1(c)}^{t_2(c)} \frac{g(c,t)\,dt}{\sqrt{f(t)+c}} = \frac{g(0,t_0)\,\pi}{\sqrt{a}}.$$

This lemma allows us to prove our main theorem:

THEOREM 4.3. *For any $n \geq 2$ and any $H \in (0, \frac{2\sqrt{n-1}}{n\sqrt{3}})$ there exists a non-isoparametric compact embedded hypersurface in S^{n+1} with constant mean curvature H. More generally, for any integer $m > 1$ and H between*

$$\cot\frac{\pi}{m} \quad \textit{and} \quad \frac{(m^2-2)\sqrt{(n-1)}}{n\sqrt{m^2-1}}$$

there exists a non isoparametric compact embedded hypersurface in S^{n+1} with constant mean curvature H whose isometry group contains $O(n)\times Z_m$.

Proof. We will consider only positive values for H. Here we will use the explicit solution for the ODE (3.13) given in section (3.1). Let us rewrite that ODE as

$$(g')^2 = q(g) \quad \text{where} \quad q(v) = C - v^{2-2n} - (1+H^2)v^2 - 2Hv^{2-n}.$$

We already pointed out in section (3.1) that for some values of C, the function q has positive values between two positive roots of q, denoted by t_1 and t_2. Let us be more precise and give an expression for how big C needs to be. A direct verification shows that

$$q'(v) = -2(1+H^2)v - (2-2n)v^{1-2n} - 2H(2-n)v^{1-n}$$

and that the only positive root of q' is

$$v_0 = \Big(\frac{\sqrt{H^2n^2 + 4(n-1)} + (n-2)H}{2+2H^2}\Big)^{\frac{1}{n}}. \tag{4.1}$$

Therefore, for positive values of v, the function q increases from 0 to v_0 and decreases for values greater than v_0. A direct computation shows that $q(v_0) = C - c_0$ where,

$$c_0 = n\,(2+2H^2)^{\frac{n-2}{n}}\,\frac{2+nH^2+H\sqrt{H^2n^2+4(n-1)}}{\big((n-2)H+\sqrt{H^2n^2+4(n-1)}\,\big)^{\frac{2n-2}{n}}}. \tag{4.2}$$

Therefore, whenever $C > c_0$ we will have exactly two positive roots of the function $q(v)$ that we will denote by $t_1(C)$ and $t_2(C)$ to emphasize its dependence on C. A direct computation shows that $q''(v_0) = -2a$ where

$$a = 2n(1+H^2)\,\frac{4(n-1)+H^2\,n^2+H\,(n-2)\,\sqrt{4(n-1)+H^2n^2}}{(\,H(n-2)+\sqrt{4(n-1)+H^2n^2}\,)^2}.$$

Using the notation and results of section (3.3), we get

$$K(H,n,C) = 2\int_0^{\frac{T}{2}} \frac{r(s)\lambda(s)}{1-r^2(s)}\,ds \tag{4.3}$$

Since $r(s) = \frac{g(s)}{\sqrt{C}}$ and $\lambda(s) = H + g(s)^{-n}$ we have

$$K(H,n,C) = 2\int_0^{\frac{T}{2}} \frac{\sqrt{C}g(s)(H+g^{-n}(s))}{c-g^2(s)}\,ds.$$

Since $g(0) = t_1(C)$ and $g(\frac{T}{2}) = t_2(C)$, by making the substitution $t = g(s)$ we get

$$K(H,n,C) = 2\int_{t_1(c)}^{t_2(c)} \frac{\sqrt{C}t(H+t^{-n})}{c-t^2}\,\frac{1}{\sqrt{q(t)}}\,dt.$$

Since $a > 0$ we can apply Corollary (4.2) to the get that

$$\lim_{C\to c_0^+} K(H,n,C) = \pi\sqrt{2-\frac{2nH}{\sqrt{4(n-1)+H^2n^2}}}.$$

It can be verified that this bound is the same bound we found for the case $n = 2$.

In order to analyze the limit of the function $K(H,n,C)$ when $C \to \infty$ we return to the expression (4.3) and we make the substitution $t = r(s)$ to obtain

$$K(H,n,C) = 2\int_{\frac{t_1(C)}{\sqrt{C}}}^{\frac{t_2(C)}{\sqrt{C}}} \frac{t(H+C^{-\frac{n}{2}}\,t^{-n})}{(1-t^2)\,\sqrt{1-t^2(1+(H+C^{-\frac{n}{2}}\,t^{-n})^2)}}\,dt.$$

In this case we have used (3.17) to change the ds to dt. Notice that the limit values $\frac{t_1(C)}{\sqrt{C}}$ and $\frac{t_2(C)}{\sqrt{C}}$ can also be characterized as the only positive roots of the function

$$\tilde{q} = 1-t^2(1+(H+C^{-\frac{n}{2}}\,t^{-n})^2) = 1-(1+H^2)t^2 - C^{-n}t^{2-2n} - 2HC^{-\frac{n}{2}}t^{2-n}$$

because of the relation $q(v) = C\tilde{q}(\frac{v}{\sqrt{C}})$. Since for every positive C we have that $\lim_{t\to 0^+}\tilde{q}(t) = -\infty$, $\tilde{q}(\frac{1}{\sqrt{1+H^2}}) < 0$ and for every positive $\epsilon < \frac{1}{\sqrt{1+H^2}}$ we have

$$\lim_{C\to\infty}\tilde{q}(\epsilon) > 0 \quad \text{and} \quad \lim_{c\to\infty}\tilde{q}(\frac{1}{\sqrt{1+H^2}} - \epsilon) > 0,$$

we conclude that the only two positive roots of $\tilde{q}$ converge to 0 and to $\frac{1}{\sqrt{1+H^2}}$ when $C \to \infty$. Therefore,

$$\lim_{C\to\infty} K(H,n,C) = 2\int_0^{\frac{1}{\sqrt{1+H^2}}} \frac{Ht}{(1-t^2)\sqrt{1-(1+H^2)t^2}}\, dt = 2\mathrm{arccot}(H).$$

Therefore, for any fixed $H > 0$, the function $K(H,n,C)$ takes all the values between

$$a_1(H) = 2\mathrm{arccot}(H) \quad \text{and} \quad a_{2,n}(H) = \pi\sqrt{2 - \frac{2nH}{\sqrt{4(n-1)+H^2n^2}}}.$$

The functions $a_1(H)$ and $a_{2,n}(H)$ are decreasing. Moreover, we have that for any $y < \sqrt{2}$

$$a_{2,n}(\frac{2\,(2-y^2)\,\sqrt{n-1}}{n\,y\,\sqrt{4-y^2}}) = \pi\, y.$$

Therefore, replacing y by $\frac{2}{m}$ in the expression above, we obtain that for values of H between

$$\cot\frac{\pi}{m} \quad \text{and} \quad \frac{(m^2-2)\sqrt{(n-1)}}{n\sqrt{m^2-1}}$$

the number $\frac{2\pi}{m}$ lies between $a_1(H)$ and $a_{2,n}(H)$, and therefore, for some constant C, we will have that $K(H,n,C) = \frac{2\pi}{m}$. Applying Theorem 3.1 concludes the proof. Notice that when $m = 2$ these two bounds are 0 and $\frac{2\sqrt{n-1}}{n\sqrt{3}}$. $\square$

Let us finish this section with a remark already pointed out by Otsuki in ([14]).

