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An inequality of Paley and convergence a.e. of Walsh–Fourier series

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Introduction

Let S_n denote the nth partial sum of the Walsh–Fourier series of a function $f \in L^1(0,1)$. Let $Mf(x) = \sup_n |S_n(x)|$. In this paper we will prove results concerning the operator M, which are analogous to the results for trigonometric Fourier series proved by Carleson [2] and Hunt [3]. In [1] Billard has shown that the Walsh–Fourier series of an L^2-function converges a.e. Billard essentially uses the same method as Carleson but makes some modifications of the proof to adapt it to Walsh series. We proceed in the same way and use Hunt's variant of Carleson's method with modifications according to Billard. The results are:

Theorem.

(A) *If* $\int_0^1 |f(x)| \log^+ |f(x)| \log^+ \log^+ |f(x)|\, dx < \infty$, *then* $S_n(x)$ *converges a.e.*

(B) $\|Mf\|_1 \leqslant \text{Const.} \int_0^1 |f(x)| (\log^+ |f(x)|)^2\, dx + \text{Const.}$

(C) $\|Mf\|_p \leqslant C_p \|f\|_p, \ 1 < p < \infty.$

(D) $m\{x \in (0,1) \mid Mf(x) > y\} \leqslant \text{Const.} \exp(-\text{Const.}\, y/\|f\|_\infty),\ y > 0.$

To be able to prove (A), (B) and (D) we estimate certain constants in a theorem of Paley ([4], p. 249). This is done in Section 1. The result needed for the proof of the theorem is the following lemma:

Lemma 1.8. *Assume* $f \in L^\infty(0,1)$. *Let* $\eta = \{\eta_k\}_{-1}^\infty$ *with* $\eta_k = 0$ *or* 1, $k = -1, 0, 1, 2, \dots$. *Let*

$$Hf(t) = \sup_n \left| \sum_{k=-1}^{n} \eta_k \Delta_k(t) \right|,$$

where $\Delta_k = S_{2^{k+1}} - S_{2^k}$ *for* $k \geqslant 0$ *and* $\Delta_{-1} = S_1$. *Then*

$$m\{t \in (0,1) \mid Hf(t) > y\} \leqslant \text{Const.} \exp\left(-\text{Const.} \frac{y}{\|f\|_\infty}\right), \quad y > 0,$$

where the constants are independent of η.

In Section 2 we use Hunt's method and Lemma 1.8 to prove the following basic result:

Lemma 2.7. *Let χ_F be the characteristic function of a measurable set $F \subset (0,1)$. Then*

$$m\{x \in (0,1) \mid M\chi_F(x) > y\} \leqslant B_p^p y^{-p} mF, \quad y > 0, 1 < p < \infty,$$

where $B_p \leqslant \text{Const.}\, p^2/p-1$.

Using Lemma 2.7 we could continue in the same way as in [3] to prove (B), (C) and (D). However, to establish (B) and (D) we use another method, shown to us by Professor Carleson, which also gives (A). This is done in Section 3. For ordinary Fourier series this method gives the result

(E) If $\int_{-\pi}^{\pi} |f(x)| \log^+ |f(x)| \log^+ \log^+ |f(x)|\, dx < \infty$, then the Fourier series of $f(x)$ converges a.e.

1. Estimation of constants in a theorem of Paley

Let $\Phi_0, \Phi_1, \ldots$ denote the Rademacher functions and $w_0, w_1, \ldots$ the Walsh functions on $(0,1)$. We introduce notations for dyadic intervals:

$$\omega_{0,-2} = (-2,2), \quad \omega_{-1,-1} = (-2,0), \quad \omega_{0,-1} = (0,2) \quad \text{and}$$

$$\omega_{j\nu} = (j \cdot 2^{-\nu}, (j+1)2^{-\nu}), \quad \nu \text{ integer} \geqslant 0, \ j \text{ integer}, \quad -2\cdot 2^{\nu} \leqslant j \leqslant 2 \cdot 2^{\nu} - 1.$$

If $f \in L^1(\omega)$, $\omega = \omega_{j\nu}$, $\nu \geqslant 0$, we define

$$a_l = a_l(\omega) = a_l(\omega, f) = \frac{1}{|\omega|} \int_{\omega} f(t)\, w_{2^{\nu}l}(t)\, dt, \quad l = 0, 1, 2, \ldots,$$

$$S_n(x) = S_n(x,\omega) = S_n(x,\omega,f) = \sum_{l=0}^{n-1} a_l(\omega,f)\, w_{2^{\nu}l}(x), \quad n \geqslant 1, \quad \text{and} \quad S_0(x) = 0,$$

$$\Delta_k(x) = \Delta_k(x,\omega) = \Delta_k(x,\omega,f) = \sum_{l=2^k}^{2^{k+1}-1} a_l(\omega,f)\, w_{2^{\nu}l}(x), \quad k = 0, 1, \ldots, \quad \text{and}$$

$$\Delta_{-1}(x) = a_0 w_0(x).$$

For $j = 0, 1, 2, \ldots$, let

$$\delta_j^*(t) = \begin{cases} 2^j & \text{if} \quad 0 < t < 2^{-j-1}, \\ -2^j & \text{if} \quad 2^{-j-1} < t < 2^{-j}, \\ 0 & \text{otherwise}. \end{cases}$$

If $$n = \sum_{i=0}^{\infty} \varepsilon_i 2^i \ (\varepsilon_i = 0,1) \quad \text{we define} \quad \delta_n(t) = \sum_{\varepsilon_i = 1} \delta_i^*(t).$$

If $$x = \sum_1^{\infty} \varepsilon_i 2^{-i} \quad \text{and} \quad t = \sum_1^{\infty} \eta_i 2^{-i} (\xi_i, \eta_i = 0,1) \quad \text{let} \quad x \dotplus t = \sum_1^{\infty} |\xi_i - \eta_i| 2^{-i}.$$

It is wellknown that for every $x \in (0,1)$ the formula $w_n(x \dotplus t) = w_n(x)\, w_n(t)$ holds for almost all $t \in (0,1)$. For $\nu \geqslant 0$ we define a mapping

$$\varphi_\nu : \omega_{0\nu} \to (0,1) \quad \text{by} \quad \varphi_\nu\left(\sum_{\nu+1}^{\infty} \xi_i 2^{-i}\right) = \sum_{1}^{\infty} \xi_{\nu+i} 2^{-i}.$$

If $n = \sum_0^\infty \varepsilon_i 2^i$ let
$$n[\omega_{j\nu}] = \begin{cases} 2^{-\nu} \sum_{i \geqslant \nu} \varepsilon_i 2^i & \text{if} \quad \nu \geqslant 0, \\ n & \text{if} \quad \nu < 0. \end{cases}$$

We define $\nu^+ = \nu$ if $\nu \geqslant 0$ and $\nu^+ = 0$ if $\nu < 0$. If $n[\omega] = 2^{-\nu^+} n$ we write $n \operatorname{int}(\omega)$ $(\omega = \omega_{j\nu})$.

We first give formulas for S_n and Δ_k.

Lemma 1.1. *Assume $\omega = \omega_{j\nu} \subset \omega_{00}$ and $x \in \omega$. Then*

$$S_n(x,\omega) = \int_\omega f(t)\, w_{2^\nu n}(x \dot{+} t)\, \delta_{2^\nu n}(x \dot{+} t)\, dt, \quad n \geqslant 0, \tag{1.1}$$

$$\Delta_k(x,\omega) = \int_\omega f(t)\, \delta^*_{k+\nu}(x \dot{+} t)\, dt, \quad k \geqslant 0. \tag{1.2}$$

Proof.
$$S_n(x,\omega) = \sum_{l=0}^{n-1} a_l(\omega, f)\, w_{2^\nu l}(x) = 2^\nu \sum_0^{n-1} \int_\omega f(t)\, w_{2^\nu l}(x \dot{+} t)\, dt$$
$$= 2^\nu \int_\omega f(t) \left(\sum_0^{n-1} w_{2^\nu l}(x \dot{+} t) \right) dt.$$

Using (1–4) in [1], p. 363, we get $(t \in \omega)$

$$\sum_0^{n-1} w_{2^\nu l}(x \dot{+} t) = \sum_0^{n-1} w_l(\varphi_\nu(x \dot{+} t)) = w_n(\varphi_\nu(x \dot{+} t))\, \delta_n(\varphi_\nu(x \dot{+} t)) = 2^{-\nu} w_{2^\nu n}(x \dot{+} t)\, \delta_{2^\nu n}(x \dot{+} t),$$

which yields (1.1).

