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On the nonexistence of uniform homeomorphisms between L_p-spaces

By PER ENFLO

The main result of this paper shows that an infinite-dimensional $L_{p_1}(\mu_1)$ is not uniformly homeomorphic with $L_{p_2}(\mu_2)$ if $p_1 \neq p_2$, $1 \leqslant p_i \leqslant 2$ (our conclusions will in fact be stronger). This gives an affirmative answer to a conjecture by Lindenstrauss [1]. The method used here is quite different from that suggested by Lindenstrauss. We will use the terminology of [1]. In the sequel we will consider $L_{p_1}(0, 1)$ but it is easy to see that with slight adjustments of the proofs the results hold for $L_{p_1}(\mu_1)$ and $L_{p_2}(\mu_2)$ as well.

1. A geometric property of L_p (0,1)

We shall say that a metric space has roundness p if p is the supremum of the set of q:s with the property: for every quadruple of points a_{00}, a_{01}, a_{11}, a_{10}

$$[d(a_{00}, a_{01})]^q + [d(a_{01}, a_{11})]^q + [d(a_{11}, a_{10})]^q + [d(a_{10}, a_{00})]^q \geqslant [d(a_{00}, a_{11})]^q + [d(a_{01}, a_{10})]^q \tag{1}$$

The triangle inequality shows that (1) is always satisfied if $q=1$. If the metric space has the property that every pair of points has a metric middle point, (1) is not satisfied for all quadruples if $q>2$. We see this by choosing a_{01} as the middle point between a_{00} and a_{11} and choosing $a_{01}=a_{10}$. Of course (1) is also satisfied for $q=p$.

Theorem 1.1. $L_p(0, 1)$, $1 \leqslant p \leqslant 2$, *has roundness* p.

Proof. We first prove that $\int_0^1(|f_{00}-f_{01}|^p + |f_{01}-f_{11}|^p + |f_{11}-f_{10}|^p + |f_{10}-f_{00}|^p - |f_{00}-f_{11}|^p - |f_{01}-f_{10}|^p)\,dt \geqslant 0$. We observe that it is enough to prove that the integrand is nonnegative. This is an inequality involving four real numbers and we can assume that the least of them is 0. Thus we have to prove $x^p + |z-x|^p + y^p + |z-y|^p - z^p - |y-x|^p \geqslant 0$. We observe that it is enough to prove this inequality for $1<p<2$. The inequality holds for $z=0$, $z=x$, $z=y$. The derivative with respect to z of the left side is not positive in the intervals $[0, \min(x, y)]$ and $[\min(x, y), \max(x, y)]$. Thus we can assume $0 \leqslant x \leqslant y \leqslant z$. We keep z fixed and observe that the inequality holds for $x=0$, $x=y$ and $y=z$. We then form the partial derivatives with respect to x and y of the left side and observe that both of them equals zero only when $x=y=z/2$. We finally observe that the inequality holds under this assumption. Thus the inequality is proved.

By choosing $f_{00}\equiv 0$, $f_{01}(x)=1$ on the interval $0\leqslant x\leqslant \frac{1}{2}$, and 0 on the interval $\frac{1}{2}\leqslant x\leqslant 1$, $f_{10}(x)=0$ on the interval $0\leqslant x<\frac{1}{2}$, and 1 on the interval $\frac{1}{2}\leqslant x\leqslant 1$, $f_{11}\equiv 1$ we see that $L_p(0, 1)$ has not roundness larger than p. The theorem is proved.

2. Spaces with roundness p

We shall say that a set of 2^n points (not necessarily different) in a metric space is an n-dimensional cube, if each of the points is indexed by an n-vector whose components are 0 and 1. There are 2^n n-vectors of this type. We shall say that a pair of points in an n-dimensional cube is an edge if the indexes of the points differ in only one component. We shall say that a pair of points is an m-diagonal if the indexes of the points differ in exactly m components. We shall say that the set of points where m components of the indexes are fixed is an $(n-m)$-dimensional side.

