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On absolute convergence of Fourier series

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§ 1. Let f be an integrable function on $(0, 2\pi)$ and periodic with period 2π, and its Fourier series be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

If $\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty$, then we say that the Fourier series of f converges absolutely and we write $f \in A$. If $f \in A$, then f must be bounded and continuous.

We define the modulus of continuity of f by

$$\omega(\delta; f) = \sup_{|x-x'| \leqslant \delta} |f(x) - f(x')|$$

and the integrated modulus of continuity of f by

$$\omega_p(\delta; f) = \sup_{0 < h \leqslant \delta} \left(\int_0^{2\pi} |f(x+h) - f(x)|^p \, dx \right)^{1/p}$$

for $p \geqslant 1$. It is known that $\omega_p(\delta; f) \leqslant \omega(\delta; f)$ and $\lim_{p \to \infty} \omega_p(\delta; f) = \omega(\delta; f)$.

Concerning absolute convergence of Fourier series there are two famous theorems, one is due to S. Bernstein and the other to A. Zygmund.

Bernstein's theorem ([1], p. 241; [2], p. 154) reads as follows:

Theorem I. *If*
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \omega\left(\frac{1}{n}; f\right) < \infty,$$
then $f \in A$.

This was generalized hy O. Szász ([2], p. 155) in the following form:

Theorem II. *If*
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \omega_2\left(\frac{1}{n}; f\right) < \infty,$$
then $f \in A$.

Zygmund's theorem ([1], p. 242; [2], p. 160)[1] reads as follows.

[1] The condition of this form was first formulated by E. Hille and J. D. Tamarkin [3].

Theorem III. *If f is of bounded variation and*

$$\sum_{n=1}^{\infty} \frac{1}{n} \sqrt{\omega\left(\frac{1}{n}; f\right)} < \infty,$$

then $f \in A$.

We shall prove a "bridge" theorem between Theorems II and III. For this purpose we need the notion of r-bounded variation which is defined as follows: f is called of r-bounded variation when

$$\exists\, M > 0 : \left(\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^r\right)^{1/r} \leqslant M$$

for all divisions $0 = x_0 < x_1 < \ldots < x_n = 2\pi$. The case $r = 1$ is ordinary bounded variation and the case $r \to \infty$ means boundedness of f.

Theorem 1. *Let $1 < p < \infty$, $1/p + 1/q = 1$ and $1 \leqslant r < 2p$. If f is of r-bounded variation and*

$$\sum_{n=1}^{\infty} \frac{1}{n^{1-1/2q}} (\omega_{r+(2-r)q}(\pi/n; f))^{1-r/2p} < \infty,$$

then $f \in A$.

If $r = 1$ and $p \to 1$ (consequently $q \to \infty$), then Theorem 1 reduces to Theorem III. If $r \to \infty$ and $p \to \infty$ such that $r/p \to 0$ (for example $r = \sqrt{p}$), then the theorem reduces to Theorem II, since $q \to 1$ and $r + (2-r)q = 2q + r(1-q) = 2q - q(r/p) \to 2$ and the boundedness of f is necessary for $f \in A$. Thus Theorem 1 is a bridge between Theorems II and III.

If we consider the cases $r = p = q = 2$ and $r = 3$, $p = q = 2$ in Theorem 1, then we get the following corollaries.

Corollary 1. *If f is of 2-bounded variation and*

$$\sum_{n=1}^{\infty} n^{-\frac{3}{4}} \sqrt{\omega_2\left(\frac{1}{n}; f\right)} < \infty,$$

then $f \in A$.

Corollary 2. *If f is of 3-bounded variation and*

$$\sum_{n=1}^{\infty} n^{-\frac{3}{4}} \sqrt[4]{\omega_1\left(\frac{1}{n}; f\right)} < \infty,$$

then $f \in A$.

Further we prove the following theorems.

Theorem 2. *Suppose that f is even and of bounded variation and put $g(x) = \int_0^x t\, df(t)$. If*

$$\int_0^{\pi} \frac{\omega(t; g)}{t^2} \log \frac{2\pi}{t}\, dt < \infty,$$

then $f \in A$.

