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A note on Chern coefficients and Cohen-Macaulay rings

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Abstract. In this paper, we investigate the relationship between the index of reducibility and Chern coefficients for primary ideals. As an application, we give characterizations of a Cohen-Macaulay ring in terms of its type, irreducible multiplicity, and Chern coefficients with respect to certain parameter ideals in Noetherian local rings.

1. Introduction

Throughout this paper, let $(R,\mathfrak{m})$ be a homomorphic image of a Cohen–Macaulay local ring with the infinite residue field k, $\dim R=d>0$, and M a finitely generated R-module of dimension s. A submodule N of M is called an *irreducible submodule* if N can not be written as an intersection of two properly larger submodules of M. The number of irreducible components of an irredundant irreducible decomposition of N, which is independent of the choice of the decomposition by E. Noether [19], is called the *index of reducibility* of N and denoted by $\mathrm{ir}_M(N)$. For an $\mathfrak{m}$-primary ideal I of M, *the index of reducibility* of I on M is the index of reducibility of IM, and denoted by $\mathrm{ir}_M(I)$. Moreover, we have $\mathrm{ir}_M(I)=\dim_k \mathrm{Soc}(M/IM)$, where we denote by $\mathrm{Soc}(N)$ the dimension of the socle of an R-module N as a k-vector space. In the case I is a parameter ideal of M, several properties of $\mathrm{ir}_M(I)$ had been found and played essential roles in the earlier stage of development of the theory of Gorenstein rings and/or Cohen-Macaulay rings. Recently, the index of

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reducibility of parameter ideals has been used to deduce a lot of information on the structure of some classes of modules, such as regular local rings by W. Gröbner [15]; Gorenstein rings by Northcott, Rees [19], [20], [26] and [28]; Cohen-Macaulay modules by D.G. Northcott, N.T. Cuong, P.H. Quy [7], [26]–[28]; Buchsbaum modules by S. Goto, N. Suzuki and H. Sakurai [12], [14]; generalized Cohen-Macaulay modules by N.T. Cuong, P.H. Quy and the second author [6], [8] (see also [25], [28] for other modules). The aim of our paper is to continue this research direction. Concretely, we will give characterizations of a Cohen-Macaulay ring in terms of its *type* and Chern coefficients with respect to g-parameter ideals (see Definition 2.4). Recall that *type* of a module M was first introduced by S. Goto and N. Suzuki [14], and is defined as the supremum

$$\mathrm{r}(M) = \sup_{\mathfrak{q}} \mathrm{ir}_M(\mathfrak{q}),$$

where $\mathfrak{q}$ runs through the parameter ideals of M. It well-known that there are integers $\mathrm{e}_i(I;M)$, called the *Hilbert coefficients* of M with respect to I such that for $n \gg 0$

$$\ell_R(M/I^{n+1}M) = \mathrm{e}_0(I;M)\binom{n+s}{s} - \mathrm{e}_1(I;M)\binom{n+s-1}{s-1} + \ldots + (-1)^s \mathrm{e}_s(I;M).$$

Here $\ell_R(N)$ denotes, for an R-module N, the length of N. In particular, the leading coefficient $\mathrm{e}_0(I)$ is said to be *the multiplicity* of M with respect to I and $\mathrm{e}_1(I)$, which Vasconcelos ([29]) refers to as the *Chern coefficient* of M with respect to I. Now our motivation stems from the work of [29]. Vasconcelos posed *the Vanishing Conjecture*: R is a Cohen–Macaulay local ring if and only if $\mathrm{e}_1(\mathfrak{q}, R) = 0$ for some parameter ideal $\mathfrak{q}$ of R. It is shown that the relation between Cohen-Macaulayness and the Chern number of parameter ideals is quite surprising. In [26], motivated by some deep results of [5], [13] and also by the fact that this is true for R is unmixed as shown in [9], it was asked whether the characterization of the Cohen-Macaulayness in terms of the Chern number of non-parameter ideals and the type of R in the case that R is mixed. Concretely, the goal of this note is to understand the nature of the following open question.

Question 1. Is R is Cohen-Macaulay if and only if there exists a parameter ideal $\mathfrak{q}$ of R such that

$$\mathrm{r}(R) \le \mathrm{e}_1(\mathfrak{q} : \mathfrak{m}) - \mathrm{e}_1(\mathfrak{q}).$$

Our main result partially answers the question in the following way.

Theorem 1.1. *Assume that $d = \dim R \ge 2$. Then the following statements are equivalent.*

(i) R is Cohen-Macaulay.
(ii) For all parameter ideals $\mathfrak{q}$, we have

$$\mathrm{r}(R)\leq \mathrm{e}_1(\mathfrak{q}:\mathfrak{m})-\mathrm{e}_1(\mathfrak{q}).$$

(iii) For some g-parameter ideal $\mathfrak{q}\subseteq\mathfrak{m}^2$, we have

$$\mathrm{r}(R)\leq \mathrm{e}_1(\mathfrak{q}:\mathfrak{m})-\mathrm{e}_1(\mathfrak{q}).$$

N.T. Cuong et al. [7] showed that there are integers $f_i(I;M)$, called the *irreducible coefficients* of M with respect to I such that for $n\gg 0$

$$\mathrm{ir}_M(I^{n+1})=\ell_R([I^{n+1}M:_M\mathfrak{m}]/I^{n+1}M)=\sum_{i=0}^{s-1}(-1)^i f_i(I;M)\binom{n+s-1-i}{s-1-i}.$$

The leading coefficient $f_0(I;R)$ is called the irreducible multiplicity of I. From the notations given above, the second main result is stated as follows.

