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Families of Gorenstein and almost Gorenstein rings

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Abstract. Starting with a commutative ring R and an ideal I, it is possible to define a family of rings $R(I)_{a,b}$, with $a,b\in R$, as quotients of the Rees algebra $\oplus_{n\geq 0} I^n t^n$; among the rings appearing in this family we find Nagata's idealization and amalgamated duplication. Many properties of these rings depend only on R and I and not on a, b; in this paper we show that the Gorenstein and the almost Gorenstein properties are independent of a, b. More precisely, we characterize when the rings in the family are Gorenstein, complete intersection, or almost Gorenstein and we find a formula for the type.

Introduction

Let R be a commutative ring with unity and let $I\neq 0$ be a proper ideal of R. In [1] the authors introduce and study the family of quotient rings

$$R(I)_{a,b}=\mathcal{R}_+/\big(I^2\big(t^2+at+b\big)\big),$$

where $\mathcal{R}_+$ is the Rees algebra associated with the ring R with respect to I, i.e. $\mathcal{R}_+=\bigoplus_{n\geq 0}(I^n t^n)$, and $(I^2(t^2+at+b))$ is the contraction to $\mathcal{R}_+$ of the ideal generated by t^2+at+b in $R[t]$.

This family provides a unified approach to Nagata's idealization (with respect to an ideal, see [11, p. 2]) and to amalgamated duplication (see [4] and [5]); they can be both obtained as particular members of it, in particular they are $R\ltimes I\cong R(I)_{0,0}$ and $R\bowtie I\cong R(I)_{0,-1}$ respectively. This fact explains why these constructions produce rings with many common properties; as a matter of fact, it is shown, in [1], that many properties of the rings in this family (like, e.g., Krull dimension, Noetherianity and local Cohen–Macaulayness) do not depend on the defining polynomial. One interesting fact about this family is that, if R is a domain, we can always find domains among its members, while the idealization is never reduced and the amalgamated duplication is never a domain.

In this paper we deepen the study of the rings in the family initiated in [1]. In particular, we characterize when these rings are Gorenstein, complete intersection, and almost Gorenstein and prove that these properties do not depend on the particular member chosen in the family, but only on R and I. The concepts of Gorenstein ring and complete intersection ring are so prominent that they do not need a presentation; as for the concept of almost Gorenstein ring, we recall that it was introduced for one-dimensional analytically unramified rings by Barucci and Fröberg in [2]; recently this definition has been generalized for local Cohen–Macaulay one-dimensional rings possessing a canonical ideal (see [8]) and successively for rings of any Krull dimension (see [9]). This class of rings has been widely studied in the last years also because of its connection with almost symmetric numerical semigroups.

The structure of the paper is the following. In the first section we recall some properties about the family $R(I)_{a,b}$ and complete the characterization of its Cohen–Macaulayness. Then we prove that $R(I)_{a,b}$ is Gorenstein if and only if I is a canonical ideal of R (see Corollary 1.4); moreover, we determine the type of $R(I)_{a,b}$ showing that it is independent of a, b (see Theorem 1.3) and, finally, we give a characterization of the complete intersection property for $R(I)_{a,b}$ (see Proposition 1.8). In Section 2, we consider the almost Gorenstein property of $R(I)_{a,b}$. As for the one-dimensional case, we give an explicit description of the canonical ideal of $R(I)_{a,b}$ (cf. Proposition 2.1) and we use it to find some characterizations that generalize the particular cases studied in [6] and [8] (cf. Theorem 2.3 and Corollary 2.4); moreover, in this case, we find a simpler formula for the type of $R(I)_{a,b}$ that depends only on I and the canonical module of R (cf. Proposition 2.6); furthermore this formula implies that, in this case, the type of $R(I)_{a,b}$ is odd and included between 1 and $2t(R)+1$, where $t(R)$ is the type of R (see Corollary 2.7). Finally, we prove that, also in the higher dimensional case, the almost Gorenstein property does not depend on a and b (see Proposition 2.12); in particular, the results about idealization proved in [9] can be generalized to all the members of the family.

1. Gorenstein property for $R(I)_{a,b}$

We start this section by recalling some basic facts on the rings $R(I)_{a,b}$ proved in [1].

Proposition 1.1. *Let $f(t)=t^2+at+b\in R[t]$ be a monic polynomial. Then*

$$f(t)R[t]\cap\mathcal{R}_+=\left\{f(t)g(t)\mid g(t)\in I^2\mathcal{R}_+\right\}.$$

If we denote this ideal by $(I^2f(t))$ and the quotient ring $\mathcal{R}_+/(I^2f(t))$ by $R(I)_{a,b}$ we have:

(1) *$R(I)_{a,b} \cong R \oplus I$ as R-module (we will denote each element of $R(I)_{a,b}$ simply by $r+it$, where $r \in R$ and $i \in I$);*

(2) *the ring extensions $R \subseteq R(I)_{a,b} \subseteq R[t]/(f(t))$ are both integral and, therefore, the three rings have the same Krull dimension;*