LEMMA 4.4. *For any integer $n \geq 2$ and any $\epsilon > 0$ there exist compact non-isoparametric minimal hypersurfaces in S^{n+1} such that $n - \epsilon \leq ||A||^2(p) \leq n + \epsilon$ for all $p \in M$.*

Proof. This is a consequence of the fact that the expression for v_0 in Equation (4.1) reduces to $(n-1)^{\frac{1}{2n}}$ when $H = 0$ and the fact that by picking C close to c_0, the roots $t_1(C)$ and $t_2(C)$ of the function q are as close as v_0 as we want. Since the range of the function g move from $t_1(C)$ to $t_2(C)$, we can make the values of g to move as close of $(n-1)^{\frac{1}{2n}}$ as we want. When $H = 0$, we have that

$$\lambda = g^{-n} \quad \mu = -(n-1)g^{-n} \quad \text{and} \quad ||A||^2 = (n-1)g^{-2n} + (n-1)^2 g^{-2n} = n(n-1)g^{-2n}.$$

Therefore, we can make $||A||^2$ as close n as we want. By density of the rational numbers and the continuity of the function $K(H,n,C)$, we can choose C so that $K(H,n,C)$ is of the form $\frac{2k\pi}{m}$ for some pair of integers m and k. This last condition guarantees the compactness of the profile curve and therefore the compactness of the hypersurface. □

5. Non isoparametric stable cones in S^{n+1}. For any compact minimal hypersurface $M \subset S^{n+1}$, let us define the operator L_1 and the number λ_1 as follows,

$$L_1(f) = -\Delta f - ||A||^2 f \quad \text{and} \quad \lambda_1 = \quad \text{first eigenvalue of} \quad L_1.$$

Moreover, let us denote by $CM = \{tm : t \in [0,1],\, m \in M\}$ the cone over M. We will say that CM is stable if every variation of CM, which holds M fixed, increases area.

In ([20], Lemma 6.1.6) Simons proved that if $\lambda_1 + (\frac{n-1}{2})^2 > 0$ then CM is stable. We will prove that for any $n \geq 6$, the cone over some non isoparametric examples studied in this paper for $H = 0$, i.e, the cone over some of the Otsuki's examples, are stable. More precisely we have,

THEOREM 5.1. *For any $n \geq 6$, there are non-isoparametric compact hypersurfaces in S^{n+1} bounding stable minimal cones.*

Proof. A direct verification shows that

$$(\frac{n-1}{2})^2 \geq n + \frac{1}{4} \quad \text{for all} \quad n \geq 6.$$

Using Lemma (4.4), let us consider a non isoparametric compact minimal hypersurface M such that $||A||^2 \leq n + \frac{1}{8}$. We have that the first eigenvalue λ_1 of the operator L_1 is greater than $-n - \frac{1}{8}$ because

$$\lambda_1 = \inf \{ \frac{\int_M (-\Delta f - ||A||^2 f) f}{\int_M f^2} : f \quad \text{is smooth and} \quad \int_M f^2 \neq 0 \}$$

and we have that,

$$\frac{\int_M (-\Delta f - ||A||^2 f) f}{\int_M f^2} = \frac{\int_M |\nabla f|^2}{\int_M f^2} - \frac{\int_M ||A||^2 f^2}{\int_M f^2} \geq -(n + \frac{1}{8}).$$

Therefore, we get that

$$\lambda_1 + (\frac{n-1}{2})^2 \geq -(n + \frac{1}{8}) + n + \frac{1}{4} = \frac{1}{8} > 0$$

which implies, by Simons' result, that the cone over M is stable. □

6. Some explicit solutions. In this section we will pick some arbitrary values of H to explicitly show the embedding, the graph of the profile curves, and the stereographic projections of some examples of surfaces with CMC in S^3.

A direct computation shows that the solution of the equation (3.13) when $n = 2$ is given by

$$g(t) = \sqrt{\frac{(C-2H) + \sqrt{-4+C^2-4CH}\,\sin(2\sqrt{1+H^2}\,t\,)}{2(1+H^2))}}\,.$$

From the expression for g we get that its period T is $\frac{\pi}{\sqrt{1+H^2}}$. Since $n=2$, the condition on C to get solutions of the ODE (3.13) reduces to $C > 2(H+\sqrt{1+H^2})$.

We can get surfaces associated with $m=2$ if we take H between 0 and $\frac{1}{\sqrt{3}} \simeq 0.57735$ and we can surfaces associated with $m=3$ if we take H between $\frac{1}{\sqrt{3}}$ and $\frac{7}{4\sqrt{2}} \simeq 1.23744$. Once we have picked the value for H in the right range, in order to get the embedded surface, we need to solve the equation

$$K(H,2,C) = \int_0^{\frac{\pi}{\sqrt{1+H^2}}} \frac{\sqrt{C}\,g(t)(H+g(t)^{-2})}{C-g(t)^2}\,dt = \frac{2\pi}{m}.$$

Finally, when we have the H and the C, the profile curve is given by

$$(\,\sqrt{1-\frac{g^2(t)}{C}}\,\cos(\theta(t)), \sqrt{1-\frac{g^2(t)}{C}}\,\sin(\theta(t))\,)$$
$$\text{where}\quad \theta(t) = \int_0^t \frac{\sqrt{C}\,g(\tau)(H+g^{-2}(\tau))}{C-g^2(\tau)}\,d\tau$$

and the embedding is given by

$$(\,\frac{g(t)}{\sqrt{C}}\cos(u), \frac{g(t)}{\sqrt{C}}\sin(u), \sqrt{1-\frac{g^2(t)}{C}}\,\cos(\theta(t)), \sqrt{1-\frac{g^2(t)}{C}}\,\sin(\theta(t))\,)$$
$$0 \le u < 2\pi \quad 0 \le t < m\,\frac{\pi}{\sqrt{1+H^2}}$$

Here are some graphics,

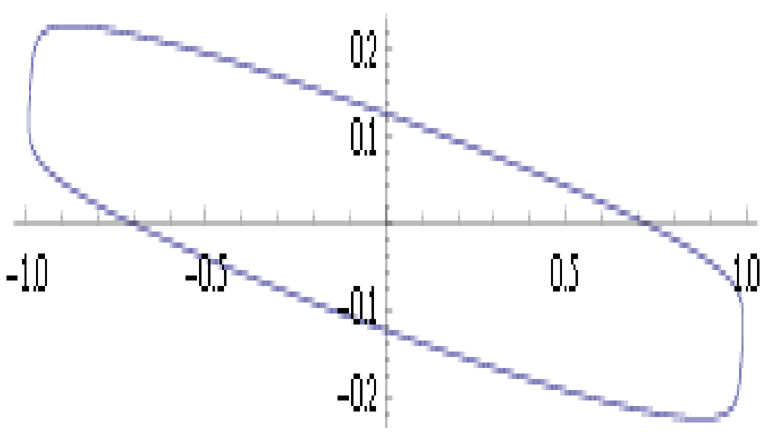

FIG. 6.1. *Profile curve for $m=2$, $H=0.1$, in this case* $C = 41.28796038772471$