Theorem 2.1. *In an n-dimensional cube in a metric space with roundness p,* $\Sigma s_\alpha^p \geqslant \Sigma (d_{\beta,n})^p$ *where* s_α *runs through the lengths of all edges and* $d_{\beta,n}$ *runs through the lengths of all n-diagonals.*

Proof. The theorem is true for $n=2$ by definition of roundness ($n=1$ is quite trivial). We assume that it is true for $n-1$ and shall prove it for n.

We consider the two $(n-1)$-dimensional sides S_1 and S_2 which are characterized by a fixed first component of the indexes. We then consider an n-diagonal. We construct a two-dimensional cube by letting two of its points be the points of the n-diagonal and two of its points be the points we get when we change the first indexes of the points of the n-diagonal.

For this two-dimensional cube we have

$$s_1^p + s_2^p + d_{1,n-1}^p + d_{2,n-1}^p \geqslant d_{1,n}^p + d_{2,n}^p \qquad (2)$$

where s_i are the lengths of the edges we get, $d_{i,n-1}$ the lengths of the $(n-1)$-diagonals and $d_{i,n}$ the lengths of the n-diagonals.

We now make a series of such constructions, so that every n-diagonal appears once on the right side of an inequality of type (2). Then every edge where the first components of the indexes differ will appear once on the left side and every $(n-1)$-diagonal of S_1 and S_2 will appear once. On S_1 and S_2 we have by the induction hypothesis $\Sigma_{s_i} s_\alpha^p \geqslant \Sigma_{s_i} d_{\beta,n-1}^p$ and thus for the n-dimensional cube we have $\Sigma s_\alpha^p \geqslant \Sigma d_{\beta,n}^p$. The theorem is proved.

Since in an n-dimensional cube there are $n\cdot 2^{n-1}$ edges and 2^{n-1} n-diagonals we get $n\cdot 2^{n-1} s_{\max}^p \geqslant 2^{n-1} d_{n,\min}^p$ which gives the

Corollary. *In an n-dimensional cube in a metric space with roundness p we have* $n^{1/p}\cdot s_{\max} \geqslant d_{n,\min}$.

We can now state the main theorem of this paper. It is also related to Smirnov's question: Is every separable metric space uniformly homeomorphic with a subset of l_2, and gives a stronger partial negative result than Theorem 12 in [1]. We shall say that a map T from a metric space into a metric space satisfies a Lipschitz condition of order α for large distances, if for every $\varepsilon>0$ there is a C such that $d(T(x), T(y)) \leqslant C\cdot (d(x, y))^\alpha$ if $d(x, y)>\varepsilon$. We recall the lemma from [1], that a uniformly continuous map from a convex set in a Banach space into a metric space satisfies a first order Lipschitz condition for large distances. (3)

Theorem 2.2. *If $L_{p_1}(0, 1)$, $1 \leqslant p_1 \leqslant 2$, is uniformly homeomorphic with a metric space with roundness $p_2 > p_1$, and T is a uniform homeomorphism, then T^{-1} does not satisfy a Lipschitz condition of order less than p_2/p_1 for large distances.*

Proof. Put $\sup_{d(x,y)\leqslant 1} d(T(x), T(y)) = K$. We construct an n-dimensional cube in $L_{p_1}(0, 1)$ in the following way. We divide $[0, 1]$ into n intervals of length $1/n$ and let the functions in the cube be either n^{1/p_1} or 0 on each interval. We give a function index by letting the mth component of the n-vector be 0 if the function is 0 on the mth interval and 1 if the function is n^{1/p_1} on this interval. In this cube every edge has length 1 and every n-diagonal length n^{1/p_1}. In the image under T of this cube every edge has length $\leqslant K$ and thus the shortest n-diagonal has length $\leqslant K \cdot n^{1/p_2}$, by the corollary of Theorem 2.1. But for every C, $C \cdot (K \cdot n^{1/p_2})^\alpha < n^{1/p_1}$ if $\alpha < p_2/p_1$ and n is sufficiently large. This completes the proof of the theorem.

Theorem 1.1, Theorem 2.2 and (3) give the

Corollary. *$L_{p_1}(0, 1)$ and $L_{p_2}(0, 1)$ are not uniformly homeomorphic if $p_1 \neq p_2$, $1 \leqslant p_i \leqslant 2$.*

REFERENCES

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