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On the existence of approximate identities in ideals of group algebras

By Haskell P. Rosenthal

ABSTRACT

A short proof is given of the theorem by Wik, who, in 1965, showed that the nowhere-dense strong Ditkin subsets of **T**, the circle group, are finite.

In [8] Ingemar Wik characterized the nowhere-dense strong Ditkin subsets of **T**, the circle group. Wik showed that such sets are finite, using an argument analogous to that given by Cohen in [2].

In this paper we give a fairly short proof of this result, based on a result of Y. Meyer (see [4]) and the statement of Cohen's theorem on idempotent measures. (A very simple and elegant demonstration of Cohen's theorem may be found in [3]. See also [6], where some of our results have appeared in a preliminary form.)

Our results generalize to the n-dimensional Euclidean group $\mathbf{R}^n$, and the n-dimensional torus $\mathbf{T}^n$, enabling us to show in Theorem 1.3 that the ideals associated with most nowhere dense subsets in these groups do not possess approximate identities. (We say that a separable commutative Banach algebra B possesses an approximate identity if there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset B$ so that $\lim_{n\to\infty} \|f_n * g - g\| = 0$ for all $g \in B$.)

Thus, in particular, these sets do not exhibit the strong form of spectral synthesis studied in [8]; i.e., these sets are not strong Ditkin sets. For example, we show that circles, half-lines, and finite intervals in the plane are not strong Ditkin sets (see the remark following Theorem 1.3).

In § 2, we establish the appropriate generalizations of the theorems in [8] on the existence of strong Ditkin sets to separable metrizable groups; for example, we show in Theorem 2.3 that closed subgroups are strong Ditkin sets.

Finally, we show in Theorem 2.5 that a closed nowhere-dense subset of the real line is a strong Ditkin set if and only if, with the possible exception of finitely many points, the set is a finite union of arithmetic progressions. Thus, even most discrete subsets of the real line are not strong Ditkin sets.

Definitions and notation

We follow [7] in most of our definitions and notation, with the exception of the two underlined paragraphs below.

Throughout this paper, unless explicitly stated otherwise, G denotes an arbitrary locally compact metrizable abelian group, E an arbitrary closed subset of G, Γ the dual group of continuous characters on G, $L^1(\Gamma)$ the Banach algebra, under convolution, of (equivalence classes of) integrable functions on Γ (with respect to a fixed Haar measure). If μ is a finite complex-valued regular measure defined on the Borel subsets of Γ (hereafter simply called a finite measure), $\hat{\mu}$ denotes the function on G which is the Fourier-Stieltjes transform of μ; given μ and ν finite measures, $\mu * \nu$ denotes the convolution of μ and ν.

Given E, we denote by $I(E)$ (resp. $I_0(E)$) the largest (resp. smallest) closed ideal in $L^1(\Gamma)$ associated with E. (We say that an ideal J is associated with E if $E = \{g \in G: \hat{f}(g) = 0 \text{ all } f \in J\}$.) Thus,

$$I(E) = \{f \in L^1(\Gamma): \hat{f}(g) = 0 \quad \text{all } f \in E\};$$

and $I_0(E)$ is the closure of the set of all $f \in L^1(\Gamma)$ such that $\hat{f} = 0$ on an open neighborhood of E.

Given E, we say that there is an approximate identity for $I_0(E)$ in $I(E)$, denoted $\exists$ App. Id. in $I(E)$, if there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset I(E)$ with $\lim_n \|v_n * f - f\|_{L^1(\Gamma)} = 0$ for all $f \in I_0(E)$.

(We note that if J is associated with E and $\nexists$ App. Id. in $I(E)$, then J does not possess an approximate identity.)

We recall that if J is a closed ideal in $L^1(\Gamma)$ and if μ is a finite measure on Γ, then $\mu * g \in J$ for all $g \in J$. (This is an easy consequence of the Hahn–Banach theorem, the associativity of convolution, and the fact that $L^{\infty}(\Gamma)$ may be identified with the conjugate space of $L^1(\Gamma)$.)