(1.2) can be proved in the same way if we use the equality

$$\sum_{2^k}^{2^{k+1}-1} w_l(x) = \delta_k^*(x).$$

If $f \in L^1(0,1)$ we define a maximal function f^*:

$$f^*(t) = \sup_I \frac{1}{mI} \int_I |f|\, dx,$$

where supremum is to be taken over all intervals I satisfying $t \in I \subset (0,1)$. It is wellknown that the following estimate holds:

Lemma 1.2. *There exists a constant $C > 1$ such that for $2 \leqslant p < \infty$*

$$\int_0^1 |f^*|^p\, dt \leqslant C \int_0^1 |f|^p\, dt. \tag{1.3}$$

For a proof see [5], p. 652.

We need some more lemmas.

Lemma 1.3. *If C is the same constant as in* (1.3) *the following inequality holds*

$$\int_0^1 (\sup_n |S_{2^n}|)^p \, dt \leqslant C \int_0^1 |f|^p \, dt, \quad 2 \leqslant p < \infty. \tag{1.4}$$

Proof. From (1.1) it follows that

$$|S_{2^n}(t)| \leqslant 2^n \int_I |f| \, dx,$$

where $t \in I \subset (0,1)$ and $mI = 2^{-n}$. We therefore have $\sup_n |S_{2^n}(t)| \leqslant f^*(t)$, which yields the lemma.

Lemma 1.4. *Let $m_1 > \max(m_2, m_3, \ldots, m_q)$ and let r be an odd positive integer. Then*

$$\int_0^1 \Delta_{m_1}^r \Delta_{m_2} \Delta_{m_3} \ldots \Delta_{m_q} \, dt = 0. \tag{1.5}$$

Proof. This lemma is proved for $r = 1$ in [4], p. 250, and the same proof holds also in this case.

Theorem 1.5. *Assume k is an even integer $\geqslant 2$ and $f \in L^k(0,1)$, f real. Then*

$$\left\| \left(\sum_{-1}^{\infty} \Delta_n^2 \right)^{1/2} \right\|_k \leqslant \text{Const.} \sqrt{k} \, \|f\|_k. \tag{1.6}$$

Proof. In [4] (pp. 253–254) Paley proves $\|(\sum_{-1}^{\infty} \Delta_n^2)^{1/2}\|_k \leqslant B_k \|f\|_k$. However, the method used by Paley gives $B_k > (\text{Const.})^k$ and we modify the proof to get (1.6).

Let $\nu = \frac{1}{2}k$ and $F_N = \sum_{-1}^{N-1} \Delta_n = S_{2^N}$. Assume $N - 1 \geqslant n_1 \geqslant n_2 \geqslant \ldots \geqslant n_{\nu-1}$. We have

$$F_N^2 = \left(F_{n_1} + \sum_{n_1}^{N-1} \Delta_n \right)^2 = F_{n_1}^2 + \sum_{n=n_1}^{N-1} \Delta_n^2 + 2 \sum_{n=n_1}^{N-1} \Delta_n F_{n_1} + \sum_{\substack{n \neq m \\ n_1 \leqslant n,\, m \leqslant N-1}} \Delta_n \Delta_m,$$

and it follows that

$$\int_0^1 \Delta_{n_1}^2 \ldots \Delta_{n_{\nu-1}}^2 F_N^2 \, dt = \int_0^1 \Delta_{n_1}^2 \ldots \Delta_{n_{\nu-1}}^2 F_{n_1}^2 \, dt + \sum_{n=n_1}^{N-1} \int_0^1 \Delta_{n_1}^2 \ldots \Delta_{n_{\nu-1}}^2 \Delta_n^2 \, dt$$
$$+ 2 \sum_{n=n_1}^{N-1} \int_0^1 \Delta_{n_1}^2 \ldots \Delta_{n_{\nu-1}}^2 \Delta_n F_{n_1} \, dt + \sum_{\substack{n \neq m \\ n_1 \leqslant n,\, m \leqslant N-1}} \int_0^1 \Delta_{n_1}^2 \ldots \Delta_{n_{\nu-1}}^2 \Delta_n \Delta_m \, dt.$$

The two last terms vanish according to Lemma 1.4 and we get

$$\sum_{n=n_1}^{N-1} \int_0^1 \Delta_{n_1}^2 \ldots \Delta_{n_{\nu-1}}^2 \Delta_n^2 \, dt \leqslant \int_0^1 \Delta_{n_1}^2 \ldots \Delta_{n_{\nu-1}}^2 F_N^2 \, dt. \tag{1.7}$$

Let $\gamma(n_0, \ldots, n_{\nu-1})$ denote the number of different permutations of $n_0, \ldots, n_{\nu-1}$. We have

$$\int_0^1 \left(\sum_{-1}^{N-1} \Delta_n^2\right)^{k/2} dt = \sum_{\substack{n_0, \ldots, n_{\nu-1} \\ N-1 \geqslant n_0 \geqslant \ldots \geqslant n_{\nu-1}}} \gamma(n_0, \ldots, n_{\nu-1}) \int_0^1 \Delta_{n_0}^2 \ldots \Delta_{n_{\nu-1}}^2 \, dt.$$

It is easy to verify that $\gamma(n_0, \ldots, n_{\nu-1}) \leqslant \nu\gamma(n_1, \ldots, n_{\nu-1})$. Using this inequality and (1.7) we get

$$\int_0^1 \left(\sum_{-1}^{N-1} \Delta_n^2\right)^{k/2} dt = \sum_{\substack{n_1, \ldots, n_{\nu-1} \\ N-1 \geqslant n_1 \geqslant \ldots \geqslant n_{\nu-1}}} \sum_{n=n_1}^{N-1} \gamma(n, n_1, \ldots, n_{\nu-1}) \int_0^1 \Delta_{n_1}^2 \ldots \Delta_{n_{\nu-1}}^2 \Delta_n^2 \, dt$$

$$\leqslant \nu \sum_{\substack{n_1, \ldots, n_{\nu-1} \\ N-1 \geqslant n_1 \geqslant \ldots \geqslant n_{\nu-1}}} \gamma(n_1, \ldots, n_{\nu-1}) \int_0^1 \Delta_{n_1}^2 \ldots \Delta_{n_{\nu-1}}^2 F_N^2 \, dt$$

$$= \nu \int_0^1 \left(\sum_{\substack{n_1, \ldots, n_{\nu-1} \\ N-1 \geqslant n_1 \geqslant \ldots \geqslant n_{\nu-1}}} \gamma(n_1, \ldots, n_{\nu-1}) \Delta_{n_1}^2 \ldots \Delta_{n_{\nu-1}}^2 \right) F_N^2 \, dt$$

$$= \nu \int_0^1 \left(\sum_{-1}^{N-1} \Delta_n^2\right)^{\nu-1} F_n^2 \, dt.$$

Hölder's inequality now gives

$$\int_0^1 \left(\sum_{-1}^{N-1} \Delta_n^2\right)^{\nu} dt \leqslant \nu \left(\int_0^1 \left(\sum_{-1}^{N-1} \Delta_n^2\right)^{\nu} dt\right)^{(\nu-1)/\nu} \left(\int_0^1 F_N^{2\nu} \, dt\right)^{1/\nu},$$

that is
$$\int_0^1 \left(\sum_{-1}^{N-1} \Delta_n^2\right)^{\nu} dt \leqslant \nu^\nu \int_0^1 F_N^{2\nu} \, dt.$$

Lemma 1.3 implies
$$\int_0^1 \left(\sum_{-1}^{N-1} \Delta_n^2\right)^{\nu} dt \leqslant \nu^\nu C \int_0^1 |f|^{2\nu} \, dt,$$

and (1.6) follows from this estimate.

Theorem 1.6. *If k is an even integer $\geqslant 2$ and f is real, then*

$$\|f\|_k \leqslant \text{Const.}\, k \left\| \left(\sum_{-1}^{\infty} \Delta_n^2\right)^{1/2} \right\|_k. \tag{1.8}$$

Proof. Paley has proved (1.8) with a constant B_k instead of Const. k ([4], pp. 251–252), but his method gives a larger value for B_k than Const. k.

We first prove

$$\int_0^1 F_{N+1}^k \, dt \leqslant (Ck)^k \int_0^1 \left(\sum_{-1}^{\infty} \Delta_n^2\right)^{k/2} dt, \quad N = -1, 0, 1, 2, \ldots, \tag{1.9}$$

where C is the same constant as in Lemma 1.2.