Theorem 1.2. *Let M be a finitely generated R-module of dimension $s\geq 2$. Then the following statements are equivalent.*
(i) M is Cohen-Macaulay.
(ii) For all parameter ideals $\mathfrak{q}$ of M, we have

$$\mathrm{r}(M)\leq f_0(\mathfrak{q},M).$$

(iii) For some g-parameter ideal $\mathfrak{q}$ of M, we have

$$\mathrm{r}(M)\leq f_0(\mathfrak{q},M).$$

From this main result, we obtain the following results.

Corollary 1.1. *Assume that R is non-Cohen-Macaulay local ring with $d\geq 2$. Then we have*

$$e_1(\mathfrak{q}:\mathfrak{m})-e_1(\mathfrak{q})\leq f_0(\mathfrak{q};R)\leq \mathrm{r}(R).$$

for all g-parameter ideals $\mathfrak{q}\subseteq\mathfrak{m}^2$.

The remainder of this paper is organized as follows. In the next section, we prove some preliminary results on the irreducible multiplicity and index of reducibility for g-parameter ideals. In the last section, we prove the main results and their consequences.

2. Notations and preliminaries

Throughout this section, assume that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, $d = \dim R > 0$ with the infinite residue field $k = R/\mathfrak{m}$ and I is an $\mathfrak{m}$-primary ideal of R. Let M be a finitely generated R-module of dimension s. We denote $H^i_{\mathfrak{m}}(M)$ as the i-th local cohomology module of M with respect to $\mathfrak{m}$. Set $r_i(M) = \dim_{R/\mathfrak{m}}((0) :_{H^i_{\mathfrak{m}}(M)} \mathfrak{m})$. Recall that a submodule N of M is irreducible if N can not be written as the intersection of two properly larger submodules of M. Every submodule N of M can be expressed as an irredundant intersection of irreducible submodules of M. The number of irreducible submodules appearing in such an expression depends only on N, but not on the expression. That number is called the index of reducibility of N and is denoted by $\mathrm{ir}_M(N)$. In particular, if $N = IM$, then we have

$$\mathrm{ir}_M(I) := \mathrm{ir}_M(IM) = \ell_R([IM :_M \mathfrak{m}]/IM).$$

It is well known that by Lemma 4.2 in [7], there exists a polynomial $p_{I,M}(n)$ of degree $s-1$ with rational coefficients such that

$$p_{I,M}(n) = \mathrm{ir}_M(I^{n+1}) = \ell_R([I^{n+1}M :_M \mathfrak{m}]/I^{n+1}M)$$

for all large enough n. Then, there are integers $f_i(I; M)$ such that

$$p_{I,M}(n) = \sum_{i=0}^{s-1} (-1)^i f_i(I; M) \binom{n+s-1-i}{s-1-i}.$$

These integers $f_i(I; M)$ are called the irreducible coefficients of M with respect to I. In particular, the leading coefficient $f_0(I; M)$ is called the irreducible multiplicity of M with respect to I. The readers may refer to [26], [27] and [7] for more characterizations of the Cohen-Macaulayness of M in terms of the coefficient $f_0(\mathfrak{q}, M)$. In [7], the authors studied the function $\mathrm{ir}_M(\mathfrak{q}^{n+1})$ when M is generalized Cohen-Macaulay and $\mathfrak{q}$ is a standard parameter ideal of M. Recall that an R-module M is said to be a *generalized Cohen-Macaulay module* if $H^i_{\mathfrak{m}}(M)$ are of finite length for all $i = 0, 1, ..., s-1$ (see [4]). This condition is equivalent to say that there exists a parameter ideal $\mathfrak{q} = (x_1, ..., x_s)$ of M such that $\mathfrak{q} H^i_{\mathfrak{m}}(M/\mathfrak{q}_j M) = 0$ for all $0 \le i + j < s$, where $\mathfrak{q}_j = (x_1, ..., x_j)$ (see [24]), and such a parameter ideal is called a *standard parameter ideal* of M. It is well-known that if M is a generalized Cohen-Macaulay module, every parameter ideal of M in a large enough power of the maximal ideal $\mathfrak{m}$ is standard (see [23], [24]).

For proving the main result in next section, we need the following auxiliary lemma, which is shown in [26, Lemma 2.1].

Lemma 2.1. *Assume that N is a submodule of M such that* $\dim N < \dim M$. *Then we have*

$$f_0(I;M) \leqslant f_0(I;M/N).$$

The following lemma shows the existence of a special superficial element which is useful in many inductive proofs in the sequel.

Lemma 2.2. *Let M be a finitely generated R-module of dimension $s>1$ and I an $\mathfrak{m}$-primary ideal of R. Assume that x is a superficial element of M with respect to I such that $H^0_{\mathfrak{m}}(M)=(0):_M x$. Then we have*

$$f_0(I;M) \leqslant f_0(I;M/xM).$$

Proof. Let $W=H^0_{\mathfrak{m}}(M)$. Since x is a superficial element of M with respect to I, we have $I^{n+1}M:_M x = I^nM + (0):_M x = I^nM+W$ for large enough n. Therefore it follows from that the sequences

$$0 \longrightarrow M/(I^nM+W) \longrightarrow M/I^{n+1}M \longrightarrow M/(x, I^{n+1})M \longrightarrow 0$$

are exact for large enough n, that we get the following exact sequence

$$\begin{aligned} 0 &\longrightarrow ((I^nM+W):_M \mathfrak{m})/(I^nM+W) \longrightarrow (I^{n+1}M:_M \mathfrak{m})/I^{n+1}M \\ &\longrightarrow ((x, I^{n+1})M:_M \mathfrak{m})/(x, I^{n+1})M. \end{aligned}$$

Claim 2.3. $(I^nM+W):_M \mathfrak{m} = (I^nM:_M \mathfrak{m})+W$ *for large enough n.*