We recall that E is said to be of *spectral synthesis* if $I_0(E) = I(E)$. Following [8], we say that E is a *strong Ditkin set* if E is of spectral synthesis and $\exists$ App. Id. in $I(E)$. (See our Lemma 2.2 for equivalent definitions.) We recall that E is said to be a *Ditkin set* (i.e., a C-set as defined on page 169 of [7]), if E is of spectral synthesis and given $f \in I(E)$, there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset I(E)$ with $v_n * f \to f$.

We denote by $\bar{\Gamma}$ the Bohr compactification of Γ; and by G_d the group G endowed with the discrete topology. $\bar{\Gamma}$ may be defined simply as the dual group of G_d; see pages 30–32 of [7] for other properties of $\bar{\Gamma}$.

Given G, we denote by $\mathcal{R}(G_d)$ the coset-ring of G_d. $\mathcal{R}(G_d)$ thus denotes the smallest ring of subsets of G (closed under finite unions and complementation), containing all cosets of arbitrary subgroups of G.

§ 1. Sets E for which $\nexists$ App. Id. in I(E)

We first state the version of Cohen's theorem that we will need (see 3.1.3, page 60, of [7]).

Cohen's theorem: E belongs to $\mathcal{R}(G_d)$ if and only if there exists a finite measure μ on Γ with $\hat{\mu} = 1$ on E, $\hat{\mu} = 0$ off E (i.e. $\hat{\mu} \equiv \chi_E$, the characteristic function of E).

Our first theorem generalizes Theorem 4 of [8].

Theorem 1.1. *Let G be arbitrary, E a not necessarily closed subset of G, and $\{v_n\}$ a sequence in $L^1(\Gamma)$ satisfying the following two properties:*

(i) $\hat{v}_n(g) \to 0$ *as* $n \to \infty$ *for all* $g \in E$.
(ii) $\hat{v}_n(g) \to 1$ *as* $n \to \infty$ *for all* $g \notin E$.

Then $\|v_n\|_{L^1(\Gamma)}$ tends to infinity with n unless $E \in \mathcal{R}(G_d)$.

Proof. Suppose that $\sup_n \|v_n\| = K < \infty$. We may regard each v_n as defining a finite measure on $\bar{\Gamma}$. By the compactness of the K-sphere of a conjugate space in the weak* topology and the Riesz representation theorem (cf. appendices C7 and F4 of [7]), there exists a finite measure μ on $\bar{\Gamma}$ that is a point of closure, in the weak* topology, of the sequence $\{v_n\}$ of linear functionals defined on the Banach space of continuous functions on $\bar{\Gamma}$. In particular, for each $g \in G$, $\hat{\mu}(g)$ is a limit point of the sequence of complex numbers $\{\hat{v}_n(g)\}_{n=1}^{\infty}$, regarding the character defined by g as a continuous function on Γ.

Hence, $\hat{\mu}(g) = 0$ for all $g \in E$, $\hat{\mu}(g) = 1$ for all $g \notin E$. Thus by Cohen's theorem the complement of E, and hence E itself, is a member of $\mathcal{R}(G_d)$. Q.E.D.

(Note that we could have assumed merely that each v_n is a finite measure on Γ, replacing $\|v_n\|_{L^1(\Gamma)}$ by $\|v_n\|_{M(\Gamma)}$.)

We now obtain Theorem 4 of [8] as a corollary of 1.1.

Corollary 1.2 (Wik). *Suppose $\Gamma = \mathbf{Z}$, the group of integers, and E is a closed subset of $\mathbf{T}$ with $\{v_n\}$ a sequence satisfying* (i) *and* (ii) *of Theorem* 1.1. *Then $\|v_n\|_{L^1(\mathbf{Z})}$ tends to infinity with n unless E is a finite set or all of* $\mathbf{T}$.

Proof. $E \in \mathcal{R}(T_d)$ by 1.1, and we have assumed that E is closed. Hence by the proof of Corollary 1.7 of [5], E is a finite set or all of $\mathbf{T}$.

Our next theorem shows that for many groups G, nowhere dense strong Ditkin sets must belong to $\mathcal{R}(G_d)$. It is in reality an easy consequence of our Theorem 1.1 and a result of Meyer (see [4]) generalizing Lemma 6 of [8].

