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Invariant sets under iteration of rational functions

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CONTENTS

Introduction

The theory of the iteration of a rational function $R(z)$ developed by Fatou [5–6] and Julia [9] treats the sequence of iterates $\{R_n(z)\}$ defined by

$$R_0(z)=z,\ R_1(z)=R(z),\ R_{n+1}(z)=R_1(R_n(z)),\quad n=0,1,2,\ldots.$$

A fundamental role is played here by the set F of those points of the complex plane where $\{R_n(z)\}$ is not normal. In the general theory a number of properties of F are deduced. Fatou and Julia have established the possible structures F can have. These structures depend in a very complicated way on the coefficients of $R(z)$. But very little is known about what the possible structures are even for the simplest classes, e.g. the second and the third degree polynomials with real coefficients. The aim of this thesis is to continue the general investigations concerning F, and to examine the structure of F for the polynomials mentioned above.

Since there exists no modern survey on this subject, we shall devote most of Chapter I to a treatment of the known properties of F and of the theory needed in what follows. A list of references for these known theorems is added at the end of the paper. Furthermore, in Chapter I we solve a problem, treated by Fatou under special conditions, concerning the Lebesgue measure of F on a line and in the plane under assumptions which imply that F is totally disconnected.

In Chapter II, we consider polynomials. We examine the structure of F for polynomials of the second and the third degree. Certain results concerning the second degree polynomials and the polynomial z^3+p, p real, have already been established by Myrberg [10–19].

In Chapter III, we define a mass distribution μ_n by placing the mass k^{-n} at the k^n roots of the equation $P_n(z)-z_0=0$, where $P(z)$ is a polynomial of degree k and z_0 any point in the plane with at most two exceptions. We prove that $\mu_n \to \mu^*$, under weak convergence, where μ^* is the equilibrium distribution of F with respect to the logarithmic potential. In proving this, we also establish that the logarithmic capacity of F is positive. Finally, by regarding $P(z)$ as a transformation T on F we prove that T preserves μ^* and that T is strongly mixing.

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Chapter I. Main results concerning the iteration of rational functions

1. *Definitions*

In the investigations of this paper we will use the extended complex plane with the usual topology and the following notation, for a set E.

$\complement E$ is the complement of E,
$\bar{E}$ is the closure of E,
∂E is the boundary of E,
$d(E_1, E_2)$ is the distance between the sets E_1 and E_2.

Henceforth $R(z)$ will always denote a non-linear rational function and $P(z)$ a non-linear polynomial. The sequence of iterates $\{R_n(z)\}$ to be studied is defined by

$$R_0(z)=z,\ R_1(z)=R(z),\ R_{n+1}(z)=R_1(R_n(z)),\quad n=0, 1, 2, \dots .$$

Definition 1.1. *If $w=R_n(z)$ we say that w is a successor of z and z is a predecessor of w, in both cases of order n.*

Since the fixpoints of the iterates play an important part in iteration theory, we need the following definition.

Definition 1.2. *If* $R_n(\alpha)=\alpha$ *and* $R_p(\alpha)\neq\alpha$ *when* $p<n$, *we say that* α *is a fixpoint of order* n. *The derivative* $R'_n(\alpha)$ *is called the multiplier of* α.

The successor of a fixpoint of order n is a new fixpoint of order n. Furthermore, the set $\{\alpha, R(\alpha), R_2(\alpha), \ldots, R_{n-1}(\alpha)\}$ is called a cycle of order n and all fixpoints of an n-cycle have the same multiplier $R'_n(\alpha)$ since $R'_n(\alpha)=\prod_{k=0}^{n-1} R'(R_k(\alpha))$.

Definition 1.3. *A fixpoint* α *(or a cycle) of order* n *is called attractive, indifferent, or repulsive according as* $|R'_n(\alpha)|<1, =1$, *or* >1, *respectively. If* $R'_n(\alpha)=e^{2\pi i\cdot p\ q}$, *where* p *and* q *are integers, we say that* α *(or the cycle) is rationally indifferent.*

In this paper a Möbius transformation of $w=R(z)$ is a transformation of the following form

$$(z, w)\rightarrow(Lz, Lw),$$

where L is linear. It is easy to see that the fixpoints and their multipliers are left invariant by this transformation. Finally, we need the following definition.

Definition 1.4. *A set* E *is said to be invariant under* $R(z)$ *if* $R(E)=E$, *and completely invariant if* $R_{-1}(E)=E=R(E)$, *where* $R_{-1}(z)$ *denotes the inverse function of* $R(z)$.

Remark. If nothing else is said, $R_{-1}(z)$ always means all the inverse branches and $R^{(\nu)}_{-1}(z)$ one branch.

2. *The set F*

We shall now introduce that set which is the principal object of our investigations.

Definition 2.1. *The set* F *consists of those points at which the sequence* $\{R_n(z)\}$ *is not normal, in the sense of Montel.*

This implies that $\complement F$ is an open set. Hence the

Lemma 2.1. *The set* F *is closed.*

Before characterizing F we must prove the

Theorem 2.1. $F\neq\phi$.

Proof. Suppose, on the contrary, that $F=\phi$. Then $\{R_n(z)\}$ is normal in the whole extended plane and there exists a subsequence $\{R_{n_\nu}(z)\}$ with a rational limit function $g(z)$. By considering the equations $R(z)-z=0$ and $R_2(z)-z=0$ it is easy to see that there exist at least two different cycles. Thus $g(z)$ is not constant. Suppose that $g(z)$ is of degree s. Choose $p>s$. From $\{R_{n_\nu-p}(z)\}$ we then extract a subsequence which tends to a rational function $h(z)$ of degree $t>0$. Thus $R_p(h(z))=g(z)$, where the degree of $R_p(h(z))$ is $>s$, which contradicts the hypothesis.
We shall now start our investigation of the properties of F.

Theorem 2.2. *The set consisting of the repulsive and rationally indifferent fixpoints is a subset of* F.

Proof. It is sufficient to consider fixpoints of order one.

(*a*) Let α be a repulsive fixpoint of order one, and for simplicity take $\alpha = 0$. Hence, in a neighbourhood of the origin,

$$R(z) = a_1 z + a_2 z^2 + \ldots, \quad \text{where } |a_1| > 1.$$

Since
$$R_n(z) = a_1^n z + \ldots$$

and
$$\lim_{n \to \infty} |a_1|^n = \infty,$$

it is easily seen that $\{R_n(z)\}$ cannot be normal at $\alpha = 0$.

(*b*) Suppose now that $\alpha = 0$ is a rationally indifferent fixpoint of order one. It is sufficient to consider the case where $R'(\alpha) = 1$, for when $R'(\alpha) = e^{2\pi i \cdot p/q}$ we consider $R_q(z)$ and its iterates. Hence, in a neighbourhood of the origin,

$$R_n(z) = z + n \cdot a_p z^p + \ldots$$

and evidently $\{R_n(z)\}$ cannot be normal at $\alpha = 0$.

Theorem 2.3. *The set F is completely invariant under $R(z)$.*

Proof. It is evident that if $\{R_{n_\nu}(z)\}$ converges uniformly in a neighbourhood of ζ, then $\{R_{n_\nu - 1}(z)\}$ and $\{R_{n_\nu + 1}(z)\}$ converge uniformly in neighbourhoods of $R(\zeta)$ and $R_{-1}(\zeta)$, respectively. Thus, if $\{R_n(z)\}$ is normal at a point ζ it is also normal at the points $R(\zeta)$ and $R_{-1}(\zeta)$. This implies that $\complement F$ is completely invariant under $R(z)$ and then F has the same property.

This theorem has the following corollary.

Corollary 2.1. *The set F does not change, if we replace $R(z)$ by any iterate $R_n(z)$.*

Lemma 2.2. *Let ζ be an arbitrary point in F. Then in every neighbourhood of ζ the functions $\{R_n(z)\}$ omit at most two values. Moreover, the exceptional points, if any, are independent of ζ and do not belong to F.*

Proof. If the lemma is not true, then there exist arbitrarily small neighbourhoods of ζ in which each $R_n(z)$ in the sequence $\{R_n(z)\}$ omits at least three values. Hence $\{R_n(z)\}$ is normal at ζ (for example see Hille [8] p. 248), which contradicts the assumption $\zeta \in F$.

Consider the possibility of exceptional points. Suppose that there exists one exceptional point a. Then a can have no predecessors other than itself. By a Möbius transformation we can move a to ∞, and then the transformed function must be a polynomial.

Suppose now that there exist two exceptional points a and b. Then the following two cases are possible.

1°. a has no predecessors other than itself and b has no other predecessors than itself.

2°. a and b make up a cycle of order two, and all their predecessors coincide with a or b. By a Möbius transformation we can move a and b to 0 and ∞. Clearly, in the first case the transformed function must be of the form Mz^k where M is a constant, and in the second case of the form Mz^{-k}.

From this we conclude that the exceptional points depend only on $R(z)$ for by considering the transformed functions above, we see that the exceptional points are attractive fixpoints of order one or two. Thus, the following quite trivial lemma completes the proof.

Lemma 2.3. *The sequence* $\{R_n(z)\}$ *is normal at an attractive fixpoint* α *of any order, i.e.* $\alpha \in \complement F$.

As we know, the set F is closed. It is, however, now possible to prove a much stronger result, namely

Theorem 2.4. *The set* F *is perfect.*

Proof. Since F is closed it is sufficient to prove that F is dense in itself.

We first observe that every $\zeta \in F$ has at least one predecessor ζ^* such that $\zeta^* \notin \bigcup_0^\infty R_n(\zeta)$. This is evidently true when ζ is not a fixpoint. If ζ is a fixpoint of order n, then ζ is in the n-cycle $\{\zeta, \zeta_1, \zeta_2, \ldots, \zeta_{n-1}\}$ and ζ has at least one predecessor, ζ^{ν}_{-n}, such that $\zeta \neq \zeta^{\nu}_{-n}$. Otherwise the equation $R_n(z) - \zeta = 0$ has a multiple root $z = \zeta$, i.e. $R_n'(\zeta) = 0$ and ζ is an attractive fixpoint. Furthermore, $\zeta^{\nu}_{-n} \neq \zeta_\nu$, $\nu = 1, 2, \ldots, n-1$, for otherwise $\zeta = \zeta_\nu$, some ν. Thus we can take ζ^{ν}_{-n} to be ζ^*.

Let $\zeta \in F$ and choose ζ^* as above. Since F is completely invariant, $\zeta^* \in F$. By Lemma 2.2 we conclude that ζ is an accumulation point of the predecessor of ζ^*, i.e. an accumulation point of the set F. Thus, F is dense in itself and theorem is proved.

Having established Theorem 2.4 we use it to prove the converse of Lemma 2.2.

Theorem 2.5. *Let* z *be any point in the plane with at most two exceptions. Then a point* ζ *belongs to* F *if and only if* ζ *is an accumulation point of the predecessors of* z.

Proof. The necessity follows from Lemma 2.2. Suppose that ζ satisfies the condition supposed to be sufficient. Choose a point $\eta \in F$. Then ζ is an accumulation point of the predecessors of η, i.e. of points in F. Since F is perfect it follows that $\zeta \in F$, which was to be proved.

We shall make use of the following corollary in later sections.

Corollary 2.2. *If* $q \in F$ *and* P_q *is the set of predecessors of* q, *then* $F = \overline{P}_q$.

Proof. This follows immediately from Theorem 2.4 and 2.5.

We end this first characterization of F with the

Theorem 2.6. *If the set* F *contains interior points, then* F *is identical with the extended plane.*

Proof. Suppose that $\zeta \in F$ is an interior point. Then there exists a neighbourhood O of ζ such that $O \subset F$. Then any point z different from the exceptional points of Theorem 2.5 has predecessors in O, which implies that $z \in F$. Furthermore, it is easy to see that exceptional points cannot exist in this case.

Remark. In 1918 Latté constructed a rational function for which the corresponding set F consists of the whole plane. Thus, this case can really occur. (For example see Cremer [3], p. 199.)

3. *On critical points*

As we have already mentioned, in this paper we shall be concerned chiefly with finding the properties of the set F. In these investigations, however, the critical points of the inverse functions $\{R_{-n}(z)\}$ will be of great importance. Therefore we have to discuss their relationship to iteration theory.

Definition 3.1. *If the equation $R_n(z) - c = 0$ has a multiple root, then c is called a critical point of the inverse function $R_{-n}(z)$. Henceforth C will denote the set of critical points of all functions $\{R_{-n}(z)\}$.*

We begin by establishing two simple but important results.

Lemma 3.1. *The critical points of $R_{-n}(z)$ consist of the critical points of $R_{-1}(z)$ and their successors of order* $1, 2, 3, \ldots, n-1$.

Proof. Divide the equation $R_n(z) = c$ as follows:

$$R_{n-1}(x) = c \tag{3.1}$$

$$R(z) = x. \tag{3.2}$$

Equation (3.2) has a multiple root if and only if x is one of the critical points of $R_{-1}(z)$. By (3.1) the successors of order $n-1$ of these points are critical points of $R_{-n}(z)$. Now treat the equation $R_{n-1}(x) = c$ in the same way as $R_n(z) = c$. Repeating this procedure $n-1$ times will complete the proof.

It is quite trivial that the critical points of $R_{-1}(z)$ are the first order successors of the zeros of $R'(z)$. This fact yields the following lemma.

Lemma 3.2. *If $R(z)$ is of degree d, then N, the number of critical points of $R_{-1}(z)$, satisfies the inequality $N \leqslant 2(d-1)$.*

In this paper a domain always means an open connected set. We need the following definition.

Definition 3.2. 1. *The immediate attractive set $A^*(\alpha)$ of a first order attractive fixpoint α is the maximal domain of normality of $\{R_n(z)\}$ which contains α. The attractive set $A(\alpha)$ of α is defined by*

$$A(\alpha) = \{z \mid \lim_{n \to \infty} R_n(z) = \alpha\}.$$

2. *Let $\{\alpha_k\}$ be an attractive cycle of order n. Then the immediate attractive set $A^*(\{\alpha_k\})$ of the cycle is defined by*

$$A^*(\{\alpha_k\}) = \bigcup_k A_n^*(\alpha_k),$$

where $A_n^*(\alpha_k)$ *is the maximal domain of normality containing* α_k, *and the attractive set* $A(\{\alpha_k\})$ *of the cycle is defined by*

$$A(\{\alpha_k\})=\{z\,|\,\{\alpha_k\}\ \text{ is the cluster set of }\ \{R_n(z)\}\}.$$

Remark. From the definition of $A^*(\alpha)$ it is obvious that if $z\in A^*(\alpha)$, then $\lim_{n\to\infty}R_n(z)=\alpha$. Furthermore, by Corollary 2.2 it is easy to see that $\partial A(\alpha)=F$.

The following theorem establishes the influence of the critical points on the number of attractive fixpoints.

Theorem 3.1. *If* $\{\alpha_k\}$ *is an attractive cycle, then there exists at least one critical point* c *of* $R_{-1}(z)$, *such that* $c\in A^*(\{\alpha_k\})$.

Proof. Suppose first that α is an attractive fixpoint of order one. Then choose a neighbourhood U of α such that $U\subset A^*(\alpha)$ and an inverse branch $R^*_{-1}(z)$ which satisfies $R^*_{-1}(\alpha)=\alpha$. Further, introduce the functions $\{R^*_{-n}(z)\}$ defined in U by $R^*_{-n}(z)=R^*_{-1}(R^*_{-(n-1)}(z))$. Thus, if no critical point of $R_{-1}(z)$ belongs to $A^*(\alpha)$, then the functions $\{R^*_{-n}(z)\}$ are meromorphic in U. Since each $R^*_{-n}(z)$ omits at least three values in U, for example the set F, $\{R^*_{-n}(z)\}$ is normal in U. That, however, contradicts the fact that α is a repulsive fixpoint of the function $R^*_{-1}(z)$.

If α belongs to an attractive cycle $\{\alpha,\alpha_1,\alpha_2,\dots,\alpha_p\}$, then we define the functions $\{R^*_{-n}(z)\}$ by

$$R^*_{-1}(\alpha)=\alpha_p,\ \ R^*_{-2}(\alpha)=\alpha_{p-1},\ \dots,\ R^*_{-p}(\alpha)=\alpha.$$

Now we use the same argument as above and the theorem is proved.

It is evident that the indifferent fixpoints must be considered as exceptional points in the iteration theory. In characterizing the set F, however, we cannot omit the rationally indifferent fixpoints, which have the same influence on the structure of F as do the attractive fixpoints. But since a complete treatment of these exceptional points takes more space than the general case and does not involve any special difficulties, we will without proof summarize some of their properties, namely those of importance for the following.

