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On the central limit theorem in R_k

By Bengt von Bahr

1. Introduction

Let $X^{(\nu)}=(X_1^{(\nu)}, \dots X_k^{(\nu)})$, $\nu=1, 2, \dots n$, be a sequence of independent and identically distributed random vectors (r.v.'s) in R_k, $k>1$, with zero mean and non-singular covariance matrix M. Then, according to the Central Limit Theorem, the normed sum $Y_n=n^{-\frac{1}{2}}\sum_{\nu=1}^{n}X^{(\nu)}$ is approximately normally distributed, with the same moments of the first and second orders as $X^{(1)}$. In the present paper, we shall consider the distribution of the norm $|Y_n|=(Y_{n1}^2+\dots+Y_{nk}^2)^{\frac{1}{2}}$, and estimate the difference

$$P(|Y_n|\leqslant a)-\int_{|x|\leqslant a} d\Phi(x), \tag{1}$$

where $\Phi(x)$, $x=(x_1, \dots x_k)$ is the corresponding normal distribution function (d.f.) and $|x|=(x_1^2+\dots+x_k^2)^{\frac{1}{2}}$. If the moments of the fourth order exist and if $M=E$ (unit matrix of order $k\times k$), then (Esseen [3])

$$|P(|Y_n|\leqslant a)-K_k(a^2)|\leqslant Cn^{-k/(k+1)}, \tag{2}$$

where $K_k(x)$ is the d.f. of the χ^2-distribution with k degrees of freedom, and C is a finite constant, only depending on the moments of $X^{(1)}$. Here we shall study the difference (1) as a function of both n and a.

2. Convergence of characteristic functions

We introduce the d.f.'s $F(x)$ and $F_n(x)$ and the characteristic functions (ch.f.'s) $f(t)$ and $f_n(t)$ of $X^{(1)}$ and Y_n respectively. We have

$$f(t)=\int_{R_k} e^{i(t,x)}\,dF(x), \quad t=(t_1, \dots t_k), \quad (t,x)=\sum_{j=1}^{k} t_j x_j$$

and $f_n(t)=f^n(t/\sqrt{n})$. If the moment $\beta_r=E|X^{(1)}|^r<\infty$, r integer $\geqslant 3$, then $\log f(t)$ has the Taylor expansion

$$\log f(t)=-\tfrac{1}{2}(t, Mt)+\sum_{\nu=3}^{r}\frac{(\varkappa, it)^{\nu}}{\nu!}+o(|t|^r), \tag{3}$$

where $(\varkappa, it)^\nu = (\varkappa_1 it_1 + \dots + \varkappa_k it_k)^\nu$, and $\varkappa_1^{i_1} \dots \varkappa_k^{i_k}$ is the semi-invariant of order $(i_1, \dots i_k)$. According to (3), the relation

$$e^{(t, Mt)/2} f_n(t) = 1 + \sum_{\nu=1}^{r=2} n^{-\nu/2} P_\nu(it) + o\left(n^{-\frac{r-2}{2}}\right) \tag{4}$$

defines a sequence of polynomials P_ν of degree 3ν, the coefficients of which are functions of the moments of $X^{(1)}$. By estimating the remainder term in (4), we obtain the following lemma.

Lemma 1. *If $\beta_r < \infty$, r integer ≥ 3, then for all t with $|t| \leq K\sqrt{n}$*

$$\left| f_n(t) - \left(1 + \sum_{\nu=1}^{r-2} n^{-\nu/2} P_\nu(it)\right) e^{-(t, Mt)/2} \right| \leq C \frac{d(n)}{n^{(r-2)/2}} |t|^r e^{-\alpha|t|^2};$$

K and α are positive constants, only depending on k, r and the moments of $X^{(1)}$; $d(n)$ is bounded by one and $\lim_{n\to\infty} d(n) = 0$. Here and in what follows, we denote by C unspecified constants, with the same properties as K and α.

A proof of the lemma in the one-dimensional case is given by Gnedenko and Kolmogorov ([5] pp. 204–208). The present case is treated in the same way.