We show (1.9) by induction with respect to N. For $N=-1$ we have

$$\int_0^1 F_0^k\,dt=\int_0^1 \Delta_{-1}^k\,dt\leqslant\int_0^1\left(\sum_{-1}^{\infty}\Delta_n^2\right)^{k/2}dt.$$

Now assume
$$\int_0^1 F_N^k\,dt\leqslant(Ck)^k\int_0^1\left(\sum_{-1}^{\infty}\Delta_n^2\right)^{k/2}dt. \tag{1.10}$$

For $0\leqslant n\leqslant N$ we have

$$\int_0^1(F_{n+1}^k-F_n^k)\,dt=\int_0^1[(F_n+\Delta_n)^k-F_n^k]\,dt=\sum_{i=1}^{k}\binom{k}{i}\int_0^1\Delta_n^i F_n^{k-i}\,dt$$
$$=\sum_{j=1}^{\frac12 k}\binom{k}{2j}\int_0^1\Delta_n^{2j}F_n^{k-2j}\,dt,$$

because the terms with odd i equal 0 according to Lemma 1.4. We therefore get

$$\begin{aligned}\int_0^1 F_{N+1}^k\,dt&=\sum_{n=0}^{N}\int_0^1(F_{n+1}^k-F_n^k)\,dt+\int_0^1 F_0^k\,dt\\&=\sum_{n=0}^{N}\sum_{j=1}^{\frac12 k}\binom{k}{2j}\int_0^1\Delta_n^{2j}F_n^{k-2j}\,dt+\int_0^1\Delta_{-1}^k\,dt\\&=\sum_{j=1}^{\frac12 k}\binom{k}{2j}\int_0^1\left(\sum_{n=0}^{N}\Delta_n^{2j}F_n^{k-2j}\right)dt+\int_0^1\Delta_{-1}^k\,dt\\&\leqslant\sum_{j=1}^{\frac12 k}\binom{k}{2j}\int_0^1\left(\sum_{n=-1}^{N}\Delta_n^{2j}\right)\max_{0\leqslant n\leqslant N}F_n^{k-2j}\,dt\\&\leqslant\sum_{j=1}^{\frac12 k}\binom{k}{2j}\int_0^1\left(\sum_{-1}^{\infty}\Delta_n^{2}\right)^{j}\max_{0\leqslant n\leqslant N}F_n^{k-2j}\,dt.\end{aligned}$$

Thus, letting
$$I_j=\int_0^1\left(\sum_{-1}^{\infty}\Delta_n^2\right)^j\max_{0\leqslant n\leqslant N}F_n^{k-2j}\,dt,$$

we have
$$\int_0^1 F_{N+1}^k\,dt\leqslant\sum_{j=1}^{\frac12 k}\binom{k}{2j}I_j. \tag{1.11}$$

For $j<\frac12 k$ Hölder's inequality implies

$$I_j\leqslant\left(\int_0^1\left(\sum_{-1}^{\infty}\Delta_n^2\right)^{k/2}dt\right)^{(2j)/k}\left(\int_0^1\left(\max_{0\leqslant n\leqslant N}|F_n|\right)^k dt\right)^{(k-2j)/k}.$$

According to Lemma 1.3 and the induction assumption (1.10) we have

$$\int_0^1\left(\max_{0\leqslant n\leqslant N}|F_n|\right)^k dt\leqslant C\int_0^1 F_N^k\,dt\leqslant C(Ck)^k\int_0^1\left(\sum_{-1}^{\infty}\Delta_n^2\right)^{k/2}dt.$$

It follows that

$$I_j \leqslant C^{(k-2j)/k}(Ck)^{k-2j}\int_0^1\left(\sum_{-1}^{\infty}\Delta_n^2\right)^{k/2}dt \leqslant C(Ck)^k(Ck)^{-2j}\int_0^1\left(\sum_{-1}^{\infty}\Delta_n^2\right)^{k/2}dt.$$

This inequality obviously holds also for $j=\frac{1}{2}k$.

Using (1.11) we get

$$\int_0^1 F_{N+1}^k\,dt \leqslant \int_0^1\left(\sum_{-1}^{\infty}\Delta_n^2\right)^{k/2}dt\,(Ck)^k C\sum_{j=1}^{\frac{1}{2}k}\binom{k}{2j}(Ck)^{-2j}.$$

We have

$$\sum_{j=1}^{\frac{1}{2}k}\binom{k}{2j}(Ck)^{-2j} \leqslant \sum_{i=2}^{k}\binom{k}{i}(Ck)^{-i} \leqslant \frac{1}{C}\sum_{i=2}^{k}\frac{k(k-1)\dots(k-i+1)}{1\cdot 2\cdot\ldots\cdot i}\cdot\frac{1}{k^i} \leqslant \frac{1}{C}\sum_{i=2}^{k}2^{-i+1} \leqslant \frac{1}{C},$$

and (1.9) follows.

We can now continue in the same way as in Paley's proof ([4], p. 252) and we get (1.8).

Remark. Theorems 1.5 and 1.6 hold even if k is not an even integer. To extend Theorem 1.5 we can use Marcinkiewicz's interpolation theorem and the extension of Theorem 1.6 follows from an argument similar to the proof of Theorem 1.7.

In the sequel let $\eta=\{\eta_k\}_{k=-1}^{\infty}$, where $\eta_k=0$ or 1.

Theorem 1.7. *Assume* $f\in L^p(0,1)$, $p\geqslant 2$, *and let* $f_\eta(t)=\sum_{-1}^{\infty}\eta_n\Delta_n(t,f)$. *Then*

$$\|f_\eta\|_p \leqslant \text{Const.}\, p\,\|f\|_p. \tag{1.12}$$

Proof. A combination of Theorems 1.5 and 1.6 gives $\|f_\eta\|_p\leqslant \text{Const.}\,p^{3/2}\|f\|_p$, $p\geqslant 2$. In the original version of this paper we used this estimate instead of (1.12) and obtained a slightly weaker theorem than the theorem in the introduction. However, the inequality (1.12) can be obtained as a consequence of a result of C. Watari [6]. Watari proves

$$m\{x\in(0,1)\mid |f_\eta(x)|>y\}\leqslant \text{Const.}\,\frac{\|f\|_1}{y},\quad y>0.$$

Using this result and Parseval's formula and applying Marcinkiewicz's interpolation theorem we obtain

$$\|f_\eta\|_p\leqslant \text{Const.}\,\frac{1}{p-1}\|f\|_p,\quad 1<p<\tfrac{3}{2}. \tag{1.13}$$

(1.12) now follows from (1.13) and a standard argument with conjugate indices.

We will now prove Lemma 1.8 (see the introduction).

Proof of Lemma 1.8. First assume $f\in L^p(0,1)$, $p\geqslant 2$. We have

$$Hf(t)=\sup_n\left|\sum_{k=-1}^{n}\eta_k\Delta_k(t,f)\right|=\sup_n\left|\sum_{k=-1}^{n}\Delta_k(t,f_\eta)\right|=\sup_n|S_{2^n}(t,f_\eta)|.$$

From Lemma 1.3 and Theorem 1.7 we get

$$\|Hf\|_p \leqslant \text{Const.}\ \|f_\eta\|_p \leqslant \text{Const.}\ p\,\|f\|_p.$$

From the proof of Theorem (4.41) in [7], p. 119, it follows that $|f| \leqslant 1$ implies

$$\int_0^1 \exp\,(\text{Const.}\ |Hf|)\,dx \leqslant \text{Const.},$$

and the lemma is an easy consequence of this estimate.

By a change of scale we can prove Lemma 1.8 with $(0, 1)$ replaced by an arbitrary interval $\omega = \omega_{j\nu}$:

Lemma 1.9. *Let* $f \in L^\infty(\omega)$, $\omega = \omega_{j\nu}$, $\nu \geqslant 0$. *Let* $Hf(t) = \sup_n |\sum_{k=-1}^{n} \eta_k \Delta_k(t, \omega, f)|$. *Then*

$$m\{t \in \omega \mid Hf(t) > y\} \leqslant \text{Const.}\ |\omega| \exp\left(-\text{Const.}\ \frac{y}{\|f\|_\infty}\right), \quad y > 0. \tag{1.14}$$

2. Proof of the basic result

We need some more notations. If $f \in L^1(\omega)$, $\omega = \omega_{j\nu}$, $\nu \geqslant 0$, we write

$$A_n(\omega) = A_n(\omega, f) = \tfrac{1}{10} \sum_{l=0}^{\infty} |a_l(\omega)| \frac{1}{1 + (l-n)^2}$$

and $A^*_{n[\omega]}(\omega) = \max_{\omega'} A_{n[\omega']}(\omega')$, where maximum is to be taken over the four subintervals ω' of ω with $4|\omega'| = |\omega|$. If $f \in L^1(0, 1)$, $n \geqslant 0$ and $x \in \omega = \omega_{j\nu} \subset \omega_{00}$ we define

$$S^*_n(x, \omega) = S^*_n(x, \omega, f) = \int_\omega f(t)\, w_n(t)\, \delta_n(x \dotplus t)\, dt.$$

For $x \in \omega_{00}$ let $S^*_n(x) = S^*_n(x, \omega_{0,-2}) = S^*_n(x, \omega_{0,-1}) = S^*_n(x, \omega_{00})$ and $S_n(x) = S_n(x, \omega_{00})$. From (1.1) it follows that

$$|S_n(x)| = |S^*_n(x)|. \tag{2.1}$$

We also define the operators M and M_N (N positive integer) by

$$Mf(x) = \sup_n |S_n(x)| \quad \text{and} \quad M_N f(x) = \sup_{n \leqslant N} |S_n(x)|.$$

We will now prove the basic result using Lemma 1.9 and the method in [3], Sections 5–11. Since our proof contains no new ideas, but is just a combination of the proofs in [1] and [3], we do not give all the details.