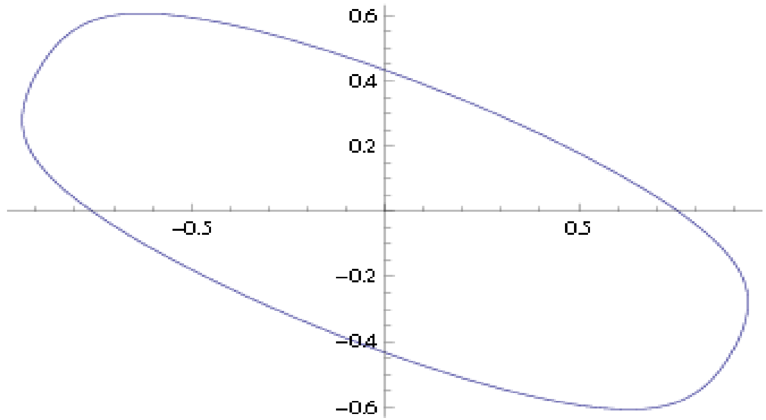

Fig. 6.2. *Profile curve for* $m = 2$*,* $H = 0.3$*, in this case* $C = 9.129645968138256$

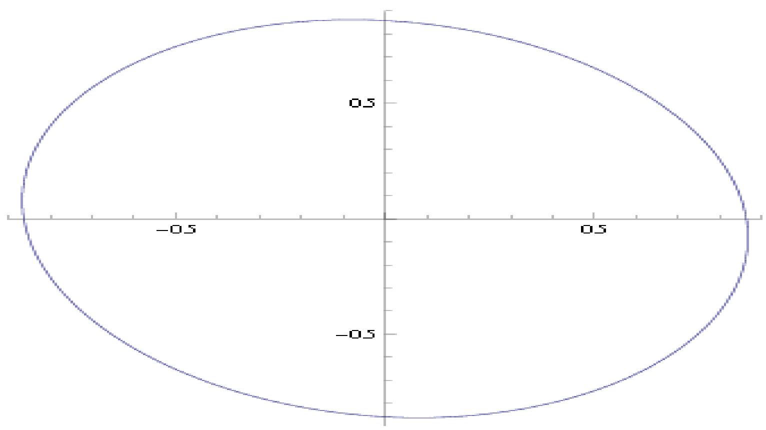

Fig. 6.3. *Profile curve for* $m = 2$*,* $H = 0.57$*, in this case* $C = 3.5313222039296357$

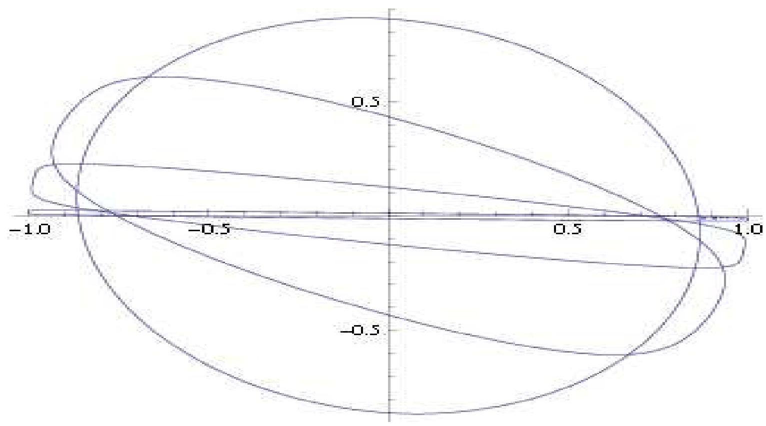

FIG. 6.4. *Profile curve for $m = 2$, $H = 0.001$, $H = 0.1$, $H = 0.3$, $H = 0.57$.*

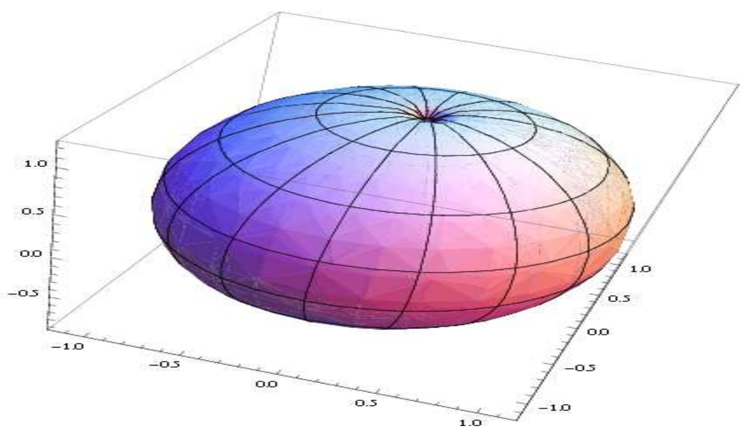

FIG. 6.5. *Stereographic projection for the surface with CMC $H = 0.1$*

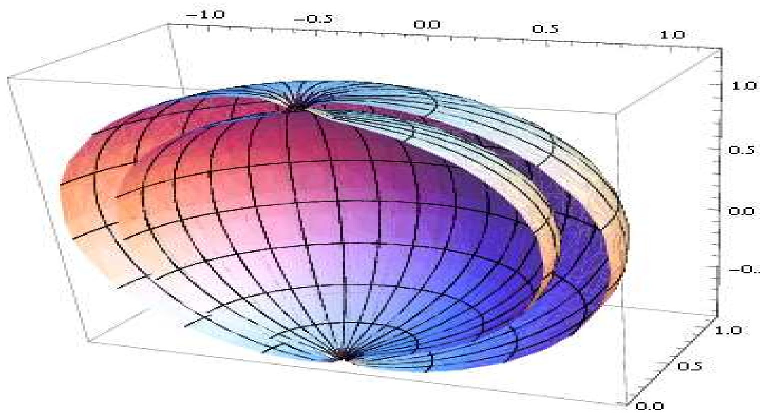

FIG. 6.6. *Stereographic projection of half the surface with CMC $H = 0.1$*

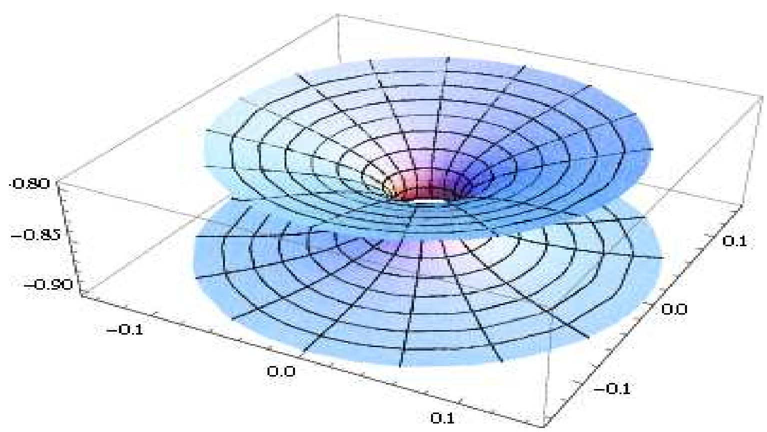

FIG. 6.7. *Stereographic projection one of the two catenoid necks of the surface with CMC $H = 0.1$*