If $g(t)$ is the Fourier–Stieltjes Transform (F.S.T.) of $G(x)$, that is

$$g(t) = \int_{R_k} e^{i(t, x)} dG(x),$$

then $-it_j g(t)$ is the F.S.T. of $\partial G(x)/\partial x_j$ and thus $P_\nu(it) e^{-(t, Mt)/2}$ is the F.S.T. of $P_\nu(-D)\Phi(x)$, where $P_\nu(-D)$ is a derivation operator obtained from $P(it)$ by replacing it_j by $-\partial/\partial x_j$. We put

$$G_n(x) = \left(1 + \sum_{\nu=1}^{r-2} n^{-\nu/2} P_\nu(-D)\right) \Phi(x) \tag{5}$$

and $H_n(x) = F_n(x) - G_n(x)$, and thus, the corresponding F.S.T.'s are

$$g_n(t) = \left(1 + \sum_{\nu=1}^{r-2} n^{-\nu/2} P_\nu(it)\right) e^{-(t, Mt)/2}$$

and

$$h_n(t) = f_n(t) - g_n(t). \tag{6}$$

3. Main formula

In order to estimate $P(|Y_n| \leq a)$, we shall use the formula (Bochner [2], p. 318)

$$\int_{R_k} U(|x|)\, dH_n(x) = (2\pi)^{-k} \int_{R_k} u(|t|)\, h_n(t)\, dt \quad (dt = dt_1 \dots dt_k), \tag{7}$$

where $U(|x|)$ and $u(|t|)$ are integrable functions in R_k, only depending on $|x|$ and $|t|$ respectively and being F.T.'s in R_k, that is (Bochner [2], p. 235)

$$u(|t|)=\int_{R_k} e^{i(t,x)}\,U(|x|)\,dx=(2\pi)^{k/2}t^{-k/2+1}\int_0^\infty x^{k/2}J_{k/2-1}(x|t|)\,U(x)\,dx;$$

$U(|x|)$ is to be approximately 1 when $|x|\leqslant a$ and 0 when $|x|>a$, and for this purpose we let $U(|x|)$ be the convolution in R_k of two functions $V(|x|)$ and $\lambda^k Q(\lambda|x|)$, $\lambda>0$:

$$U(|x|)=\int_{R_k} V(|y|)\,\lambda^k Q(\lambda|x-y|)\,dy$$

$$=2\lambda^k\pi^{(k-1)/2}(\Gamma((k-1)/2))^{-1}\iint_{v\geqslant 0} V(\sqrt{u^2+v^2})\,Q(\lambda\sqrt{(|x|-u)^2+v^2})\,v^{k-2}\,du\,dv. \quad (8)$$

We take
$$V(x)=\begin{cases}1, & x\leqslant b\\ 0, & x>b\end{cases}$$

and choose the function $Q(|x|)$ with compact support and rapidly decreasing F.T., the existence of which is guaranteed by the following lemma.

Lemma 2. *If $\varepsilon(t)$ is a positive function monotonically decreasing to zero when $t\to\infty$ and if $\int_1^\infty \varepsilon(t)/t\,dt<\infty$, then there exist two functions $Q(x)$* and $q(t)$, *defined for $x\geqslant 0$ and $t\geqslant 0$ respectively, being* F.T.'s *in R_k, that is*

$$q(|t|)=\int_{R_k} e^{i(t,x)}Q(|x|)\,dx,\quad t\in R_k \quad (9)$$

and satisfying
$$Q(x)\geqslant 0,\quad 0\leqslant q(t)\leqslant q(0)=1$$
$$Q(x)=0 \quad \textit{when} \quad x\geqslant 1$$
$$q(t) \textit{ and } q'(t) \textit{ are } O(e^{-t\varepsilon(t)}) \quad \textit{when} \quad t\to\infty.$$

In the one-dimensional case, the lemma follows from theorems proved by Paley and Wiener [7] and Ingham [6]. In the present case it can be proved by putting

$$q(t)=\prod_{n=1}^{\infty}\Gamma(k/2+1)\,2^{k/2}(\varrho_n t)^{-k/2}J_{k/2}(\varrho_n t),$$

the quantities ϱ_n being suitably chosen and satisfying $\varrho_n>0$ and $\sum_{n=1}^\infty \varrho_n\leqslant 1$. We put $P(|Y_n|\leqslant a)=\mu(a)$ and

$$\eta(a)=\int_{|x|\leqslant a} dH_n(x)=\int_{|x|\leqslant a} dF_n(x)-\int_{|x|\leqslant a} dG_n(x)=\mu(a)-\psi(a)$$

and thus the formula (7) becomes

$$\int_0^\infty U(x)\,d\eta(x) = \int_{R_k} (b/2\pi|t|)^{k/2} J_{k/2}(b|t|)\, q(|t|/\lambda)\, h_n(t)\, dt \tag{10}$$

which is the starting-point for our estimations.