This is not contained in Theorem III. For example, let

$$f(x)=\frac{1}{(\log(2\pi/x))^2} \text{ on } (0,\pi), \quad f(x)=f(-x) \text{ on } (-\pi,0),$$

then

$$\omega(t;f)=\frac{1}{(\log(2\pi/t))^2}$$

and hence

$$\sum_{n=1}^{\infty}\frac{1}{n}\sqrt{\omega\left(\frac{1}{n};f\right)}\geqslant A\int_0^{\pi}\sqrt{\omega(t;f)}\,\frac{dt}{t}=A\int_0^{\pi}\left(\log\frac{2\pi}{t}\right)^{-1}\frac{dt}{t}=\infty.$$

Therefore the condition of Theorem III is not satisfied. But, the conditions of Theorem 2 are satisfied. For,

$$tf'(t)=\frac{-2}{(\log(2\pi/t))^3}, \quad g(t)=\frac{-2t(1+0(t))}{(\log(2\pi/t))^3}$$

and then[1]

$$\omega(t;g)\leqslant\frac{Ct}{(\log(2\pi/t))^3},$$

$$\int_0^{\pi}\frac{\omega(t;g)}{t^2}\log\frac{2\pi}{t}\,dt\leqslant C\int_0^{\pi}\frac{dt}{t(\log(2\pi/t))^2}<\infty.$$

Theorem 3. *Suppose that f is an odd function of bounded variation and vanishing at 0 and π.*[2] *If*

$$\int_0^{\pi}\log\frac{2\pi}{t}\,|df(t)|<\infty$$

and

$$\int_{2h}^{\pi}t^{\Delta}\,|d(f(t)-f(t+h))|\leqslant Ch^{\alpha} \tag{1}$$

for some $\Delta>1$ and some $\alpha(0<\alpha\leqslant 1)$, then $f\in A$.

The condition of the theorem is satisfied by the odd function f such that

$$f(x)=\frac{1}{(\log(2\pi/x))^{1+\varepsilon}} \quad (\varepsilon>0)$$

in the right neighbourhood of $t=0$ and is very smooth otherwise.

§ 2. *Proof of Theorem 1.* We have

$$f(x+h)-f(x-h)\sim 2\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)\sin nh,$$

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[1] We denote by C an absolute constant, different in different occurrences.

[2] The continuity of f at the point π is contained in the last assumption (1).

and then, by the Parseval formula, we get

$$\frac{1}{\pi}\int_0^{2\pi}[f(x+h)-f(x-h)]^2\,dx = 4\sum_{n=1}^{\infty}\varrho_n^2\sin^2 nh \quad \text{where } \varrho_n^2 = a_n^2+b_n^2.$$

We have to prove that $\sum \varrho_n < \infty$.

Now we write

$$2 = \frac{r}{p} + \left(2-\frac{r}{p}\right) = \frac{r}{p} + \frac{2q-r(q-1)}{q} = \frac{r}{p} + \frac{(2-r)\,q+r}{q}$$

and let us consider the sum (cf. [1], p. 242)

$$\sum_{k=1}^{2N}\left(\int_0^{2\pi}[f(x+k\pi/N)-f(x+(k-1)\pi/N)]^2\,dx\right)^p$$
$$= \sum_{k=1}^{2N}\left(\int_0^{2\pi}[f(x+k\pi/N)-f(x+(k-1)\pi/N)]^{\frac{r}{p}+\frac{(2-r)\,q+r}{q}}\,dx\right)^p.$$

By Hölder's inequality, this is less than

$$\sum_{k=1}^{2N}\left(\int_0^{2\pi}|f(x+k\pi/N)-f(x+(k-1)\pi/N)|^r\,dx\right.$$
$$\times\left(\int_0^{2\pi}|f(x+k\pi/N)-f(x+(k-1)\pi/N)|^{r+(2-r)\,q}\,dx\right)^{p/q}$$
$$= \sum_{k=1}^{2N}\left(\int_0^{2\pi}|f(x+k\pi/N)-f(x+(k-1)\pi/N)|^r\,dx\right)$$
$$\times\left(\int_0^{2\pi}|f(x+\pi/N)-f(x)|^{r+(2-r)\,q}\,dx\right)^{p/q}$$
$$= \int_0^{2\pi}\left(\sum_{k=1}^{2N}|f(x+k\pi/N)-f(x+(k-1)\pi/N)|^r\right)dx\,(\omega_{r+(2-r)\,q}(\pi/N;f))^{2p-r}$$
$$\leqslant 2V_r^r\,(\omega_{r+(2-r)\,q}(\pi/N;f))^{2p-r},$$