Theorem 1.3. *Let G be $\mathbf{R}^n$, $\mathbf{T}^n$, or any compact metrizable group such that the union of all of its finite subgroups is everywhere dense. Then if E is a closed subset of G without interior and $\exists$ App. Id. in $I(E)$, E is a member of $\mathcal{R}(G_d)$.*

Proof. It is shown by Meyer in [4] that under our hypotheses, the norm of a finite measure μ on Γ is equal to its operator norm, by convolution, on $I_0(E)$. That is,

$$\sup_{\substack{\|f\| \leq 1 \\ f \in I_0(E)}} \|\mu * f\| = \|\mu\|.$$

Now suppose $\{v_n\}_{n=1}^{\infty}$ is an App. Id. in $I(E)$. As was observed by Wik, by a general theorem of Banach on convergence of operators (page 80 of [1]), there exists a finite constant M so that $\|v_n * f\| \leq M\|f\|$ for all $f \in I_0(E)$ and all n. Hence by Meyer's theorem, $\|v_n\|_{L^1(\Gamma)} \leq M$ for all n. Since we have assumed that $v_n \in I(E)$, $\hat{v}_n(g) = 0$ for all $g \in E$ and all n. Finally, given $g \notin E$, we may choose $f \in I_0(E)$ with $\hat{f}(g) = 1$ (cf. page 49 of [7]); hence since $v_n * f \to f$, $\hat{v}_n \cdot \hat{f}$ tends to $\hat{f}$ uniformly as n tends to infinity, so $\hat{v}_n(g) \to 1$. Thus $E \in \mathcal{R}_d(G)$ by Theorem 1.1.

Remark. Theorem 1.3 yields the following examples of sets that are not strong Ditkin sets, because they are nowhere dense and do not belong to $\mathcal{R}(G_d)$.

(i) Any infinite closed nowhere-dense subset of $\mathbf{T}$.
(ii) Circles in the plane, e.g. $E = \{\langle x, y \rangle : x^2 + y^2 = 1\}$.
(iii) Bounded intervals and half lines in the plane, e.g.

$$E = \{\langle x, y\rangle : y = 0, \quad 0 \leqslant x \leqslant 1\},$$

and
$$E = \{\langle x, y\rangle : y = 0, \quad x \leqslant 1\}.$$

(iv) Appropriate circles and intervals in $\mathbf{T}^2$, e.g.

$$E = \{\langle e^{ix}, e^{iy}\rangle : x^2 + y^2 = 1\},$$

and
$$E = \{\langle e^{ix}, e^{iy}\rangle : x = 1, \quad 1 \leqslant y \leqslant 2\}.$$

We note that the sets in (iii), and intervals in $\mathbf{T}^2$ are Ditkin sets by 7.5.2(e) of [7]; moreover, the sets in (iii), as subsets of $\mathbf{R}$, are strong Ditkin sets in $\mathbf{R}$ (see our Theorem 2.4, below).

To see that these sets do not belong to $\mathcal{R}(G_d)$, see the proof of Corollary 1.7 of [5] for (i); (ii) and (iii) both follow from the fact that if G is torsion free, an infinite subset of $\mathcal{R}(G_d)$ must contain, with the possible exception of a finite set, the coset of an infinite subgroup (see A_0 of the Appendix, page 71 of [5]). (iv) may be reduced to the same argument by observing that if $\pi : \mathbf{R}^2 \to \mathbf{T}^2$ is the homomorphism defined by $\pi(\langle x, y\rangle) = \langle e^{ix}, e^{iy}\rangle$, then if E is closed and in $\mathcal{R}(\mathbf{T}^2_d)$, then $\pi^{-1}(E)$ is closed and in $\mathcal{R}(\mathbf{R}^2_d)$. Now if E is as in (iv), E (or a translate of E) will have the property that $\pi^{-1}(E)$ has an infinite compact component F. We may choose $f \in L^1(\mathbf{R}^2)$ so that $\hat{f} = 1$ on F and $\hat{f} = 0$ on $\pi^{-1}(E) \cap \complement F$, from which it follows that $\chi_F = \chi_E \hat{f}$; thus by Cohen's theorem, if $E \in \mathcal{R}(\mathbf{R}^2_d)$, we would also have $F \in \mathcal{R}(R^2_d)$, since χ_F would be the product of two Stieltjes transforms of measures on the Bohr compactification of $\mathbf{R}^2$.