Theorem 3.2. 1°. *If* α *is a rationally indifferent fixpoint, then there exists an immediate attractive set* $A^*(\alpha)$ *which is a union of maximal domains where* $R_n(z)$ *is normal, each of which has* α *as a boundary point.*

2°. $A^*(\alpha)$ *contains at least one critical point of* $R_{-1}(z)$.

3°. *The number of indifferent fixpoints is finite.*

4. *The set F is homogenous*

We shall now prove an equivalent definition of F, which was the start point of Julia's investigations.

Theorem 4.1. *The set* F *is identical with the closure of the set of repulsive fixpoints.*

We shall need the following lemma in the proof.

Lemma 4.1. *Every $\zeta\in F$ is an accumulation point of fixpoints.*

Proof of Lemma 4.1. Let $\zeta\in F$, $\neq\infty$, be different from the poles and the critical points of $R_{-2}(z)$. Then there exists a neighbourhood U of ζ where the d^2 inverse branches of $R_{-2}(z)$ are bounded, holomorphic and have different ranges. Choose three of these branches $R^{(1)}_{-2}(z)$, $R^{(2)}_{-2}(z)$ and $R^{(3)}_{-2}(z)$.

Now suppose, on the contrary, that $R_n(z)-z\neq 0$, every $z\in U$, and every n. This implies that

$$R_n(z)\neq R^{(1)}_{-2}(z),\quad R_n(z)\neq R^{(2)}_{-2}(z),\quad R_n(z)\neq R^{(3)}_{-2}(z),$$

if $z\in U$, every n. Otherwise $R_{n+2}(z)=z$ for some $z\in U$. We introduce the functions

$$\varphi_n(z)=\frac{R_n(z)-R^{(1)}_{-2}(z)}{R_n(z)-R^{(2)}_{-2}(z)}\cdot\frac{R^{(3)}_{-2}(z)-R^{(2)}_{-2}(z)}{R^{(3)}_{-2}(z)-R^{(1)}_{-2}(z)}=\frac{R_n(z)-R^{(1)}_{-2}(z)}{R_n(z)-R^{(2)}_{-2}(z)}\cdot Q(z).$$

Each function $\varphi_n(z)$ omits in U the values 0, 1, ∞. Thus the sequence $\{\varphi_n(z)\}$ is normal in U. But

$$R_n(z)=R^{(2)}_{-2}(z)+Q(z)\cdot\frac{R^{(2)}_{-2}(z)-R^{(1)}_{-2}(z)}{\varphi_n(z)-Q(z)}$$

and we conclude that $\{R_n(z)\}$ is normal in U, contradicting $\zeta\in F$. Thus, the lemma is proved.

Proof of Theorem 4.1. By Theorem 3.1 and 3.2 the number of attractive and indifferent fixpoints is finite. Since the repulsive fixpoints belong to F an application of Lemma 4.1 proves the theorem.

By using the previous theorem, we get a simple proof of the following fundamental result concerning F.

Theorem 4.2. *Let E be a closed set containing none of the exceptional points of Theorem 2.5. If $\zeta\in F$, then there exists for every neighbourhood U of ζ an N such that $E\subset R_N(U)$.*

Proof. According to Theorem 4.1 it is sufficient to consider the repulsive fixpoints. Let ζ be a repulsive fixpoint of order n and choose a neighbourhood U of ζ such that

$$U\subset R_n(U)\subset R_{2n}(U)\subset\ldots\subset R_{\nu n}(U)\subset.$$

We see from Theorem 2.5 that every $z\in E$ belongs to some $R_{\nu n}(U)$, and since E is closed we can extract a finite subcovering from $\{R_{\nu n}(U)\}$. If we then choose N equal to the largest index used in this covering, we get $E\subset R_N(U)$, and the theorem is proved.

Since F is invariant under $R(z)$ this theorem yields the

Theorem 4.3. *If D is any domain such that $D\cap F=F^*\neq\phi$, then there exists an integer N such that $F=R_N(F^*)$.*

Remark. Instead of the formulation above we might say that F is *"rationally homogenous"*.

5. *On limit functions of the iterates $\{R_n(z)\}$ in the complement of F*

Henceforth G and G_ν will always denote maximal domains, where $\{R_n(z)\}$ is normal.

Lemma 5.1. *If the number of limit functions of $\{R_n(z)\}$ is finite, then every limit function is a constant.*

Proof. If $\lim_{n\to\infty} R_{\alpha_n}(z)=f(z)$, uniformly, in some G_ν, then $\lim_{n\to\infty} R_{\alpha_n+h}(z)=R_h(f(z))$. According to the assumption there exist integers h and N such that $R_h(f(z))=R_{h+N}(f(z))$. We conclude that $f(z)$ is a constant.

Theorem 5.1. *If $\lim_{n\to\infty} R_{\alpha_n}(z)=a$, uniformly, in a domain D and if $a\notin F$, then a is an attractive fixpoint.*

Proof. If $D_{\alpha_n}=R_{\alpha_n}(D)$, then the sequence $\{D_{\alpha_n}\}$ converges uniformly to $z=a$. Thus there exist two domains $D_{\alpha_{p-q}}$ and D_{α_p} such that $D_{\alpha_{p+q}}\subset D_{\alpha_p}$. By taking

$$h=\alpha_{p+q}-\alpha_p,\quad \beta_n=\alpha_n-\alpha_{p+q}$$

we get, for $z\in D_{\alpha_{p-q}}$,

$$\lim_{n\to\infty} R_{\beta_n}(z)=\lim_{n\to\infty} R_{h+\beta_n}(z)=a.$$

Observing that

$$R_{h+\beta_n}(z)=R_h(R_{\beta_n}(z)) \tag{5.1}$$

we obtain, by taking limits in (5.1),

$$a=R_h(a).$$

Thus, a is a fixpoint of order h. Since $a\notin F$ it cannot, however, be a repulsive or rationally indifferent fixpoint. Moreover, if a is an indifferent fixpoint, not rational, then

$$R_{nh}(a)=a,\quad |R'_{nh}(a)|=1$$

and no constant limit function can exist in a neighbourhood of $z=a$. We conclude that a must be an attractive fixpoint and the theorem is proved.

A more difficult problem has been to decide whether non-constant limit functions can exist. It was finally proved in 1942 by Carl Ludwig Siegel [20] that this case actually can occur. In the next section we shall establish a condition due to Fatou, which excludes the possibility of non-constant limit functions. Other results needed are stated in the following theorem.

Theorem 5.2. *If $\lim_{\nu\to\infty} R_{n_\nu}(z)=f(z)$ in a domain G and if $f(z)$ is not constant and $f(G)=G^*$, then there exists a subsequence which converges to the limit function $F(z)=z$ in G^*. Furthermore, there exists an iterate $R_k(z)$, which maps G^* one to one onto itself.*

Proof. Since $\{R_n(z)\}$ is normal in G^*, we can extract from the subsequence $\{R_{n_{\nu+1}-n_\nu}(z)\}$ another subsequence $\{R_{n'_{\nu+1}-n'_\nu}(z)\}=\{R_{m_\nu}(z)\}$ which tends uniformly to a function $F(z)$ on every compact subset of G^*. Observing that

$$R_{n'_{\nu+1}-n'_\nu}(R_{n'_\nu}(z))=R_{n'_{\nu+1}}(z) \tag{5.2}$$

we obtain, by taking the limits in (5.2),

$$F(f(z)) = f(z)$$

and we conclude that $F(z) = z$.

Now if $z_0 \in G^*$, there exists an iterate $R_k(z)$ such that $R_k(z_0) \in G^*$. If $R_k(G^*) = G_k$, then $G_k \cap G^* \neq \phi$. Since G_k and G^* are maximal domains of normality of $\{R_n(z)\}$, we have $G_k \equiv G^*$.

It remains to prove that the mapping is one to one. Since $\lim_{\nu\to\infty} R_{m_\nu}(z) = z$, the assumption $R_k(z_1) = R_k(z_2)$ implies

$$z_1 = \lim_{\nu\to\infty} R_{m_\nu}(z_1) = \lim_{\nu\to\infty} R_{m_\nu - k}(R_k(z_1)) = \lim_{\nu\to\infty} R_{m_\nu}(z_2) = z_2$$

and the theorem is proved.

Remark. A domain such as G^* is called a *singular domain.*

6. *On the inverse functions $\{R_{-n}(z)\}$ of the iterates $\{R_n(z)\}$*

Clearly, a more detailed investigation of F has to make use of Theorem 2.5. Then a good knowledge of the behavior of the inverse functions $\{R_{-n}(z)\}$ is needed. Therefore this section is devoted to these functions.

We begin, however, by treating the following closely related question.

Let a be any point in the plane other than the exceptional points of Theorem 2.5, and form the set P_a of predecessors of a. Let P'_a be the derived set of P_a and include a in P'_a when a has an infinite number of predecessors which coincide with a. If $a \in F$ then by Theorem 2.4 and 2.5 $F = P'_a$ and if $a \notin F$ then at least $F \subset P'_a$. The question now is whether or not $F = P'_a$ can occur when $a \notin F$. The following result holds.

Theorem 6.1. *$F = P'_a$ if and only if a does not belong to the set of attractive fixpoints or to a singular domain.*

Proof. Consider two points a and b such that $b \in P'_a$ and $b \notin F$. Let $\{a_{-n_\nu}\}$ be a sequence of predecessors of a such that $\lim_{\nu\to\infty} a_{-n_\nu} = b$. Since $b \notin F$, the sequence $\{R_{n_\nu}(z) - a\}$ is normal in a neighbourhood U of b.

We can extract a subsequence $\{R_{n'_\nu}(z) - a\}$ which converges uniformly in U. Since $R_{n'_\nu}(a_{-n'_\nu}) - a = 0$ we conclude that

$$\lim_{\nu\to\infty} R_{n'_\nu}(b) - a = 0.$$

Thus a is an accumulation point of the successors of b. Moreover, since $b \notin F$, it follows that $a \notin F$.

Hence, if b belongs to a domain where the iterates $\{R_n(z)\}$ have only constant limit functions, then by Theorem 5.1 a is an attractive fixpoint of some order. If, however, b belongs to a domain where there exist non-constant limit functions, then by Theorem 5.2 a belongs to a singular domain. The necessity is obvious from Theorem 5.2 and the properties of attractive fixpoints.

We shall now consider the inverse functions $\{R_{-n}(z)\}$ which are algebraic functions. Before proving some fundamental lemmas we recall that C denotes the set of critical points of the functions $\{R_{-n}(z)\}$.

Lemma 6.1. *Any infinite set of branches $\{R^{(\nu)}_{-\lambda_p}(z)\}$, meromorphic in a domain D, is normal in D.*

Proof. By considering the equation $R(z)-z=0$ it is easy to see that there exists at least one fixpoint α different from the exceptional points of Theorem 2.5. This point α has two predecessors α_{-1} and α_{-2} of order one and two such that $\alpha\neq\alpha_{-1}\neq\alpha_{-2}\neq\alpha$. If $\alpha\notin D$, then each function $R^{(\nu)}_{-\lambda_p}(z)$, $\lambda_p>2$, evidently omits the values α, α_{-1} and α_{-2} in D, and thus $\{R^{(\nu)}_{-\lambda_p}(z)\}$ is normal in D. If $\alpha\in D$, then by considering the equations $R(z)-z=0$ and $R_2(z)-z=0$, it is easy to see that there exists at least one more fixpoint β of order one or a cycle (γ_1, γ_2) of order two, in both cases different from the exceptional points (cf. Baker [8]). We can repeat the discussion concerning α, and consequently there remains only the case where all the fixpoints mentioned above belong to D. But then we can divide D into a finite number of overlapping subsets, such that $\{R^{(\nu)}_{-\lambda_p}(z)\}$ is normal in each of these sets. Since this implies the normality of $\{R^{(\nu)}_{-\lambda_p}(z)\}$ in all D, the lemma is proved.

Lemma 6.2. *If the domain D is simply connected and if $D\cap\bar{C}=\phi$, then the set of functions $\{R_{-n}(z)\}$ is a normal family in D.*

Proof. This follows immediately from Lemma 6.1.

Lemma 6.3. *Let E be a closed set which contains no accumulation point of the successors of a point outside F. If $E_n=R_{-n}(E)$, then the sequence $\{E_n\}$ converges uniformly to F.*

Proof. Suppose that the lemma is false. Then there exists a sequence of increasing integers $\{\lambda_n\}$ and a sequence of points $\{z^{(n)}\}$ outside an ε-neighbourhood U of F such that $R_{\lambda_n}(z^{(n)})=\xi^{(n)}$, where $\xi^{(n)}\in E$. Evidently $\{z^{(n)}\}$ has an accumulation point $z^{(0)}$, also outside U. Thus $\{R_{\lambda_n}(z)\}$ is normal in a neighbourhood of $z^{(0)}$. It is then easy to see that there exists a subsequence of $\{R_{\lambda_n}(z)\}$ which, according to uniform convergence and the fact that E is closed, in $z^{(0)}$ tends to a point $\xi^{(0)}\in E$. This contradicts our assumption and so the lemma is proved.

We now state the main result of this section.

Theorem 6.2. *Let $\{R^{(\nu)}_{-\lambda_n}(z)\}$ be any infinite set of inverse branches, which are meromorphic in a domain D. We suppose that D is not a subset of a singular domain and that F is not identically equal to the whole plane. Then $\{R^{(\nu)}_{-\lambda_n}(z)\}$ is normal in D and every convergent subsequence of $\{R^{(\nu)}_{-\lambda_n}(z)\}$ tends to a constant.*

Proof. By Lemma 6.1 $\{R^{(\nu)}_{-\lambda_n}(z)\}$ is normal in D. Furthermore, it is evident that there exists a domain $D_1\subset D$ such that D_1 satisfies the conditions of Lemma 6.3. Thus, in D_1 the values of the functions $\{R^{(\nu)}_{-\lambda_n}(z)\}$ converge uniformly to F, i.e. to a set containing no interior points. Since the convergent subsequences tend to meromorphic functions, they must be constants and the theorem is proved.

It is now possible to prove the theorem mentioned earlier concerning the non-existence of singular domains.

Theorem 6.3. *If the set $\bar{C}$ does not divide the plane, then there exist no singular domains.*

Proof. Suppose, on the contrary, that there exists a singular domain G^*, although $\bar{C}$ does not divide the plane. Furthermore, let G^*_{-1} be any of the non-singular domains $R_{-1}(G^*)$. If we choose $z' \in G^*$ and $z'' \in G^*_{-1}$ we can, according to the assumption, find a simply connected domain D containing z' and z'' and such that $D \cap \bar{C} = \phi$.

In D the functions $\{R_{-n}(z)\}$ make a normal family. Since G^* is a singular domain there exists a subsequence $\{R^{(\nu)}_{-n_\nu}(z)\}$, which in a neighbourhood of z' tends to a non-constant limit function. In a neighbourhood of z'', however, $\{R^{(\nu)}_{-n_\nu}(z)\}$ tends to a constant. This is impossible and the theorem is proved.

However, we can state a more useful condition, which also can be used to prove a theorem concerning the values of $R'_n(z)$ on F.

Theorem 6.4. *If $F \cap \bar{C} = \phi$, then there exist no singular domains.*

Proof. Suppose there exists a singular domain G^* and that $\bar{C}$ divides the plane. Since there always exist non-singular domains, by Theorem 4.2 any neighbourhoods of a point $\zeta \in \partial G^*$ contain non-singular components. Thus we can choose z', z'' and D as in the proof of Theorem 6.3 and then use the same argument.

Theorem 6.5. *If $F \cap \bar{C} = \phi$, then for each $k > 1$ there exists an integer h such that $|R'_n(z)| > k > 1$ if $z \in F$ and $n \geq h$.*

Proof. If $d(F, \bar{C}) = \delta > 0$, we cover F by a finite number of circles D_ν with radii of length $r < \delta$. Set $D = \bigcup D_\nu$ and suppose that F is bounded. The functions $\{R_{-n}(z)\}$ are meromorphic in D_ν and thus constitute a normal family. According to Theorem 6.4 no singular domains can exist, so then all limit functions are constants. Consequently we conclude that the functions $\{R'_{-n}(z)\}$ converge uniformly to zero in D_ν. This implies that for each $k > 1$, there exists an integer h such that

$$|R'_{-n}(z)| < k^{-1}, \quad \text{if} \quad z \in D, \ n \geq h, \quad \text{i.e.} \quad |R'_n(z)| > k, \quad \text{if} \quad z \in F, \ n \geq h,$$

which was to be proved.

Remark. If $R_{n_\nu}(z) \to a$ in a domain then it follows that $a \in \bar{C}$, ([6], pp. 60–61.) Thus if $F \cap \bar{C} = \phi$, then by Theorem 5.1 and 6.4 the limit functions of $\{R_n(z)\}$ are attractive fixpoints.