where $V_r = \sup\,(\sum_{i=1}^n|f(x_i)-f(x_{i-1})|^r)^{1/r}$, sup being taken about all divisions $0 = x_0 < x_1 < \ldots < x_n = 2\pi$. Therefore

$$\sum_{n=1}^{N}\varrho_n^2\sin^2\frac{n\pi}{2N} \geqslant C(\omega_{r+(2-r)\,q}(\pi/N;f))^{2-r/p}\cdot N^{-1/p}$$

and then
$$\sum_{n=1}^{N'} n^2\varrho_n^2 \leqslant CN^{1+1/q}\,(\omega_{r+(2-r)\,q}(\pi/N;f))^{2-r/p}.$$

Let us put $\varphi_n = \sum_{k=1}^n k\varrho_k$, then $\varphi_n \leqslant \sqrt{n}\,(\sum_{k=1}^n k^2\varrho_k^2)^{\frac{1}{2}}$ by the Schwarz inequality and then

$$\varphi_n \leqslant Cn^{1+1/2q}\,(\omega_{r+(2-r)\,q}(\pi/n;f))^{1-r/2p}.$$

Now

$$\sum_{n=1}^{N}\varrho_n=\sum_{n=1}^{N}(\varphi_n-\varphi_{n-1})/n$$

$$=\sum_{n=1}^{N-1}\varphi_n\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{\varphi_N}{N}<\sum_{n=1}^{N-1}\frac{\varphi_n}{n^2}+\frac{\varphi_N}{N}$$

$$\leqslant C\sum_{n=1}^{N-1}\frac{1}{n^{1-1/2q}}\,(\omega_{r+(2-r)q}(\pi/n;f))^{1-r/2p}+CN^{1/2q}(\omega_{r+(2-r)q}(\pi/N;f))^{1-r/2p}$$

$$\leqslant C\sum_{n=1}^{N-1}\frac{1}{n^{1-1/2q}}\,(\omega_{r+(2-r)q}(\pi/n;f))^{1-r/2p}<C$$

for all N. Thus we have proved that $\sum\varrho_n<\infty$.

§ 3. *Proof of Theorem 2.* Let $f(x)\sim\sum a_n\cos nx$, then

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=-\frac{2}{\pi n}\int_0^{\pi}\sin nx\,df(x).$$

We shall prove that

$$\sum_{n=1}^{\infty}|a_n|=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^{\pi}\sin nx\,df(x)\right|$$

is finite. Now we write

$$\int_0^{\pi}\sin nx\,df(x)=\int_0^{\pi/n}\sin nx\,df(x)+\int_{\pi/n}^{\pi}\frac{\sin nx}{x}\,dg(x)=I_n+J_n.$$

Then

$$\sum_{n=1}^{\infty}\frac{1}{n}|I_n|\leqslant\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=n}^{\infty}\int_{\pi/(k+1)}^{\pi/k}nx\,|df(x)|$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{k}\int_{\pi/(k+1)}^{\pi/k}x\,|df(x)|=\sum_{k=1}^{\infty}k\int_{\pi/(k+1)}^{\pi/k}x\,|df(x)|\leqslant C\int_0^{\pi}|df(x)|<\infty.$$