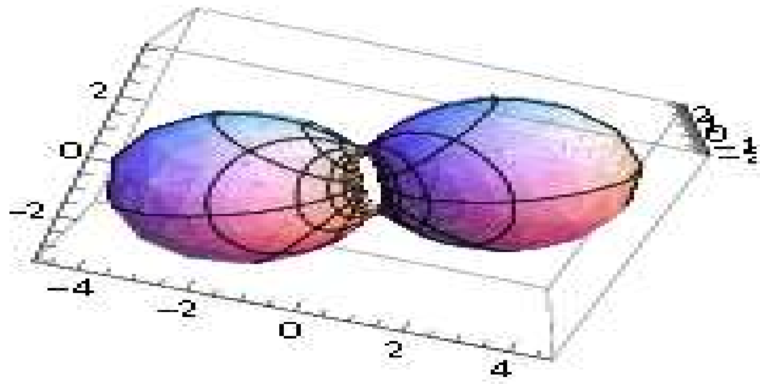

Fig. 6.8. *Stereographic projection of the surface with CMC $H = 0.3$ and $m = 2$*

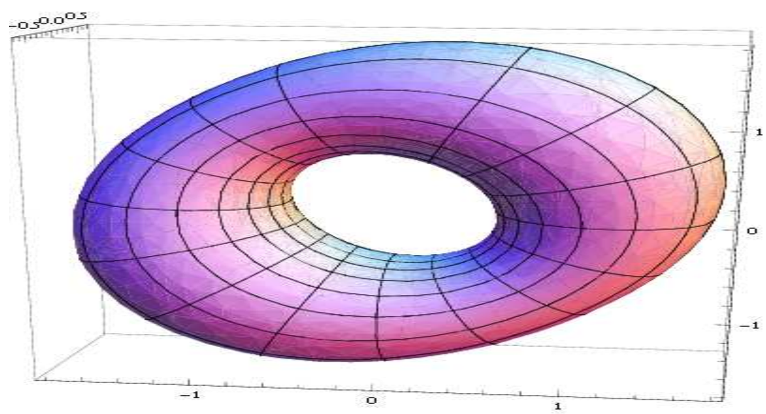

Fig. 6.9. *Stereographic projection of the surface with CMC $H = 0.57$ and $m = 2$*

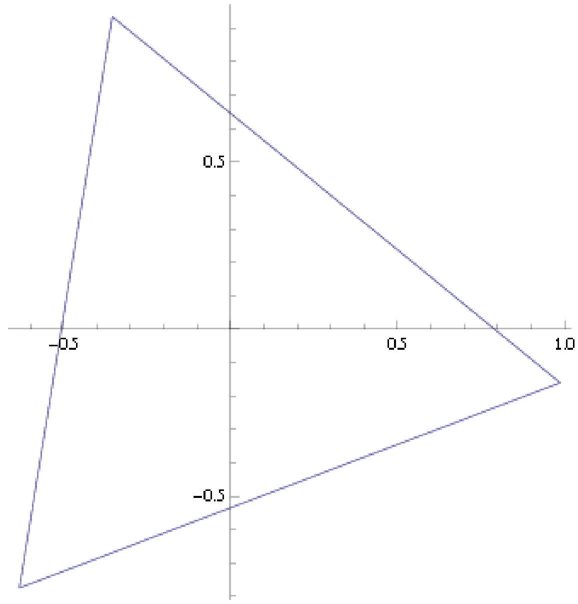

Fig. 6.10. *Profile curve for* $m = 3$ *and* $H = 0.5774$*, in this case* $C = 346879.6632142387$

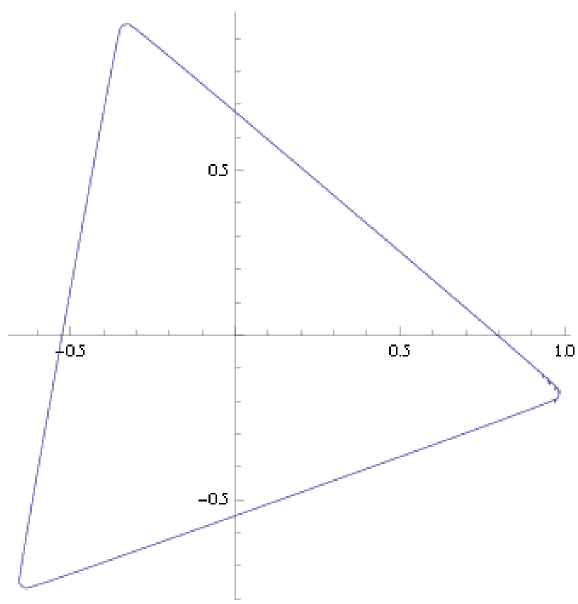

Fig. 6.11. *Profile curve for* $m = 3$ *and* $H = 0.6$*, in this case* $C = 365.3705636110441$

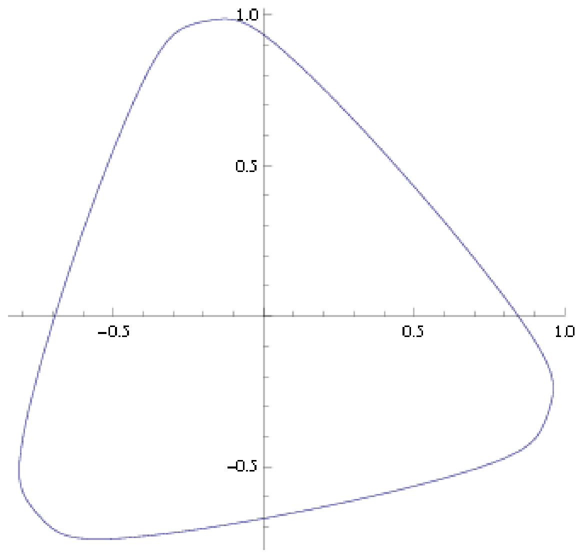

FIG. 6.12. *Profile curve for $m = 3$ and $H = 0.8$, in this case $C = 22.320379289179478$*