7. *On the structure of the complement of F*

In this section we shall discuss how the set F divides the plane. We need the following theorem.

Theorem 7.1. *The number of simply connected domains, which are completely invariant under $R(z)$ is at most 2.*

For the proof we need the following lemma.

Lemma 7.1. *If the domain D is simply connected and completely invariant under a rational function $R(z)$ of degree d, then D contains at least $d-1$ critical points of $R_{-1}(z)$.*

Remark. It is always understood that the critical points have to be repeated as many times as their order indicates.

Proof of Lemma 7.1. We can omit the two quite trivial cases where D is identical with the whole plane and where D has only one boundary point. Suppose further that $z=\infty \notin D$.

If $a \in D$, then D contains the d roots of the equation $R(z)-a=0$. Evidently there exists a Jordan curve γ sufficiently close to ∂D for $\gamma_{-1}=\bigcup R_{-1}^{(\nu)}(\gamma)$ to enclose all these roots. From the argument principle it follows that the curve γ_{-1} is generated by d inverse branches, which are permuted cyclically when z runs through γ d times. Thus γ must enclose at least $d-1$ critical points of $R_{-1}(z)$ and the lemma is proved.

Proof of Theorem 7.1. We know that if $R(z)$ is a rational function of degree d, then $R_{-1}(z)$ has at most $2(d-1)$ critical points. The conclusion of the theorem now follows from Lemma 7.1.

We now consider the possible number of components G_ν of $\complement F$, i.e. the number of maximal domains of normality of $\{R_n(z)\}$.

Theorem 7.2. *If the number of disjoint components of $\complement F$ is finite, then it is either 1 or 2.*

Proof. If the number N of disjoint components of $\complement F$ is finite, it follows that every component G_ν must be completely invariant under some iterate $R_n(z)$. Furthermore, if $N \geqslant 2$, then every component is simply connected, or else there exists at least one multiply connected component G_ν which contains a closed curve separating boundary points of another connected domain $G_\mu \neq G_\nu$. The theorem then follows from Theorem 7.1.

To get further information about the components of $\complement F$ we now return to the immediate attractive set $A^*(\alpha)$ and the attractive set $A(\alpha)$, and consider their connectivity. We recall that $A^*(\alpha)$ denotes the largest connected set containing the first order attractive fixpoint α where $\lim_{n\to\infty} R_n(z)=\alpha$ and that $A(\alpha)=\{z \mid \lim_{n\to\infty} R_n(z)=\alpha\}$.

Theorem 7.3. *The immediate attractive set $A^*(\alpha)$ is either simply connected or of infinite connectivity.*

Proof. Since α is a first order attractive fixpoint there exists a circular disk ω with α as centre and such that for $z\in\omega$

$$|R(z)-\alpha| < k|z-\alpha|, \quad 0<k<1,$$

If $\omega_{-n}=R_{-n}(\omega)$, then

$$\omega \subset \omega_{-1} \subset \omega_{-2} \subset \ldots \subset \omega_{-n} \subset .$$

Let E_n be the largest connected subset of ω_{-n} which contains α. Hence

$$E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots$$

and it follows that

$$A^*(\alpha) = \lim_{n\to\infty} E_n.$$

Suppose now that $A^*(\alpha)$ is multiply connected, i.e. that some E_h is multiply connected. Then the boundary of E_h consists of q disjoint closed curves. For simplicity we assume that $z=\infty \in E_h$. We may also assume that $\partial\omega \cap C=\phi$. If the boundary curves are denoted by $\gamma^{(1)}_{-h}, \gamma^{(2)}_{-h}, \ldots, \gamma^{(q)}_{-h}$, then $R_h(\gamma^{(\nu)}_{-h})=\partial\omega$, $\nu=1, 2, \ldots, q$. Thus, if $\zeta \in \gamma^{(\nu)}_{-h}$, we have $\zeta_1=R_h(\zeta)\in\partial\omega$. Let ζ_1 describe a curve arc outside ω which terminates at a point $\eta_1 \in F$. Then it follows that ζ describes a curve arc inside $\gamma^{(\nu)}_{-h}$ terminating at a point $\eta \in F$. Thus we conclude that each $\gamma^{(\nu)}_{-h}$ encloses points belonging to F.

In an analogous way it is then possible to prove that if ∂E_{h+n-1} consists of q^n closed curves, each enclosing points belonging to F, then the same holds for ∂E_{h+n} with q^n replaced by q^{n+1}. By induction the theorem then follows.

Since $F=\partial A(\alpha)=\partial A(\beta)=\partial A(\gamma)=\ldots$, where $\alpha, \beta, \gamma \ldots$ are attractive fixpoints, this theorem has the following corollary.

Corollary 7.1. *Suppose that* α *is an attractive fixpoint of order one and that* $A^*(\alpha)=A(\alpha)$ *is not simply connected. Then if there exist first order attractive fixpoints* $\beta, \gamma, \ldots$ *other than* α, $A^*(\beta)\neq A(\beta)$, $A^*(\gamma)\neq A(\gamma)$, ... *and each* $A(\beta)$, $A(\gamma)$, ... *consists of simply connected components.*

8. *The structure of F, when the number of attractive fixpoints is $\geqslant 2$*

The last sections of the first chapter will be devoted to a more detailed investigation of the structure of F. We begin here by proving that under quite general conditions F consists of Jordan curves. The number of these curves can be either one or infinity.

Theorem 8.1. *If* $R(z)$ *has two first order attractive fixpoints* α *and* β *and if* $A^*(\alpha)=A(\alpha)$ *and* $A^*(\beta)=A(\beta)$, *then* F *is a Jordan curve.*

Proof. From Theorem 7.3 and Corollary 7.1 it follows that both $A(\alpha)$ and $A(\beta)$ are simply connected. If the degree of $R(z)$ is d then, by Lemma 7.1, both $A(\alpha)$ and $A(\beta)$ contain $d-1$ critical points and consequently there exist no attractive or rationally indifferent fixpoint other than α and β. Since $F\cap\bar{C}=\phi$, then, according to Theorem 6.5, given $k>1$, we can find an integer h such that

$$|R_n'(z)|>k>1 \quad \text{for} \quad z\in F, \quad n\geqslant h.$$

It is no restriction to suppose that $h=1$, i.e.

$$|R'(z)|>k>1 \quad \text{for} \quad z\in F \tag{8.1}$$

and to take $\alpha=0$ and $\beta=\infty$. We can choose two Jordan curves γ and ω in the following way (for example see the proof of Theorem 7.3)

(i) $\gamma\subset A(0)$ and $\omega\subset A(\infty)$.
(ii) The critical points belonging to $A(0)$ are inside γ and those belonging to $A(\infty)$ are outside ω.
(iii) $\gamma_{-1}=R_{-1}(\gamma)$ encloses γ and ω encloses $\omega_{-1}=R_{-1}(\omega)$.

By Lemma 6.3 the sequences $\{\gamma_{-n}\}$ and $\{\omega_{-n}\}$ of the predecessors of γ and ω both tend uniformly to F.

Now consider the Jordan curves $\{\gamma_{-n}\}$. To get a parametric representation of them we proceed as follows. Map the doubly connected domain D bounded by γ_{-1} and γ_{-2} conformally and one to one onto a circular ring bounded by C_1 and C_2. Let a radius r of C_2 cut C_1 at a and C_2 at b. The inverse mapping function maps the subarc r_{ab} of r onto an arc λ_{AB} which consequently cuts γ_{-1} and γ_{-2} at right angles. By letting r run through the circular ring we get a corresponding covering of D by orthogonal trajectories λ_{AB}. By successively mapping these trajectories by $R_{-1}(z)$ we get a covering of the domains between the curves $(\gamma_{-2}, \gamma_{-3})$, $(\gamma_{-3}, \gamma_{-4})$, ..., so that it corresponds to every point on every γ_{-n} one and only one orthogonal trajectory, i.e. an arc that cuts γ_{-n} at right angles.

Let γ_{-1} have the parametric form $z=z_1(t)$, $0\leqslant t\leqslant 1$ and $z(0)=z(1)$. Then give every γ_{-n} a parametric form such that points on the same trajectory have the same t-value. If the maximal length of the trajectories between γ_{-n} and $\gamma_{-(n+1)}$ is l_{n+1}, then by (8.1) $l_{n+1}<k^{-1}\cdot l_n$, and we obtain

$$\lim_{n\to\infty} z_n(t)=z(t), \quad \text{uniformly.}$$

Since $\{z_n(t)\}$ are continuous functions it follows that $z(t)$ is continuous and thus $F=\{z(t)\,|\,0\leqslant t\leqslant 1\}$ is a continuous curve.

It remains to prove that $z=z(t)$ is simple. Suppose that $\zeta=z(t_1)=z(t_2)$ and that $t_1\neq t_2$. Then there exist two different trajectories λ_{t_1} and λ_{t_2} which terminate at ζ. Hence λ_{t_1} and λ_{t_2} together with γ_{-1} bound a simply connected domain Ω and $\lambda_t\subset\Omega$, $t_1<t<t_2$, whence $\{z(t)\,|\,t_1<t<t_2\}\subset\Omega\cup\{\zeta\}$. We now observe that we can treat the curves $\{\omega_{-n}\}$ in the same way as $\{\gamma_{-n}\}$, i.e. ω_{-n} has a parametric form $y=y_n(t)$ and $\lim_{n\to\infty}y_n(t)=y(t)$, uniformly, where $F=\{y(t)\,|\,0\leqslant t\leqslant 1\}$. Since $\{y(t)\,|\,0\leqslant t\leqslant 1\}\cap\Omega=\phi$ we obtain that $\{z(t)\,|\,t_1<t<t_2\}=\{\zeta\}$. Thus ζ is not a double-point. We conclude that F is a Jordan curve, which is what we wished to show.

Theorem 8.2. *Suppose that*

(i) *the number of attractive fixpoints is* $\geqslant 2$,
(ii) *one and only one of them,* β, *has* $A^*(\beta)=A(\beta)$,
(iii) $F\cap\bar{C}=\phi$.

Then F contains an infinite number of Jordan curves.

Proof. From Theorem 7.3 and Corollary 7.1 it follows that the assumptions imply the existence of at least one attractive fixpoint α such that $A^*(\alpha)\neq A(\alpha)$ and such that $A^*(\alpha)$ is simply connected. Let $\partial A^*(\alpha)=F_\alpha$ and let $R^\alpha_{-1}(z)$ be the branches of $R_{-1}(z)$ for which $R_{-1}(A^*(\alpha))\subset A^*(\alpha)$. Then $R^\alpha_{-1}(F_\alpha)=F_\alpha$ and furthermore $R(F_\alpha)=F_\alpha$. Thus by using $R^\alpha_{-1}(z)$ instead of $R_{-1}(z)$ we can prove, as in the proof of Theorem 8.1, that F_α is a Jordan curve. By then taking all the predecessors of F_α we get an infinite number of Jordan curves that belong to F.

Remark 8.1. Fatou [6], pp. 300–303 proved that F is a Jordan curve, if $R(z)$ has a first order attractive fixpoint α and a first order rationally indifferent fixpoint β such that $A^*(\alpha)=A(\alpha)$ and $A^*(\beta)=A(\beta)$. In this case β must satisfy $R'(\beta)=+1$ and $R''(\beta)\neq 0$, i.e. $A^*(\beta)$ consists of only one maximal domain of normality of $\{R_n(z)\}$. See Fatou [5], pp. 191–221 and Julia [9], pp. 223–237.

It is, however, possible to get more detailed information about the Jordan curves in Theorem 8.1 and 8.2. The next section shows that these curves do not have tangents at any point.

9. *On the existence of tangents to the curves that lie in F*

Theorem 9.1. *Let α be an attractive first order fixpoint of $R(z)$. Suppose that $A^*(\alpha)$ is simply connected and that $F_\alpha \cap \bar{C} = \phi$, where $F_\alpha = \partial A^*(\alpha)$. Then, if F_α is not a circle or a straight line F_α does not have a tangent at any point.*

For the proof we need the following important lemma.

Lemma 9.1. *Let α be an attractive first order fixpoint of $R(z)$. If $A^*(\alpha)$ is simply connected and if $\partial A^*(\alpha) = F_\alpha$ is an analytic Jordan curve or arc, then F_α is either a circle or an arc of a circle (straight line or a segment).*

Proof. As usual $R(z)$ is of degree d. We begin by mapping $A^*(\alpha)$ conformally and one to one onto $|t| < 1$ and so that $z = \alpha \leftrightarrow t = 0$. Let the inverse mapping function be $z = h(t)$. Since F is an analytic Jordan curve or arc, $h(t)$ is meromorphic in $|t| < r_1$ where $r_1 > 1$. Moreover, if $\omega = \{t \mid |t| = 1\}$ then $F_\alpha = h(\omega)$. Furthermore, $R(z)$ maps $A^*(\alpha)$ onto itself q times. Thus

$$h_{-1}(R(h(t))) = \varphi(t) \tag{9.1}$$

maps the unit disk onto itself q times. Then $\varphi(t)$ must be a Blaschke product, i.e.

$$\varphi(t) = A \cdot t^p \cdot \prod_{\nu=1}^{q-p} \frac{a_\nu - t}{1 - t \cdot \bar{a}_\nu}, \quad |A| = 1, \ 1 \leqslant p \leqslant q \leqslant d. \tag{9.2}$$

We wish to prove that $h(t)$ is a rational function. From (9.1) we get

$$R(h(t) = h(\varphi(t)). \tag{9.3}$$

For $|t| > r > 1$ and $r < r_1$ we have $|\varphi(t)| > k \cdot |t|$, where $k > 1$ is a constant independent of r_1. Since $\{t \mid |t| < kr_1\} \subset \{\varphi(t) \mid |t| < r_1\}$, by (9.3) we can continue $h(t)$ analytically to $|t| < kr_1$, where any singularities of $h(t)$ are poles. For if $\varphi_{-1}(t)$ has a critical point in $1 \leqslant |t| < kr_1$, let t move along a closed path in $1 \leqslant |t| < kr_1$ such that the critical point is inside the path. Assume that the path starts and ends at $t_0 \in \omega$. Then a branch $\varphi_{-1}^{(\nu)}(t)$ moves along a path in $1 \leqslant |t| < r_1$ from $\varphi_{-1}^{(\nu)}(t_0) = t_1 \in \omega$ to $\varphi_{-1}^{(\nu)}(t_0) = t_2 \in \omega$, where $t_1 \neq t_2$. But $R(h(\varphi_{-1}(t)) = h(t)$ and thus for each $t_0 \in \omega$ we have $R(h(t_1)) = R(h(t_2)) = h(t_0)$ and we conclude that $h(t)$ has no algebraic singularities in $|t| < kr_1$. Thus, by (9.3), we can continue $h(t)$ analytically to the whole plane.

Consider the behaviour of $h(t)$ at $t = \infty$. Suppose first that $\varphi(t)$ has some finite poles, i.e. $\varphi(t) \neq At^q$. If $\varphi(t)$ has a pole at $z = b$, then in a neighbourhood O of $z = \infty$ one branch $\varphi_{-1}^{(\nu)}(t)$ takes its values in a neighbourhood U of $z = b$. Thus, $h(t) = R(h(\varphi_{-1}^{(\nu)}(t))$ has the range $R(h(U))$ in O, i.e. $h(t)$ has at most a pole of finite order at $t = \infty$.

It remains to study the behaviour of $h(t)$ at $t = \infty$, when $\varphi(t) = At^q$. We can take $A = 1$ and thus we have the functional equation

$$R(h(t)) = h(t^q). \tag{9.4}$$

Suppose that $h(t_0) = \alpha$, where $t_0 \neq 0$. Thus, by (9.4)

$$h(t_0^{q^n}) = R_n(h(t_0)) = R_n(\alpha) = \alpha.$$

If θ satisfies $\theta^{q^n} = 1$, then

$$h[(\theta t_0)^{q^n}] = R_n(h(\theta t_0)) = \alpha. \tag{9.5}$$

It follows from (9.5) that α has an infinite number of predecessors $h(\theta t_0)$ on an arc containing α. These predecessors have α as an accumulation point, which is impossible since α is an attractive fixpoint. Thus, $h(t) = \alpha$ has no roots other than $t = 0$. Since, however, this equation has roots $t \neq 0$ when α has predecessors other than itself, we conclude that α can have no predecessor other than itself. By a Möbius transformation we move α to $z = \infty$. Then the transformed function will be a polynomial $P(z)$. From the discussion above we conclude that $h(t)$ has only one finite pole, namely $t = 0$. Thus in $|t| > 1$, $h(t)$ has no singularities.