4. Point estimations

We first show two theorems, which are generalizations to the multi-dimensional case of results given by Esseen [3].

Theorem 1. *If $\beta_r < \infty$, r integer $\geqslant 3$, and if m is the largest eigenvalue of the matrix M, then*

$$\left| P(|Y_n| \leqslant a) - \int_{|x| \leqslant a} d\Phi(x) \right| \leqslant C \cdot a^{-r} \cdot \frac{d(n)}{n^{(r-2)/2}}$$

for $$a \geqslant (\tfrac{5}{4} m(r-2) \log n)^{\frac{1}{2}}.$$

Proof. We take $G_n(x)$ according to (5) and obtain

$$\eta(a) = P(|Y_n| \leqslant a) - \int_{|x| \leqslant a} d\Phi(x) + \sum_{\nu=1}^{r-2} n^{-\nu/2} \int_{|x| \leqslant a} dP_\nu(-D)\, \Phi(x).$$

In order to estimate $\eta(a)$, we choose $Q(|x|)$ and $q(|t|)$ according to Lemma 2, such that $q(|t|) \leqslant C(1+|t|^{r+k/2})^{-1}$ and distinguish between the two cases $\eta(a) \geqslant 0$ and $\eta(a) < 0$. If $\eta(a) \geqslant 0$, we put $b = a + \lambda^{-1}$, and thus $U(x) = 1$ when $x \leqslant a$, $0 \leqslant U(x) \leqslant 1$ when $a \leqslant x \leqslant a + 2/\lambda$ and $U(x) = 0$ when $x > a + 2/\lambda$. Since $d\eta(x) = d\mu(x) - d\psi(x)$ and $d\mu(x) \geqslant 0$, we obtain from (10)

$$\eta(a) \leqslant |I| + \int_a^{a+2/\lambda} |d\psi(x)|, \tag{11}$$

where I is the integral of the right-hand side of (10). We put $2/\lambda = a/2$ and divide I into two parts:

$$I = \int_{|t| \leqslant K\sqrt{n}} + \int_{|t| > K\sqrt{n}} .$$

In the first integral, we use Lemma 1 and in the second one the inequality $|h(t)| \leqslant C$ for estimating $h(t)$. Easy calculations now give

$$|I| \leqslant Ca^{-r} \frac{d(n)}{n^{(r-2)/2}}.$$

The last term of (11) is at most equal to

$$\int_{a<|x|<a+2/\lambda} |dG(x)| \leqslant \int_{a<|x|<a+2/\lambda} p(|x|)\, e^{-\frac{1}{2}(x, M^{-1}x)}\, dx,$$

where $p(y)$, $y>0$, is a positive polynomial. Now $(x, M^{-1}x) \geqslant m^{-1}|x|^2$ for all $x \in R_k$ and consequently

$$\int_a^{a+2/\lambda} |d\psi(x)| \leqslant C \int_a^{a+2/\lambda} p(x)\, e^{-x^2/2m}\, x^{k-1}\, dx \leqslant C \cdot a^{-r} \frac{d(n)}{n^{(r-2)/2}}$$

since $a^2/m \geqslant \frac{5}{4}(r-2)\log n$.

It now remains to estimate $\int_{|x|\leqslant a} dP_\nu(-D)\,\Phi(x)$, $\nu = 1, 2, \ldots, r-2$, but since $\int_{R_k} dP_\nu(-D)\,\Phi(x) = 0$, they can be treted in exactly the same way as the last term of (11), and thus the theorem is proved if $\eta(a) \geqslant 0$. If $\eta(a) < 0$, we choose $b = a - \lambda^{-1}$ and proceed in a similar way.

The proof is concluded.

In the remaining interval $a \leqslant \sqrt{\frac{5}{4}m(r-2)\log n}$ the estimations are more complicated, and the convergence of $P(|Y| \leqslant a)$ towards $\int_{|x|\leqslant a} d\Phi(x)$ is slower. In the following theorem we shall make use of Esseen's result (2), and thus we have to assume that $M = E$.

Theorem 2. *If $\beta_4 < \infty$ and if $M = E$, then for $a \leqslant \sqrt{\frac{5}{2}\log n}$*

$$|P(|Y| \leqslant a) - K_k(a^2)| \leqslant C n^{-k/(k+1)}(1 + a^{k+2})\, e^{-\delta a^2} + O\left(\frac{(\log n)^{(k-1)/4}}{n}\right),$$

where $\delta = \frac{1}{8}$ if $k=2$, and $\delta = (k-1)/2(k+1)$ if $k \geqslant 3$.