§ 2. Sets E which are strong Ditkin sets in separable metrizable groups G

Most of our theorems constitute the appropriate generalizations of Theorems 1, 2, and 7 of [8]. Throughout this section, we assume that G and Γ are both metrizable, i.e. that G is separable metrizable; δ denotes the finite measure on Γ assigning point mass 1 to the zero of Γ; i.e. $\hat{\delta} \equiv 1$.

Let us recall first that strong Ditkin sets, like Ditkin sets, have the following property:

Theorem 2.1 (Wik). *Finite unions of strong Ditkin sets are strong Ditkin sets.*

For the proof, see Theorem 3 of [8]. (Although Wik stated this result only for $\mathbf{T}$, his short, elegant argument holds word for word in complete generality.)

For most of the theorems in this section, we will have need of the following characterization of strong Ditkin sets:

Lemma 2.2.(a) *E is a strong Ditkin set if and only if there exists a sequence $\{v'_n\}_{n=1}^{\infty} \subset L^1(\Gamma)$ so that for all n, $\hat{v}'_n$ has compact support disjoint from E, and $v'_n * f \to f$ for all $f \in I(E)$.*

(b) *E is a strong Ditkin subset of G if and only if there exists a sequence $\{\mu_n\}$ of finite measures on Γ so that for all n, $\hat{\mu}_n = 1$ in a neighborhood of E, and $\|\mu_n * f\| \to 0$ for all $f \in I(E)$.*

(Hereafter, if $\{\mu_n\}_{n=1}^{\infty}$ has these properties, we shall simply say that $\{\mu_n\}$ *obeys* 2.2 *for* $I(E)$.)

Proof. (a) Assuming that E is a strong Ditkin set, let $\{v_n\} \subset I(E)$ with $v_n * f \to f$ for all $f \in I(E)$. Now by definition, E is also of spectral synthesis, hence, $J = \{g \in L^1(\Gamma): \hat{g}$ has compact support disjoint from $E\}$ must be dense in $I(E)$. (Indeed J is an ideal associated with E, and hence the closure of J must be equal to $I(E)$.) So, for each n, choose $v_n' \in J$ with $\|v_n' - v_n\|_{L^1(\Gamma)} \leq 1/n$. Then $\{v_n'\}_{n=1}^{\infty}$ has the desired properties. The converse is trivial.

(b) Suppose first that E is a strong Ditkin set. Let $\{v_n'\}$ be the sequence given by (a), and define $\mu_n = \delta - v_n'$ for all n; then $\{\mu_n\}$ has the desired properties.

Conversely, given the sequence $\{\mu_n\}$, let $\{f_n\}$ be an approximate identity for $L^1(\Gamma)$, with $\|f_n\| \leq 1$ for all n. (For example, letting $\{U_n\}_{n=1}^{\infty}$ be a nested base for the open neighborhoods of 0, simply let f_n be a non-negative continuous function with compact support in U_n and $\|f_n\| = \hat{f}_n(0) = 1$. A countable nested base $U_1 \supset U_2 \supset \ldots$ exists because we assume that Γ is metrizable.)

Now set $v_n = f_n * (\delta - \mu_n)$; then $v_n * f \to f$ all $f \in I(E)$; moreover, $\hat{v}_n = 0$ in a neighborhood of E for all n, showing that $I_0(E) = I(E)$ and hence that E is of spectral synthesis. Q.E.D.

Theorem 2.3. *Every closed coset of G is a strong Ditkin set; i.e. if H is a closed subgroup of G and g is an element of G, then $g + H$ is a strong Ditkin set.*