Now set
$$\max_{|t|=r} |h(t)| = M(r).$$

Since in (9.2) $q = d$, we get from (9.4)

$$M(r^d) \leqslant B \cdot (M(r))^d, \tag{9.6}$$

where B is a positive constant. Choose $\lambda > 1$ and an integer m such that $\lambda > B$ and $M(\lambda) < \lambda^m$. Then by (9.6)

$$M(\lambda^d) \leqslant B \cdot (\lambda^d)^m < (\lambda^d)^{m+1}$$

and
$$M(\lambda^{d^\nu}) \leqslant B^{1+d+\dots+d^{\nu-1}} \cdot (\lambda^{d^\nu})^m < (\lambda^{d^\nu})^{m+1}.$$

Thus we conclude that $h(t)$ has at most a pole of order $m+1$ at $t = \infty$ and $h(t)$ is a rational function.

We now consider the possible cases:

(*a*) $h(t)$ is a linear function. Then F_α is a circle or a straight line and the lemma is proved in this case.

(*b*) $h(t)$ is of degree $p \geqslant 2$. Let the complement of the unit disk in the t-plane be ω_e and set $h(\omega_e) = D$. By (9.3)

$$R_n(h(t)) = h(\varphi_n(t)).$$

Since $\varphi_n(t) \to \infty$ in ω_e we have

$$\lim_{n \to \infty} R_n(z) = h(\infty) = \beta, \quad z \in D.$$

If now $h(t)$ is of degree $\geqslant 2$, then $A^*(\alpha) \cap D \neq \phi$. Thus $\beta = \alpha$ and we obtain $A^*(\alpha) = D$, i.e. $F \cup A^*(\alpha)$ is equal to the extended plane.

Moreover, we assert that $h(t)$ is of degree 2. Let $z_0 \in F$ be no critical point of $h_{-1}(z)$. If $h(t)$ is of degree $p > 2$ then z_0 has the p predecessors $t_0, t_0', t_0'', \dots$ on ω. Move z along the normal to F at z_0. Then the corresponding t-values move along the normals to ω at $t_0, t_0', t_0'', \dots$, respectively. If $p > 2$, then there exists a $z \in A^*(\alpha)$ to which at least two t-values in $|t| < 1$ correspond. That is impossible and thus the degree of $h(t)$ is 2.

By a Möbius transformation, we move the endpoints of F to $z=0$ and $z=\infty$. Then map $A^*(\alpha)$ onto $\operatorname{Im} t>0$ instead of $|t|<1$ and so that $z=0\leftrightarrow t=0$ and $z=\infty\leftrightarrow t=\infty$. Thus we conclude that $h(t)=At^2$ and consequently F is a segment. That completes the proof.

Proof of Theorem 9.1. We may assume that $A^*(\alpha)=A(\alpha)$. If $A^*(\alpha)\neq A(\alpha)$, then in our proof below, we merely replace F with F_α and $R_{-n}(z)$ with $R^{\alpha}_{-n}(z)$, where $R^{\alpha}_{-n}(z)$ is the restriction of $R_{-n}(z)$ to $A^*(\alpha)$.

Let D be a domain such that $D\cap\bar{C}=\phi$ and $\alpha\notin D$. By Lemma 6.2 all the functions $\{R_{-n}(z)\}$ make a normal family in D. Moreover, by Theorem 6.2, every convergent subsequence of $\{R_{-n}(z)\}$ tends to a constant, which is a point in F. By Theorem 6.1, such a subsequence corresponds to every point in F.

Let $\zeta\in F$ and let $\{R^{(\nu)}_{-n_\nu}(z)\}$ be a sequence which converges to ζ in D. Furthermore, let ζ^* be an accumulation point of the successors $\{\zeta_{n_\nu}\}=\{R_{n_\nu}(\zeta)\}$ of ζ. Since $\bar{C}\cap F=\phi$ we can deform D so that $\zeta^*\in D$. Thus if $\zeta_m\to\zeta^*$, $m\to\infty$, then $R_{-m}(\zeta_m)-\zeta=0$ for every m, where $\{R_{-m}(z)\}$ is extracted from $\{R^{(\nu)}_{-n_\nu}(z)\}$. Set

$$f_m(z)=\frac{R_{-m}(z)-\zeta}{R'_{-m}(\zeta_m)}. \tag{9.7}$$

The functions $\{f_m(z)\}$ are univalent in D and $f_m(\zeta_m)=0$ $f'_m(\zeta_m)=1$. We obtain by a distortion theorem by Koebe (see for example Hille [8], p. 351), that $\{f_m(z)\}$ is normal in D. Extract a subsequence $\{f_{m_\nu}(z)\}$ which tends uniformly to $\varphi(z)$ in D. Obviously, $\varphi(z)$ is univalent and $\not\equiv$ constant in D. By (9.7)

$$R_{-m_\nu}(z)-\zeta=\mu_{m_\nu}(\varphi(z)+\varepsilon_{m_\nu}(z)),$$

where the constants $\mu_{m_\nu}\to 0$, $\nu\to\infty$, and $\varepsilon_{m_\nu}(z)\to 0$ uniformly in D.

If $D\cap F=\gamma$ then take $z\in\gamma$ such that $z\neq\zeta^*$. Consider

$$\lim_{\nu\to\infty}\arg\,(R_{-m_\nu}(z)-\zeta). \tag{9.8}$$

If $\overline{\lim}_{\nu\to\infty}\arg\mu_{m_\nu}=\theta$ and $\arg\varphi(z)=\psi$, then $\theta+\psi$ is one of the limits of (9.8). Hence, for (9.8) to have a unique limit, it is necessary that $\arg\varphi(z)$ be constant when z moves along γ. Thus, the image $\gamma^*=\varphi(\gamma)$ must be a straight line. Since $\varphi(z)$ is univalent, γ must be an analytic arc. By Theorem 4.3, there exists an integer N such that $F=R_N(\gamma)$. Thus if γ is analytic, then F is analytic and it follows from Lemma 9.1 that F is a circle or a straight line.

Since ζ was arbitrary, no points of F have tangents, except when F is a circle or a straight line. Thus the theorem is proved.

10. *The structure of F when the number of attractive fixpoints equals* 1

As usual a set is said to be totally disconnected if all its components are single points. The following theorem, stated by Fatou, shows that the set F can have this structure under quite general conditions. Fatou only outlined the proof, but we shall give a detailed proof here, since both the theorem and some details of the proof will be of great importance in our investigations.

Theorem 10.1. *If α is a first order attractive fixpoint of a rational function $R(z)$ and if $\bar{C} \subset A^*(\alpha)$ then the set F is totally disconnected.*

Proof. From the assumption we conclude that there exists no attractive or rationally indifferent fixpoint other than α and that $A^*(\alpha) = A(\alpha)$. Thus $A(\alpha) = \complement F$ and $\complement F$ is a connected set. For simplicity we move α to $z = \infty$ by a Möbius transformation. According to the assumption, it is possible to cover F by a simply connected closed set E_0 such that

$$E_0 \cap \bar{C} = \phi, \quad \partial E_0 \cap F = \phi.$$

If $\complement E_0 = B$ then $B \subset A(\infty)$. Since $R_n(B)$ tends uniformly to $z = \infty$ there exists an integer p such that

$$R_n(B) \subset B \quad \text{if} \quad n \geqslant p.$$

Now set

$$R_p(z) = p(z)$$

and consider the iterates $\{p_n(z)\}$. Since every inverse branch is holomorphic in E_0, we can use the same arguments as in the proof of Theorem 6.5. Thus, $\{p_{-n}(z)\}$ is normal in E_0 and every convergent subsequence tends to a constant. Furthermore, the functions $\{p'_{-n}(z)\}$ tend uniformly to zero in E_0 and thus there exists an integer h such that

$$|p'_{-n}(z)| < k < 1$$

if $z_0 \in E_0$ and $n \geqslant h$. Set

$$p_h(z) = h(z),$$

where the degree of $h(z)$ is $m = d^{p \cdot h}$ if d is the degree of $R(z)$. By mapping E_0 by the inverse function $h_{-1}(z)$ we obtain m simply connected sets $\{E_{1\nu}\}$. Because of the choices of E_0 and $h(z)$ these sets satisfy

$$F \subset \bigcup_{\nu=1}^{m} E_{1\nu} = E_1 \subset E_0; \quad E_{1\nu} \cap E_{1\mu} = \phi, \quad \text{if} \quad \nu \neq \mu.$$

Map E_1 by $h_{-1}(z)$ and then $E_2 = h_{-1}(E_1)$ by $h_{-1}(z)$ and so on. After n such mappings we obtain m^n simply connected closed sets $\{E_{n\nu}\}$ satisfying

$$F \subset \bigcup_{\nu=1}^{m^n} E_{n\nu} = E_n \subset E_{n-1} \subset \ldots \subset E_0; \quad E_{n\nu} \cap E_{n\mu} \neq \phi, \quad \text{if} \quad \nu \neq \mu.$$

If the boundaries $\{\partial E_{n\nu}\}$ have the lengths $\{l_{n\nu}\}$ the condition $|h'_{-1}(z)| < k < 1$, $z \in E_0$, implies

$$l_{n\nu} < k l_{(n-1)\mu} < \ldots < k^n l_0 \tag{10.1}$$

and

$$\lim_{n \to \infty} l_{n\nu} = 0 \quad \text{for every } \nu.$$

Thus, every component of F is a single point and the proposition is established.

We shall denote by m_1 and m_2 Lebesgue measure on a line and in the plane respectively. It is now natural to ask if it is possible to obtain results concerning $m_1 F$ and $m_2 F$ under the conditions of Theorem 11.1. Fatou gave some results but we can now give a more complete solution.

Theorem 10.2. *If α is a first order attractive fixpoint of a rational function $R(z)$ and if $\bar{C} \subset A^*(\alpha)$, then F is totally disconnected and we also have*

(i) $m_2 F = 0$,
(ii) $m_1 F = 0$ *if* $F \subset L$, *where L is a straight line.*

Proof (i). We use the same coverings $\{E_n\}_1^\infty$ of F as in the proof of Theorem 10.1 and introduce a new sequence of sets $\{O_n\}$ defined by

$$O_n = E_n - E_{n+1} \tag{10.2}$$

To prove the theorem it is sufficient to show that there exists a fixed number $\lambda > 0$ independent of n, such that

$$\frac{m_2 O_n}{m_2 E_n} > \lambda.$$

As in (10.2) we introduce

$$O_{n\nu} = E_{n\nu} - \bigcup_{\mu=1}^{m} E_{(n+1)\mu}, \quad \text{where} \quad E_{(n+1)\mu} \subset E_{n\nu}, \quad \mu = 1, 2, \ldots, m,$$

and

$$\lambda_{n\nu} = \frac{m_2 O_{n\nu}}{m_2 E_{n\nu}}.$$

In forming the sets $\{E_n\}$ we used a function $h(z)$ satisfying

$$0 < K_1 \leqslant |h'(z)| \leqslant K_2 < \infty, \quad z \in E_0, \tag{10.3}$$

where K_1 and K_2 are constants, and also

$$h(E_{n\nu}) = E_{(n-1)\nu}, \quad h(O_{n\nu}) = O_{(n-1)\nu}, \tag{10.4}$$

where for simplicity, we keep the index ν. If

$$\max_{z \in E_{n\nu}} |h'(z)| = |h'(\zeta_{n\nu})| \quad \text{and} \quad \min_{z \in E_{n\nu}} |h'(z)| = |h'(z_{n\nu})|$$

we get from (10.4) that

$$\lambda_{n\nu} = \frac{m_2 O_{n\nu}}{m_2 E_{n\nu}} \geqslant \frac{|h'(z_{n\nu})|^2}{|h'(\zeta_{n\nu})|^2} \cdot \frac{m_2 O_{(n-1)\nu}}{m_2 E_{(n-1)\nu}}.$$

After repeating this procedure n times we obtain

$$\lambda_{n\nu} \geqslant \frac{m_2 O_0}{m_2 E_0} \prod_{k=1}^{n} \frac{|h'(z_{k\nu})|^2}{|h'(\zeta_{k\nu})|^2} = K_3 \cdot \prod_{k=1}^{n} \frac{|h'(z_{k\nu})|^2}{|h'(\zeta_{k\nu})|^2}.$$

To verify the existence of a $\lambda > 0$, such that $\lambda_{n\nu} > \lambda$ for all n and ν, it is sufficient to prove that the product

$$\prod_{n=1}^{\infty} \frac{|h'(z_{n\nu})|^2}{|h'(\zeta_{n\nu})|^2} \tag{10.5}$$

is uniformly bounded for all ν. But a sufficient condition for (10.5) to be uniformly bounded is that the series

$$S=\sum_{n=1}^{\infty}\frac{|h'(z_{n\nu})^2-h'(\zeta_{n\nu})^2|}{|h'(\zeta_{n\nu})|^2}$$

is uniformly bounded for all ν. On account of (10.3) we get

$$S\leqslant K_4\sum_{n=1}^{\infty}|h'(z_{n\nu})-h'(\zeta_{n\nu})|$$

and since $h'(z)$ is a rational function

$$S\leqslant K_5\sum_{n=1}^{\infty}|z_{n\nu}-\zeta_{n\nu}|.$$

According to the proof of Theorem 11.1 the length $l_{n\nu}$ of $\partial E_{n\nu}$ satisfies $l_{n\nu}<l_0\cdot k^n$, where $0<k<1$ (see (10.1)). Hence we get

$$S\leqslant K_5\sum_{n=1}^{\infty}|z_{n\nu}-\zeta_{n\nu}|\leqslant K_5\sum_{n=1}^{\infty}l_{n\nu}\leqslant K_6\sum_{n=1}^{\infty}k^n\leqslant K_7<\infty.$$

We have thus proved that there exists a $\lambda>0$ such that

$$\frac{m_2 O_{n\nu}}{m_2 E_{n\nu}}>\lambda$$

for all n and ν. But then we obtain

$$\frac{m_2 O_n}{m_2 E_n}=\frac{\sum_{\nu=1}^{m^n} m_2 O_{n\nu}}{\sum_{\nu=1}^{m^n} m_2 E_{n\nu}}>\lambda.$$

Since $O_n=E_n-E_{n+1}$ we have

$$m_2 E_n=m_2 O_n+m_2 E_{n+1}>\lambda\cdot m_2 E_n+m_2 E_{n+1}$$

and $$m_2 E_{n+1}<(1-\lambda)m_2 E_n<\ldots<(1-\lambda)^{n+1}m_2 E_0.$$

Thus $$m_2 F\leqslant \lim_{n\to\infty} m_2 E_n\leqslant \lim_{n\to\infty}(1-\lambda)^n m_2 E_0=0$$

and the first part of the theorem is proved.

(ii) Suppose now that $F\subset L$ where L is a straight line and let

$$E_0\cap L=L_0;\quad E_{n\nu}\cap L=\omega_{n\nu},\quad O_{n\nu}\cap L=\alpha_{n\nu}.$$

We then define in analogy to the proof of (i)

$$\lambda_{n\nu}=\frac{m_1\,\alpha_{n\nu}}{m_1\,\omega_{n\nu}}$$

and again we have to verify the existence of a $\lambda > 0$ such that $\lambda_{n\nu} > \lambda$ for all n and ν. We estimate $\lambda_{n\nu}$ in the following way

$$\lambda_{n\nu} = \frac{m_1\,\alpha_{n\nu}}{m_1\,\omega_{n\nu}} = \frac{m_1(\bigcup\limits_{\mu=1}^{m} \alpha_{n\nu}^{(\mu)})}{m_1\,\omega_{n\nu}} > \frac{m_1\,\alpha_{n\nu}^{(1)}}{m_1(\partial E_{n\nu})} \geqslant \frac{|h'(z_{k\nu})|}{|h'(\zeta_{k\nu})|} \cdot \frac{m_1\,\gamma_{(n-1)\nu}^{(1)}}{m_1(\partial E_{(n-1)\nu})},$$

where $\gamma_{(n-1)\nu}^{(1)} = h(\alpha_{n\nu}^{(1)})$. We can now repeat this mapping and if we estimate $m_1\gamma_{0\nu}^{(1)}$ by $d(\partial E_0, \partial E_1) = d_1$ in the last step we get

$$\lambda_{n\nu} > \frac{d_1}{m_1(\partial E_0)} \cdot \prod_{k=1}^{n} \frac{|h''(z_{k\nu})|}{|h'(\zeta_{k\nu})|} = K_1 \cdot \prod_{k=1}^{n} \left|\frac{h'(z_{k\nu})}{h'(\zeta_{k\nu})}\right|.$$

The remaining part of the proof then follows as in the proof of (i). We only replace m_2 by m_1.

Chapter II. On the iteration of polynomials

11. *General results*

For polynomials the point at infinity has a special character. This follows from the following theorem.