Proof. Because of (2), we can assume $a \geqslant 1$. We put

$$G_n(x) = \Phi(x) + n^{-\frac{1}{2}} P_1(-D)\,\Phi(x),$$

and then $\eta(a) = P(|Y| \leqslant a) - K_k(a^2)$, since $dP_1(-D)\,\Phi(x)$ is odd. According to Lemma 2, we can find two functions $Q(x)$ and $q(t)$ defined for $x \geqslant 0$ and $t \geqslant 0$, satisfying (9), and

$$Q(x) \geqslant 0, \quad 0 \leqslant q(t) \leqslant q(0) = 1,$$

$$q(t) = 0 \quad \text{when} \quad t \geqslant 1,$$

$$Q(x) \leqslant C e^{-x^{\frac{3}{4}}}, \quad |Q'(x)| \leqslant C e^{-x^{\frac{3}{4}}}.$$

As in the proof of Theorem 1, we must consider separately the two cases $\eta(a) \geqslant 0$ and $\eta(a) < 0$. If $\eta(a) \geqslant 0$, we take $\varepsilon > 0$ (to be determined later), put $b = a + \varepsilon$ and use (10). After dividing the left-hand integral into three parts, corresponding to the intervals $[0, a)$, $[a, a+2\varepsilon)$ and $(a+2\varepsilon, \infty)$, we obtain

$$\eta(a) = (1 - U(a))\,\eta(a) + U(a+2\varepsilon)\,\eta(a+2\varepsilon) - \int_a^{a+2\varepsilon} U(x)\, d\eta(x)$$
$$+ \left(\int_0^a + \int_{a+2\varepsilon}^\infty\right) U'(x)\,\eta(x)\, dx + I, \quad (12)$$

where I is the integral on the right-hand side of (10). Using (2), we get, since $d\mu(x) \geqslant 0$,

$$\eta(a) \leqslant \int_a^{a+2\varepsilon} |d\psi(x)| + Cn^{-k/(k+1)} \left\{1 - U(a) + U(a+2\varepsilon) + \left(\int_0^a + \int_{a+2\varepsilon}^{\infty}\right) |U'(x)|\, dx\right\} + |I|.$$

We now put $\lambda = n^{k/(k+1)}\, e^{-2\delta a^2/(k-1)}$, $\varepsilon = a^3/\lambda$, and easily obtain

$$\int_a^{a+2\varepsilon} |d\psi(x)| \leqslant C\varepsilon a^{k-1}\, e^{-a^2/2} \leqslant Cn^{-k/(k+1)}\, a^{k+2}\, e^{-\delta a^2}.$$

$V(y) = 0$ when $y > a + \varepsilon$, and thus we obtain from (8)

$$U(a) \geqslant 2\lambda^k \pi^{(k-1)/2} (\Gamma((k-1)/2))^{-1} \iint\limits_{\substack{v \geqslant 0 \\ (u-a)^2 + v^2 \leqslant \varepsilon^2}} Q(\lambda \sqrt{(u-a)^2 + v^2})\, v^{k-2}\, du\, dv$$

$$= 1 - 2\pi^{k/2} (\Gamma(k/2))^{-1} \int_{\lambda\varepsilon}^{\infty} Q(\varrho)\, \varrho^{k-1}\, d\varrho,$$

and

$$1 - U(a) \leqslant C \int_{a^3}^{\infty} e^{-\varrho^{\frac{1}{2}}} \varrho^{k-1}\, d\varrho \leqslant C \cdot a^{k+2}\, e^{-\delta a^2}.$$

$U(a + 2\varepsilon)$ is estimated in the same way.

By taking the derivative with respect to $|x|$ in (8), we obtain

$$U'(x) = 2\lambda^{k+1} \pi^{(k-1)/2} (\Gamma((k-1)/2))^{-1} \iint\limits_{\substack{u^2 + v^2 \leqslant b^2 \\ v \geqslant 0}} Q'(\lambda \sqrt{(x-u)^2 + v^2})\, \frac{x-u}{\sqrt{(x-u)^2 + v^2}}\, v^{k-2}\, du\, dv.$$

We first take $x \leqslant a$. The integrand is an odd function with respect to $u - x$, and thus there is no contribution to the integral from the region $(u-x)^2 + v^2 \leqslant (b-x)^2$, $v \geqslant 0$. After change of variables, we get

$$|U'(x)| \leqslant C\lambda \int_{\lambda(b-x)}^{\lambda(b+x)} |Q'(\varrho)|\, \varrho^{k-1}\, d\varrho, \quad x \leqslant a.$$