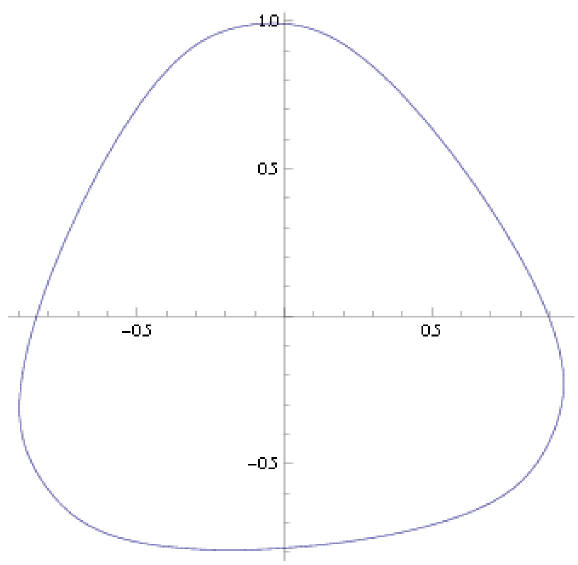

FIG. 6.13. *Profile curve for $m = 3$ and $H = 1.0$, in this case $C = 9.908469426660892$*

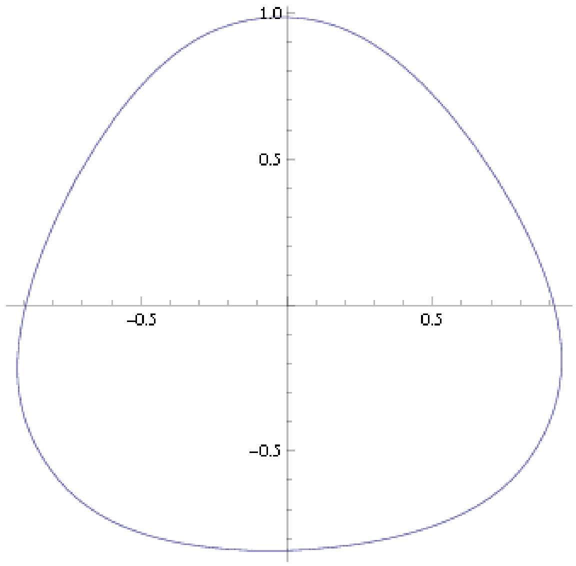

FIG. 6.14. *Profile curve for* $m = 3$ *and* $H = 1.2$, *in this case* $C = 6.084010495710457$

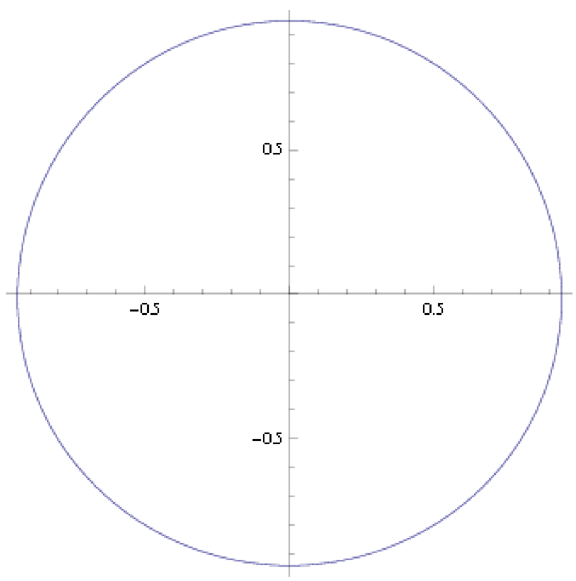

FIG. 6.15. *Profile curve for* $m = 3$ *and* $H = 1.237$, *in this case* $C = 5.6615177218839605$

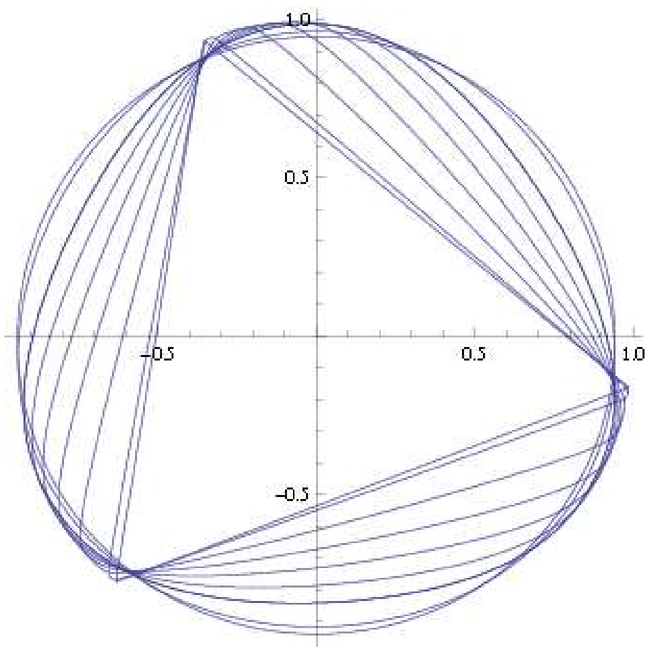

Fig. 6.16. *Profile curve for* $m = 3$, $H = 0.5774$, $H = 0.6$, $H = 0.7$, $H = 0.8$, $H = 1.0\,H = 1.1\,H = 1.2$, $H = 1.22\,H = 1.237$.

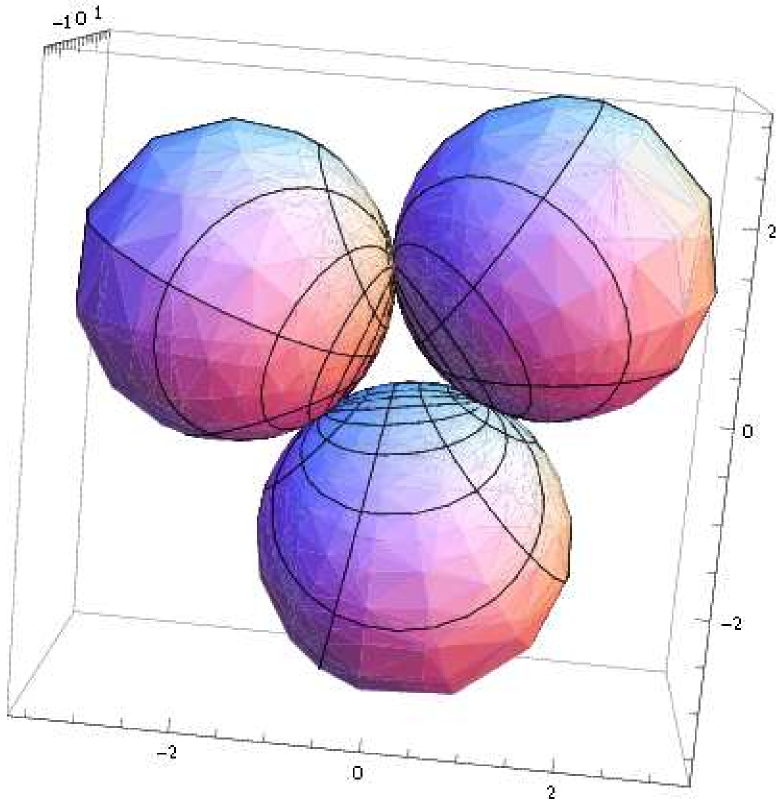

FIG. 6.17. *Stereographic projection of a surface with CMC $H = 0.5774$ and $m = 3$*