Starting from a function $f = \chi_F$ and numbers $1 < p < \infty$, $y > 0$, $N > 0$ and $k \geqslant 0$, we define the Walsh polynomials $P_k(x, \omega)$ ($\omega = \omega_{j\nu}$, $\nu \geqslant 0$) and the set G_k ([3], Section 5). We get

$$\text{If } a'_n \text{ is a coefficient of } P_k(x, \omega) \text{ then } |a'_n| \geqslant b_k y^{p/2}\ (b_k = 2^{-k}), \tag{2.2}$$

$$|a_m(\omega, f-P(\cdot, \omega))| < b_k y^{p/2} \text{ for all } m, \tag{2.3}$$

$$\sum_{(n,\omega)\in G_k} |\omega| \leqslant 4 b_k^{-2} y^{-p} mF. \tag{2.4}$$

We define $A_k(x)$ and X_k as in [3], and take X_k^* as the union of all intervals $\omega_{j',\nu-2}$ such that $\omega_{j',\nu-2} \supset \omega_{j\nu}$ for some $\omega_{j\nu} \subset X_k$. We then have

$$mX_k^* \leqslant 16 b_k y^{-p} mF, \tag{2.5}$$

$$\text{If } \omega \not\subset X_k \text{ then } P_k(x,\omega) \text{ has at most } b_k^{-3} \text{ terms}, \tag{2.6}$$

$$\text{If } \omega \not\subset X_k \text{ and } P_k(x,\omega) = \sum a_n' w_{\lambda_n}(x) \text{ then } |P_k(x,\omega)| \leqslant \sum |a_n'| \leqslant b_k^{-2} y^{p/2}. \tag{2.7}$$

Defining $\tilde{G}_k$ as in [3] and then G_k^* as in [1], p. 373, we get

$$\sum_{(n,\omega)\in G_k^*} |\omega| \leqslant \text{Const. } b_k^{-16} y^{-p} mF. \tag{2.8}$$

The following two lemmas are consequences of the definitions made above (see [1], pp. 373–374).

Lemma 2.1. *Assume* $\omega = \omega_{j\nu}$, $\nu \geqslant 0$, $\omega \not\subset X_k$ *and* $(n, \omega) \notin \tilde{G}_k$ $(k \geqslant 1)$. *Let* $Q_1(x)$ *denote the sum of those* $aw_\lambda(x)$ *in* $P_k(x,\omega)$ *for which* $|n - \lambda[\omega]| \geqslant b_k^{-10}$ *holds. Let*

$$P_k(x,\omega) = Q_0(x) + Q_1(x). \tag{2.9}$$

For $x \in \omega$ *we then have*

$$Q_0(x) = a_1 w_{\lambda_1}(x) + a_2 w_{\lambda_2}(x), \text{ where } |a_2| \leqslant |a_1| \leqslant b_k^{-2} y^{p/2} \text{ and } |\lambda_1[\omega] - \lambda_2[\omega]| = 1. \tag{2.10}$$

Lemma 2.2. *Assume* $\omega \not\subset X_k^*$ *and* $(n[\omega], \omega) \notin G_k^*$. *Let* ω_j, $j = 1, 2, 3, 4$ *satisfy* $4|\omega_j| = |\omega|$, $\omega_j \subset \omega$. *Let* $P_k(x, \omega_j) = Q_0(x, \omega_j) + Q_1(x, \omega_j)$, *where* Q_1 *contains those terms* aw_λ *in* P_k *for which* $|n[\omega_j]| - \lambda[\omega_j]| \geqslant b_k^{-10}$.

Then $Q_0(x, \omega_1) = Q_0(x, \omega_2) = Q_0(x, \omega_3) = Q_0(x, \omega_4)$.

We now define the set S as in [3] and take S^* as the union of all $\omega_{j',\nu-2}$ for which $\omega_{j',\nu-2} \supset \omega_{j\nu}$ for some $\omega_{j\nu} \subset S$. We get

$$mS^* \leqslant 16 y^{-p} mF, \tag{2.11}$$

$$\omega_0 \not\subset S^* \text{ implies } A_n^*(\omega_0) < y \text{ and } A_n^*(\omega_0) < y^p \text{ for all } n, \tag{2.12}$$

$$\text{If } \omega_0 \not\subset S^* \text{ then } b_{k-1} y > A_n^*(\omega_0) \geqslant b_k y \text{ for some } k \geqslant 1, \text{ unless } f = 0 \text{ a.e. on } \omega_0. \tag{2.13}$$

According to [3] there exists an integer $L = L(p)$ such that

$$L(p) \leqslant \text{Const. } \frac{p^2}{p-1} \tag{2.14}$$

and

$$\omega_0 \not\subset S^* \quad \text{and} \quad A_n^*(\omega_0) \geqslant b_k y \quad \text{imply} \quad y^{p/2} \leqslant b_{kL}^{-1/4} y. \tag{2.15}$$

Given $(n[\omega_0], \omega_0)$, $\omega_0 = \omega_{j_0, \nu_0}$, $-2 \leqslant \nu_0 \leqslant N-2$, (if $\omega_0 \not\subset \omega_{00}$ we assume that ω_0 equals $\omega_{0,-1}$ or $\omega_{0,-2}$) and $k \geqslant 1$ satisfying the condition

$$\Omega(k) : (n[\omega_0], \omega_0) \in G^*_{kL} \quad \text{and} \quad A^*_{n[\omega_0]}(\omega_0) < b_{k-1} y,$$

we construct the partition $\Omega_N((n[\omega_0], \omega_0), k)$ of ω_0 as in [3]. If $x \in \omega_0 \cap \omega_{00}$ and x is not an endpoint of an $\omega_{j\nu}$ we define $\omega_0(x)$ as in [1], p. 374.

Now assume n int (ω_0) and $\omega_0(x) \subset \omega' \subset \omega_0$, $\omega' = \omega_{j',\nu'}$. Let $n_1 = 2^{\nu'+} n[\omega']$. As in [1], pp. 375–376, we get

$$\Big| |S^*_n(x, \omega_0)| - |S^*_{n_1}(x, \omega')| \Big| \leqslant R_n(x, \omega_0), \tag{2.16}$$

where
$$R_n(x, \omega_0) = \sup_k \left| \sum_{j=0}^{k} \zeta_{j+\nu_0} \Delta_j(x, \omega_0, E_n) \right| \left(n = \sum_{i=0}^{N_n} \zeta_i 2^i \right) \tag{2.17}$$

and
$$E_n(t, \omega_0) = \frac{1}{|\omega_0(t)|} \int_{\omega_0(t)} f(u)\, w_n(u)\, du. \tag{2.18}$$

Now let
$$T^*(n[\omega_0], \omega_0) = \{x \in \omega_0 \mid R_n(x, \omega_0) > CLk\, b_{k-1} y\}, \tag{2.19}$$

where C is a constant. (If $\nu_0 < 0$ let

$$R_n(x, \omega_0) = \sup_k \left| \sum_{j=0}^{k} \zeta_j \Delta_j(x, \omega_{00}, E_n(\cdot, \omega_0)) \right|$$

and
$$T^*(n[\omega_0], \omega_0) = \{x \in \omega_{00} \mid R_n(x, \omega_0) > CLk\, b_{k-1} y\}.)$$

Observing that $|E_n(t, \omega_0)| \leqslant \text{Const.}\, b_{k-1} y$ and using Lemma 1.9 we get

$$mT^*(n[\omega_0], \omega_0) \leqslant \text{Const.}\, |\omega_0| \exp(-\text{Const.}\, CLk). \tag{2.20}$$

We sum up the results in the following lemma:

Lemma 2.3. *If* n int (ω_0), $n[\omega_0]$ *and* ω_0 *satisfy* $\Omega(k)$, $x \in \omega_{00}$ *and* $\omega_0(x) \subset \omega' \subset \omega_0$, *then for* $x \notin T^*(n[\omega_0], \omega_0)$

$$\Big| |S^*_n(x, \omega_0)| - |S^*_{2^{\nu'+} n[\omega']}(x, \omega')| \Big| \leqslant CLk\, b_{k-1} y. \tag{2.21}$$

We define the sets X^*, T^* and W^* as in [3]. Letting $E = S^* \cup X^* \cup T^* \cup W^*$ we then get

$$mE \leqslant \text{Const.}\, y^{-p} mF. \tag{2.22}$$

We now prove the analogue of Lemma 10.2 in [3]:

Lemma 2.4. *Assume* $\omega_0 = \omega_{j_0, \nu_0}$, $x \in \omega_0 \cap \omega_{00}$, $x \notin E$ *and* $(n_0[\omega_0], \omega_0) \notin G^*_{kL}$, *where* k *is defined by* $b_{k-1} y > A^*_{n_0[\omega_0]}(\omega_0) \geqslant b_k y$. *Then there exist* ω_1 *and* n_1 *such that*

$$\omega_1 \supset \omega_0, \tag{2.23}$$

$$(n_1[\omega_1], \omega_1) \in G^*_{kL},\ n_1 \operatorname{int}(\omega_1'), \quad \textit{where} \quad 4|\omega_1'| = |\omega_1|,$$

and
$$|n_1[\omega_0] - n_0[\omega_0]| < Ab_k^{-1}, \quad \textit{where } A \textit{ is a constant.} \tag{2.24}$$

If n satisfies $|n_0[\omega_0] - n[\omega_0]| \leqslant 2Ab_k^{-2}$, then

$$\left| |S^*_{2^{\nu_0} n_0[\omega_0]}(x, \omega_0)| - |S^*_{2^{\nu_0} n[\omega_0]}(x, \omega_0)| \right| \leqslant \text{Const.}\, \{A^*_{n_1[\omega_0]}(\omega_0) + b_k y\}. \tag{2.25}$$

Proof. Defining ω_0', ω', P_0, P as in [3] we get

$$A_n(\omega', f - P) \leqslant b_{kL}^{1/2} y \quad \text{for all } n \tag{2.26}$$

and
$$A_{n_0[\omega_0']}(\omega_0', P_0) \geqslant (b_k - b_{kL}^{1/2})\, y. \tag{2.27}$$

P_0 must contain an index λ with $|\lambda[\omega_0'] - n_0[\omega_0']| < b_{kL}^{-4}$, because otherwise we would have ($P_0 = \sum a_\mu' w_{\lambda_\mu}$)

$$A_{n_0[\omega_0']}(\omega_0', P_0) \leqslant \sum |a_\mu'|\, A_{n_0[\omega_0']}(\omega_0', w_{\lambda_\mu}) = \sum |a_\mu'| \tfrac{1}{10} \frac{1}{1 + (\lambda_\mu[\omega_0'] - n_0[\omega_0'])^2}$$
$$\leqslant \sum |a_\mu'|\, b_{kL}^8 \leqslant b_{kL}^8 b_{kL}^{-2} y^{p/2} \leqslant b_{kL}^5 y,$$

which contradicts (2.27).

Since $(n_0[\omega_0], \omega_0) \notin G^*_{kL}$ we have according to Lemma 2.1 $P_{kL}(x, \omega') = Q_0(x, \omega') + Q_1(x, \omega')$, where

$$Q_0 = \varrho_1 w_{\lambda_1} + \varrho_2 w_{\lambda_2} \quad \text{with} \quad |\varrho_2| \leqslant |\varrho_1| \leqslant b_{kL}^{-2} y^{p/2} \quad \text{and} \quad |\lambda_1[\omega'] - \lambda_2[\omega']| = 1 \tag{2.28}$$

and

All indices λ' of Q_1 satisfy $|\lambda'[\omega'] - n_0[\omega']| \geqslant b_{kL}^{-10}$. (2.29)

Lemma 2.2 implies

The polynomials Q_0 corresponding to different subintervals ω' are equal. (2.30)

(2.27) implies $(b_k - b_{kL}^{1/2})\, y \leqslant A_{n_0[\omega_0']}(\omega_0', Q_0) + A_{n_0[\omega_0']}(\omega_0', Q_1)$, and using (2.28) and (2.29) we get

$$(b_k - b_{kL}^{1/2})\, y \leqslant \text{Const.} \frac{|\varrho_1|}{1 + (\lambda_1[\omega_0'] - n_0[\omega_0'])^2} + b_{kL}^{17} y.$$

We define $n_1 = \lambda_1$. $(n_1[\omega_1'], \omega_1') \in G_{kL}$ for some $\omega_1' \supset \omega_0'$. If $\omega_1 \supset \omega_1'$, $4|\omega_1'| = |\omega_1|$, then $\omega_1 \supset \omega_0$, $n_1 \operatorname{int}(\omega_1')$, $(n_1[\omega_1], \omega_1) \in G^*_{kL}$ and $|n_1[\omega_0'] - n_0[\omega_0']| < b_{kL}^{-5}$. Using this estimate together with (2.26) and (2.29) we get

$$A_{n_1[\omega_0']}(\omega_0', \varrho_1 w_{\lambda_1} + \varrho_2 w_{\lambda_2}) \leqslant A_{n_1[\omega_0']}(\omega_0', f) + A_{n_1[\omega_0']}(\omega_0', f - P_0) + A_{n_1[\omega_0']}(\omega_0', Q_1)$$
$$\leqslant A^*_{n_1[\omega_0]}(\omega_0, f) + b_{kL}^{1/2} y + b_{kL}^{17} y.$$

We also have

$$A_{n_1[\omega_0']}(\omega_0', \varrho_1 w_{\lambda_1} + \varrho_2 w_{\lambda_2}) \geqslant |\varrho_1| \tfrac{1}{10} - |\varrho_2| \cdot \tfrac{1}{10} \cdot \tfrac{1}{2} \geqslant \tfrac{1}{20} |\varrho_1|,$$

and a combination of these inequalities yields

$$|\varrho_1| \leqslant \text{Const.}\, \{A^*_{n_1[\omega_0]}(\omega_0) + b_{kL}^{1/2} y\}. \tag{2.31}$$

If we use (2.12), (2.24) now follows from the above estimates.

It remains to prove (2.25) and we therefore assume that n satisfies

$$|n_0[\omega_0] - n[\omega_0]| \leqslant 2Ab_k^{-2}. \tag{2.32}$$

Assume $x \in \omega''$, $4|\omega''| = |\omega_0|$. According to [1], p. 381, we have

$$\Big| |S^*_{2^{\nu_0} n[\omega_0]}(x, \omega_0)| - |S_{n[\omega'']}(x, \omega'')| \Big| \leqslant \text{Const.}\, A^*_{n[\omega_0]}(\omega_0). \tag{2.33}$$

Using (2.26), (2.29) and (2.32) we get

$$\begin{aligned} A_{n[\omega']}(\omega', f) &\leqslant A_{n[\omega']}(\omega', f-P) + A_{n[\omega']}(\omega', \varrho_1 w_{\lambda_1} + \varrho_2 w_{\lambda_2}) + A_{n[\omega']}(\omega', Q_1) \\ &\leqslant b_{kL}^{1/2} y + 2|\varrho_1| + b_{kL}^{17} y. \end{aligned}$$

A combination of this inequality with (2.33) gives

$$\begin{aligned} \Big| |S^*_{2^{\nu_0} n_0[\omega_0]}(x, \omega_0)| - |S^*_{2^{\nu_0} n[\omega_0]}(x, \omega_0)| \Big| &\leqslant \text{Const.}\, (b_{kL}^{1/2} y + |\varrho_1|) \\ &+ \left| \sum_{l=n_0[\omega'']}^{n[\omega'']-1} a_l(\omega'', f)\, w_{2^{\nu_0+2}l}(x) \right| \quad (\text{if } n_0[\omega''] < n[\omega'']). \end{aligned} \tag{2.34}$$

The last sum is less than or equal to

$$\begin{aligned} &|\textstyle\sum a_l(f-P)\, w_{2^{\nu_0+2}l}(x)| + |\sum a_l(Q_1)\, w_{2^{\nu_0+2}l}(x)| \\ &\qquad + |\textstyle\sum a_l(\varrho_1 w_{\lambda_1} + \varrho_2 w_{\lambda_2})\, w_{2^{\nu_0+2}l}(x)| = d_1 + d_2 + d_3. \end{aligned}$$

(2.3), (2.15) and (2.32) yield

$d_1 \leqslant 2Ab_k^{-2} b_{kL}^{3/4} y \leqslant b_k y \quad (L \text{ large}),$

$d_2 = 0$ because of the definition of Q_1 and (2.32), and

$d_3 \leqslant 2|\varrho_1|.$

(2.25) therefore follows from (2.31) and (2.34) and the lemma is proved.

We can now conclude the proof of the basic result in the same way as in [3]. Since we need not make any essential changes of the argument in [3], we omit this part of the proof. We finally get

$$|S_n^*(x)| \leqslant \text{Const.} \sum_{i=1}^{\infty} i\, b_{i-1} L y = \text{Const.}\, Ly, \qquad x \notin E, \quad n \leqslant N.$$

Hence $m\{x \mid M_N f(x) > \text{Const.}\, Ly\} \leqslant mE$ and the basic result follows if we use (2.22) and (2.14).