Proof. Since W has finite length, there exists an integer n_0 such that $\mathfrak{m}^{n_0}M \cap W=(0)$. There exists a superficial element x_1 of $\overline{M}=M/W$ with respect to I. Therefore, there exists an integer n_1 such that

$$(I^nM+W):_M \mathfrak{m} \subseteq (I^nM+W):_M x_1 = I^{n-1}M+W,$$

for all $n>n_1$. Let n be an integer such that $n>\max\{n_0, n_1\}+1$. Let $a\in(I^nM+W):_M \mathfrak{m}$, we have $a=b+c$ with $b\in I^{n-1}M$ and $c\in W$. Hence $\mathfrak{m}b \in I^{n-1}M \cap (I^nM+W)=I^nM$. So $b\in I^nM:_M \mathfrak{m}$. Then

$$(I^nM+W):_M \mathfrak{m} = (I^nM:_M \mathfrak{m})+W$$

for all $n>\max\{n_0, n_1\}+1$. □

We have

$$\begin{aligned} &\ell_R(((x, I^{n+1})M:_M \mathfrak{m})/(x, I^{n+1})M) \\ &\geq \ell_R((I^{n+1}M:_M \mathfrak{m})/I^{n+1}M) - \ell_R(((I^nM+W):_M \mathfrak{m})/(I^nM+W)) \\ &= \ell_R((I^{n+1}M:_M \mathfrak{m})/I^{n+1}M) - \ell_R((I^nM:_M \mathfrak{m}+W)/(I^nM+W)) \\ &= \ell_R((I^{n+1}M:_M \mathfrak{m})/I^{n+1}M) - \ell_R((I^nM:_M \mathfrak{m})/I^nM) + \ell_R((I^nM:_M \mathfrak{m})\cap W). \end{aligned}$$

Since W has finite length and $s>1$, we have

$$f_0(I;M)\leqslant f_0(I;M/xM),$$

as required. □

In [28] the second author proved important properties of g-parameter ideals which we will need to prove our main result. In the last part of this section we recall part of these results. We refer to [28] and to [26] for details. Put $\operatorname{Assh}_R M=\{\mathfrak{p}\in\operatorname{Supp}_R M\,|\dim R/\mathfrak{p}=s\}$, where $\operatorname{Supp}_R M$ is the support of M. Compared with the set of associated primes, we have $\operatorname{Assh}_R M\subseteq\operatorname{Ass}_R M$. Let $\Lambda(M)=\{\dim_R N\,|\, N$ is an R-submodule of $M, N\neq(0)\}$. Then we have

$$\Lambda(M)=\{\dim R/\mathfrak{p}\,|\,\mathfrak{p}\in\operatorname{Ass}_R M\}.$$

We put $\mathfrak{t}=\sharp\Lambda(M)$, and

$$\Lambda(M)=\{0\leq d_1<d_2<\ldots<d_{\mathfrak{t}}=s\}.$$

Because R is Noetherian, M contains the largest submodule D_i with $\dim_R D_i=d_i$, for all $1\leq i\leq\mathfrak{t}$. Then the filtration

$$\mathcal{D}: D_0=(0)\subsetneq D_1\subsetneq D_2\subsetneq\ldots\subsetneq D_{\mathfrak{t}}=M$$

of submodules of M is called *the dimension filtration* of M. The notion of dimension filtration was first given by P. Schenzel [22]. Our notion of dimension filtration is different from that of [3], [22], however throughout this paper let us unite the above definition. Notice that, if $(0)=\bigcap_{\mathfrak{p}\in\operatorname{Ass}_R M} N(\mathfrak{p})$ is a reduced primary decomposition of the submodule (0) of M, then $D_i=\bigcap_{\mathfrak{p}\in\operatorname{Ass}_R M,\ \dim R/\mathfrak{p}\geq d_{i+1}} N(\mathfrak{p})$ for all $1\leq i\leq\mathfrak{t}-1$, and so $D_{\mathfrak{t}-1}$ is also called the *unmixed component* of M. For all $1\leq i\leq\mathfrak{t}$, we put $C_i=D_i/D_{i-1}$. Then $\operatorname{Ass}_R C_i=\{\mathfrak{p}\in\operatorname{Ass}_R M\,|\dim R/\mathfrak{p}=d_i\}$, and $\operatorname{Ass}_R M/D_i=\{\mathfrak{p}\in\operatorname{Ass}_R M\,|\dim R/\mathfrak{p}\geq d_{i+1}\}$.

A system $x_1,x_2,\ldots,x_s$ of parameters of M is called *distinguished*, if

$$(x_j\,|\,d_i<j\leq s)D_i=(0),$$

for all $1\leq i\leq\mathfrak{t}$. A parameter ideal $\mathfrak{q}$ of M is called *distinguished*, if there exists a distinguished system $x_1,x_2,\ldots,x_s$ of parameters of M such that $\mathfrak{q}=(x_1,x_2,\ldots,x_s)$. Notice that, if M is a Cohen-Macaulay R-module, every parameter ideal of M is distinguished. It is well-known that distinguished systems of parameters always exist (see [22]). If $x_1,x_2,\ldots,x_s$ is a distinguished system of parameters of M, so are $x_1^{n_1},x_2^{n_2},\ldots,x_s^{n_s}$ for all integers $n_j\geq1$.

We denote by $\mathfrak{q}_i$ the ideal $(x_1,...,x_i)$ for $i=1,...,d$ and stipulate that $\mathfrak{q}_0$ is the zero ideal of R. Then the sequence $x_1,x_2,...,x_m\in\mathfrak{m}$ is called a d-sequence on M if

$$\mathfrak{q}_iM:_M x_{i+1}x_j=\mathfrak{q}_iM:_M x_j,$$

for all $0\leq i<j\leq m$. The concept of a d-sequence is given by Huneke [16] and it plays an important role in the theory of Blow up algebras, i.e. Rees algebras. We now present the main object of this paper.