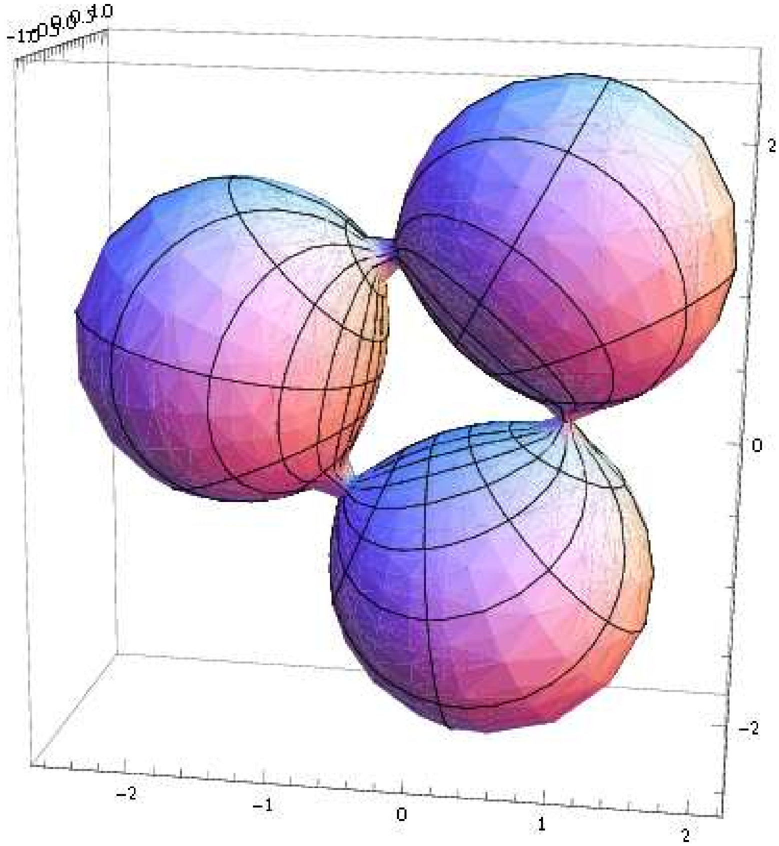

FIG. 6.18. *Stereographic projection of a surface with CMC $H = 0.8$ and $m = 3$*

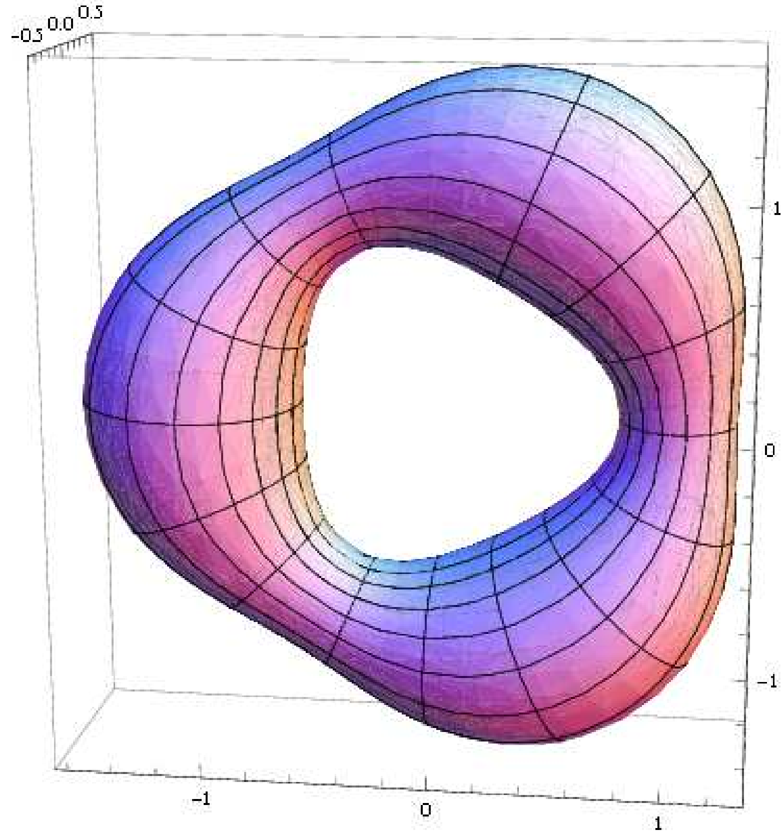

FIG. 6.19. *Stereographic projection of a surface with CMC $H = 1.2$ and $m = 3$*

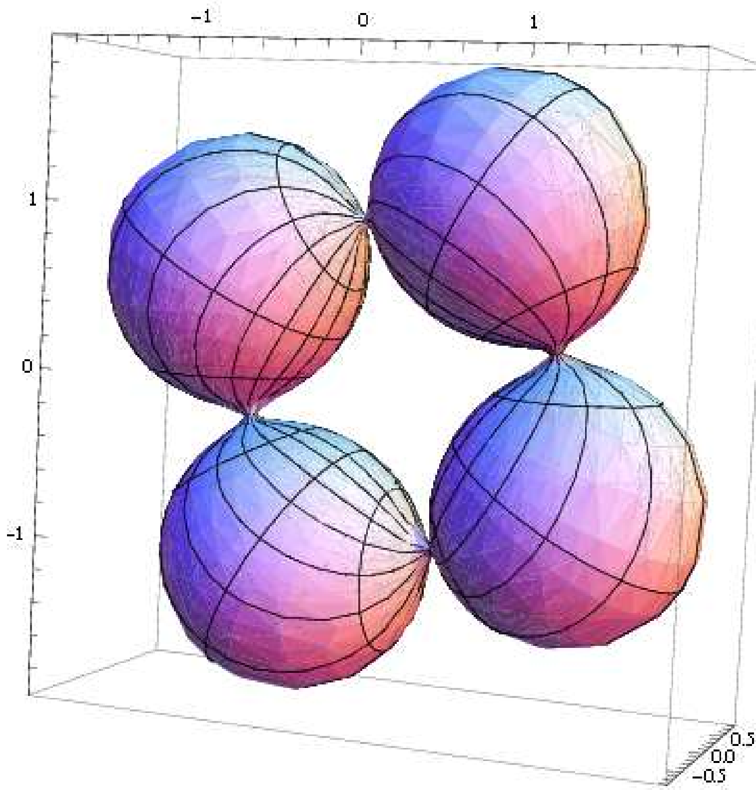

FIG. 6.20. *Stereographic projection of a surface with CMC $H = 1.2$ and $m = 4$*

6.1. Embedded solutions in hyperbolic spaces. Here we show that the theorem above can be adapted to hyperbolic spaces. In this case we get the embedded hypersurfaces without much effort since Hyperbolic space is not compact. We will use the following model for hyperbolic space:

$$H^{n+1} = \{\, x \in \mathbf{R}^{n+2} : x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 = -1 \,\}.$$

The following notation will only be used in this subsection. For any pair of vectors $v = (v_1, \ldots, v_{n+2})$ and $w = (w_1, \ldots, w_{n+2})$, $\langle v, w\rangle = v_1 w_1 + v_{n+1} w_{n+1} - v_{n+2} w_{n+2}$.