3. Proof of the theorem

Proof of (A). From the basic result we get

$$m\{x\in(0,1)\,|\,M\chi_F(x)>y\}\leqslant \text{Const.}\left(\frac{1}{p-1}\right)^p y^{-p}mF,\qquad 1<p<3,\quad y>0.\tag{3.1}$$

For $0<y<\frac{1}{2}$ we substitute $p=1+(\log(1/y))^{-1}$ in (3.1). After some computation we conclude

$$m\{x\in(0,1)\,|\,M\chi_F(x)>y\}\leqslant \text{Const.}\,\frac{1}{y}\log\frac{1}{y}\,mF,\quad 0<y<\tfrac{1}{2}.\tag{3.2}$$

Taking $p=2$ in (3.1) we get

$$m\{x\in(0,1)\,|\,M\chi_F(x)>y\}\leqslant \text{Const.}\,y^{-2}mF,\quad y>0.\tag{3.3}$$

Let N denote the set of all functions of $L^1(0,1)$ which take only finitely many values, where the taken values are zero or of the type $2^{N_1}+2^{N_2}+\ldots+2^{N_n}$, $N_n<N_{n-1}<\ldots<N_1$, N_i integers. Also let

$$J(f)=\int_0^1|f(x)|\{\log^+|f(x)|\log^+\log^+|f(x)|+1\}\,dx.$$

For $f\in N, J(f)<\frac{1}{2}$, we define a function $\tilde f\in N$ in the following way:

(a) If $f\equiv 0$ we let $\tilde f\equiv 0$.

(b) If $f\not\equiv 0$, let α be the largest number of the type 2^{-n}, $n\geqslant 2$, which satisfies $5\alpha\leqslant J(f)$. From this it follows that $(1/\alpha)f\in N$ and

$$\alpha<\tfrac{1}{4},\tag{3.4}$$

$$(1/\alpha)f \text{ takes some value } \geqslant 4 \quad ((1/\alpha)f<4 \text{ implies } J(f)<4\alpha),\tag{3.5}$$

$$J(f)<10\alpha.\tag{3.6}$$

Let $G_n=\{x\in(0,1)\,|\,2^n\leqslant(1/\alpha)f(x)<2^{n+1}\}$, $n=2,3,\ldots$, and let χ_n be the characteristic function for G_n. We now define $\tilde f$ by

$$\frac{1}{\alpha}\tilde f=\frac{1}{\alpha}f-\sum_2^\infty 2^n\chi_n.\tag{3.7}$$

From (3.7) it follows that

$$f=\tilde f+\alpha\sum_2^\infty 2^n\chi_n,\tag{3.8}$$

$$\tilde f(x)\leqslant\tfrac{1}{2}f(x)\quad\text{for}\quad x\in\bigcup_2^\infty G_n,\ \tilde f(x)=f(x)\quad\text{otherwise.}\tag{3.9}$$

We will now prove

$$f\in N,\ J(f)<\tfrac{1}{2}\quad\text{implies}\quad J(\tilde f)\leqslant\tfrac{9}{10}J(f).\tag{3.10}$$

(3.10) is obvious if $J(f)=0$. For $J(f)>0$ we set $f=g+h$, where

$$g(x)=\begin{cases} f(x), & x\in \bigcup_2^\infty G_n,\\ 0 & \text{otherwise,}\end{cases}\qquad h(x)=\begin{cases} f(x), & x\notin \bigcup_2^\infty G_n,\\ 0 & \text{otherwise.}\end{cases}$$

We have $J(f)=J(g)+J(h)\geqslant 5\alpha$. ($G_n$ and α defined as in the definition of $\tilde{f}$.) From $h<4\alpha$ it follows that $J(h)<4\alpha$, which gives $J(g)>\alpha>\frac{1}{4}J(h)$. Using (3.9) we get

$$J(\tilde{f})\leqslant J(\tfrac{1}{2}g)+J(h)\leqslant \tfrac{1}{2}J(g)+J(h)\leqslant \tfrac{1}{2}J(g)+\tfrac{1}{10}4J(g)+\tfrac{9}{10}J(h)=\tfrac{9}{10}(J(g)+J(h))=\tfrac{9}{10}J(f),$$

and (3.10) is proved.

We now prove the following lemma:

Lemma 3.1. *Assume* $f\in N$ *and* $J(f)<\frac{1}{2}$. *Then there exists a set* E *with* $mE\leqslant$ Const. $J(f)^{1/5}$ *such that*

$$x\notin E \quad \text{implies} \quad Mf(x)\leqslant M\tilde{f}(x)+J(f)^{2/5}. \tag{3.11}$$

Proof. This lemma is obviously true if $J(f)=0$. For $J(f)>0$ we use the same notations as in the definition of $\tilde{f}$. Let ϱ be a number larger than 1 and define

$$\left.\begin{aligned} F_n&=\{x\in(0,1)\,|\,M\chi_n(x)>\varrho 2^{-n}n^2\},\quad F=\bigcup_2^\infty F_n,\\ K_n&=\{x\in(0,1)\,|\,\varrho 2^{-n}n^{-2}<M\chi_n(x)\leqslant \varrho 2^{-n}n^2\}\\ \text{and}\qquad L_n&=\{x\in(0,1)\,|\,M\chi_n(x)\leqslant\varrho 2^{-n}n^{-2}\}.\end{aligned}\right\} \tag{3.12}$$

From (3.2) and (3.3) it follows that

$$mF_n\leqslant \text{Const.}\,\frac{1}{\varrho}2^n n^{-2}\log\frac{1}{\varrho}2^n n^{-2}mG_n \quad \text{if} \quad \varrho 2^{-n}n^2<\tfrac{1}{4}$$

and
$$mF_n\leqslant\text{Const.}\left(\frac{1}{\varrho}2^n n^{-2}\right)^2 mG_n\leqslant \text{Const.}\,\frac{1}{\varrho}2^n n^{-2}mG_n \quad \text{otherwise.}$$

We now estimate the measure of F:

$$\begin{aligned} mF\leqslant \sum_2^\infty mF_n &\leqslant \text{Const.}\sum_{\varrho 2^{-n}n^2<\frac14}\frac{1}{\varrho}2^n n^{-2}\log\frac{1}{\varrho}2^n n^{-2}mG_n\\ &+\text{Const.}\sum_{\varrho 2^{-n}n^2\geqslant\frac14}\frac{1}{\varrho}2^n n^{-2}mG_n\leqslant \text{Const.}\sum_{\varrho 2^{-n}n^2<\frac14}\int_{G_n}\frac{1}{\varrho\alpha}f\log\frac{1}{\varrho\alpha}f\,dx\\ &+\text{Const.}\sum_{\varrho 2^{-n}n^2\geqslant\frac14}\int_{G_n}\frac{1}{\varrho\alpha}f\,dx.\end{aligned}$$

From the definition of $J(f)$ it follows that

$$mF\leqslant\text{Const.}\,J\left(\frac{1}{\varrho\alpha}f\right). \tag{3.13}$$

We now want an estimate of $M(\sum_2^\infty 2^n \chi_n)$ outside F. We have

$$\int_{CF} M\left(\sum_2^\infty 2^n \chi_n\right) dx \leqslant \sum_2^\infty \int_{CF_n} M(2^n \chi_n)\, dx \leqslant \sum_2^\infty 2^n \int_{K_n} M\chi_n\, dx + \sum_2^\infty 2^n \int_{L_n} M\chi_n\, dx.$$

The definition of L_n implies

$$\int_{CF} M\left(\sum_2^\infty 2^n \chi_n\right) dx \leqslant S + \sum_2^\infty \varrho n^{-2} = S + \text{Const. } \varrho, \tag{3.14}$$

where

$$S = \sum_2^\infty 2^n \int_{K_n} M\chi_n\, dx. \tag{3.15}$$

Introducing the distribution function $\mu_n(\lambda) = m\{x \in (0,1) \mid M\chi_n(x) > \lambda\}$ we get

$$2^n \int_{K_n} M\chi_n\, dx = -2^n \int_{\varrho 2^{-n} n^{-2}}^{\varrho 2^{-n} n^2} \lambda\, d\mu_n(\lambda) = -2^n \left[\lambda \mu_n(\lambda)\right]_{\varrho 2^{-n} n^{-2}}^{\varrho 2^{-n} n^2} + 2^n \int_{\varrho 2^{-n} n^{-2}}^{\varrho 2^{-n} n^2} \mu_n(\lambda)\, d\lambda.$$

We will now prove

$$2^n \int_{K_n} M\chi_n\, dx \leqslant \text{Const.} \int_{G_n} \frac{1}{\alpha} f \log \frac{1}{\alpha} f \log\log \frac{1}{\alpha} f\, dx. \tag{3.16}$$

We consider three cases:

Case 1. $\varrho 2^{-n} n^2 \leqslant \frac{1}{4}$.