(3) *let Q be the total ring of fractions of $R(I)_{a,b}$; then each element of Q is of the form $\frac{r+it}{u}$, where u is a regular element of R;*

(4) *assume that I is a regular ideal, i.e. I contains a regular element; the rings $R(I)_{a,b}$ and $R[t]/(f(t))$ have the same total ring of fractions and the same integral closure;*

(5) *R is a Noetherian ring if and only if $R(I)_{a,b}$ is a Noetherian ring for some $a, b \in R$ if and only if $R(I)_{a,b}$ is a Noetherian ring for all $a, b \in R$;*

(6) *$(R, \mathfrak{m})$ is local if and only if $R(I)_{a,b}$ is local. In this case the maximal ideal of $R(I)_{a,b}$ is $M = \{m+it \mid m \in \mathfrak{m}, i \in I\}$ and it is isomorphic to $\mathfrak{m} \oplus I$ as R-module.*

Throughout the rest of this paper we will assume that R is Noetherian, that $I \neq 0$ is a proper ideal of R and we fix all the notation above.

In order to study the Gorenstein property for $R(I)_{a,b}$, we have to look first at Cohen–Macaulayness (briefly CM). A weaker formulation of the following result is Proposition 2.7 of [1]. For the convenience of the reader we include here the complete proof.

Proposition 1.2. *Assume that R is a local ring. The following conditions are equivalent:*

(1) *R is a CM ring and I is a maximal CM R-module;*

(2) *$R(I)_{a,b}$ is a CM R-module;*

(3) *$R(I)_{a,b}$ is a CM ring;*

(4) *R is a CM ring and each regular R-sequence of R is also an $R(I)_{a,b}$-regular sequence.*

Proof. We set $\dim R = \dim R(I)_{a,b} = d$ (cf. Proposition 1.1 (2)) and observe that also $\dim_R R(I)_{a,b} = d$ as R-module, because $\operatorname{Ann} R(I)_{a,b} = 0$.

(1) $\Leftrightarrow$ (2): Since $R(I)_{a,b}$ is isomorphic to $R \oplus I$ as R-module, we have that $\operatorname{depth} R(I)_{a,b} = \min\{\operatorname{depth} R, \operatorname{depth} I\}$ and so $\operatorname{depth} R(I)_{a,b} = d$ (i.e. $R(I)_{a,b}$ is a maximal CM R-module) if and only if $\operatorname{depth} R = d$ (i.e. R is a CM ring) and $\operatorname{depth} I = d$. This last equality holds if and only if $\dim_R I = \operatorname{depth} I = d$, i.e. I is a maximal CM R-module.

(2) $\Leftrightarrow$ (3): We know that R and $R(I)_{a,b}$ have the same Krull dimension. Moreover, since the extension $R \subset R(I)_{a,b}$ is finite, the depth of $R(I)_{a,b}$ as R-module coincides with the depth of $R(I)_{a,b}$ as $R(I)_{a,b}$-module (see [3, Exercise 1.2.26]).

(3) ⇒ (4): We have already proved that (3) is equivalent to (1) and thus also R is a CM ring. Since in a CM ring $\mathbf{x}$ is a regular sequence if and only if it is part of a system of parameters (cf. [3, Theorem 2.1.2(d)]), it is enough to recall that if $\mathbf{x}$ is part of a system of parameters of R, then it is also part of a system of parameters of $R(I)_{a,b}$, because the extension $R \subseteq R(I)_{a,b}$ is integral.

(4) ⇒ (2): We know that there exists an R-regular sequence of R of length d. It is also an $R(I)_{a,b}$-regular sequence and so $R(I)_{a,b}$ is a CM R-module. □

We recall (following [7, Sects. 21.1 and 21.3]) that a canonical module of a zero-dimensional local ring is defined as the injective hull of its residue class field; if R is a local CM ring of dimension $d>0$, then a finitely generated R-module ω_R is a canonical module of R if there exists a non-zerodivisor $x \in R$ such that $\omega_R/x\omega_R$ is a canonical module of $R/(x)$. If R has a canonical module ω_R and this is isomorphic to an ideal of R, we say that ω_R is a canonical ideal of R. It is well known that a CM local ring R has a canonical module if and only if it is the homomorphic image of a Gorenstein local ring (cf. e.g. [3, Proposition 3.3.6]) and the canonical module is isomorphic to an ideal I if and only if R is generically Gorenstein, i.e. $R_{\mathfrak{p}}$ is Gorenstein for each minimal prime $\mathfrak{p}$ of R (see [7, Exercise 21.18]).

The authors proved in [1, Corollary 3.3] that, if R is a one-dimensional Noetherian local ring and I is a regular ideal of R, then $R(I)_{a,b}$ is Gorenstein if and only if I is a canonical ideal of R.

The main goal of this section is to generalize this result to any dimension $d \geq 0$. More generally in the next theorem we compute the type of $R(I)_{a,b}$ generalizing [1, Theorem 3.2].