In a similar way, we can estimate $U'(x)$ for $x \geqslant a + 2\varepsilon$, and integration gives

$$\left(\int_0^a + \int_{a+2\varepsilon}^{\infty}\right) |U'(x)|\, dx \leqslant Ca^{k+2}\, e^{-\delta a^2}.$$

It remains to estimate I. Now $q(|t|/\lambda) = 0$ when $|t| > \lambda$, and consequently

$$I = \int\limits_{|t| \leqslant K\sqrt{n}} + \int\limits_{K\sqrt{n} \leqslant |t| < \lambda} = I_1 + I_2.$$

We use Lemma 1 with $r=4$ for estimating I_1. Our choice of δ implies that ε is finite and therefore $b \leqslant C(\log n)^{\frac{1}{4}}$. We obtain

$$|I_1| \leqslant Cn^{-1}(\log n)^{(k-1/4}.$$

We divide I_2 into two parts according to (6): $I_2 = I_{21} - I_{22}$, where

$$I_{21} = \int\limits_{K\sqrt{n} \leqslant |t| < \lambda} (b/2\pi|t|)^{k/2} J_{k/2}(b|t|)\, q(|t|/\lambda)\, f^n(t/\sqrt{n})\, dt. \tag{13}$$

After change of variables we get

$$|I_{21}| < C(b\sqrt{n})^{(k-1)/2} \int\limits_{K < |t| \leqslant \lambda/\sqrt{n}} |f^n(t)|\,|t|^{-(k+1)/2}\, dt.$$

In order to estimate this integral, we need a more detailed knowledge of the value distribution of ch.f.'s.

Following Esseen ([3], pp. 94–98 and 107–108), we obtain

$$|I_{21}| \leqslant C(b\sqrt{n})^{(k-1)/2} n^{-k/2} (\lambda/\sqrt{n})^{(k-1)/2} \leqslant Ca^{(k-1)/2} e^{-\delta a^2} n^{-k/(k+1)}$$

I_{22} is $o(n^{-1})$, and thus the theorem is proved in the case $\eta(a) \geqslant 0$. If $\eta(a) < 0$, we put $b = a - \varepsilon$, and obtain instead of (12)

$$\eta(a) = (1 - U(a-2\varepsilon))\,\eta(a-2\varepsilon) + U(a)\,\eta(a) + \int_{a-2\varepsilon}^{\infty} (1 - U(x))\, d\eta(x)$$
$$+ \left(\int_0^{a-2\varepsilon} + \int_a^{\infty}\right) U'(x)\,\eta(x)\, dx + I,$$

and proceed in the same way as when $\eta(a) \geqslant 0$.

When $M = E$, it is thus possible to express the probability $P(|Y| \leqslant a)$ in terms of the normal d.f. and associated functions, except for quantities of the magnitude $o(n^{-(r-2)/2})$ for large values of a and $O(n^{-k/(k+1)})$ for small values of a. It is not possible in the general case to improve the latter result much. In fact, Esseen [3] has shown that, if $F(x)$ is a lattice distribution and if $k > 4$, then $\mu(a)$ may have discontinuities of the magnitude $O(n^{-1})$. However, if the ch.f. of $X^{(1)}$ satisfies Cramér's condition

$$\overline{\lim_{|t| \to \infty}}\, |f(t)| < 1, \tag{C}$$

the following theorem holds.

Theorem 3. *If $\beta_r < \infty$, r integer $\geqslant 3$, and if $f(t)$ satisfies the condition (C), then uniformly in a*

$$P(|Y| < a) = \int_{|x| \leqslant a} d\Phi(x) + \sum_{1 \leqslant \mu \leqslant (r-3)/2} n^{-\mu} \int_{|x| \leqslant a} dP_{2\mu}(-D)\,\Phi(x) + o\left(\frac{(\log n)^{(k-1)/4}}{n^{(r-2)/2}}\right).$$

Proof. Owing to Theorem 1, we can restrict ourselves to $a \leqslant \sqrt{\frac{5}{4} m(r-2)\log n}$. We define $G_n(x)$, $Q(x)$ and $q(t)$ in the same way as in the proof of that theorem, and thus

$$\eta(a) = P(|Y| \leqslant a) - \int_{|x|\leqslant a} d\Phi(x) - \sum_{1\leqslant \mu \leqslant (r-3)/2} n^{-\mu} \int_{|x|\leqslant a} dP_{2\mu}(-D)\,\Phi(x) + O(n^{-(r-2)/2}),$$

since $dP_\nu(-D)\,\Phi(x)$ is odd when ν is odd.