Theorem 11.1. *If $P(z)$ is a polynomial of degree d, then*

(i) *$z = \infty$ is an attractive fixpoint of order* 1.
(ii) *$z = \infty$ is a critical point of $P_{-1}(z)$ of order $d-1$.*
(iii) *$A^*(\infty) = A(\infty)$.*

Proof. Let the polynomial be

$$z_1 = a_d z^d + a_{d-1} z^{d-1} + \ldots + a_1 z + a_0$$

and move $z = \infty$ to $w = 0$ by the Möbius transformation $z_1 = 1/w_1$, $z = 1/w$. The transformed function then has the form

$$w_1 = \frac{w^d}{a_0 w^d + a_1 w^{d-1} + \ldots + a_d} \tag{11.1}$$

By (11.1) we conclude that $w = 0$ is a fixpoint of order 1 and a zero of dw_1/dw of order $d-1$. Since all predecessors of $z = \infty$ coincide with $z = \infty$, it follows that $A^*(\infty) = A(\infty)$ and the theorem is proved.

Henceforth the the set F will correspond to a polynomial of degree d, if nothing else is said. The fact that a polynomial always has $z = \infty$ as an attractive fixpoint yields the

Corollary 11.1. *The set F is bounded and not equal to the whole plane, i.e. F contains no interior points.*

For simplicity we introduce the

Definition 11.1. *C_1 is the set of finite critical points of $P_{-1}(z)$.*

Theorem 11.2. *The set F is connected if and only if $A(\infty) \cap C_1 = \phi$.*

Proof. Sufficiency. Let $D = \{z \,|\, |z| > R\}$ be such that $P(D) \subset D \subset A(\infty)$. Thus, if z moves around ∂D d times, then the d inverse branches of $P_{-1}(z)$ permute cyclically. If $P_{-n}(D) = D_{-n}$, then

$$D \subset D_{-1} \subset \ldots \subset D_{-n} \subset A(\infty).$$

Since D_{-n} is simply connected for every n and $A(\infty) = \lim_{n \to \infty} D_{-n}$ we conclude that $A(\infty)$ is simply connected. Thus $F = \partial A(\infty)$ is a connected set.

Necessity. If $A(\infty) \cap C_1 \neq \phi$ then there exists an N, such that D_{-n} contains at least one finite critical point for every $n \geqslant N$. Thus D_{-n} is multiply connected for $n \geqslant N$ and by Theorem 7.3, $A(\infty)$ is then of infinite connectivity. Thus $F = \partial A(\infty)$ is disconnected and the theorem is proved.

Corollary 11.2. *If $C_1 \subset F$, then F is a connected set and $A(\infty)$ is simply connected.*

Since a polynomial always has $z = \infty$ as an attractive fixpoint the Theorems 8.1, 10.1, and 10.2 can be reformulated.

Theorem 11.3. *If a polynomial $P(z)$ has a finite first order attractive fixpoint α such that $C_1 \subset A^*(\alpha)$, then F is a Jordan curve.*

Theorem 11.4. *If $P(z)$ is a polynomial such that $C_1 \subset A(\infty)$, then*

(i) *F is totally disconnected.*
(ii) $m_2 F = 0$.
(iii) *$m_1 F = 0$ if $F \subset L$ where L is a straight line.*

12. *On the iteration of polynomials of the second degree with real coefficients*

Let the polynomial be

$$t_1 = at^2 + 2bt + c, \tag{12.1}$$

where a, b, c are real numbers. By a Möbius transformation of the form $t_1 = z_1/a - b/a$, $t = z/a - b/a$, we get from (12.1)

$$z_1 = z^2 - p, \tag{12.2}$$

where $p = b^2 - b - ac$. Thus we can consider the simpler function (12.2) instead of (12.1). The polynomial $P(z) = z^2 - p$ has the finite first order fixpoints q and q_1 and the inverse function $P_{-1}(z)$ has the only finite critical point c_1. These are

$$q = \tfrac{1}{2} + (\tfrac{1}{4} + p)^{\frac{1}{2}}, \quad q_1 = \tfrac{1}{2} - (\tfrac{1}{4} + p)^{\frac{1}{2}}, \quad c_1 = -p.$$

We will be concerned chiefly with the problem of finding the structure of F for each real value of p. Certain results have here also been established by Myrberg (see [10–11, 13, 16–17, 19]). We need the following lemma.

Lemma 12.1. *$c_1 \notin A(\infty)$ if and only if $-\frac{1}{4} \leqslant p \leqslant 2$.*

Proof. If $p < -\frac{1}{4}$, then $x^2 - p > |x|$, x real, and thus $P_n(c_1) \to \infty$, i.e. $c_1 \in A(\infty)$. Consider $p \geqslant -\frac{1}{4}$. Then q is real and $P_n(x) \to \infty$ if $x > q$. Since $P(c_1) > q$ for $|p| > q$, it follows that $c_1 \in A(\infty)$ when $p > 2$. Furthermore, if $-\frac{1}{4} \leqslant p \leqslant 2$, then $|p| \leqslant q$ and it follows that $|P_n(c_1)| \leqslant q$ for every n, i.e. $c_1 \notin A(\infty)$.

Theorem 12.1. *Let $P(z)=z^2-p$ be a polynomial with p real.*

1°. *If $-\frac{1}{4}\leqslant p\leqslant 2$ then F is connected. Furthermore, F is a Jordan curve if and only if $-\frac{1}{4}\leqslant p<\frac{3}{4}$ and F is the real interval $[-2, 2]$ if $p=2$.*

2°. *If $p<-\frac{1}{4}$ then F is totally disconnected and $m_2F=0$.*

3°. *If $p>2$ then F is real and totally disconnected. Furthermore, $F\subset[-q, q]$ and $m_1F=0$.*

Proof. 1°. Since $c_1\notin A(\infty)$ when $-\frac{1}{4}\leqslant p\leqslant 2$, it follows from Theorem 11.2 that F is connected. Furthermore, $P(z)$ has one finite attractive fixpoint q_1 if and only if $-\frac{1}{4}<p<\frac{3}{4}$, and only for $p=-\frac{1}{4}$, $P(z)$ has a rationally indifferent fixpoint which satisfies Remark 8.1. Since F is symmetric with respect to the real axis and since for $\frac{3}{4}\leqslant p\leqslant 2$, $\pm q$ and $q_1\in F$ and are real F is not a Jordan curve. Thus, by Theorem 11.3 and Remark 8.1, F is a Jordan curve if and only if $-\frac{1}{4}\leqslant p<\frac{3}{4}$. Finally, if $p=2$, then $q=2$ is repulsive and thus $2\in F$. Since $c_1=-2$ and $P(-2)=2$ we conclude that $c_1\in F$. By Corollary 11.2 F is connected and since the interval $[-2, 2]$ is completely invariant under $P(x)=x^2-2$, we obtain that $F=[-2, 2]$.

2°. If $p<-\frac{1}{4}$ then by Lemma 12.1 $c_1\in A(\infty)$. It follows from Theorem 11.4 that F is totally disconnected and that $m_2F=0$.

3°. Now consider $p>2$. By Lemma 12.1 $c_1\in A(\infty)$ and thus from Theorem 11.4 it follows that F is totally disconnected. For $p>2$, q is a repulsive fixpoint and hence $q\in F$. Consider the set P_q of predecessors of q, i.e.

$$P_q=\{q,\ \pm\sqrt{p+q},\ \pm\sqrt{p\pm\sqrt{p+q}}\ldots\}.$$

Since
$$q^2=p+q,\quad 2<q<p$$

it follows that P_q is a real point set and that $P_q\subset[-q, q]$. By Corollary 2.2, $F=\overline{P}_q$ and hence $F\subset[-q, q]$. On account of Theorem 11.4, $m_1F=0$ and the theorem is proved.

We shall now prove the last assertion, i.e. that $m_1F=0$ for $p>2$ by using explicite estimates. This proof will also give an upper bound of the Hausdorff dimension of F.

Explicit construction of F, $p>2$.

We introduce the set $E_0=[-q, q]$. Then by Theorem 12.1, $F\subset E_0$. Now map E_0 by the inverse function $P_{-1}(x)=\pm(x+p)^{\frac{1}{2}}$. Then map the inverse image by $P_{-1}(x)$ and so on. This gives us a sequence of coverings $\{E_n\}$ of F such that

$$F\subset\ldots\subset E_{n+1}\subset E_n\subset\ldots\subset E_0.$$

Set $q'=(p-q)^{\frac{1}{2}}$. If $E_1=\bigcup_{\nu=1}^{2}\omega_{1\nu}$, then $\omega_{11}=[-q, -q']$, $\omega_{12}=[q', q]$ and if $E_2=\bigcup_{\nu=1}^{2^2}\omega_{2\nu}$, then

$$\omega_{21}=[-q, -(p+q')^{\frac{1}{2}}]=-\omega_{24};\quad \omega_{22}=[-(p-q')^{\frac{1}{2}}, -q']=-\omega_{23}.$$

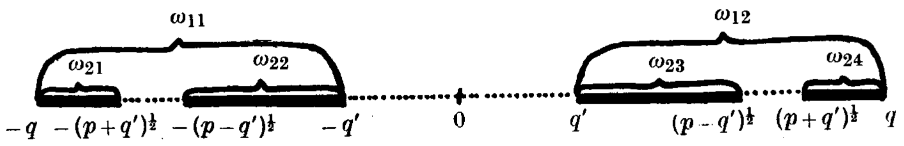

Fig. 12.1.

Observe that since $q \in F$ the endpoints of the intervals belong to F. After n mappings we get

$$E_n = \bigcup_{\nu=1}^{2^n} \omega_{n\nu}$$

and evidently
$$F = \bigcap_{n=1}^{\infty} E_n.$$

Thus F is a generalized Cantor set on the real axis, symmetric with respect to the origin. We shall now estimate the lengths of the intervals $\{\omega_{n\nu}\}$.

Lemma 12.2. *Let $\{E_n\}$ be the coverings of F obtained by the mapping process described above. If $E_n = \bigcup_{\nu=1}^{2^n} \omega_{n\nu}$ and $m_1 \omega_{n\nu} = r_{n\nu}$, then given $p > 2$ there exist two constants A and k, $0 < k < 1$, such that $r_{n\nu} < Ak^n$ for every ν.*

Proof. We shall prove the assertion by induction. Thus suppose that there exist constants A and k, where $0 < k < 1$, such that for every ν

$$r_{\mu\nu} < A \cdot k^{\mu}, \quad \mu \leqslant n-1. \tag{12.3}$$

We have to prove that A and k can be chosen so that (12.3) holds and so that (12.3) implies that

$$r_{n\nu} < A \cdot k^n, \quad \text{every } \nu.$$

Since $P(\omega_{n\nu}) = \omega_{(n-1)\mu}$, where $P(x) = x^2 - p$, it follows that

$$2\,|x_{n\nu}|\,r_{n\nu} = r_{(n-1)\mu}; \quad x_{n\nu} \in \omega_{n\nu}.$$

Thus the existence of k is evident for $|x_{n\nu}| > \frac{1}{2}$. By symmetry it is sufficient to consider the intervals on the positive real axis. Hence, after mapping $\omega_{n\nu}$ twice by $P(x)$, we get

$$4\,x_{n\nu} \cdot x_{(n-1)\nu}\, r_{n\nu} = r_{(n-2)\nu}. \tag{12.4}$$

We keep the index ν for simplicity. Now it is easy to see that if $x_{n\nu} \leqslant \frac{1}{2}$, then

$$x_{(n-1)\nu} > (p+q')^{\frac{1}{2}} > \sqrt{2} \tag{12.5}$$

(see Fig. 12.1). Thus by (12.4) and (12.5), the existence of k is evident for $x_{n\nu} \geqslant \frac{1}{4}$. It remains to investigate the values of p for which $q' < \frac{1}{4}$ and then the intervals $\{\omega_{n\nu}\}$ such that

$$\omega_{n\nu} \subset [q', \tfrac{1}{4}]. \tag{12.6}$$

After an a-fold mapping of $\omega_{n\nu}$ by $P(x)$, we obtain

$$2^a \cdot x_{n\nu} \cdot x_{(n-1)\nu} \cdot \ldots \cdot x_{(n-a+1)\nu}\, r_{n\nu} = r_{(n-a)\nu}. \tag{12.7}$$

Consider the product

$$Q = 2^a \cdot \prod_{k=0}^{a-1} x_{(n-k)\nu}.$$

From (12.6) it follows that $\omega_{n\nu} \subset [q', (p-(p+q')^{\frac{1}{2}})^{\frac{1}{2}}]$. Now set

$$\omega_{n\nu} = [y_0', y_0], \quad \omega_{(n-k)\nu} = [y_k, y_k'], \quad k = 1, 2, \ldots, (a-1).$$

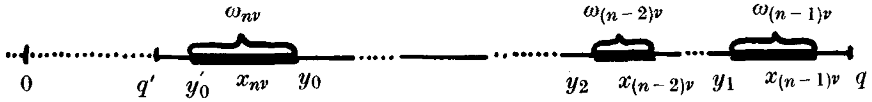

Fig. 12.2.

Then
$$Q = 2^a \prod_{k=0}^{a-1} x_{(n-k)\nu} > 2^{a-1} y_0 \cdot \prod_{k=1}^{a-1} y_k. \tag{12.8}$$

If we put $y_1 = q - \varepsilon$, then

$$\begin{aligned} y_2 &= y_1^2 - q = (q-\varepsilon)^2 - p > q - 2q\cdot\varepsilon \\ &\vdots \\ y_k &> q - (2q)^{k-1}\cdot\varepsilon. \end{aligned}$$

Since $q < p$, $\varepsilon = q - p + y_0^2 < y_0^2$ and

$$y_k > q - (2q)^{k-1} y_0^2. \tag{12.9}$$

We now associate with every interval $\omega_{n\nu}$ an integer $a_{n\nu}$ satisfying

$$(2q)^{a_{n\nu}-2}\, y_0^2 \leqslant \tfrac{1}{2}, \qquad (12.10.$$

$$(2q)^{a_{n\nu}-1}\, y_0^2 > \tfrac{1}{2}. \tag{12.11}$$

Thus, since $q > 2$, we get form (12.9)

$$y_k > \tfrac{3}{2},\; k = 1, 2, \ldots, a_{n\nu} - 1. \tag{12.12}$$

Choose $a = a_{n\nu}$ in (12.8). By (12.11) and by (12.12)

$$Q^2 > \Bigl(2^{a_{n\nu}-1} y_0 \prod_{k=1}^{a_{n\nu}-1} y_k\Bigr)^2 > \tfrac{1}{2}\cdot\left(\frac{9}{2q}\right)^{a_{n\nu}-1}. \tag{12.13}$$

Since we consider $q' < \frac{1}{4}$ we have $q < 2.05$. Moreover, by (12.10) and (12.11) $a_{n\nu} \geqslant 3$. Inserting these estimates in (12.13) yields

$$Q^2 > 2.4. \tag{12.14}$$

Now return to formula (12.7). By putting $a = a_{n\nu}$ and using (12.14) we get

$$r_{n\nu} = \frac{r_{(n-a_{n\nu})\nu}}{Q} < \frac{A\cdot k^n}{\sqrt{2.4}\cdot k^{a_{n\nu}}} \tag{12.15}$$

Thus, the existence of k is evident if the integers $a_{n\nu}$ are uniformly bounded. This, however, is easily verified. For since $y_0 \geqslant q'$ for every $\omega_{n\nu}$, we have $a_{n\nu} \leqslant b$, where

$$(2q)^{b-2}\cdot(q')^2 = \tfrac{1}{2}. \tag{12.16}$$

Hence, by taking $$k=(\tfrac{2}{3})^{1/b}$$

it follows from (12.15) that $r_{n\nu}<A\cdot k^n$. Finally, we choose A such that

$$r_{n\nu}<A\cdot(\tfrac{2}{3})^{n/b},\quad n=1,2,\ldots,[b].$$

Thus the induction argument is also valid for $x_{n\nu}<\frac{1}{4}$. Moreover, it is easy to see that when $q'<\frac{1}{4}$ we can take $k=(\frac{2}{3})^{1/b}$ even if $x_{n\nu}\geqslant\frac{1}{4}$. We can also take $k=(\frac{2}{3})^{1/b}$ when $q'\leqslant\sqrt{3}$, but for $q'>\frac{1}{2}$ it is simpler to take $k=1/2q'$.