Using (3.2) we get

$$2^n \int_{K_n} M\chi_n\, dx \leqslant \text{Const. } 2^n \log \frac{1}{\varrho} 2^n n^2 mG_n$$

$$+ \text{Const. } 2^n \int_{\varrho 2^{-n} n^{-2}}^{\varrho 2^{-n} n^2} \frac{\log(1/\lambda)}{\lambda}\, d\lambda\, mG_n = \text{Const. } 2^n \log \frac{1}{\varrho} 2^n n^2 mG_n$$

$$+ \text{Const. } 2^n \left\{\left(\log \frac{1}{\varrho} 2^n n^2\right)^2 - \left(\log \frac{1}{\varrho} 2^n n^{-2}\right)^2\right\} mG_n.$$

According to the mean value theorem there exists a number ξ, $\log(1/\varrho)\, 2^n n^{-2} < \xi < \log(1/\varrho)\, 2^n n^2$, such that

$$\left(\log \frac{1}{\varrho} 2^n n^2\right)^2 - \left(\log \frac{1}{\varrho} 2^n n^{-2}\right)^2 = \left(\log \frac{1}{\varrho} 2^n n^2 - \log \frac{1}{\varrho} 2^n n^{-2}\right) 2\xi$$

$$\leqslant \text{Const. } \log n \log \frac{1}{\varrho} 2^n n^2 \leqslant \text{Const. } \log 2^n \log n.$$

(3.16) now follows from the above estimates and the definition of G_n.

Case 2. $\varrho 2^{-n} n^{-2} < \frac{1}{4} < \varrho 2^{-n} n^2$.

Using (3.2) and (3.3) we have the following estimates:

$$2^n \int_{K_n} M\chi_n \, dx \leqslant \text{Const. } 2^n \log \frac{1}{\varrho} 2^n n^2 mG_n$$

$$+ \text{Const. } 2^n \int_{\varrho 2^{-n} n^{-2}}^{1/4} \frac{\log (1/\lambda)}{\lambda} d\lambda \, mG_n + \text{Const. } 2^n \int_{1/4}^{\varrho 2^{-n} n^2} \frac{1}{\lambda^2} d\lambda mG_n$$

$$\leqslant \text{Const. } 2^n \log \frac{1}{\varrho} 2^n n^2 mG_n + \text{Const. } 2^n \left\{ \left(\log \frac{1}{\varrho} 2^n n^2 \right)^2 - (\log 4)^2 \right\} mG_n.$$

The mean value theorem implies $(\log 4 < \xi < \log (1/\varrho) \, 2^n n^2)$

$$\left(\log \frac{1}{\varrho} 2^n n^2 \right)^2 - (\log 4)^2 = \left(\log \frac{1}{\varrho} 2^n n^2 - \log 4 \right) 2\xi$$

$$\leqslant \text{Const.} \left(\log \frac{1}{\varrho} 2^n n^2 - \log \frac{1}{\varrho} 2^n n^{-2} \right) \log \frac{1}{\varrho} 2^n n^2$$

$$\leqslant \text{Const.} \log 2^n \log n$$

and it is easy to see that (3.16) holds.

Case 3. $\varrho 2^{-n} n^{-2} \geqslant \frac{1}{4}$.

(3.3) implies

$$2^n \int_{K_n} M\chi_n \, dx \leqslant \text{Const. } 2^n mG_n + \text{Const. } 2^n \int_{\varrho 2^{-n} n^{-2}}^{\varrho 2^{-n} n^2} \frac{1}{\lambda^2} d\lambda \, mG_n \leqslant \text{Const. } 2^n mG_n,$$

which yields (3.16) also in this case.

(3.15) and (3.16) imply $S \leqslant \text{Const. } J[(1/\alpha) f]$. From (3.14) we now get

$$\int_{CF} M \left(\alpha \sum_2^\infty 2^n \chi_n \right) dx \leqslant \text{Const. } \alpha J \left(\frac{1}{\alpha} f \right) + \text{Const. } \alpha\varrho.$$

Let γ be a positive number and let

$$H = \left\{ x \in (0, 1) \mid x \in CF, \, M \left(\alpha \sum_2^\infty 2^n \chi_n \right)(x) > \gamma \right\} \quad \text{and} \quad E = F \cup H.$$

We obviously have

$$mH \leqslant \text{Const.} \frac{\alpha}{\gamma} J \left(\frac{1}{\alpha} f \right) + \text{Const.} \frac{\alpha\varrho}{\gamma},$$

and if we use (3.13) we get

$$mE \leqslant \text{Const.} \left\{ J \left(\frac{1}{\varrho\alpha} f \right) + \frac{\alpha}{\gamma} J \left(\frac{1}{\alpha} f \right) + \frac{\alpha\varrho}{\gamma} \right\}.$$

If we choose $\varrho = J(f)^{-2/5}$ and $\gamma = J(f)^{2/5}$, we obtain

$$mE \leqslant \text{Const. } J(f)^{1/5}.$$

If $x \notin E$ we have $Mf(x) \leqslant M\tilde{f}(x) + M(\alpha \sum_2^\infty 2^n \chi_n)(x) \leqslant M\tilde{f}(x) + J(f)^{2/5}$, and the lemma is proved.

We need another lemma:

Lemma 3.2. *There exist constants C_1 and C_2 with the following property:*

If $J(f) < \frac{1}{2}$ we can find a set E such that $mE \leqslant C_1 J(f)^{1/5}$ and $Mf(x) \leqslant C_2 J(f)^{2/5}$ if $x \notin E$. (3.17)

Proof. We first consider the case when $f \in N$. We define a sequence $\{f_i\}_{i=0}^\infty$ of functions belonging to N by $f_0 = f, f_{i+1} = \tilde{f}_i$, $i = 0, 1, 2, \dots$. Using Lemma 3.1 and (3.10) n times we get

$$Mf(x) \leqslant Mf_n(x) + J(f)^{2/5} \sum_{i=0}^{n-1} \left(\frac{9}{10}\right)^{2i/5} \tag{3.18}$$

outside a set of measure smaller than Const. $J(f)^{1/5} \sum_{i=0}^{n-1} (9/10)^{i/5}$. From (3.5) and the definition of $\tilde{f}$ it follows that $f_n \equiv 0$ if n is sufficiently large and it is easy to see that (3.17) holds for $f \in N$.

Now assume $f \geqslant 0$ and $J(f) < \frac{1}{2}$. We can then find a sequence $\{f_n\}_{n=1}^\infty$ such that $f_n \in N$, $J(f_n) \leqslant J(f)$ and $\lim_{n\to\infty} J(f - f_n) = 0$. For fixed K there exists an n such that $M_K(f - f_n) = \sup_{k \leqslant K} |S_k(f - f_n)| \leqslant J(f)^{2/5}$, since $M_K(f - f_n)$ tends to zero uniformly when n tends to infinity. We therefore have

$$M_K(f) \leqslant M_K(f_n) + M_K(f - f_n) \leqslant \text{Const. } J(f_n)^{2/5} + J(f)^{2/5} \leqslant C_0 J(f)^{2/5},$$

$C_0 = $ Const., outside a set of measure smaller than Const. $J(f_n)^{1/5} \leqslant$ Const. $J(f)^{1/5}$. If $E_K = \{x \in (0,1) \mid M_K f(x) > C_0 J(f)^{2/5}\}$ this estimate gives $mE_K \leqslant$ Const. $J(f)^{1/5}$. $M_K \leqslant M_{K+1}$ implies $E_K \subset E_{K+1}$, and we get $m(\bigcup_1^\infty E_K) = \lim_{K\to\infty} mE_K \leqslant$ Const. $J(f)^{1/5}$. From the definition of E_K it follows that (3.17) holds with $E = \bigcup_1^\infty E_K$. Thus we have proved the lemma for positive f and it is easy to see that it holds also for general complex f.

We are now able to prove (A). We assume that (A) is not true. We can then find a real function f with $J(f) < \infty$ and $S_n(f)$ diverging on a set of positive measure. This means that there exist numbers $\varepsilon > 0$ and $\delta > 0$ such that $\overline{\lim}\, S_n(f) - \underline{\lim}\, S_n(f) \geqslant \varepsilon$ on a set of measure δ. We now choose a Walsh polynomial P such that $2C_2 J(f-P)^{2/5} < \varepsilon$, $C_1 J(f-P)^{1/5} < \delta$ and $J(f-P) < \frac{1}{2}$ (C_1 and C_2 as in Lemma 3.2). Using (3.17) we get that $\overline{\lim}\, S_n(f) - \underline{\lim}\, S_n(f) = \overline{\lim}\, S_n(f-P) - \underline{\lim}\, S_n(f-P) \leqslant 2M(f-P) \leqslant 2C_2 J(f-P)^{2/5} < \varepsilon$ holds outside a set of measure smaller than $C_1 J(f-P)^{1/5} < \delta$, and this gives a contradiction. This completes the proof of (A).

The proof of (E) is the same.

Proof of (B). We first use the basic result to deduce the following lemma.

