ARKIV FÖR MATEMATIK Band 8 nr 12

1.69 06 1 Communicated 9 April 1969 by LENNART CARLESON

# On a problem of Smirnov

By PER ENFLO

From the theorem that every separable metric space is isometric with a subset of $C(0, 1)$ and the theorem that all separable Banach spaces are homeomorphic it follows that every separable metric space is homeomorphic with a subset of $L_2(0, 1)$. In this paper we shall construct a countable metric space which is not uniformly homeomorphic with any subset of $L_2(0, 1)$. This gives a negative answer to a question asked by Smirnov. This question is the theme in [3] where among other things it is proved that Euclidean $n$-space is uniformly homeomorphic with a bounded subset of $L_2(0, 1)$. The question is also treated in [2] where a result in the negative direction is obtained and in [1] where a stronger result is obtained.

## 1. A geometric property of $L_2(0,1)$

We shall say that a set of $2n+2$ points in a metric space is a double $n$-simplex if the points are written $a_1, a_2, \dots a_{n+1}, b_1, b_2, \dots b_{n+1}$. We shall call a pair of points $(a_i, a_j)$ or $(b_i, b_j)$ $i \neq j$ an edge and a pair of points $(a_i, b_k)$ a connecting line. We shall say that a metric space $M$ has generalised roundness $p$, if $p$ is the supremum of the $q$'s with the property: for every $n \geqslant 1$ and every double $n$-simplex in $M, \Sigma c_\alpha^q \geqslant \Sigma s_\beta^q$ where $c_\alpha$ runs through the lengths of all connecting lines and $s_\beta$ runs through the lengths of all edges. In [1] we defined roundness to be the supremum of the $q$'s for which the inequality holds for double 1-simplexes. It is obvious that the generalised roundness is not larger than the roundness. In [1] it was proved that $L_p(0, 1)$ $1 \leqslant p \leqslant 2$ has rundness $p$. If a metric space has the property that some pair of points $(a_1, a_2)$ has a metric middle point $m$ then its roundness and thus its generalised roundness is not larger than 2. We see this by choosing $b_1 = b_2 = m$.

Since in every double $(n-1)$-simplex there are $n^2$ connecting lines and $n(n-1)$ edges the generalised roundness of a metric space is $\geqslant 0$. If in a double $(n-1)$-simplex we put the lengths of all connecting lines $=\frac{1}{2}$ and the lengths of all edges $=1$ then it is easy to see that we get a metric space with generalised roundness $-{}^2\log(1-1/n)$ which tends to 0 as $n \to \infty$. If in a double $(n-1)$-simplex we put instead the lengths of all connecting lines $=(1-1/n)^{1/q}$, $(1-1/n)^{1/q} \geqslant \frac{1}{2}$, $q>0$, it is easy to see that we get a metric space with generalised roundness $q$.

**Theorem 1.1.** $L_2(0, 1)$ *has generalised roundness* 2.

*Proof.* Since $L_2(0,1)$ has roundness 2, the generalised roundness is not larger than 2. Thus it is enough to prove $\Sigma c_\alpha^2 \geqslant \Sigma s_\beta^2$ for all double $n$-simplexes in $L_2(0, 1)$. This inequality is for a double $(n-1)$-simplex equivalent with the inequality

$$\int_0^1 \Big( \sum_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant n}} (a_i-b_j)^2 - \sum_{1\leqslant i<j\leqslant n} (a_i-a_j)^2 - \sum_{1\leqslant i<j\leqslant n} (b_i-b_j)^2 \Big)\, dx \geqslant 0$$

which holds since the integrand is equal to

$$\Big( \sum_{1\leqslant i\leqslant n} a_i - \sum_{1\leqslant i\leqslant n} b_i \Big)^2.$$

*Remark.* The identity above gives $\sum c_\alpha^2 = \sum s_\beta^2$ when the two simplexes have the same centre of gravity. For double 1-simplexes this is the parallellogram theorem.

## 2. Universal uniform embedding spaces

We shall say that a metric space $M$ is a universal uniform embedding space if every separable metric space is uniformly homeomorphic with a subset of $M$. $C(0, 1)$ is a universal uniform embedding space since every separable metric space is isometric with a subset of $C(0, 1)$. We now prove

**Theorem 2.1.** *Every universal uniform embedding space has generalised roundness* 0.

*Proof.* We prove that a metric space with generalised roundness $p>0$ is not a universal uniform embedding space.

Consider the metric space $M=\{\exp((2\pi k i)/2^{n+1}),\ k=0, 1, 2, \ldots, (2^{n+1}-1)\}$ where $n$ is an even number. Take the product $M_n$ of $n^n$ such spaces and define a metric in $M_n$ by letting the distance between two points in $M_n$ be the largest of the distances in the coordinate spaces. We shall say that a pair of points $(a, b)$ in $M_n$ is an $m$-segment if the coordinates of $a$ and $b$ are different in exactly $n^m$ coordinate spaces, and the difference between the coordinates in each of these spaces is $\exp((2\pi k i)/2^n) \times (\exp(\pi i/2^m)-1)$ where $k$ may depend on the coordinate space.