Having established the lemma above, we can now use some details from the proof to give an upper bound of the Hausdorff dimension of F. We recall that

$$q=\tfrac{1}{2}+(\tfrac{1}{4}+p)^{\frac{1}{2}},\quad q'=(p-q)^{\frac{1}{2}},\quad 2<q<p.$$

Thus $$\lim_{p\to\infty} q=\infty,\quad \lim_{p\to\infty} q'=\infty,$$

$$\lim_{p\to 2} q=2,\quad \lim_{p\to 2} q'=0.$$

Theorem 12.2. *Let $\alpha(p)$ denote the Hausdorff dimension of F for a fixed p.*

1°. *If $p\geqslant 2+\sqrt{2}$ then* $\alpha(p)<\dfrac{\log 2}{\log 2q'}$.

2°. *If $p\leqslant 6$ then* $\alpha(p)<\dfrac{\log 2}{\exp(-60(\log q'/5)^2)+\log 2}$.

Remark. We shall later prove that the logarithmic capacity of F is positive for each p.

Proof. 1°. We use the same coverings of F as in the proof of Lemma 13.2 and introduce

$$m_\alpha(E_n)=\sum_{\nu=1}^{2^n} r_{n\nu}^\alpha=2\sum_{\nu=2^{n-1}+1}^{2^n} r_{n\nu}^\alpha,$$

where the symbol $m_\alpha(E_n)$ is a slight abuse of notation. Since $r_{(n-1)\nu}=2x_{n\nu}r_{n\nu}$ and $x_{n\nu}>q'$ we get

$$m_\alpha(E_n)=2\cdot\sum_{\nu=2^{n-1}+1}^{2^n}\frac{r_{(n-1)\nu}^\alpha}{(2x_{n\nu})^\alpha}<\frac{2}{(2q')^\alpha}\cdot m_\alpha(E_{n-1}).$$

An n-fold application of this procedure gives

$$m_\alpha(E_n)<\left(\frac{2}{(2q')^\alpha}\right)^n\cdot(2q)^\alpha.$$

Since $p>2+\sqrt{2}$, then $q'>1$ and thus

$$\alpha(p)<\alpha_1(p)=\frac{\log 2}{\log 2q'}.$$

2°. By Theorem 10.2 there exists for each p a $\lambda > 0$ such that

$$m_1 E_n \leqslant K(1-\lambda)^n, \tag{12.17}$$

where K is a constant. Thus, by Hölder's inequality and (12.17)

$$m_\alpha(E_n) \leqslant \Big(\sum_{\nu=1}^{2^n} r_{n\nu}\Big)^\alpha \Big(\sum_{\nu=1}^{2^n} 1\Big)^{1-\alpha} = (m_1 E_n)^\alpha \cdot 2^{n(1-\alpha)} \leqslant K^\alpha \cdot [(1-\lambda)^\alpha \cdot 2^{1-\alpha}]^n$$

and we have

$$\alpha(p) < \alpha_2(p) = \frac{\log 2}{\log \dfrac{2}{1-\lambda}} \leqslant \frac{\log 2}{\lambda + \log 2}. \tag{12.18}$$

Hence we have to consider λ. Using our notation from Lemma 13.4, we get as in Theorem 10.2, that

$$\lambda_{n\nu} \geqslant \frac{q'}{q} \cdot \prod_{k=1}^{n} \frac{x_{k\nu}}{y_{k\nu}} \geqslant \frac{q'}{q} \cdot \prod_{k=1}^{N-1} \frac{x_{k\nu}}{y_{k\nu}} \prod_{k=N}^{n} \left(1 - \frac{|x_{k\nu} - y_{k\nu}|}{y_{k\nu}}\right) \geqslant$$

$$\geqslant \frac{q'}{q} \left(\prod_{k=1}^{N-1} \frac{x_{k\nu}}{y_{k\nu}}\right) \left(1 - \frac{1}{q'} \sum_{k=N}^{n} r_{k\nu}\right). \tag{12.19}$$

On account of Lemma 12.2, we have $r_{n\nu} < A(\frac{2}{3})^{n/b}$, where b is determined by (12.16). Furthermore, it is easy to see that we can choose $A = 2q$. By (12.19)

$$\lambda_{n\nu} \geqslant \frac{q'}{2q} \prod_{k=1}^{N-1} \frac{x_{k\nu}}{y_{k\nu}} > \frac{q'}{2} \left(\frac{1}{q}\right)^N \prod_{k=1}^{N-1} x_{k\nu}, \tag{12.20}$$

if

$$\frac{2q}{q'} \frac{(\frac{2}{3})^{N/b}}{(1 - (\frac{2}{3})^{1/b})} < \frac{1}{2}. \tag{12.21}$$

For (12.21) to hold it is sufficient that

$$N > \frac{b \cdot \log \dfrac{4q}{q'(1 - (\frac{2}{3})^{1/b})}}{\log \frac{3}{2}} = N(q').$$

Since $p \leqslant 6$, we have $q \leqslant 3$. By (12.16), $1/q'^2 = 2 \cdot (2q)^{b-2}$. A simple calculation gives $N(q') < 10b^2$. From (12.16) it follows that $b < \frac{3}{2} \log 5/q'$. Thus, by taking

$$N = \left[\frac{45}{2} \cdot \left(\log \frac{q'}{5}\right)^2\right] + 1 \tag{12.22}$$

the estimate (12.20) holds. Hence in (12.20) it remains to consider the product $\prod_{k=1}^{N-1} x_{k\nu}$. If here $x_{k\nu} \leqslant \frac{1}{4}$, we use the estimate (12.13). If $k \geqslant \alpha_{k\nu}$, then

$$\prod_{\mu = k - \alpha_{k\nu} + 1}^{k} x_{\mu\nu} > (2, 4)^{\frac{1}{2}} \cdot 2^{-\alpha_{k\nu}}$$

and if $k < \alpha_{k\nu}$, then

$$\prod_{\mu=1}^{k} x_{\mu\nu} > (\tfrac{3}{2})^{k-1} \cdot q'.$$

It follows that

$$\prod_{k=1}^{N-1} x_{k\nu} > q' \cdot (\tfrac{1}{4})^{N-2}. \tag{12.23}$$

By (12.20), (12.22), and (12.23)

$$\lambda_{n\nu} > (q')^2 \cdot \left(\frac{1}{4q}\right)^N > \exp\left(-60\left(\log\frac{q'}{5}\right)^2\right).$$

We choose

$$\lambda = \exp\left(-60\left(\log\frac{q'}{5}\right)^2\right)$$

in (12.18), i.e.

$$\alpha(p) < \alpha_2(p) < \frac{\log 2}{\exp\left(-60\left(\log\frac{q'}{5}\right)^2\right) + \log 2},$$

if $p \leqslant 6$. Our theorem is thus proved.

We end our discussion about second degree polynomials by observing that since the only critical point c_1 is real, the set $\bar{C}$ cannot divide the plane. Thus, Theorem 6.3 yields the

Theorem 12.3. *If $P(z) = z^2 - p$, p real, then the iterates $\{P_n(z)\}$ have only constant limit functions in their domains of normality.*

13. *On the iteration of polynomials of the third degree with real coefficients*

Let the polynomial be

$$t_1 = at^3 + bt^2 + ct + d, \tag{13.1}$$

where a, b, c and d are real numbers. By a Möbius transformation of the form

$$t_1 = \frac{z_1}{|a|^{\frac{1}{2}}} - \frac{b}{3a}, \quad t = \frac{z}{|a|^{\frac{1}{2}}} - \frac{b}{3a}$$

we get from (13.1)

$$z_1 = \pm z^3 + pz + r, \tag{13.2}$$

where the sign of z^3 is the same as the sign of a and where

$$p = -\frac{b^2}{3a} + c, \quad r = |a|^{\frac{1}{2}}\left(\frac{b}{3a} + \frac{2b^3}{27a^2} - \frac{bc}{3a} + d\right).$$

Thus we can consider the simpler function (13.2) instead of (13.1).

Case A: $P(z) = \pm z^3 + pz$.

First consider the polynomial $z_1 = -z^3 + pz$. By the Möbius transformation $z_1 = iw_1$, $z = iw$, we get the transformed function $w_1 = w^3 + pw$. Hence in Case A it is sufficient to consider the polynomial

$$P(z) = z^3 + pz. \tag{13.3}$$

$P(z)$ has three finite first order fixpoints q_1, q_2 and q_3 and $P_{-1}(z)$ has two finite critical points c_1 and c_2. These are

$$q_1 = 0,\ q_2, q_3 = \pm(1-p)^{\frac{1}{2}},\ c_1, c_2 = \pm\frac{2p}{3}\left(\frac{-p}{3}\right)^{\frac{1}{2}} \tag{13.4}$$

Furthermore, we shall need the fixpoints of order two of (13.3), i.e. those roots of the equation

$$(z^3 + pz)^3 + p(z^3 + pz) - z = 0$$

which are not fixpoints of order one. A simple calculation gives the following three cycles:

$$\zeta_1, \zeta_2 = \pm(-p-1)^{\frac{1}{2}},\ \zeta_3, \zeta_4 = \left(-\frac{p}{2} \pm \left(\frac{p^2}{4} - 1\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = -\zeta_5, -\zeta_6. \tag{13.5}$$

Lemma 13.1. *If* $|p| \leqslant 1$ *then* c_1, $c_2 \in A^*(0)$.

Proof. Since $q_1 = 0$ is an attractive fixpoint for $|p| < 1$ and a rationally indifferent one for $|p| = 1$, at least one critical point belongs to $A^*(0)$ (see Theorem 3.1 and 3.2). But since $c_1 = -c_2$ and $A^*(0)$ is symmetric with respect to the origin, $A^*(0)$ must contain both c_1 and c_2.

Lemma 13.2. c_1, $c_2 \in A(\infty)$ *if and only if* $|p| > 3$.

Proof. Suppose first that $p \leqslant 0$. Then, for $x > q_2$, we have $P(x) > x$ and hence $P_n(x) \to +\infty$. Analogously, for $x < q_3$, we have that $P_n(x) \to -\infty$. Thus c_1, $c_2 \in A(\infty)$ if $|c_1| > q_2$, i.e. when $p < -3$. If $-3 \leqslant p \leqslant 0$, then $|c_1| \leqslant q_2$ and $|P_n(c_1)| = |P_n(c_2)| \leqslant q_2$ for every n, i.e. c_1, $c_2 \notin A(\infty)$.

Suppose now that $p > 0$. Since c_1, $c_2 = \pm i(2p/3)(p/3)^{\frac{1}{2}}$ and $P(iy) = i(-y^3 + py)$ we have to consider the behavior of $c_1', c_2' = \pm(2p/3)(p/3)^{\frac{1}{2}}$ under the iterates of $P^*(y) = -y^3 + py$. Since we have $P^*(y) + y = 0$ for $y = 0$ and $y = \pm(p+1)^{\frac{1}{2}}$, it follows that for every n, $|P_n^*(c_1')| \leqslant (p+1)^{\frac{1}{2}}$, if $|c_1'| \leqslant (p+1)^{\frac{1}{2}}$, i.e. when $0 \leqslant p \leqslant 3$. Now if $p > 3$, then $|c_1'| > (p+1)^{\frac{1}{2}}$ and it is easy to see that $P_{2n}(c_1') > P_{2n-2}(c_1')$ and that $P_{2n+1}(c_1') < P_{2n-1}(c_1')$. By the Möbius transformation $iP(z) = P^*(y)$, $iz = y$, $P^*(y)$ can be transformed to $P(z) = z^3 + pz$. Then, on account of (13.5), $P^*(y)$ has no real fixpoint ζ of order two such that $|\zeta| > (p+1)^{\frac{1}{2}}$ when $p > 3$.

Thus we conclude that $P_{2n}(c_1') \to +\infty$ and that $P_{2n+1}(c_1') \to -\infty$. Clearly $P_{2n}(c_2') \to -\infty$ and $P_{2n+1}(c_2') \to +\infty$ and the lemma is proved.

We shall also need the following bound on the set F.

Lemma 13.3. $F \subset \{z \,|\, |z| \leqslant (1 + |p|)^{\frac{1}{2}}\}$.

Proof. If $|z| > (k(1 + |p|))^{\frac{1}{2}}$, $k > 1$, then $|P(z)| = |z^3 + pz| > k|z|$. Thus $|P_n(z)| > k^n|z|$ and we have $|P_n(z)| \to \infty$, i.e. $z \in A(\infty)$.

Remark. By (13.4) and (13.5) there exists for each p a repulsive fixpoint ζ, i.e. $\zeta \in F$, such that $|\zeta| = (1+|p|)^{\frac{1}{2}}$.

Theorem 13.1. *Let $P(z)=z^3+pz$ be a polynomial with p real.*

1°. *If $|p| \leqslant 3$ then F is connected. Furthermore, F is a Jordan curve if and only if $|p|<1$, F is the real interval $[-2, 2]$ if $p=-3$, and finally, F is the imaginary interval $[-2i, 2i]$ if $p=3$.*

2°. *If $|p|>3$ then F is totally disconnected and $m_1 F=0$. Furthermore, if $p<-3$ then F is real and $F \subset [-q_2, q_2]$ and if $p>3$ then F is purely imaginary and $F \subset [-\zeta_1, \zeta_1]$.*

Proof. 1°. If $|p| \leqslant 3$ then by Lemma 13.2 $c_1, c_2 \notin A(\infty)$ and hence by Theorem 11.2 F is connected. Moreover, $P(z)$ has one and only one finite attractive fixpoint, $q_1=0$, if and only if $|p|<1$. By Lemma 13.1 $c_1, c_2 \in A^*(0)$ for $|p|<1$ and then it follows from Theorem 11.3 that F is a Jordan curve when $|p|<1$. Furthermore, if $|p| \geqslant 1$ then $P_{-1}(q_1)=\{0, \pm\sqrt{-p}\} \subset F$ and hence by the symmetry F is not a Jordan curve. Thus, F is a Jordan curve if and only if $|p|<1$. Finally, if $p=-3$ then $c_1=2 \in F$ and $c_2=-2 \in F$. Since the interval $[-2, 2]$ is completely invariant under $P(z)$, it follows from Theorem 11.2 that $F=[-2, 2]$. Analogously, if $p=3$ then $F=[-2i, 2i]$.

2°. Suppose now that $|p|>3$. By Lemma 13.2 $c_1, c_2 \in A(\infty)$ and thus, according to Theorem 11.4, the set F is totally disconnected. We then have to verify that F lies on the appropriate intervals.

(a) Suppose that $p<-3$ and consider the equation $z^3+pz=x$ where x is real and $|x| \leqslant (1-p)^{\frac{1}{2}}$. This equation has the discriminant $D>0$ when $p<-3$ and thus the equation has three distinct real roots. Since $q_2=(1-p)^{\frac{1}{2}} \in F$ when $p<-3$ we consider the set P_{q_2} of the predecessors of q_2. From Lemma 13.3 and from the discussion above it follows that the set P_{q_2} is real and that $P_{q_2} \subset [-q_2, q_2]$. By Corollary 2.2 $F=\bar{P}_{q_2}$ and hence F is real and $F \subset [-q_2, q_2]$. Thus, by Theorem 11.4 $m_1 F=0$ if $p<-3$.

(b) Suppose now that $p>3$. We can proceed analogously to (a). Since $\zeta_1 = (-1-p)^{\frac{1}{2}} \in F$ we form P_{ζ_1} and it follows that $P_{\zeta_1} \subset [-\zeta_1, \zeta_1]$. Thus $F \subset [-\zeta_1, \zeta_1]$ and by Theorem 11.4 $m_1 F=0$ if $p>3$.

Since the set of critical points C is either real or purely imaginary and since the point at infinity is a first order attractive fixpoint, the set $\bar{C}$ cannot divide the plane. Then Theorem 6.3 yields the

Theorem 13.2. *If $P(z)=z^3+pz$, where p is real, then the iterates $\{P_n(z)\}$ have only constant limit functions in their domains of normality.*

Case B: $P(z)=z^3+r$.

If $|r|<2\sqrt{3}/9$, then $P(z)=z^3+r$ has three real first order fixpoints q_1, q_2, q_3 satisfying

$$q_3<0<r<q_2<q_1; \quad 0<r<\frac{2\sqrt{3}}{9} \tag{13.6}$$

$$q_3<q_2<r<0<q_1; \quad -\frac{2\sqrt{3}}{9}<r<0. \tag{13.7}$$

Here q_1 and q_3 are repulsive and q_2 is attractive. If $|r| = 2\sqrt{3}/9$, then two of these fixpoints coincide and this point is a rationally indifferent fixpoint, while the other fixpoint is repulsive. If $|r| > 2\sqrt{3}/9$ then there exists only one real first order fixpoint, namely

$$q_3 < 0, \text{ when } r > \frac{2\sqrt{3}}{9}; \quad q_1 > 0, \text{ when } r < -\frac{2\sqrt{3}}{9}. \tag{13.8}$$

This fixpoint is always repulsive. The inverse function $P_{-1}(z)$ has one finite critical point $c_1 = r$ and we have the

Lemma 13.4. *$c_1 \in A(\infty)$ if and only if $|r| > 2\sqrt{3}/9$.*

Proof. If $r > 2\sqrt{3}/9$ then by (13.8), $P(x) > x$ for $x > 0$. Since $c_1 = r$, $P_n(c_1) \to \infty$ when $r > 2\sqrt{3}/9$. Analogously, if $r < -2\sqrt{3}/9$ then by (13.8), $P_n(c_1) \to -\infty$. Finally, it follows immediately from (13.6) and (13.7) that for every n, $P_n(c_1) \in [q_3, q_1]$ when $|r| \leqslant 2\sqrt{3}/9$.