Definition 2.4. (cf. [28, Definition 2.3]) A distinguished system $x_1,x_2,...,x_s$ of parameters of M is called a *g-system of parameters* on M, if it is a d-sequence, and we have

$$\operatorname{Ass}(C_i/\mathfrak{q}_jC_i)\subseteq\operatorname{Assh}(C_i/\mathfrak{q}_jC_i)\cup\{\mathfrak{m}\},$$

for all $0\leq j\leq s-1$ and $0\leq i\leq\mathfrak{t}$. A parameter ideal $\mathfrak{q}$ of M is called *g-parameter ideal*, if there exists a g-system $x_1,x_2,...,x_s$ of parameters of M such that $\mathfrak{q}=(x_1,x_2,...,x_s)$.

The readers can refer some facts about g-systems of parameters in [28]. Notice that g-system of parameters always exist. Moreover if M is a generalized Cohen-Macaulay module, then by the definition of g-parameter ideal, every g-parameter ideal of M is standard. Besides, we have following property.

Lemma 2.5. *Let $x_1,x_2,...,x_s$ form a g-system of parameters on M with $s\geq2$. Let $0\leq j\leq s-2$ and $N/\mathfrak{q}_jM$ denote the unmixed component of $M/\mathfrak{q}_jM$. Assume that M/N is Cohen-Macaulay. Then $C_\mathfrak{t}$ is Cohen-Macaulay.*

Proof. We may assume that $s\geq3$ and $j=1$. For a submodule L of M, we denote $\overline{L}=(L+xM)/xM$ with $x:=x_1$. By the definition of g-system of parameters, we have $\operatorname{Ass}(C_\mathfrak{t}/xC_\mathfrak{t})\subseteq\operatorname{Assh}(C_\mathfrak{t}/xC_\mathfrak{t})\cup\{\mathfrak{m}\}$. Therefore $\overline{N}/\overline{D}_{\mathfrak{t}-1}$ has finite length. Since M/N is a Cohen-Macaulay module, $\mathrm{H}^i_\mathfrak{m}(M/D_{\mathfrak{t}-1}+xM)=0$ for all $0<i<s-1$. Therefore, we derive from the exact sequence

$$0\longrightarrow C_\mathfrak{t}\xrightarrow{.x}C_\mathfrak{t}\longrightarrow M/D_{\mathfrak{t}-1}+xM\longrightarrow0$$

the following exact sequence:

$$0\longrightarrow\mathrm{H}^0_\mathfrak{m}(M/D_{\mathfrak{t}-1}+xM)\longrightarrow\mathrm{H}^1_\mathfrak{m}(C_\mathfrak{t})\xrightarrow{.x}\mathrm{H}^1_\mathfrak{m}(C_\mathfrak{t})\longrightarrow0.$$

Thus $\mathrm{H}^1_\mathfrak{m}(C_\mathfrak{t})=0$, so $\overline{N}/\overline{D}_{\mathfrak{t}-1}=\mathrm{H}^0_\mathfrak{m}(M/D_{\mathfrak{t}-1}+xM)=0$. Hence $N=D_{\mathfrak{t}-1}+xM$. Moreover, since x is $C_\mathfrak{t}$-regular and $C_\mathfrak{t}/xC_\mathfrak{t}\cong\overline{M}/\overline{D}_{\mathfrak{t}-1}=M/N$ is a Cohen-Macaulay module, $C_\mathfrak{t}$ is Cohen-Macaulay. $\square$

3. Proof of main theorems and corollaries

In this section we prove the main results of this paper. Recall that

$$\mathrm{r}(M)=\sup\{\mathrm{ir}_M(\mathfrak{q})\mid \mathfrak{q} \text{ is a parameter ideal for } M\},$$

and it is called *the type* of M. In particular, if $M=R$, we simply denote by $\mathrm{r}(R)$ the type of the local ring R as a module over itself. The notion of the type of a module M was first introduced by Goto and Suzuki ([14]). With notation as above we have the following lemma.

Lemma 3.1. *Assume that M/N is Cohen-Macaulay with* $\dim N<\dim M=s$. *Then there exists a parameter ideal Q such that*

$$\mathrm{ir}_N(Q)+r_s(M)\le \mathrm{r}(M).$$

Proof. By Lemma 3.6 in [12] there exists a parameter ideal Q of M such that the canonical map

$$\phi_M : M/QM \longrightarrow \mathrm{H}^s_{\mathfrak{m}}(M)$$

is surjective on the socles. Put $L=M/N$. Then we look at the exact sequence

$$0\longrightarrow N \xrightarrow{\ \iota\ } M \xrightarrow{\ \varepsilon\ } L \longrightarrow 0$$

of R-modules, where ι (resp. ε) denotes the canonical embedding (resp. the canonical epimorphism). Then, since $\dim N<\dim M$ and since L is Cohen-Macaulay, we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N/QN & \xrightarrow{\ \overline{\iota}\ } & M/QM & \xrightarrow{\ \overline{\varepsilon}\ } & L/QL & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \phi_M} & & \downarrow{\scriptstyle \phi_L} & & \\ & & & & \mathrm{H}^s_{\mathfrak{m}}(M) & \xrightarrow{\ =\ } & \mathrm{H}^s_{\mathfrak{m}}(L) & & \end{array}$$

with exact first row on socles. In fact, let $x\in(0):_{L/QL}\mathfrak{m}$. Then, since ϕ_M is surjective on the socles, we get an element $y\in(0):_{M/\mathfrak{q}M}\mathfrak{m}$ such that $\phi_L(x)=\phi_M(y)$. Thus $\overline{\varepsilon}(y)=x$, because the canonical map ϕ_L is injective. Therefore, we have

$$\mathrm{ir}_M(Q)=\mathrm{ir}_N(Q)+\mathrm{ir}_{M/N}(Q)=\mathrm{ir}_N(Q)+r_s(M).$$

Thus by the definition of type, we have

$$\mathrm{ir}_N(Q)+r_s(M)\le \mathrm{r}(M). \quad \square$$

Recall that M is *sequentially Cohen-Macaulay*, if $C_i=D_i/D_{i-1}$ is Cohen-Macaulay for all $1\le i\le \mathfrak{t}$. Notice that every module of dimension 1 is sequentially Cohen-Macaulay.