Proof. It suffices to prove this for a closed subgroup H, since translates of strong Ditkin sets are again strong Ditkin sets. Let $U_1 \supset U_2 \supset \ldots$ be a nested base for the open neighborhoods of 0 in G, and let $\pi: G \to G/H$ be the natural continuous homomorphism from G onto G/H. Let Λ be the annihilator of H in Γ, i.e.,

$$\Lambda = \{\gamma \in \Gamma: (\gamma, h) = 1 \text{ all } h \in H\};$$

and let m_Λ be a haar measure on Λ. By 2.6.3, page 49, of [7] we may choose for each n, an $f_n \in L^1(\Lambda)$ with $\|f_n\|_{L^1(\Lambda)} \leq 2$, $\hat{f}_n$ supported in $\pi(U_n)$, and $\hat{f}_n = 1$ on a neighborhood of $\pi(0)$. Now set

$$\mu_n = f_n dm_\Lambda.$$

Then μ_n is a finite measure on Γ, $\hat{\mu}_n = 1$ on a neighborhood of H (in fact, if $w_n = \{\tau \in G/H: \hat{f}_n(\tau) = 1\}$, then $\hat{\mu}_n = 1$ on $H + \pi^{-1}(w_n)$), and $\hat{\mu}_n$ vanishes off the set $H + U_n$. Now, let

$$J = \{g \in L^1(\Gamma): \hat{g} \text{ has compact support disjoint from } H\}.$$

Now $\bar{U}_n$ must be compact for all sufficiently large n. Hence, if $g \in J$, then there is an N so that $\hat{\mu}_n \cdot \hat{g} \equiv 0$ all $n \geq N$; whence $\lim_{n \to \infty} \mu_n * g = 0$. Now J is dense in $I(H)$, since J is associated with H and H is of spectral synthesis (cf. 7.5.2 of [7]). Finally, since $\|\mu_n\| \leq 2$ for all n, it follows that $\{\mu_n\}$ obeys 2.2 for $I(E)$. Q.E.D.

Our next theorem generalizes Theorems 7 and 2 of [8], where it is established for $G = T$.

Theorem 2.4. (a) *If E is a strong Ditkin set and F is a compact subset of E so that $E \cap \complement F$ is closed, then both $E \cap \complement F$ and F are strong Ditkin sets.*

(b) *If the boundary of E is a strong Ditkin set, then E is a strong Ditkin set.*

Proof. (a) Our hypotheses imply that there exists an open set U so that $F \subset U$ and $E \cap \complement F \subset \complement \bar{U}$. Since F is compact we may choose a compact neighborhood C

of F so that $C \subset U$. Then by 2.6.2 of [7] there exists an $h \in L^1(\Gamma)$ so that $\hat{h}=1$ on C and $\hat{h}=0$ outside U. Hence $\hat{h}=1$ on a neighborhood of F and $\hat{h}=0$ on a neighborhood of $E \cap \complement F$.

Now, let $\{\mu_n\}_{n=1}^{\infty}$ obey 2.2 for $I(E)$. Then setting $\mu_n' = \mu_n * h$ for all n, $\{\mu_n'\}$ obeys 2.2 for $I(F)$; setting $\mu_n'' = \mu_n * (\delta - h)$ for all n, $\{\mu_n''\}$ obeys 2.2 for $I(E \cap \complement F)$.

(*b*) Let FE denote the boundary, or frontier, of E, and let $\{v_n\}_{n=1}^{\infty}$ satisfy the conditions of 2.2(*a*) for the set FE. That is, letting W_n be the support of v_n, we have for all n that W_n is compact and disjoint from FE, and that $v_n * f \to f$ for all $f \in I(FE)$. Fixing n, it follows that $W_n \cap \complement E$ is a compact set, and hence we may choose a function $h_n \in L^1(\Gamma)$ with $\hat{h}_n = 1$ on $W_n \cap \complement E$ and $\hat{h}_n = 0$ on a neighborhood of E; then $h_n * v_n * f = v_n * f$ for all $f \in I(E)$, by uniqueness of Fourier transform. Hence the sequence $\{h_n * v_n\}_{n=1}^{\infty}$ satisfies the conditions of Lemma 2.2(*a*) for the set E. Q.E.D.