For the following theorem see also Myrberg [19].

Theorem 13.3. *Let $P(z) = z^3 + r$ be a polynomial with r real.*

1°. *F is a Jordan curve if and only if $|r| \leqslant 2\sqrt{3}/9$.*

2°. *If $|r| > 2\sqrt{3}/9$ then F is a totally disconnected set and $m_2 F = 0$.*

Proof. 1°. From Theorem 3.1 and Lemma 13.4 it follows that there exists one and only one attractive fixpoint, q_2, if and only if $|r| < 2\sqrt{3}/9$. Furthermore, for $r = \pm 2\sqrt{3}/9$ the fixpoints $\pm\sqrt{3}/3$ satisfy $P'(\pm\sqrt{3}/3) = +1$ and $P''(\pm\sqrt{3}/3) \neq 0$. Thus, by Theorem 11.3 and Remark 8.1, F is a Jordan curve if and only if $|r| \leqslant 2\sqrt{3}/9$ for:

2°. Suppose now that $|r| > 2\sqrt{3}/9$. Then, according to Lemma 13.4 and Theorem 11.4, F is totally disconnected and $m_2 F = 0$.

Case C: $P(z) = -z^3 + r$.

The polynomial $P(z) = -z^3 + r$ has for each r only one real first order fixpoint q. This is attractive, rationally indifferent, or repulsive, according as $|r| < 4\sqrt{3}/9$, $|r| = 4\sqrt{3}/9$, or $|r| > 4\sqrt{3}/9$. Moreover, if $4\sqrt{3}/9 < |r| \leqslant 4\sqrt{6}/9$ then $P(z)$ has four real fixpoints of order two $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ satisfying

$$\zeta_1 < \zeta_2 < q < r < \zeta_3 < \zeta_4, \quad 1 < r < \frac{4\sqrt{6}}{9} \tag{13.9}$$

$$\zeta_1 < \zeta_2 < r < q < \zeta_3 < \zeta_4, \quad -\frac{4\sqrt{6}}{9} < r < -1.$$

For $|r| = 4\sqrt{6}/9$, ζ_1, ζ_2 and ζ_3, ζ_4 coincide and are rationally indifferent fixpoints. The inverse function $P_{-1}(z)$ has only one finite critical point $c_1 = r$ and we have the

Lemma 13.5. *$c_1 \in A(\infty)$ if and only if $|r| > 4\sqrt{6}/9$.*

Proof. By symmetry it is sufficient to consider $r \geqslant 0$. If $r \leqslant 1$ then it follows that $0 \leqslant P_n(r) \leqslant 1$ for every n, i.e. $c_1 \notin A(\infty)$. If $r > 1$, then $P_{2n}(r) > P_{2n-2}(r) > r$ and $P_{2n+1}(r) < P_{2n-1}(r) < 0$. Thus by (13.9) $P_{2n}(r) \to \zeta_3$ and $P_{2n+1}(r) \to \zeta_2$ if $1 < r \leqslant 4\sqrt{6}/9$, and we have $c_1 \notin A(\infty)$. Since for $r > 4\sqrt{6}/9$, $P(z)$ has no real fixpoints of order two $P_{2n}(r) \to +\infty$ and $P_{2n-1}(r) \to -\infty$, i.e. $c_1 \in A(\infty)$.

Theorem 13.4. *Let $P(z) = -z^3 + r$ be a polynomial with r real.*

1°. If $|r| \leqslant 4\sqrt{6}/9$, *then F is connected. Furthermore, F is a Jordan curve if and only if $|r| < 4\sqrt{3}/9$.*

2°. *If $|r| > 4\sqrt{6}/9$, then F is totally disconnected and $m_2 F = 0$.*

Proof. By Lemma 13.5 and Theorem 11.2 F is connected only when $|r| < 4\sqrt{6}/9$. If $4\sqrt{3}/9 < |r| \leqslant 4\sqrt{6}/9$ then ζ_1, ζ_4, $q \in F$ and are real so by symmetry F is no Jordan curve. However, $P(z)$ has one and only one attractive first order fixpoint, q, if and only if $|r| < 4\sqrt{3}/9$. For $|r| = 4\sqrt{3}/9$, q is rationally indifferent and does not satisfy Remark 8.1. Then, by Theorem 11.3, F is a Jordan curve if and only if $|r| < 4\sqrt{3}/9$. Finally, if $|r| > 4\sqrt{6}/9$ then, by Lemma 13.5 and Theorem 11.4, F is totally disconnected and $m_2 F = 0$.

Since c_1 is real, we have, analogously to the Theorems 12.3 and 13.2,

Theorem 13.5. *If $P(z) = \pm z^3 + r$, where r is real, then the iterates $\{P_n(z)\}$ have only constant limit functions in their domains of normality.*

Case D: $P(z) = \pm z^3 + pz + r$, $p \neq 0$, $r \neq 0$.

In the cases A, B, C the critical points were distributed so that either $C_1 \subset A(\infty)$ or $C_1 \cap A(\infty) = \phi$. With the aid of general results it was then possible to determine the structure of F. We are not going to state detailed conditions under which $C_1 \subset A(\infty)$ or $C_1 \cap A(\infty) = \phi$ in case D. By using known algebraic formulas and the methods of this paper, it is easy to decide whether or not a given numerical example satisfies one of the conditions above. We illustrate this with the following simple case.

Theorem 13.6. *Let $P(z) = z^3 - pz + r$ be a polynomial, where $p > 0$ and r are real. If $27r^2 > 4(p+1)^3$, then the set F is totally disconnected and $m_2 F = 0$.*

Proof. By assumption, the equation $z^3 - (p+1)z + r = 0$ has the discriminant $D = 4(p+1)^3 - 27r^2 < 0$ and consequently there exists only one real fixpoint q of order one. This point can be explicitly expressed by a known algebraic formula, from which it is easy to see that either $q < 0 < r$ or $r < 0 < q$. The inverse function $P_{-1}(z)$ has two finite critical points, namely c_1, $c_2 = \pm(2p/3) \cdot (p/3)^{\frac{1}{2}} + r$. Our assumption implies that if $r > 0$, then c_1, $c_2 > 0$ and if $r < 0$, then c_1, $c_2 < 0$. Now consider $r > 0$. Then $P(x) > x$ if $x > 0$ and thus $P_n(c_1), P_n(c_2) \to +\infty$, i.e. c_1, $c_2 \in A(\infty)$. Analogously, if $r < 0$ then $P(x) < x$ when $x < 0$ and $P_n(c_1), P_n(c_2) \to -\infty$, i.e. c_1, $c_2 \in A(\infty)$. The theorem then follows from Theorem 11.4.

However, there exist functions $P(z) = \pm z^3 + pz + r$ such that $C_1 \cap A(\infty) \neq \phi$ and $C_1 \cap \complement A(\infty) \neq \phi$. If for such a function we can prove that there exist finite attractive or rationally indifferent fixpoints, then the general results are applicable. If, however, $C_1 \cap \complement A(\infty) \subset F$, then there seems to be no results concerning the structure of F. We shall now establish a general result for such third degree polynomials. Furthermore, we shall prove that two different structures can occur.

Theorem 13.7. *Let $P(z)$ be a polynomial of the third degree and let $P_{-1}(z)$ have the finite critical points c_1 and c_2. If $c_1 \in F$ and $c_2 \in A(\infty)$ then F contains an infinite number of single point components.*

Proof. We observe that the assumption implies that there is no finite attractive or rationally indifferent fixpoint. As in the proof of Theorem 10.1 we can cover F by a simply connected closed set E_0 such that $\partial E_0 \cap F = \phi$ and $P_n(c_2) \notin E_0$, $n = 0, 1, 2, \ldots$. Moreover, we first suppose that $P\,(\complement E_0) \subset \complement E_0$. Since $c_2 \notin E_0$, there exists an inverse branch of $P_{-1}(z)$, e.g. $P^{(3)}_{-1}(z)$, which is holomorphic in E_0. Consider the functions $\{P^{(3)}_{-n}(z)\}$ defined by $P^{(3)}_{-n}(z) = P^{(3)}_{-1}(P^{(3)}_{-(n-1)}(z))$. By repeating the argument in the proof of Theorem 10.1, we see that $\{P^{(3)}_{-n}(z)\}$ is normal in E_0 and that the convergent subsequences tend to constants.

Now map E_0 by the functions $\{P^{(3)}_{-n}(z)\}$. The images satisfy $P^{(3)}_{-n}(E_0) \subset P^{(3)}_{-(n-1)}(E_0)$, $n = 1, 2, 3, \ldots$. If $\partial P^{(3)}_{-n}(E_0)$ has the length $l^{(3)}_{-n}$, then, according to the properties of $\{P^{(3)}_{-n}(z)\}$, $\lim_{n \to \infty} l^{(3)}_{-n} = 0$. Thus the component of F, which belongs to all $\{P^{(3)}_{-n}(E_0)\}$, is a single point. Hence, by taking its predecessors, we get an infinite number of single point components of F.

It remains to prove that we need not assume $P(\complement E_0) \subset \complement E_0$; the choice of E_0 does not guarantee that this assumption is satisfied. But since $\complement E_0 \subset A(\infty)$ the uniform convergence of $\{P_n(\complement E_0)\}$ to $z = \infty$ implies the existence of an integer h such that $P_n(\complement E_0) \subset \complement E_0$ if $n \geq h$. Clearly, that is sufficient for our proof to work.

Remark. It seems to be an open question whether this theorem is valid for an arbitrary polynomial which satisfies $C_1 \cap A(\infty) \neq \phi$, $C_1 \cap \complement A(\infty) \neq \phi$ and $C_1 \cap \complement A(\infty) \subset F$.

We are now going to prove that Theorem 13.7 is the most general possible under the given assumptions. First we prove the

Theorem 13.8. *Let $P(z)$ be a polynomial of the third degree and let $P_{-1}(z)$ have the finite critical points c_1 and c_2. If c_1 is a repulsive fixpoint of order one, i.e. $c_1 \in F$, and $c_2 \in A(\infty)$, then F is totally disconnected.*

Remark. An example is the polynomial $P(z) = 18(z^3 - 2z^2 + z)$. In fact, $c_1 = P(1) = 0$ and $P(0) = 0$, $|P'(0)| = 18$; $c_2 = P(\frac{1}{3}) = \frac{8}{3} \in A(\infty)$.

Proof of Theorem 13.8. As in the preceeding proof we cover F by a simply connected closed set E_0 such that $\partial E_0 \cap F = \phi$ and $P_n(c_2) \in \complement E_0$, $n = 0, 1, 2, \ldots$. We add further the assumption $P(\complement E_0) \subset \complement E_0$. Let the inverse branches be distributed so that

$$c_1 \leftrightarrow (P^{(1)}_{-1}(z),\ P^{(2)}_{-1}(z)),\quad c_2 \leftrightarrow (P^{(1)}_{-1}(z),\ P^{(3)}_{-1}(z)).$$

Then $P^{(3)}_{-1}(z)$ is holomorphic in E_0. Now map E_0 by $P_{-1}(z)$. The branches $P^{(1)}_{-1}(z)$

and $P^{(2)}_{-1}(z)$ permute cyclically as z runs through ∂E_0 twice. Put $P^{(1)}_{-1}(E_0)\cup P^{(2)}_{-1}(E_0)=E_1^{(1,2)}$ and $P^{(3)}_{-1}(E_0)=E_1^{(3)}$, which thus are simply connected sets such that

$$E_1^{(1,2)}\subset E_0,\quad E_1^{(3)}\subset E_0,\quad E_1^{(1,2)}\cap E_1^{(3)}=\phi.$$

Since c_1 is both a critical point and a repulsive fixpoint, it follows that

$$P^{(3)}_{-1}(c_1)=c_1\in E_1^{(3)};\quad P^{(1)}_{-1}(c_1)=P^{(2)}_{-1}(c_1)=\zeta\in E_1^{(1,2)}, \tag{13 10}$$

where evidently $P'(\zeta)=0$. We map $E_1^{(1,2)}$ and $E_1^{(3)}$ and their successively obtained images by the three inverse branches. After an n-fold mapping, we get a number of simply connected closed sets $\{E_n^{(\nu)}\}$ such that

$$F\subset\bigcup_{\nu}E_n^{(\nu)}\subset\bigcup E_{n-1}^{(\nu)},\quad n=1,2,\ldots;$$

$$E_n^{(\nu)}\cap E_n^{(\mu)}=\phi\quad\text{if}\quad \nu\neq\mu.$$

If we denote the length of $\partial E_n^{(\nu)}$ by $l_n^{(\nu)}$we have to prove that $l_n^{(\nu)}\to 0$ for every ν.

First consider the functions $\{P^{(3)}_{-n}(z)\}$, defined by $P^{(3)}_{-n}(z)=P^{(3)}_{-1}(P^{(3)}_{-(n-1)}(z))$. These functions are holomorphic in E_0 and thus it follows, as in the proof of Theorem 13.7, that if $P^{(3)}_{-n}(E_0)=E_n^{(3)}$, then

$$\lim_{n\to\infty}l_n^{(3)}=0. \tag{13.11}$$

From (13.10) we see that $E_n^{(3)}\to c_1$. Now consider the sets $\{E_n^{(1,2)}\}$ defined by

$$E_n^{(1,2)}=P^{(1)}_{-1}(E_{n-1}^{(3)})\cup P^{(2)}_{-1}(E_{n-1}^{(3)}). \tag{13.12}$$

These sets are simply connected and $E_n^{(1,2)}\subset E_{n-1}^{(1,2)}$, $n=2, 3, \ldots$. According to (13.11) and (13.12), we obtain that $l_n^{(1,2)}\to 0$.

It is evident that for every n, each inverse branch $P^{(\nu)}_{-1}(z)$ is holomorphic in $\bigcup_\nu E_n^{(\nu)}-E_n^{(3)}$. After making the usual arguments concerning normal families and their limit functions, it follows that, for every ν, $l_n^{(\nu)}\to 0$.

Finally, the assumption $P(\complement E_0)\subset\complement E_0$ can be excluded by the same argument as in the proof of Theorem 13.7. Thus the theorem is proved.

Remark. Fatou conjectured ([6], p. 84) that if a critical point belongs to F, then F cannot be totally disconnected. Our Theorem 13.8, however, gives a counter-example to this.

We have now seen that if $P(z)$ is a polynomial of the third degree and is such that the critical points $c_1\in F$ and $c_2\in A(\infty)$, then the set F contains one-point components, and furthermore can be totally disconnected. It is thus natural to ask whether the assumptions above always imply that F is totally disconnected. The answer is in the negative, as the following example shows.

Example. If $P(z)=(3\sqrt{3}/2)(z^3+3z^2+2z)$ then $P_{-1}(z)$ has the critical points $c_1=-1\in F$ and $c_2=1\in A(\infty)$. Moreover, the set F contains both an infinite number of single point components and an infinite number of connected components. This can be seen as follows.

Since $P(-1)=0$, $P(0)=0$ and $|P'(0)|=3\sqrt{3}$ we conclude that $c_1 \in F$. Furthermore, since $P(z)$ has no positive, real fixpoint of order one, $P_n(c_2) \to \infty$, i.e. $c_2 \in A(\infty)$. Suppose then that c_1 is a critical point of the branches $P_{-1}^{(1)}(z)$ and $P_{-1}^{(2)}(z)$. Then we see that

$$P_{-1}^{(1)}[-1,0] \cup P_{-1}^{(2)}[-1,0] = [-1,0] \quad \text{and} \quad P[-1,0]=[-1,0].$$

Thus the closed interval $[-1,0]$ is completely invariant under $P(z)$ if we use only the branches $P_{-1}^{(1)}(z)$ and $P_{-1}^{(2)}(z)$. It follows that $[-1,0] \subset F$. Hence F contains an infinite number of connected components, namely $[-1,0]$ and its predecessors. By Theorem 13.7, F contains an infinite number of single point components. Such a component is the repulsive first order fixpoint $q=\frac{1}{6}(-9-(9+8\sqrt{3})^{\frac{1}{2}})$. By taking its predecessors, we get an infinite number of point components.