We shall estimate J_n. We suppose first that n is odd, then

$$J_n=\int_{\pi/n}^{\pi}\frac{\sin nx}{x}\,dg(x)=\sum_{k=1}^{n-1}\int_{k\pi/n}^{(k+1)\pi/n}\frac{\sin nx}{x}\,dg(x)$$

$$=\sum_{k=1}^{n-1}(-1)^k\int_0^{\pi/n}\sin nt\,\frac{dg(t+k\pi/n)}{t+k\pi/n}$$

$$=\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\sin nt\left(\frac{dg(t+2k\pi n)}{t+2k\pi/n}-\frac{dg(t+(2k-1)\pi/n)}{t+(2k-1)\pi/n}\right)$$

$$= \sum_{k=1}^{(n-1)/2} \left[\int_0^{\pi/n} \frac{\sin nt}{2k\pi/n} (dg(t+2k\pi/n) - dg(t+(2k-1)\pi/n)) \right.$$

$$- \int_0^{\pi/n} \frac{t \sin nt}{(t+2k\pi/n)\cdot 2k\pi/n} (dg(t+2k\pi/n) - dg(t+(2k-1)\pi/n))$$

$$\left. - \frac{\pi}{n} \int_0^{\pi/n} \frac{\sin nt}{(t+2k\pi/n)(t+(2k-1)\pi/n)} dg(t+(2k-1)\pi/n) \right]$$

$$= \sum_{k=1}^{(n-1)/2} (J_{n,k}^1 - J_{n,k}^2 - J_{n,k}^3) = J_n^1 - J_n^2 - J_n^3.$$

We have

$$J_{n,k}^1 = \frac{n}{2k\pi} \int_0^{\pi/n} \sin nt \, [dg(2k\pi/n + t) - dg(2k\pi/n - (\pi/n - t))]$$

$$= \frac{n}{2k\pi} \int_0^{\pi/n} \sin nt \, d\, [g(2k\pi/n + t) - g(2k\pi/n - t)]$$

$$= \frac{n^2}{2k\pi} \int_0^{\pi/n} \cos nt \, [g(2k\pi/n + t) - g(2k\pi/n - t)] \, dt.$$

Writing $\omega(t) = \omega(t; g)$, we get

$$|J_{n,k}^1| \leqslant \frac{n^2}{k\pi} \int_0^{\pi/n} \omega(t)\, dt,$$

$$|J_n^1| \leqslant \sum_{k=1}^{(n-1)/2} |J_{n,k}^1| \leqslant Cn^2 \log n \int_0^{\pi/n} \omega(t)\, dt$$

and then

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} |J_n^1| \leqslant C \sum_{n=1}^{\infty} n \log n \int_0^{\pi/n} \omega(t)\, dt$$

$$= C \sum_{n=1}^{\infty} n \log n \sum_{n=1}^{\infty} \int_{\pi/(k+1)}^{\pi/k} \omega(t)\, dt$$

$$= C \sum_{k=1}^{\infty} \int_{\pi/(k+1)}^{\pi/k} \omega(t)\, dt \sum_{n=1}^{k} n \log n$$

$$\leqslant C \sum_{k=1}^{\infty} k^2 \log k \int_{\pi/(k+1)}^{\pi/k} \omega(t)\, dt$$

$$\leqslant C \sum_{k=1}^{\infty} \int_{\pi/(k+1)}^{\pi/k} \frac{\omega(t)}{t^2} \log \frac{2\pi}{t}\, dt$$

$$= C \int_0^{\pi} \frac{\omega(t)}{t^2} \log \frac{2\pi}{t}\, dt < \infty$$

by the assumption.

Secondly we have

$$J_{n,k}^2=\frac{n}{2k\pi}\int_0^{\pi/n}\frac{t\sin nt}{t+2k\pi/n}[dg(2k\pi/n+t)-dg((2k-1)\pi/n+t)]$$

$$=-\frac{n}{2k\pi}\int_0^{\pi/n}\left(\frac{t\sin nt}{t+2k\pi/n}\right)'[g(2k\pi/n+t)-g((2k-1)\pi/n+t)]\,dt$$

and then
$$|J_{n,k}^2|\leqslant\frac{Cn}{k^2}\,\omega\left(\frac{\pi}{n}\right),$$

$$|J_n^2|\leqslant\sum_{k=1}^{(n-1)/2}|J_{n,k}^2|\leqslant C\sum_{k=1}^{(n-1)/2}\frac{n}{k^2}\,\omega\left(\frac{\pi}{n}\right)\leqslant Cn\omega\left(\frac{\pi}{n}\right)$$

and
$$\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{n}|J_n^2|\leqslant C\sum_{n=1}^{\infty}\omega\left(\frac{\pi}{n}\right)\leqslant C\int_0^{\pi}\frac{\omega(t)}{t^2}\,dt<\infty.$$