It thus suffices to estimate $\eta(a)$, and we get in the case $\eta(a) \geqslant 0$

$$\eta(a) \leqslant |I| + \int_a^{a+2/\lambda} |d\psi(x)|.$$

By putting $\lambda = n^{(r-2)/2}$, we can easily estimate the second term. We divide I into two parts

$$I = \int_{|t|<K\sqrt{n}} + \int_{|t|>K\sqrt{n}} = I_1 + I_2,$$

where, because of Lemma 1 and since $b \leqslant C(\log n)^{\frac{1}{2}}$,

$$I_1 = o(n^{-(r-2)/2}(\log n)^{(k-1)/4}).$$

We also put $I_2 = I_{21} - I_{22}$ according to (6), where

$$I_{21} = \int_{|t|>K\sqrt{n}} (b/2\pi|t|)^{k/2} J_{k/2}(b|t|)\, q(|t|/\lambda)\, f^n(t/\sqrt{n})\, dt.$$

Now, since $f(t)$ satisfies $(\bar{C})$, there exists a constant $\gamma > 0$ such that $|f(t)| \leqslant e^{-\gamma}$ when $|t| > K$, that is $|f^n(t/\sqrt{n})| \leqslant e^{-\gamma n}$ in I_{21}, and thus after some calculation

$$|I_{21}| \leqslant C\, e^{-\gamma n}(\lambda b)^{(k-1)/2} \int_0^\infty q(t)\, t^{(k-3)/2}\, dt = o(n^{-(r-2)/2}).$$

Finally it is easy to show that $I_{22} = o(n^{-(r-2)/2})$ and the theorem is proved when $\eta(a) \geqslant 0$. The case $\eta(a) < 0$ is treated in a similar way.

Remark. R. R. Rao [8] has announced without proof a corresponding expansion for $P(Y \in A)$, where A is an arbitrary convex subset of R_k, but with the remainder term $O(n^{-(r-2)/2}(\log n)^{(k-1)/2})$.

5. Mean estimations

In the one-dimensional case, it is known (Agnew [1] and Esseen [4]) that $F_n(x)$ converges towards $\Phi(x)$ in L_p-mean, $p \geqslant 1$.

The two following theorems concerning the mean convergence of $P(|Y| \leqslant x)$

towards $\int_{|y|\leqslant x} d\Phi(y)$ are immediate consequences of the theorems of the previous section. We define the L_p-norm

$$\| u(x) \|_p = \left(\left(\int_0^\infty |u(x)|^p dx \right)^{1/p} \text{ for every function } u(x) \in L_p(0, \infty) \right).$$

Theorem 4. *If $\beta_5 < \infty$ and if $f(t)$ satisfies (C), then for $p \geqslant 1$*

$$\left\| P(|Y| \leqslant x) - \int_{|y|\leqslant x} d\Phi(y) \right\|_p = \frac{1}{n} \left\| \int_{|y|\leqslant x} dP_2(-D)\,\Phi(y) \right\|_p \left(1 + O\frac{(\log n)^\beta}{\sqrt{n}}\right),$$

where $\beta = (k-1)/4 + 1/(2p)$.

Proof. Putting
$$u_1(x) = P(|Y| \leqslant x) - \int_{|y|\leqslant x} d\Phi(y)$$

and
$$u_2(x) = \frac{1}{n} \int_{|y|\leqslant x} dP_2(-D)\,\Phi(x)$$

and using Minkowsky's inequality

$$\| u_2(x) \|_p - \| u_1(x) - u_2(x) \|_p \leqslant \| u_1(x) \|_p \leqslant \| u_2(x) \|_p + \| u_1(x) - u_2(x) \|_p$$

we thus have to show that

$$\| u_1(x) - u_2(x) \|_p = O(n^{-\frac{3}{2}} (\log n)^\beta)$$

but this easily follows from Theorem 1 with $r = 5$ and Theorem 3.

In the same way we obtain from Theorem 1 with $r = 4$ and Theorem 2 the following theorem.

Theorem 5. *If $\beta_4 < \infty$ and if $M = E$, then*

$$\| P(|Y| \leqslant x) - K_k(x^2) \|_p \leqslant C n^{-k/(k+1)}.$$

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