Chapter III. Asymptotic distribution of predecessors

14. *Definitions*

Let E be a bounded closed set in the z-plane and let μ be a positive mass distribution on E of finite total mass. The logarithmic potential to be considered is then defined by

$$u(z) = \int_E \log \frac{1}{|z-\zeta|}\, d\mu(\zeta).$$

We also consider the energy integral

$$I(\mu) = \iint_{EE} \log \frac{1}{|z-\zeta|}\, d\mu(\zeta)\, d\mu(z)$$

and set

$$V = \inf_{\mu,\, \mu(E)=1} I(\mu).$$

Then we define the capacity $\gamma(E)$ of E by

$$\gamma(E) = e^{-V}.$$

The carrier of a mass distribution μ is denoted by S_μ. In this chapter, we will only consider polynomials of the form

$$P(z) = z^k + a_{k-1}z^{k-1} + \ldots + a_0, \quad k \geq 2, \tag{14.1}$$

and their iterates $\{P_n(z)\}$. Thus the set F will correspond to a polynomial of the form (14.1).

15. *The capacity of the set F*

Lemma 15.1. $\gamma(F)=1$.

Proof. Let E_0 be a simply connected closed set such that $\complement E_0 \subset A(\infty)$ and $P(\complement E_0) \subset \complement E_0$, i.e. $F \subset E_0$. Furthermore, we may assume that ∂E_0 is a Jordan curve and that

$\partial E_0 \cap C_1 = \phi$. Now set $P_{-n}(E_0) = E_n$. Then $F \subset E_n \subset E_{n-1}$, $n = 1, 2, \ldots$, and $\partial E_n \to \partial A(\infty) = F$.

Let $g_n(z, \infty)$ be the Green's function for the complement of E_n singular at infinity. If $\gamma(E_n) = e^{-V_n}$, then at $z = \infty$,

$$g_n(z,\infty) = \log|z| + V_n + o(1). \qquad (15.1)$$

By making the substitution $z \to P(z)$ in (15.1), we get at $z = \infty$

$$\frac{1}{k} g_n(P(z), \infty) = \log|z| + \frac{V_n}{k} + o(1).$$

Since $P(E_{n+1}) = E_n$ and $P_{-1}(E_n) = E_{n+1}$ and since the Green's function is unique, we conclude that

$$g_{n+1}(z, \infty) = \frac{1}{k} g_n(P(z), \infty).$$

Thus $V_{n+1} = V_n/k$ and by repeating the procedure above, we get $V_{n+1} = V_0/k^n$. Now, by Tsuji [21] p. 57, 79

$$\gamma(F) = \lim_{n\to\infty} \gamma(E_n) = \lim_{n\to\infty} e^{-V_n} = 1$$

and the lemma is proved.

Denote the equilibrium distribution of F by μ^*, i.e. $\mu^*(F) = 1$ and $I(\mu^*) = V$. The following important lemma holds.

Lemma 15.2. $S_{\mu^*} = F$.

For the proof of this lemma we need two more lemmas.

Lemma 15.3. $\gamma(F - S_{\mu^*}) = 0$.

Proof of Lemma 15.3. Since $F = \partial A(\infty)$ this is the Theorem III: 31 in Tsuji [21] p. 79.

Lemma 15.4. *Let $f(z)$ be a mapping on the bounded closed set E satisfying the inequality $|f(z_1) - f(z_2)| \leq M|z_1 - z_2|$, where M is constant. If $\gamma(E) = 0$, then $\gamma(f(E)) = 0$.*

Proof of Lemma 15.4. We shall here use the transfinite diameter as an equivalent notion of capacity. See Tsuji [21], pp. 71–75. Given $\varepsilon > 0$, there exists an N such that for $n \geq N$ and any points $w_i \in f(E)$, $w_i = f(z_i)$, $i = 1, 2, 3, \ldots, n$,

$$\prod_{i<j}^{1,\ldots,n} |w_i - w_j| \leq M^{\binom{n}{2}} \cdot \prod_{i<j}^{1,\ldots,n} |z_i - z_j| \leq \varepsilon^{\binom{n}{2}}.$$

Hence the transfinite diameter of $f(E)$ equals 0 and the lemma is proved.

Proof of Lemma 15.2. Suppose on the contrary, that $F - S_{\mu^*} = F_1 \neq \phi$. Then by Lemma 15.3, $\gamma(F_1) = 0$. Choose a closed subset H of F_1. Since S_{μ^*} is closed and $S_{\mu^*} \cap H = \phi$, we have $d(H, S_{\mu^*}) = 2\delta > 0$. Let $z_0 \in H$ and set $C_\delta = \{z \,|\, |z - z_0| \leq \delta\}$ and

$C_\delta \cap F_1 = F_1^*$. Since $F_1^* \subset F$ and $C_\delta \cap S_{\mu^*} = \phi$, it follows that F_1^* is perfect. Moreover, since $F_1^* \subset F_1$, we have $\gamma(F_1^*) = 0$. Now, according to Theorem 4.3, there exists an integer N such that $F = P_N(F_1^*)$. Thus by Lemma 15.4, $\gamma(F) = 0$. This contradicts our Lemma 15.1 and the lemma is established.

The following lemma has no connexion with the iteration theory but will be of use later.

Lemma 15.5. *Let E and H be two closed sets such that $E \subset H$ and $\gamma(E) = e^{-V} > 0$. Furthermore, let $\{\mu_n\}$ be a sequence of distributions on H with unit mass such that $\mu_n \to \mu$, weakly, where μ distributes unit mass on E.*

If $u_n(z)$ denotes the logarithmic potential with respect to μ_n and μ^* denotes the equilibrium distribution of E, then suppose

1°. $\underline{\lim}_{n \to \infty} u_n(z) \geqslant V$ *if* $z \in E$.
2°. $\overline{S_{\mu^*}} = E$.

The assertion is that $\mu = \mu^*$.

Proof. By Fatou's lemma and assumption 1°

$$\underline{\lim}_{n \to \infty} \int_E u_n(z)\, d\mu^*(z) \geqslant \int_E \underline{\lim}_{n \to \infty} u_n(z)\, d\mu^*(z) \geqslant V. \tag{15.2}$$

Let $u^*(z)$ be the equilibrium potential corresponding to μ^*, so that $u^*(z) \leqslant V$ everywhere. Then by Fubini's theorem

$$\underline{\lim}_{n \to \infty} \int_E u_n(z)\, d\mu^*(z) = \underline{\lim}_{n \to \infty} \int_H u^*(\zeta)\, d\mu_n(\zeta) \leqslant V. \tag{15.3}$$

By (15.2) and (15.3)

$$u(z) \leqslant \underline{\lim}_{n \to \infty} u_n(z) = V,$$

except on a set where $\mu^* = 0$. Since $S_{\mu^*} = E$, the neighbourhoods of each point $z_0 \in E$ contain points where $u(z) \leqslant V$. Since

$$u(z_0) \leqslant \underline{\lim}_{z \to z_0} u(z) \leqslant V,$$

we have $u(z) \leqslant V$, every $z \in E$. The uniqueness of μ^* then implies that $\mu = \mu^*$ and the lemma is proved.

16. *Mass distributions produced by iteration of polynomials*

We now return to the polynomial $P(z) = z^k + a_{k-1} z^{k-1} + \ldots + a_0$ and introduce a sequence $\{\mu_n\}$ of mass distributions defined as follows:

μ_0 places the mass 1 at a fixed point z_0 in the plane except the exceptional points of Theorem 2.5.
μ_1 places the mass k^{-1} at the k predecessors of order 1 of z_0.
⋮
μ_n places the mass k^{-n} at the k^n predecessors of order n of z_0.

We shall need the following

Lemma 16.1. *Every weakly convergent subsequence extracted from* $\{\mu_n\}$ *tends to a distribution of unit mass on* F.

Proof. If z_0 is not an attractive fixpoint and does not belong to any singular domain, then by Theorem 6.1, the lemma holds. Hence suppose that z_0 is an attractive fixpoint. If then O is an arbitrary neighbourhood of F, there evidently exists an integer N such that $\complement O$ contains exactly p predecessors of order n of z_0, if $n \geqslant N$. Thus $\mu_n(\complement O) = p \cdot k^{-n}$, $n \geqslant N$ and $\mu_n(\complement O) \to 0$ and the lemma holds in this case too.

There remains the case where z_0 belongs to a singular domain G^*. By Theorem 5.2, however, there exists an iterate $P_h(z)$, which maps G^* one to one onto itself. Thus z_0 has only one predecessor $P^*_{-h}(z_0)$ belonging to G^*. Since the other predecessors of order h, $\{P^{(\nu)}_{-h}(z_0)\}$, do not belong to a singular domain, we can proceed in the same way as above and the lemma is proved.

We can now state the main theorem of this chapter.

Theorem 16.1. *If* $\{\mu_n\}$ *is the sequence of mass distributions defined above and* μ^* *denotes the equilibrium distribution of* F, *then* $\lim_{n\to\infty} \mu_n = \mu^*$, *weak convergence.*

Proof. We shall prove that the assumptions of Lemma 15.5 are satisfied.

1°. Let E be a closed simply connected set such that $\complement E \subset A(\infty)$. Thus, if z_0 is an attractive fixpoint, not exceptional, or belongs to a singular domain, all its predecessors are in E. Furthermore, if O is an ε-neighbourhood of $z=\infty$, then there exists an integer N such that every predecessor of order $n \geqslant N$ of any point $w \in \complement O$ belongs to E.

Extract a weakly convergent subsequence $\{\mu_{n_\nu}\}$ from $\{\mu_n\}$ and suppose that $\mu_{n_\nu} \to \mu$, where by Lemma 16.1 $\mu(F)=1$. Since F is bounded in the case of a polynomial, and since F is completely invariant under $P(z)$, it follows that

$$|P_n(z)| \leqslant M, \quad z \in F \quad \text{every} \quad n. \tag{16.1}$$

The predecessors of order n_ν of z_0 are the roots of the equation $P_{n_\nu}(z) - z_0 = 0$. Let these roots be $z_1, z_2, \dots . z_{k^{n_\nu}}$ and take $n_\nu \geqslant N$, i.e. so that $\{z_\nu\}_1^{k^{n_\nu}} \subset E$. Hence

$$|P_{n_\nu}(z) - z_0| = \prod_{\nu=1}^{k^{n_\nu}} |z - z_\nu|. \tag{16.2}$$

If $z \in F$, then by (16.1) and (16.2)

$$\sum_{\nu=1}^{k^{n_\nu}} \log |z - z_\nu| \leqslant M_1$$

and

$$\frac{1}{k^{n_\nu}} \sum_{\nu=1}^{k^{n_\nu}} \log \frac{1}{|z - z_\nu|} \geqslant -\frac{M_1}{k^{n_\nu}}. \tag{16.3}$$

However, (16.3) can be written as a potential, namely

$$u_{n_\nu}(z)=\int_E \log\frac{1}{|z-\zeta|}\,d\mu_{n_\nu}(\zeta)\geqslant -\frac{M_1}{k^{n_\nu}}$$

and thus
$$\varliminf_{\nu\to\infty} u_{n_\nu}(z)\geqslant 0,\quad z\in F.$$

Since by Lemma 15.1, $\gamma(F)=1$, i.e. $V=0$, the sequence $\{\mu_{n_\nu}\}$ satisfies the assumption 1° of Lemma 15.5.

2°. The assumption 2°, $S_{\mu^*}=F$, was proved in Lemma 15.2. Thus, by Lemma 15.5 $\mu_{n_\nu}\to\mu^*$, weak convergence.

But the same argument can be used for every convergent subsequence extracted from $\{\mu_n\}$ and consequently

$$\lim_{n\to\infty}\mu_n=\mu^*;\quad \text{weak convergence.}$$

Remark 16.1. Let $\{\mu_n(\cdot, w)\}$ be the mass distributions produced by the start point w. Then if we allow w to be a function of n, we get a sequence $\{\mu_n(\cdot, w_n)\}$. It follows from the proof of Theorem 16.1 that if w_n varies in a bounded domain, then $\mu_n(\cdot, w_n)\to\mu^*$, weakly.

17. *Ergodic and mixing properties of polynomials*

Since F is completely invariant under the corresponding polynomial $P(z)$, we can regard $P(z)$ as a transformation T of F onto itself. Adler and Rivlin [1] have considered the transformation T_n which corresponds to the Chebyshev polynomial of degree n for the interval $[-1, 1]$. They proved that T preserves the equilibrium distribution μ^* of $[-1, 1]$ and that the sequence $\{T_n\}$ is strongly mixing. We shall now prove a similar theorem for the more general set F. (For definitions see Halmos [7].)

Theorem 17.1. *T preserves the measure μ^*. Furthermore, T is strongly mixing.*

Proof. If $E\subset F$, then it follows from Theorem 16.1 that $\mu^*(T^{-1}E)=\mu^*(E)$, i.e. T preserves the measure μ^*.

To establish that T is strongly mixing, we have to prove.

$$\lim_{n\to\infty}\int_F f(T^n z)g(z)\,d\mu^*(z)=\int_F f(z)\,d\mu^*(z)\cdot\int_F g(z)\,d\mu^*(z),\tag{17.1}$$

where $f(z)$, $g(z)\in L^2(F, u^*)$. Let $\{\mu_n(\cdot, w_n)\}$ be the mass distributions defined in Remark 16.1. Cover F by a finite number of squares $\{Q_j\}_{j=1}^k$ with small diameters and such that $\mu^*(\partial Q_j)=0$, $j=1, 2, \dots, k$. Given $\varepsilon>0$, we assert that there exists an N such that for $n\geqslant N$

$$|\mu_n(Q_j, w_n)-\mu^*(Q_j)|<\varepsilon,\quad j=1,2,\dots,k\tag{17.2}$$

uniformly in $|w_n|<M$. For if this is not true, then for every n there exists a square

Q_j for which (17.2) does not hold. But since the number of squares is finite, this implies the existence of a square Q_s and a subsequence $\{\mu_{n_\nu}(Q_s, w_{n_\nu})\}$ such that $\mu_{n_\nu}(Q_s, w_{n_\nu}) \nrightarrow \mu^*(Q_s)$ which contradicts Remark 16.1.

If $\zeta \in F$ and has the predecessors $\{\zeta^{(\nu)}_{-n}\}$ of order n, then it follows that for any function $g(z)$, which is constant on each square,

$$\lim_{n\to\infty} \sum_{\nu=1}^{kn} g(\zeta^{(\nu)}_{-n})\, k^{-n} = \int_F g(z)\, d\mu^*(z)$$

uniformly in $\zeta \in F$. This yields for functions $f(z)$ and $g(z)$ which are constant on each square

$$\lim_{n\to\infty} \int_F f(T^n z)\, g(z)\, d\mu^*(z) = \lim_{n\to\infty}\ \lim_{m\to\infty} \sum f(\zeta^{(\nu)}_{-m})\, g(\zeta^{(\nu)}_{-(m+n)})\, k^{-m-n}$$

$$= \lim_{n\to\infty}\ \lim_{m\to\infty} \sum f(\zeta^{(\nu)}_{-m}) \cdot k^{-m} \cdot \sum_{(\zeta^{(\nu)}_{-m}\ \text{fixed})} g(\zeta^{(\nu)}_{-(m+n)})\, k^{-n} = \int_F f(z)\, d\mu^*(z) \cdot \int_F g(z)\, d\mu^*(z).$$

By a standard approximation argument (17.1) holds for $f(z)$, $g(z) \in L^2(F, \mu^*)$ and the theorem is proved.

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Abbreviations: L = Lemma, Th = Theorem, C = Corollary

L 2.1–Th 2.6 [3] pp. 189–199, [6] pp. 34–41. — L 3.1–L 3.2 [5] pp. 180–181. — Th 3.1 [3] pp. 199–200, [6] pp. 60–63. — Th 3.2 [5] pp. 191–221, [6] pp. 63–69, [9] pp. 222–243. — Th 4.1–Th 4.3 [3] pp. 197–203, [6] pp. 38–47. — L 5.1–L 6.2 [4] pp. 317–318, [6] pp. 52–60. — L 6.3–Th 6.5 [6] pp. 69–73. — Th 7.1–L 7.1 [5] pp. 183–185. — Th 7.2–C 7.1 [6] pp. 50–51, 74–79. — Th 8.1–Th 8.2 [5] pp. 260–267, [6] pp. 80–84, [9] pp. 188–198, 213–218. — Th 9.1–L 9.1 [6] pp. 208–240. — Th 10.1 [6] pp. 84–85. — Th 11.1–Th 11.2 [6] p. 85. — Th 12.1 [10], [11], [13], [16], [17], [19]. — Th 13.3 [19].

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