Finally,

$$J_{n,k}^3=\frac{\pi}{n}\int_0^{\pi/n}\frac{\sin nt}{(t+2k\pi/n)\,(t+(2k-1)\,\pi/n)}\,dg(t+(2k-1)\,\pi/n)$$

$$=\frac{\pi}{n}\int_0^{\pi/n}\frac{\sin nt}{t+2k\pi/n}\,df(t+(2k-1)\,\pi/n)$$

and then

$$|J_{n,k}^3|\leqslant\frac{C}{k}\int_0^{\pi/n}|df(t+(2k-1)\,\pi/n)|\leqslant\frac{C}{k}\int_{(2k-1)\pi/n}^{2k\pi/n}|df(t)|,$$

$$|J_n^3|\leqslant\sum_{k=1}^{(n-1)/2}|J_{n,k}^3|\leqslant\frac{C}{n}\sum_{k=1}^{k}\int_{k\pi/n}^{(k+1)\pi/n}\frac{|df(t)|}{t}$$

and

$$\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{n}|J_n^3|\leqslant C\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n}\int_{k\pi/n}^{(k+1)\pi/n}\frac{|df(t)|}{t}$$

$$=C\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{1}{n^2}\int_{k\pi/n}^{(k+1)\pi/n}\frac{|df(t)|}{t}\leqslant C\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\pi}t\,|df(t)|\leqslant C\int_0^{\pi}|df(t)|<\infty.$$

Hence we have
$$\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{n}|J_n|<\infty.$$

For even n, we get the same estimation. Thus we have proved the theorem.

§ 4. *Proof of Theorem 3.* We write $f(x)\sim\sum b_n\sin nx$, then[1]

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin nx\,[f(x)-f(x-\pi/n)]\,dx$$

[1] Cf. [3], p. 533, Theorem (A), VI.

$$= \frac{1}{2\pi n} \int_{-\pi}^{\pi} \cos nx \, d[f(x) - f(x - \pi/n)]$$

$$= \frac{1}{2\pi n} \left(\int_{0}^{\pi} + \int_{-\pi}^{0} \right) = \frac{1}{2\pi n} (b_n' + b_n'').$$

We shall estimate b_n'. Let $r = \alpha/2\Delta$, then $0 < r < 1$. We put

$$b_n' = \int_{0}^{\pi/n^r} + \int_{\pi/n^r}^{\pi} = I_n + J_n.$$

Since f is continuous at the origin, we have

$$\sum_{n=1}^{\infty} \frac{1}{n} |I_n| \leq 3 \sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{2\pi/n^r} |df(x)|$$

$$\leq 3 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=[n^r]}^{\infty} \int_{2\pi/(k+1)}^{2\pi/k} |df(x)|$$

$$\leq 3 \sum_{k=1}^{\infty} \int_{2\pi/(k+1)}^{2\pi/k} |df(x)| \sum_{n=1}^{[k^{1/r}]+1} \frac{1}{n}$$

$$\leq C \sum_{k=1}^{\infty} \log(k+1) \int_{2\pi/(k+1)}^{2\pi/k} |df(x)|$$

$$\leq C \int_{0}^{\pi} \log \frac{2\pi}{x} |df(x)| < \infty.$$

Further

$$\sum_{n=1}^{\infty} \frac{1}{n} |J_n| \leq \sum_{n=1}^{\infty} \frac{1}{n} \int_{\pi/n^r}^{\pi} |d(f(x) - f(x - \pi/n))|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^{1-\alpha/2}} \int_{\pi/n^r}^{\pi} x^{\Delta} |d(f(x) - f(x - \pi/n))|$$

$$\leq C \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha/2}} < \infty.$$

Thus we have proved that $\sum |b_n'|/n < \infty$. Similarly we have $\sum |b_n''|/n < \infty$ and then $\sum |b_n| < \infty$. The theorem is now completely proved.

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