Corollary 3.2. *Assume that M is sequentially Cohen-Macaulay of dimension $s\geq 2$. Then we have*

$$\sum_{i\in\mathbb{Z}} r_i(M)\leq \mathrm{r}(M).$$

Proof. It follows from Lemma 3.1 and the definition of sequentially Cohen-Macaulay modules. □

Lemma 3.3. *Let M be a finitely generated R-module of dimension $s\geq 2$ and $\mathfrak{q}$ a g-parameter ideal of M such that*

$$\mathrm{r}(M)\leq f_0(\mathfrak{q},M).$$

Then $C_{\mathfrak{t}}$ is Cohen-Macaulay.

Proof. Assume that $\mathfrak{q}$ is generated by the g-system $x_1,\ldots,x_s$ of parameters of M. Put $A=M/\mathfrak{q}_{s-2}M$, and let N denote the unmixed component of A. Korollar 2.2.4 in [21] say that if R is a homomorphic image of a Cohen-Macaulay local ring with the infinite residue field k and M is a finitely generated R-module then $\dim R/\mathfrak{a}_i(M)\leq i$, where $\mathfrak{a}_i(M)=\mathrm{Ann}\, H^i_{\mathfrak{m}}(M)$ for all i. Moreover, since $\dim A=2$ and $\dim R/\mathfrak{a}_1(A)\leq 1$, we can choose z such that z is a parameter element of A and $z\in\mathfrak{a}_1(A)$. Therefore $zH^1_{\mathfrak{m}}(A)=0$. Now, we derive from that N is the unmixed component of A and the exact sequence

$$0\longrightarrow N\longrightarrow A\longrightarrow A/N\longrightarrow 0$$

that the map $H^1_{\mathfrak{m}}(N)\to H^1_{\mathfrak{m}}(A)$ is injective. Therefore $zH^1_{\mathfrak{m}}(N)=0$, so $z\in\mathrm{Ann}\, N$. Thus, x_{s-1},z is a distinguished system of parameters for A. Since $\mathrm{Ann}\, D_{\mathfrak{t}-1}+\mathfrak{q}_{s-1}$ is an $\mathfrak{m}$-primary ideal of R, we can choose $y:=z^m\in\mathrm{Ann}(D_{\mathfrak{t}-1})$ for large enough m such that $yH^1_{\mathfrak{m}}(A)=0$ and $(0):_A y=(0):_A y^2$.

Since x_{s-1},y is a distinguished system of parameters for A, by Lemma 2.3 in [25], we have $N=(0):_A y^n$ for all n. Hence we have that the following commutative diagram with exact rows for all $n\geq 2$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A/N & \xrightarrow{\cdot y} & A & \longrightarrow & A/yA & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cdot y^n} & & \downarrow & & \\
0 & \longrightarrow & A/N & \xrightarrow{\cdot y^{n+1}} & A & \longrightarrow & A/y^{n+1}A & \longrightarrow & 0.
\end{array}$$

We apply the functor $\mathrm{H}^1_{\mathfrak{m}}(\bullet)$ to the above diagram to get the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^1_{\mathfrak{m}}(A/N) & \xrightarrow{\alpha_1} & \mathrm{H}^1_{\mathfrak{m}}(A)\\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cdot y^n}\\
\mathrm{H}^1_{\mathfrak{m}}(A/N) & \xrightarrow{\alpha_{n+1}} & \mathrm{H}^1_{\mathfrak{m}}(A),
\end{array}$$

where α_1, α_{n+1} are canonical homomorphisms. Thus, $\alpha_{n+1}=y^n\circ\alpha_1=0$ for all $n\geq 3$, because of the choice of y. Therefore we get the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0_{\mathfrak{m}}(A) & \longrightarrow & \mathrm{H}^0_{\mathfrak{m}}(A/y^nA) & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(A/N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle y} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{H}^0_{\mathfrak{m}}(A) & \longrightarrow & \mathrm{H}^0_{\mathfrak{m}}(A/y^{n+1}A) & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(A/N) & \longrightarrow & 0. \end{array}$$

By applying the functor $\mathrm{Hom}(k;\bullet)$ to this diagram, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(k,\mathrm{H}^1_{\mathfrak{m}}(A/N)) & \xrightarrow{\beta_n} & \mathrm{Ext}^1(k,\mathrm{H}^0_{\mathfrak{m}}(A)) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cdot y} \\ \mathrm{Hom}(k,\mathrm{H}^1_{\mathfrak{m}}(A/N)) & \xrightarrow{\beta_{n+1}} & \mathrm{Ext}^1(k,\mathrm{H}^0_{\mathfrak{m}}(A)), \end{array}$$

where β_n, β_{n+1} are connecting homomorphisms. Thus, $\beta_{n+1}=y\circ\beta_n=0$, since $y\in\mathfrak{m}$. Therefore the sequence

$$0\longrightarrow(0):_{\mathrm{H}^0_{\mathfrak{m}}(A)}\mathfrak{m}\longrightarrow(0):_{\mathrm{H}^0_{\mathfrak{m}}(A/y^nA)}\mathfrak{m}\longrightarrow(0):_{\mathrm{H}^1_{\mathfrak{m}}(A/N)}\mathfrak{m}\longrightarrow 0$$

is exact, so $r_0(A/y^nA)=r_0(A)+r_1(A/N)$ for all $n\geq 3$.