We shall consider double $(n-1)$-simplexes in $M_n$ where every connecting line is an $(m+1)$-segment and every edge is an $m$-segment. In such a double $(n-1)$-simplex every connecting line has length $|\exp(\pi i/2^{m+1})-1|$ and every edge has length $|\exp(\pi i/2^m)-1|$. If there exists such a double $(n-1)$-simplex then, by symmetry, it follows that for fixed $m$, all $m$-segments in $M_n$ are edges in the same number $N_1$ of double $(n-1)$-simplexes of that type and all $(m+1)$-segments in $M_n$ are connecting lines in the same number $N_2$ of double $(n-1)$-simplexes of that type. For if $s_1$ and $s_2$ are two $m$-segments then there is an isometry of $M_n$ onto itself by which $s_2$ is the image of $s_1$ and the image of each $k$-segment is a $k$-segment. We now prove the existence of such a double $(n-1)$-simplex, $1\leqslant m\leqslant n-1$.

We consider an ordering of the coordinate spaces and divide the first $n^{m+1}$ coordinate spaces into $2n$ groups with $n^m/2$ coordinate spaces in each. We let the coordinates of a point in one of the simplexes be $\exp(\pi i/2^m)$ in all coordinate spaces of one of the first $n$ groups and be 1 in the remaining coordinate spaces of the first $n$ groups. The coordinates are $\exp(\pi i/2^{m+1})$ in all the remaining coordinate spaces of $M_n$. We let the coordinates of a point in the other simplex be $\exp(\pi i/^{2m})$ in all coordinate spaces of one of the last $n$ groups and be 1 in the remaining coordinate spaces of the last $n$ groups. The coordinates are $\exp(\pi i/2^{m+1})$ in all the remaining coordinate spaces of $M_n$. This double $(n-1)$-simplex has the required properties.

If $M_n$ is embedded in a metric space $B$ with generalised roundness $p>0$ then we get $\sum d^p_{\alpha_1, m+1} \geqslant \sum d^p_{\alpha_2, m}$ for every double $(n-1)$-simplex of the type described above, where $d_{\alpha_j,k}$ runs through the lengths of the images of $k$-segments. We add all these inequalities for a fixed $m$. Then we get an inequality where all lengths of images of $m$-segments appear $N_1$ times on the right side and all lengths of images of $(m+1)$-segments appear $N_2$ times on the left side. Since there are $n^2$ connecting lines and $n(n-1)$ edges in each double $(n-1)$-simplex we get $S(d^p_{\alpha,m+1}) \geqslant [(n-1)/n] S(d^p_{\alpha,m})$ where $S(d^p_{\alpha,k})$ is the arithmetic mean of the $p$th powers of the lengths of the images of the $k$-segments. If we apply this last inequality $n-1$ times we get $S(d^p_{\alpha,n}) \geqslant [n-1/n]^{n-1} S(d^p_{\alpha,1})$. From this we get $\sup d^p_{\alpha,n} \geqslant 1/e \inf d^p_{\alpha,1}$ and $\sup d_{\alpha,n} \geqslant (1/e)^{(1/p)}$ $(\inf d_{\alpha,1})$.

Now we take the union of a countable family of sets $M_n$ where we let $n$ tend to infinity. We put the distance between two points in different $M_n$'s $=2$. If this metric space were uniformly homeomorphic with some subset of $B$ and $T$ were a uniform homeomorphism then $\inf d_{\alpha,1} \geqslant \varepsilon$ for some $\varepsilon > 0$ and all spaces $M_n$. But then $\sup d_{\alpha,n} \geqslant (1/e)^{1/p} \cdot \varepsilon$ for all $n$ and this contradicts that $T$ is uniformly continuous. The theorem is proved.

*Remark.* By a modification of the last construction of the proof we can get a metric space in which no non-void open set is uniformly homeomorphic with a subset of $B$.

*Department of Mathematics, University of Stockholm, 113 85 Stockholm, Sweden*

## REFERENCES

1. ENFLO, P., On the non-existence of uniform homeomorphisms between $L_p$-spaces, Arkiv för matematik. (To appear.)
2. LINDENSTRAUSS, J., On nonlinear projections in Banach spaces. The Michigan Math. Journal, *11*, 263–287 (1964).
3. GORIN, E. A., On uniformly topological embedding of metric spaces in Euclidean and in Hilbert spaces. Uspehi Mat. Nauk *14*, no. 5 (89), 129–134 (1959).

Tryckt den 18 december 1969

Uppsala 1969. Almqvist & Wiksells Boktryckeri AB