Remark. (1) We thus obtain by 2.2(*b*) that certain closed proper neighborhoods in $\mathbf{T}^n$ are strong Ditkin sets. For example, for α a fixed real number, $0 < \alpha < 2\pi$, we have that $E = \{\langle e^{ix}, e^{iy}\rangle : 0 \leqslant x \leqslant 2\pi,\ 0 \leqslant y \leqslant \alpha\}$ is a strong Ditkin subset of $\mathbf{T}^2$, since the boundary of E consists of the union of a closed subgroup and a closed coset of $\mathbf{T}^2$, namely

$$\{\langle e^{ix}, 1\rangle : 0 \leqslant x \leqslant 2\pi\} \quad \text{and} \quad \{\langle e^{ix}, e^{i\alpha}\rangle : 0 \leqslant x \leqslant 2\pi\}.$$

Remark. (2) It is proved in 7.5.2(*d*) of [7] that if $E \subset H$ where H is a closed subgroup of G, and if the boundary of E relative to H is a Ditkin set in G, then E is a Ditkin set in G. Note that this does not hold for strong Ditkin sets, i.e. our 2.4(*b*) cannot be sharpened to this extent. Indeed, we have shown (see the remark following Theorem 1.3) that $E = \{\langle x, y\rangle : y = 0,\ 0 \leqslant x \leqslant 1\}$ is not a strong Ditkin set in $\mathbf{R}^2$, yet the boundary of this set relative to $\mathbf{R}$ consists of two points, and hence is a strong Ditkin set.

For the final result of this paper, we give a complete characterization of those nowhere dense subsets of $\mathbf{R}$ which are strong Ditkin sets. (It is perhaps surprising that even most discrete subsets are not; for example $E = \{n^2 : n \text{ an integer}\}$ is not a strong Ditkin subset of $\mathbf{R}$; note that this shows that one cannot replace the hypothesis "F is compact" by the hypothesis "F is closed" in 2.4(*a*).)

Theorem 2.5. *Let E be a closed nowhere-dense subset of $\mathbf{R}$. Then E is a strong Ditkin set if and only if there exists a finite set F, and numbers $\tau_1, \ldots, \tau_n$; $\beta_1, \ldots, \beta_n$; so that*

$$E \cup F = \bigcup_{i=1}^{n} \tau_i \mathbf{Z} + \beta_i.$$

Moreover, if E is not of this form, then $\nexists$ App. Id. in $I(E)$.

Proof. Suppose first that E is of the above form. Then by Theorem 2.3, each set of the form $\tau_i \mathbf{Z} + \beta_i$ is a closed coset of $\mathbf{R}$, hence is a strong Ditkin set; by Theorem 2.1, $\bigcup_{i=1}^{n} \tau_i \mathbf{Z} + \beta_i$ is a strong Ditkin set. Finally, we may assume that $F \cap E = \phi$; F is then a compact subset of $\bigcup_{i=1}^{n} \tau_i \mathbf{Z} + \beta_i$ with

$$E = \left(\bigcup_{i=1}^{n} \tau_i \mathbf{Z} + \beta_i\right) \sim F \text{ being a closed set.}$$

Hence by Theorem 2.4(*a*), E is a strong Ditkin set.

Conversely, if $\exists$ App. Id. in $I(E)$, then $E \in \mathcal{R}(\mathbf{R}_d)$ by Theorem 1.3. Thus by the proof of Theorem 1.6 of [5], E must be of the above form. Q.E.D.

Final remark. We note that all the results of Section 2 hold for unrestricted locally compact abelian groups G, if we replace all sequences by nets uniformly bounded in convolution operator norm. Thus, we say that an arbitrary commutative Banach algebra B has an approximate identity if there exists a net $\{f_\alpha\}_{\alpha \in D} \subset B$, where D is a directed set, such that there exists a constant M, so that for all $g \in B$,

$$\|f_\alpha * g\| \leqslant M \|g\| \text{ for all } \alpha, \tag{$\blacktriangle$}$$

and

$$\lim_{\alpha \in D} \|f_\alpha * g - g\| = 0.$$

Of course, we then say that $\{f_\alpha\}_{\alpha \in D} \subset I(E)$ is an approximate identity for $I_0(E)$ in $I(E)$, if there exists a constant M so that for all $g \in I_0(E)$, ($\blacktriangle$) holds.

Note that if $\{v_\alpha\}_{\alpha \in D}$ is an App. Id. in $I(E)$, then the appropriate versions of (i) and (ii) of Theorem 1.1 hold. *Thus if we make the stronger assumption that $\|v_\alpha\| \leqslant M$ for all $\alpha \in D$, we obtain that $E \in \mathcal{R}(G_d)$ by an argument identical to that of Theorem* 1.1.

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University of California, Berkeley, Calif., U.S.A.

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