On the other hand, since $\alpha_{n+1}=0$ for all $n\geq 1$, we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(A) & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(A/y^nA) & \longrightarrow & (0):_{\mathrm{H}^2_{\mathfrak{m}}(A/N)}y^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle y} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(A) & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(A/y^{n+1}A) & \longrightarrow & (0):_{\mathrm{H}^2_{\mathfrak{m}}(A/N)}y^{n+1} & \longrightarrow & 0, \end{array}$$

whose rows are exact sequences. By applying the functor $\mathrm{Hom}(k;\bullet)$ to this diagram, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(k,(0):_{\mathrm{H}^2_{\mathfrak{m}}(A/N)}y^n) & \xrightarrow{\gamma_n} & \mathrm{Ext}^1(k,\mathrm{H}^1_{\mathfrak{m}}(A)) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cdot y} \\ \mathrm{Hom}(k,(0):_{\mathrm{H}^2_{\mathfrak{m}}(A/N)}y^n) & \xrightarrow{\gamma_{n+1}} & \mathrm{Ext}^1(k,\mathrm{H}^1_{\mathfrak{m}}(A)), \end{array}$$

where γ_n, γ_{n+1} are connecting homomorphisms. Thus, $\gamma_{n+1}=y\circ\gamma_n=0$, since $y\in\mathfrak{m}$. Therefore the sequence

$$0\longrightarrow(0):_{\mathrm{H}^1_{\mathfrak{m}}(A)}\mathfrak{m}\longrightarrow(0):_{\mathrm{H}^1_{\mathfrak{m}}(A/y^nA)}\mathfrak{m}\longrightarrow(0):_{\mathrm{H}^2_{\mathfrak{m}}(A/N)}\mathfrak{m}\longrightarrow 0$$

is exact. Since $\dim N<2$, so we have $r_1(A/y^nA)=r_1(A)+r_2(A/N)=r_1(A)+r_2(A)$ for all $n\geq 3$. Since $\dim A/y^nA=1$, by Lemma 3.1, we have

$$r_0(A/y^nA)+r_1(A/y^nA)\leq \mathrm{r}(A/y^nA).$$

By the definition of type of local rings and the hypothesis, we have

$$\begin{aligned} r_0(A)+r_1(A/N)+r_1(A)+r_2(A) &= r_0(A/y^nA)+r_1(A/y^nA) \\ &\leq \mathrm{r}(A/y^nA)\leq \mathrm{r}(M)\leq f_0(\mathfrak{q},M). \end{aligned}$$

On the other hand, by Lemma 2.2 and Lemma 2.1, we have

$$f_0(\mathfrak{q},M)\leq f_0(\mathfrak{q},A)\leq f_0(\mathfrak{q},A/N).$$

Since $\dim A/N=2$ and N is the unmixed component of A, A is generalized Cohen-Macaulay. Thus we can choose a superficial element y_0 of A/N with respect to $\mathfrak{q}$ such that $y_0\,\mathrm{H}^1_{\mathfrak{m}}(A/N)=0$. It follows from the exact sequence

$$0\longrightarrow A/N \xrightarrow{.y_0} A/N \longrightarrow A/(y_0A+N)\longrightarrow 0$$

and $y_0H^1_{\mathfrak{m}}(A/N)=0$ that the sequence

$$0\longrightarrow \mathrm{H}^1_{\mathfrak{m}}(A/N)\longrightarrow \mathrm{H}^1_{\mathfrak{m}}(A/(y_0A+N))\longrightarrow \mathrm{H}^2_{\mathfrak{m}}(A/N)\longrightarrow 0$$

is exact. By applying the functor $\mathrm{Hom}(k;\bullet)$, we have

$$r_1(A/(y_0A+N))\leq r_1(A/N)+r_2(A/N).$$

Since y_0 is A/N-regular, by Lemma 2.2, we have

$$f_0(\mathfrak{q};A/N)\leq f_0(\mathfrak{q};A/(y_0A+N)).$$

Because $\dim(A/(y_0A+N))=1$, we have $f_0(\mathfrak{q};A/(y_0A+N))\leq r_1(A/(y_0A+N))$ and then

$$f_0(\mathfrak{q};A/N)\leq r_1(A/N)+r_2(A/N).$$

Thus, $r_1(A)=0$, so $\mathrm{H}^1_{\mathfrak{m}}(A)=0$. It follows from the exact sequence

$$0\longrightarrow N\longrightarrow A\longrightarrow A/N\longrightarrow 0$$

that the sequence

$$0\longrightarrow \mathrm{H}^1_{\mathfrak{m}}(N)\longrightarrow \mathrm{H}^1_{\mathfrak{m}}(A)\longrightarrow \mathrm{H}^1_{\mathfrak{m}}(A/N)\longrightarrow 0$$

is exact and $\mathrm{H}^1_{\mathfrak{m}}(A/N)=0$. Hence, C_{t} is Cohen-Macaulay, because of Lemma 2.5. □

We are now ready to prove the main theorems of this section.

Proof of Theorem 1.2. $(i)\Rightarrow(ii)$ follows from Theorem 5.2 in [7].
$(ii)\Rightarrow(iii)$ is trivial.
$(ii)\Rightarrow(i)$ By Lemma 3.3, we have $C_{\mathfrak{t}}=M/D_{\mathfrak{t}-1}$ is Cohen-Macaulay. By Lemma 3.1, there exists a parameter ideal Q such that

$$\mathrm{ir}_{D_{\mathfrak{t}-1}}(Q)+r_s(M)\leq \mathrm{r}(M).$$

It follows from Lemma 2.1 and that $M/D_{\mathfrak{t}-1}$ is Cohen-Macaulay, that we have

$$\mathrm{r}(M)\leq f_0(\mathfrak{q};M)\leq f_0(\mathfrak{q};M/D_{\mathfrak{t}-1})=r_s(M).$$

Therefore, we have $\mathrm{ir}_{D_{\mathfrak{t}-1}}(Q)=0$, so $D_{\mathfrak{t}-1}=0$. Hence, M is Cohen-Macaulay, as required. ☐