Theorem 6.1. *Let $g_{C,H} : \mathbf{R} \to \mathbf{R}$ be a positive solution of the equation*

$$(g')^2 + g^{2-2n} + (H^2 - 1) g^2 + 2 H g^{2-n} = C \tag{6.1}$$

associated with a non negative H and a positive constant C. If $\mu, \lambda, r, \theta : \mathbf{R} \to \mathbf{R}$ and are defined by

$$r = \frac{g_{C,H}}{\sqrt{C}}, \quad \lambda = H + g_{C,H}^{-n},\ \mu = nH - (n-1)\lambda = H - (n-1) g_{C,H}^{-n}$$
$$\text{and} \quad \theta(u) = \int_0^u \frac{r(s)\lambda(s)}{1 + r^2(s)} ds$$

then, the map $\phi : S^{n-1} \times \mathbf{R} \to H^{n+1}$ given by

$$\phi(y,u) = (\, r(u)\, y, \sqrt{1 + r(u)^2}\, \sinh(\theta(u)), \sqrt{1 + r(u)^2}\, \cosh(\theta(u))\,) \tag{6.2}$$

defines an embedded hypersurface in H^{n+1} with constant mean curvature H. Moreover, if $H^2 > 1$, the embedded manifold defined by (6.2) *admits $O(n) \times Z$ in its group of isometries, where Z is the group of integers.*

Remark. *Arguments similar to those in section (3.1) show that it is not difficult to find positive values C that lead to positive solutions of the equation (6.1) in terms of the inverse of a function defined by an integral.*

Proof. A direct computation shows the following identities,

$$(r')^2 + \lambda^2 r^2 = 1 + r^2, \quad \text{and} \quad \lambda r' + r \lambda' = \mu r'.$$

Let us define

$$B_2(u) = (0, \ldots, 0, \sinh(\theta(u)), \cosh(\theta(u)))$$
$$\text{and} \quad B_3(u) = (0, \ldots, 0, \cosh(\theta(u)), \sinh(\theta(u))).$$

Notice that $\langle B_2, B_2\rangle = -1$, $\langle B_3, B_3\rangle = 1$, $\langle B_2, B_3\rangle = 0$, $B_2' = \frac{r\lambda}{1+r^2} B_3$ and $B_3' = \frac{r\lambda}{1+r^2} B_2$, moreover, the map ϕ can be written as

$$\phi = r(y, 0, 0) + \sqrt{1 + r^2}\, B_2.$$

A direct verification shows that $\langle \phi, \phi\rangle = -1$ and that

$$\frac{\partial \phi}{\partial u} = r'\,(y,0,0) + \frac{r r'}{\sqrt{1+r^2}}\,B_2 + \frac{r\lambda}{\sqrt{1+r^2}}\,B_3$$

is a unit vector, i.e, $\langle \frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial u}\rangle = 1$. The tangent space of the immersion at (y,u) is given by

$$T_{\phi(y,u)} = \{(v,0,0) + s\,\frac{\partial \phi}{\partial u} : \langle v, y\rangle = 0 \quad \text{and} \quad s \in \mathbf{R}\}.$$

A direct verification shows that the map

$$\nu = -r\lambda\,(y,0,0) - \frac{r^2\,\lambda}{\sqrt{1+r^2}}\,B_2 + \frac{r'}{\sqrt{1+r^2}}\,B_3$$

satisfies that $\langle \nu, \nu\rangle = 1$, $\langle \nu, \frac{\partial \phi}{\partial u}\rangle = 0$ and for any $v \in \mathbf{R}^n$ with $\langle v, y\rangle = 0$ we have that $\langle \nu, (v,0,0)\rangle = 0$. It then follows that ν is a Gauss map of the immersion ϕ. The fact that the immersion ϕ has constant mean curvature H follows because for any unit vector v in $\mathbf{R}^n$ perpendicular to y, we have that

$$\beta(t) = (r\cos(t)y + r\sin(t)v, 0, 0) + \sqrt{1+r^2}\,B_2 = \phi(\cos(t)y + r\sin(t)v, u)$$

satisfies that $\beta(0) = \phi(y,u)$, $\beta'(0) = rv$ and

$$\frac{d\nu(\beta(t))}{dt}\Big|_{t=0} = d\nu(rv) = -r\lambda\, v.$$

Therefore, the tangent vectors of the form $(v,0,0)$ are principal directions with principal curvature λ and multiplicity $n-1$. Now, since $\langle \frac{\partial \nu}{\partial u}, (v,0,0)\rangle = 0$, we have that $\frac{\partial \phi}{\partial u}$ defines a principal direction, i.e. we must have that $\frac{\partial \nu}{\partial u}$ is a multiple of $\frac{\partial \phi}{\partial u}$. A direct verification shows that,

$$\langle \frac{\partial \nu}{\partial u}, y\rangle = -\lambda'\,r - \lambda r' = -\mu\, r' = -(nH - (n-1)\lambda)r'.$$

We also have that $\langle \frac{\partial \phi}{\partial u}, y\rangle = r'$, therefore,

$$\frac{\partial \nu}{\partial u} = d\nu(\frac{\partial \phi}{\partial u}) = -\mu\,\frac{\partial \phi}{\partial u} = -(nH - (n-1)\lambda)\frac{\partial \phi}{\partial u}.$$

It follows that the other principal curvature is $nH - (n-1)\lambda$. Therefore ϕ defines an immersion with constant mean curvature H, this proves the first item in the Theorem. This immersion is embedded because the immersion ϕ is one to one as we can easily check using the fact that whenever $H \geq 0$, the function θ is strictly increasing. In order to prove the condition on the isometries of the immersion when $H > 1$ we notice first that the ODE (6.1) can be written as

$$(g')^2 = g^{2-2n}\,q(g) \quad \text{where} \quad q(v) = Cv^{2n-2} - (H^2-1)v^{2n} - 2Hv^n - 1.$$

Since $q(0) = -1$ and the leading coefficient of q is negative under the assumption that $H > 1$, then by the arguments used in section (3.1) we conclude that a positive solution g of (6.1) must be periodic, moreover the values of g must move from two positive roots t_1 and t_2. Now if T is the period of g and we define

$$K = \int_0^T \frac{r(u)\lambda(u)}{1 + r^2(u)} du.$$

We then have:

For any integer j and $u \in [jT, (j+1)T]$ we have that $\quad \theta(u) = jK + \theta(u - jT).$

Using the equation above we get that the immersion ϕ is invariant under the group generated by hyperbolic rotations of the angle K in the x_{n+1}-x_{n+2} plane. This concludes the theorem. □

6.2. Solutions in Euclidean spaces. In this section we point out that the same kind of theorem can be adapted to Euclidean spaces. In this case we get the Delaunay hypersurfaces.