Lemma 3.3. We have

$$\int_0^1 (M\chi_F)^p \, dx \leqslant \text{Const.}^p \left(\frac{p^2}{p-1}\right)^{2p} mF, \quad 1 < p < \infty. \tag{3.19}$$

Proof. Let $1 < p_1 < p < p_2 < \infty$ and let

$$\mu(\lambda) = m\{x \in (0,1) \mid M\chi_F(x) > \lambda\}.$$

The basic result gives the following estimates of $\mu(\lambda)$:

$$\mu(\lambda) \leqslant B_{p_1}^{p_1} \lambda^{-p_1} mF \quad \text{and} \quad \mu(\lambda) \leqslant B_{p_2}^{p_2} \lambda^{-p_2} mF.$$

Using the first estimate for $\lambda \leqslant 1$ and the second for $\lambda \geqslant 1$ we get

$$\int_0^1 (M\chi_F)^p\,dx = -\int_0^\infty \lambda^p\,d\mu(\lambda) = p\int_0^\infty \lambda^{p-1}\mu(\lambda)\,d\lambda \leqslant pB_{p_1}^{p_1} mF \int_0^1 \lambda^{p-p_1-1}\,d\lambda$$
$$+ pB_{p_2}^{p_2} mF \int_1^\infty \lambda^{p-p_2-1}\,d\lambda = \frac{p}{p-p_1} B_{p_1}^{p_1} mF + \frac{p}{p_2-p} B_{p_2}^{p_2} mF.$$

If we select p_1 and p_2 suitably (e.g., for p near 1 we can take $p_1 = \frac{1}{2}(p+1)$ and $p_2 = p + \frac{1}{2}(p-1)$) and use the estimate of B_p given in Lemma 2.7, (3.19) easily follows from the last inequality.

For $f \in N$ we now define a function $\tilde{f}$ in the same way as in the proof of (A) but with $J(f)$ replaced by

$$I(f) = \int_0^1 |f(x)|\,\{(\log^+|f(x)|)^2 + 1\}\,dx.$$

This time we need no restriction of the type $J(f) < \frac{1}{2}$ but define $\tilde{f}$ for all $f \in N$. Using the same notations we get

$$\alpha \leqslant \tfrac{1}{4}. \tag{3.20}$$

$(1/\alpha)f$ takes some value $\geqslant 4$. (3.21)

$I(f) < \frac{5}{4}$ implies $I(f) < 10\alpha$ and $I(f) \geqslant \frac{5}{4}$ implies $\alpha = \frac{1}{4}$. (3.22)

$$f = \tilde{f} + \alpha \sum_2^\infty 2^n \chi_n, \quad \text{where } \chi_n = \chi_{G_n}. \tag{3.23}$$

$$I(\tilde{f}) \leqslant \tfrac{9}{10} I(f). \tag{3.24}$$

Now let $p_n = 1 + (1/n)$, $n = 2, 3, \ldots$. Using Hölder's inequality and (3.19) we get

$$\int_0^1 M\chi_n\,dx \leqslant \left(\int_0^1 (M\chi_n)^{p_n}\,dx\right)^{1/p_n} \leqslant \text{Const.}\left(\frac{p_n^2}{p_n-1}\right)^2 (mG_n)^{1/p_n}$$
$$\leqslant \text{Const. } n^2 (mG_n)^{n/(n+1)}, \quad n = 2, 3, \ldots. \tag{3.25}$$

We have $(mG_n)^{n/(n+1)} \leqslant \text{Const.}\, mG_n + 2^{-2n}$, since $mG_n > 2^{-2n}$ implies $(mG_n)^{n/(n+1)} \leqslant \text{Const.}\, mG_n$. Combining these estimates we get

$$\int_0^1 M(f-\tilde{f})\,dx = \int_0^1 M(\alpha \sum_2^\infty 2^n \chi_n)\,dx \leqslant \alpha \sum_2^\infty 2^n \int_0^1 M\chi_n\,dx$$
$$\leqslant \text{Const.}\,\alpha \sum_2^\infty 2^n n^2 (\text{Const. } mG_n + 2^{-2n}) = \text{Const.}\,\alpha \sum_2^\infty 2^n n^2 mG_n$$
$$+ \text{Const.}\,\alpha \sum_2^\infty 2^n n^2 2^{-2n} \leqslant \text{Const.}\,\alpha I\left(\frac{1}{\alpha} f\right) + \text{Const.}\,\alpha.$$

Using (3.22) we can prove $\alpha I[(1/\alpha) f] \leqslant \text{Const.}\,(I(f)+I(f)^{1/2})$, so we get

$$\int_0^1 M(f-\tilde{f})\,dx \leqslant \text{Const.}\,(I(f)+I(f)^{1/2}). \tag{3.26}$$

It follows that

$$\int_0^1 Mf\,dx \leqslant \int_0^1 M\tilde{f}\,dx + \int_0^1 M(f-\tilde{f})\,dx \leqslant \int_0^1 M\tilde{f}\,dx + \text{Const.}\,(I(f)+I(f)^{1/2}). \tag{3.27}$$

We now define $f_0 = f$, $f_{i+1} = \tilde{f}_i$, $i = 0, 1, 2, \ldots$. Repeated use of (3.27) and (3.24) gives

$$\begin{aligned}\int_0^1 Mf\,dx &\leqslant \int_0^1 Mf_K\,dx + \sum_{\nu=0}^{K-1} \text{Const.}\,[(\tfrac{9}{10})^\nu I(f) + (\tfrac{9}{10})^{\nu/2} I(f)^{1/2}]\\ &\leqslant \text{Const.} \sum_{\nu=0}^{\infty} [(\tfrac{9}{10})^\nu I(f) + (\tfrac{9}{10})^{\nu/2} I(f)^{1/2}] \leqslant \text{Const.}\,(I(f)+I(f)^{1/2})\\ &\leqslant \text{Const.}\, I(f) + \text{Const.}\end{aligned}$$

This estimate implies that (B) holds for $f \in N$. For positive f (B) now follows by approximation of f with functions belonging to N, and it is easy to see that (B) holds also for complex f.

Proof of (C). We use the method in [3]. We get

$$\|Mf\|_{p\infty} \leqslant A_p \|f\|^*_{p1}, \quad \text{where} \quad A_p \leqslant \text{Const.}\,\frac{p}{p-1} B_p \leqslant \text{Const.}\,\frac{p^3}{(p-1)^2}.$$

From this (C) follows with

$$C_p \leqslant \text{Const.}\,\frac{p^2}{p-1} A_p \leqslant \text{Const.}\,\frac{p^5}{(p-1)^3}.$$

Proof of (D). The basic result implies

$$m\{x \in (0,1) \mid M\chi_F(x) > y\} \leqslant C^p p^p y^{-p} mF \quad \text{for} \quad p > \text{Const.}, \tag{3.28}$$

where C is a constant. Taking $p = e^{-1} y C^{-1}$ in (3.28) we get

$$m\{x \in (0,1) \mid M\chi_F(x) > y\} \leqslant \text{Const.} \exp(-\text{Const.}\, y)\, mF,\ y > \text{Const.} \tag{3.29}$$

Now assume $0 \leqslant f(x) \leqslant 1$, $x \in (0,1)$. We then have $f = \sum_1^\infty 2^{-n} \chi_n$, where χ_n are characteristic functions for measurable sets. Letting $A = \sum_1^\infty 2^{-n/2}$ and using $Mf \leqslant \sum_1^\infty 2^{-n} M\chi_n$ and (3.29) we get

$$\begin{aligned} m\{x \mid Mf(x) > Ay\} &\leqslant \sum_1^\infty m\{x \mid 2^{-n} M\chi_n(x) > 2^{-n/2} y\}\\ &= \sum_1^\infty m\{x \mid M\chi_n(x) > 2^{n/2} y\} \leqslant \text{Const.} \sum_1^\infty \exp(-\text{Const.}\, 2^{n/2} y)\\ &\leqslant \text{Const.} \exp.(-\text{Const.}\, y),\ y > \text{Const.}\end{aligned}$$

Since $m\{x \mid Mf(x) > Ay\} \leqslant 1$ for $y > 0$, the restriction $y >$ Const. is unnecessary, and we get

$$0 \leqslant f(x) \leqslant 1 \quad \text{implies} \quad m\{x \mid Mf(x) > y\} \leqslant \text{Const.} \exp(-\text{Const.}\, y), \ y > 0. \qquad (3.30)$$

It is easy to see that the condition $0 \leqslant f(x) \leqslant 1$ in (3.30) can be replaced by $\|f\|_\infty \leqslant 1$, and from this (D) follows.

Note on departure from first version

The original version of this paper was Communicated 10 January 1968, as indicated above the title. That version had $(\log^+|f(x)|)^{3/2}$ instead of $\log^+|f(x)|$ in (A), it had $(\log^+|f(x)|)^{5/2}$ instead of $(\log^+|f(x)|)^2$ in (B) and, finally, it had $(y/\|f\|_\infty)^{2/3}$ instead of $y/\|f\|_\infty$ in (*D*). The change to the presentation now adopted was made in proof, 11 June 1968.

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