Proof of Theorem 1.1. $(i)\Rightarrow(ii)$ follows from Theorem 1.1 in [27].
$(ii)\Rightarrow(iii)$ is trivial.
$(iii)\Rightarrow(i)$ Let $I=\mathfrak{q}:\mathfrak{m}$. First, assume that $e_0(\mathfrak{m};R)>1$. Then by Proposition 2.3 in [12], we get that $\mathfrak{m}I^n=\mathfrak{m}\mathfrak{q}^n$ for all n. Thus, $I^n\subseteq\mathfrak{q}^n:\mathfrak{m}$ for all n. It follows that

$$\ell_R(R/\mathfrak{q}^{n+1})-\ell_R(R/I^{n+1})=\ell_R(I^{n+1}/\mathfrak{q}^{n+1})\leq\ell_R((\mathfrak{q}^{n+1}:\mathfrak{m})/\mathfrak{q}^{n+1}).$$

Therefore, $e_1(I;R)-e_1(\mathfrak{q};R)\leq f_0(\mathfrak{q};R)$, so $\mathrm{r}(M)\leq f_0(\mathfrak{q};R)$. By Theorem 1.2, R is Cohen-Macaulay, as required.

Now assume that $e_0(\mathfrak{m};R)=1$. Let $\mathfrak{u}$ denote the unmixed component of R, and put $S=R/\mathfrak{u}$. Since $\dim\mathfrak{u}<\dim R$ we have $e_0(\mathfrak{m};S)=1$. By Theorem 40.6 in [18], S is Cohen-Macaulay, so S is a regular local ring. By Lemma 3.1, there exists a parameter ideal Q of R such that

$$\mathrm{ir}_{\mathfrak{u}}(Q)+r_d(R)\leq\mathrm{r}(R).$$

It follows from $\mathrm{r}(R)\leq e_1(\mathfrak{q}:\mathfrak{m})-e_1(\mathfrak{q})$ and the following claim that $\mathrm{ir}_{\mathfrak{u}}(Q)=0$, so $\mathfrak{u}=0$. Hence R is Cohen-Macaulay, as required.

Claim 3.4. $e_1(\mathfrak{q}:\mathfrak{m})-e_1(\mathfrak{q})\leq r_d(R)$.

Proof. By Theorem 1.1 in [27], we have

$$e_1(\mathfrak{q}S:\mathfrak{m}S;S)-e_1(\mathfrak{q}S;S)=r_d(S)=r_d(R).$$

Since $(\mathfrak{q}:\mathfrak{m})S\subset\mathfrak{q}S:\mathfrak{m}S$, we have $\ell_R(S/(\mathfrak{q}:\mathfrak{m})^{n+1}S)\geq\ell_R(S/(\mathfrak{q}S:\mathfrak{m}S)^{n+1})$, for all $n\geq 0$. Since S is a regular local ring and $\mathfrak{q}\subseteq\mathfrak{m}^2$, by Theorem 2.1 in [2], we have $\mathfrak{q}S:\mathfrak{m}S$

is integral over $\mathfrak{q}S$ and so $e_0(\mathfrak{q}S{:}\mathfrak{m}S;S)=e_0((\mathfrak{q}{:}\mathfrak{m})S;S)=e_0(\mathfrak{q}S;S)$. Therefore, we have

$$e_1((\mathfrak{q}:\mathfrak{m})S;S)\le e_1(\mathfrak{q}S:\mathfrak{m}S;S).$$

Hence

$$e_1((\mathfrak{q}:\mathfrak{m})S;S)-e_1(\mathfrak{q}S;S)\le r_d(R).$$

On the other hand, by Lemma 3.4 in [5], we have

$$e_1(\mathfrak{q})=\begin{cases} e_1(\mathfrak{q},S) & \text{if } \dim\mathfrak{u}\le d-2\\ e_1(\mathfrak{q},S)-e_0(\mathfrak{q},\mathfrak{u}) & \text{if } \dim\mathfrak{u}=d-1,\end{cases}$$
$$e_1(\mathfrak{q}{:}\mathfrak{m})=\begin{cases} e_1(\mathfrak{q}{:}\mathfrak{m},S) & \text{if } \dim\mathfrak{u}\le d-2\\ e_1(\mathfrak{q}{:}\mathfrak{m},S)-e_0(\mathfrak{q}{:}\mathfrak{m},\mathfrak{u}) & \text{if } \dim\mathfrak{u}=d-1.\end{cases}$$

If $\dim\mathfrak{u}<d-1$, then we have $e_1(\mathfrak{q}{:}\mathfrak{m})-e_1(\mathfrak{q})=e_1(\mathfrak{q}{:}\mathfrak{m},S)-e_1(\mathfrak{q},S)\le r_d(R)$, as required.

Now, we can assume that $\dim\mathfrak{u}=d-1$. Then we have

$$e_1(\mathfrak{q}:\mathfrak{m})-e_1(\mathfrak{q})\le r_d(R)-e_0(\mathfrak{q}:\mathfrak{m},\mathfrak{u})+e_0(\mathfrak{q},\mathfrak{u}).$$

By the Prime Avoidance Theorem, we can choose a parameter element $x\in\mathfrak{q}$ of R such that $(x)\cap\mathfrak{u}=0$. Put $\overline{R}=R/(x)$, and $\overline{S}=S/(x)$. Then we have the following exact sequence

$$0\longrightarrow\mathfrak{u}\longrightarrow\overline{R}\longrightarrow\overline{S}\longrightarrow 0.$$