THEOREM 6.2. *Let $g_{C,H} : \mathbf{R} \to \mathbf{R}$ be a positive solution of the equation*

$$(g')^2 + g^{2-2n} + H^2 g^2 + 2Hg^{2-n} = C \tag{6.3}$$

associated with a real number H and a positive constant C. If $\mu, \lambda, r, R : \mathbf{R} \to \mathbf{R}$ and are defined by

$$r = \frac{g_{C,H}}{\sqrt{C}}, \quad \lambda = H + g_{C,H}^{-n}, \quad \mu = nH - (n-1)\lambda = H - (n-1)g_{C,H}^{-n}$$
$$\text{and} \quad R(u) = \int_0^u r(s)\lambda(s) ds$$

then, the map $\phi : S^{n-1} \times \mathbf{R} \to \mathbf{R}^{n+1}$ given by

$$\phi(y, u) = (\, r(u)\, y, R(u)) \tag{6.4}$$

defines an immersed hypersurface in $\mathbf{R}^{n+1}$ with constant mean curvature H. Moreover, if $H \geq 0$, the manifold defined by (6.4) is embedded. We also have that when $n > 2$, up to rigid motions they are the only non isoparametric CMC hypersurfaces with exactly two principal curvatures.

Proof. A direct computation shows the following identities,

$$(r')^2 + \lambda^2 r^2 = 1, \quad \text{and} \quad \lambda r' + r\lambda' = \mu r'.$$

In this case we have that the map

$$\nu(y, u) = (-r(u)\lambda(u)\, y, r'(u))$$

is a Gauss map of the immersion. A direct computation shows that indeed this immersion has constant mean curvature H. The fact that the immersion is an embedding when $H \geq 0$ follows from the fact that $\lambda > 0$ in this case and therefore the function R is strictly increasing. For the last part of the theorem we will use the same notation used in the previous sections, and in particular we define the functions w, λ on the whole manifold as before, and we extend the function r to the manifold by defining it as $r = \frac{w}{\sqrt{c}}$, we will also assume that γ will denote a geodesic defined by the vector field e_n. We have that,

1. the vector $\lambda\, r e_n + e_n(r)\, \nu$ is a unit constant vector on the whole manifold, we can assume that this vector is the vector $(0,\dots,0,1)$
2. The vector $\eta = -e_n(r)\, e_n + \lambda\, r\, \nu$ is constant along a geodesic γ, i.e, we can prove that $\bar{\nabla}_{e_n}\eta$ vanishes. We also have that η is a unit vector perpendicular to the vector defined in the previous item.
3. From the last items we can solve for e_n in terms of the vectors $(0,\dots,0,1)$ and η along a geodesic γ, and then, integrate in order to get the one of this geodesics. Using the differential equation for r at the beginning of the proof, we get that $e_n = \lambda\, r(0,\dots,0,1) - e_n(r)\eta$
4. Similar to the case of S^n, we can show that $\nabla_v e_n = \frac{e_n(r)}{r} e_n$ for every $v \in \mathrm{Span}\{e_1,\dots,e_{n-1}\}$. Therefore, the vector field $x + r\eta$ is independent of the integral submanifolds of the distribution $\mathrm{Span}\{e_1,\dots,e_{n-1}\}$.
5. The previous considerations and the fact that the vectors $e_1,\dots e_{n-1}$ are perpendicular to the vector η and $(0,\dots,0,1)$ imply that the integral submanifolds of the distribution $\mathrm{Span}\{e_1,\dots,e_{n-1}\}$ are spheres with center at $x + r\eta$ and radius r. Notice that $||x - (x + r\eta)|| = r$.
6. If we fix a point p_0 and we define the geodesic $\gamma(u)$ as before, then, without loss of generality we may assume that $\eta(p_0) = (0,\dots,0,1,0) = \eta(u)$ and therefore, we can also assume by doing a translation, if necessary, that

$$\gamma(u) = \int_0^u e_n(u) = \int_0^u (0,\dots,0,-r'(u),\lambda(u)\, r(u)) = (0,\dots,-r(u),R(u))$$

   Where $R(u) = \int_0^u \lambda(t)\, r(t) dt$. The theorem follows by noticing that the center of the integral submanifolds take the form $\gamma(u) + r(u)\eta(\gamma(u)) = (0,\dots,0,R(u))$

□

In the case $n = 2$ we can find explicit solutions. For any positive $C > 4H$, they look like,

$$\phi(u,v) = (\, r(u)\cos(v), r(u)\sin(v), R(u))$$

where,

$$R(u) = \int_0^u \frac{C + \sqrt{C(C-4H)}\cos(2Hy)}{\sqrt{2C}\sqrt{C - 2H + \sqrt{C(C-4H)}\cos(2Hy)}}\, dy$$

$$\text{and} \quad r(u) = \frac{\sqrt{C - 2H + \sqrt{C(C-4H)}\cos(2Hu)}}{\sqrt{2}\sqrt{C}H}.$$

Here there is the graph of a non embedded Delaunay surface,

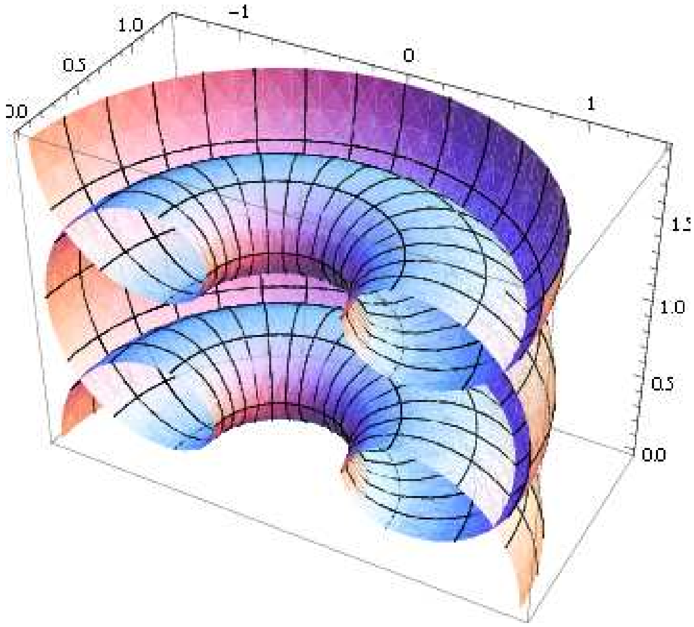

FIG. 6.21. *Half rotation of a non embedded Delaunay surface with CMC $H = -1$, here $C = 2$*

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