Consequently,

$$e_0(\mathfrak{m},\overline{R})=e_0(\mathfrak{m},\overline{S})+e_0(\mathfrak{m},\mathfrak{u})\ge 2,$$

so that by Proposition 2.3 in [12], $\mathfrak{q}\overline{R}{:}\mathfrak{m}=(\mathfrak{q}{:}\mathfrak{m})\overline{R}$ is integral over $\mathfrak{q}\overline{R}$. Thus, $e_0(\mathfrak{q}{:}\mathfrak{m};\overline{R})=e_0(\mathfrak{q},\overline{R})$. Since x is regular in S, $e_0(\mathfrak{q}{:}\mathfrak{m};\overline{S})=e_0(\mathfrak{q},\overline{S})$. Hence, we have

$$\begin{aligned}&e_1(\mathfrak{q}:\mathfrak{m})-e_1(\mathfrak{q})\\ &\le r_d(R)-e_0(\mathfrak{q}:\mathfrak{m},\mathfrak{u})+e_0(\mathfrak{q},\mathfrak{u})\\ &=r_d(R)-(e_0(\mathfrak{q}:\mathfrak{m},\overline{R})-e_0(\mathfrak{q}:\mathfrak{m},\overline{S}))+(e_0(\mathfrak{q};\overline{R})-e_0(\mathfrak{q},\overline{S}))=r_d(R). \quad\square\end{aligned}$$

Let us note the following example of non-Cohen-Macaulay local rings R where we have $f_0(\mathfrak{q};R)=e_1(\mathfrak{q}{:}\mathfrak{m})-e_1(\mathfrak{q})$ for all parameter ideals $\mathfrak{q}$.

Example 3.5. Let $d\geqslant 3$ be an integer and let $U=k[[X_1,X_2,...,X_d,Y]]$ be the formal power series ring over a field k. We look at the local ring $R=U/[(X_1,X_2,...,X_d)\cap(Y)]$. Then R is a reduced ring with $\dim R=d$. We put $A=U/(Y)$ and $D=U/(X_1,X_2,...,X_d)$. Let $\mathfrak{q}$ be a parameter ideal in R. Then, since D is a DVR and

A is a regular local ring with $\dim A{=}d$, thanks to the exact sequence $0{\to}D{\to}R{\to}A{\to}0$, we get that $\operatorname{depth} R{=}1$ and the sequence

$$0 \longrightarrow D/\mathfrak{q}^{n+1}D \longrightarrow R/\mathfrak{q}^{n+1}R \longrightarrow A/\mathfrak{q}^{n+1}A \longrightarrow 0$$

is exact. By applying the functor $\operatorname{Hom}_R(R/\mathfrak{m},\bullet)$, we obtain the following exact sequence

$$0 \longrightarrow [\mathfrak{q}^{n+1}{:_D}\mathfrak{m}]/\mathfrak{q}^{n+1} \longrightarrow [\mathfrak{q}^{n+1}{:_R}\mathfrak{m}]/\mathfrak{q}^{n+1} \longrightarrow [\mathfrak{q}^{n+1}{:_A}\mathfrak{m}]/\mathfrak{q}^{n+1} \longrightarrow 0\ .$$

Therefore, we have

$$\begin{aligned}\ell_R([\mathfrak{q}^{n+1}:_R\mathfrak{m}]/\mathfrak{q}^{n+1}) &= \ell_R([\mathfrak{q}^{n+1}:_A\mathfrak{m}]/\mathfrak{q}^{n+1}) + \ell_R([\mathfrak{q}^{n+1}:_D\mathfrak{m}]/\mathfrak{q}^{n+1}) \\ &= \binom{d-1+n-1}{d-1} + 1\end{aligned}$$

for all integers $n\geqslant 0$, whence $f_0(\mathfrak{q};R){=}1{=}r_d(R)$ for every parameter ideal $\mathfrak{q}$ in $\mathfrak{m}$. Since A is regular local ring and $d{\geq}3$, we have

$$e_1(\mathfrak{q}:\mathfrak{m}) - e_1(\mathfrak{q}) = r_d(R).$$

However R is not a Cohen-Macaulay ring, since $H^1_\mathfrak{m}(R){=}H^1_\mathfrak{m}(D)$ is not a finitely generated R-module, where $\mathfrak{m}$ denotes the maximal ideal in R, but R is sequentially Cohen-Macaulay.

Now let us note the following example of non-Cohen-Macaulay local rings R of dimension 1 where we have $f_0(\mathfrak{q};R){=}\mathrm{r}(R)$ for all parameter ideals $\mathfrak{q}$.

Example 3.6. Let $R{=}k[[X,Y,Z]]/((X^a)(X,Y,Z){+}(Z^b))$ with a, $b{\geq}2$, $A{=}k[[X,Y,Z]]$, $\mathfrak{n}$ and $\mathfrak{m}$ be the maximal ideal of A and R, respectively. Then we have

$$\dim R{=}1,\ \mathrm{H}^0_\mathfrak{m}(R){=}(X^a,Z^b)/((X^a)\mathfrak{n}{+}(Z^b)){\cong}A/\mathfrak{n} \text{ and } R/\,\mathrm{H}^0_\mathfrak{m}(R){=}A/(X^a,Z^b).$$

Therefore R is not a Cohen-Macaulay ring. Now we claim that for any parameter ideal $\mathfrak{q}$ of R, we have

$$\mathrm{ir}_R(\mathfrak{q}) = 2.$$

Moreover $\mathrm{r}(R){\leq}f_0(\mathfrak{q};R)$ for any parameter ideal $\mathfrak{q}$ of R. Indeed, let $F{=}cX{+}dY{+}eZ$ be an element of A having $\mathfrak{q}$ is generated by its image in R. Then F, X^a, Z^b form an A-regular, so we have an exact sequence

$$0 \longrightarrow A/\mathfrak{n} \longrightarrow R/\mathfrak{q}R \longrightarrow A/(X^a,Z^b,F) \longrightarrow 0.$$

Consider $\Delta{=}dX^{a-1}Z^{b-1}$, then Δ is a generator for both of the socles of $R/\mathfrak{q}R$ and $A/(X^a,Z^b,F)$. Therefore $\mathrm{ir}_R(\mathfrak{q}){=}2$, as required.

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