Theorem 1.3. *Let $(R,\mathfrak{m})$ be a local CM ring of Krull dimension $d \geq 1$ and let I be a regular ideal and a maximal CM R-module. Then the CM type of $R(I)_{a,b}$ is*

$$t(R(I)_{a,b}) = \lambda_R\left(\frac{(J:\mathfrak{m})\cap(JI:I)}{J}\right) + \lambda_R\left(\frac{(JI:\mathfrak{m})}{JI}\right),$$

where $\lambda_R(.)$ denotes the length of an R-module and $J=(x_1,x_2,...,x_d)$ is an ideal of R generated by an R-regular sequence.

In particular, the type of $R(I)_{a,b}$ is independent of a, b.

Proof. Let M be the maximal ideal of $R(I)_{a,b}$. It is well known that

$$\begin{aligned} t(R(I)_{a,b}) &= \lambda_{R(I)_{a,b}}\left(\mathrm{Ext}^d_{R(I)_{a,b}}(R(I)_{a,b}/M, R(I)_{a,b})\right) \\ &= \lambda_{R(I)_{a,b}}\left(\mathrm{Hom}_{R(I)_{a,b}}(R(I)_{a,b}/M, R(I)_{a,b}/H)\right) = \lambda_{R(I)_{a,b}}\left(\frac{H:M}{H}\right) \end{aligned}$$

for any ideal H generated by an $R(I)_{a,b}$-regular sequence (see [12, Theorem 3.1(ii)]). By Proposition 1.2 we can choose H generated by an R-regular sequence $\mathbf{x}=x_1,...,x_d$. This means that $H=JR(I)_{a,b}=\{j_1+ij_2t|\ j_1,j_2\in J, i\in I, t^2=-at-b\}$, where J is the ideal of R generated by $\mathbf{x}$. Moreover, since $(H:M)/H$ is annihilated by $\mathfrak{m}$, its length as $R(I)_{a,b}$-module coincides with its length as R-module (see [1, Remark 2.2]). Hence

$$t(R(I)_{a,b})=\lambda_R\left(\frac{(JR(I)_{a,b}:M)}{JR(I)_{a,b}}\right).$$

We want to show that

$$(JR(I)_{a,b}:M)=\left\{\frac{r}{s}+\frac{i}{s}\ t\ ;\ \frac{i}{s}\in(JI:\mathfrak{m}), \frac{r}{s}\in(JI:I)\cap(J:\mathfrak{m})\right\}.$$

Since I is a regular ideal, a generic element of $Q(R(I)_{a,b})$ is of the form $r/s+(i/s)t$, where $r,s\in R$, $i\in I$ and s is regular (cf. Proposition 1.1(3)). Therefore it is an element of $(JR(I)_{a,b}:M)$ if and only if

$$\begin{aligned}\big(r/s+(i/s)t\big)(m+jt)&=rm/s+(im/s)t+(rj/s)t+(ij/s)t^2\\&=rm/s-ijb/s+(im/s+rj/s-ija/s)t\end{aligned}$$

is an element of $JR(I)_{a,b}$, for any $m\in\mathfrak{m}$ and for any $j\in I$; that is $(rm/s-ijb/s)\in J$ and $(im/s+rj/s-ija/s)\in JI$.

Suppose that $r/s+(i/s)t\in(JR(I)_{a,b}:M)$; in particular, if $j=0$ we have $rm/s\in J$ and $im/s\in JI$, that is $r/s\in(J:\mathfrak{m})$ and $i/s\in(JI:\mathfrak{m})$. Moreover, since $ja\in I\subseteq\mathfrak{m}$ and $i/s\in(JI:\mathfrak{m})$, we have $im/s, ija/s\in JI$, hence $rj/s\in JI$ for any $j\in I$ and then $r/s\in(JI:I)$.

Conversely, suppose that $i/s\in(JI:\mathfrak{m})$ and $r/s\in(JI:I)\cap(J:\mathfrak{m})$. Then $rm/s-ijb/s\in J+JI=J$ and $im/s+rj/s-ija/s\in JI+JI+JI=JI$, consequently $r/s+(i/s)t\in(JR(I)_{a,b}:M)$.

Now it is straightforward to see that the homomorphism of R-modules

$$(JR(I)_{a,b}:M)\ \longrightarrow\ \left(\frac{(J:\mathfrak{m})\cap(JI:I)}{J}\right)\times\left(\frac{(JI:\mathfrak{m})}{JI}\right)$$

defined by $r/s+(i/s)t\mapsto(r/s+J, i/s+JI)$ is surjective and its kernel is $JR(I)_{a,b}$. The thesis follows immediately. □

Corollary 1.4. *Let R be a local ring of dimension $d\geq 1$ and let I be a regular ideal of R. Then, for every $a,b\in R$, the ring $R(I)_{a,b}$ is Gorenstein if and only if R is a CM ring and I is a canonical ideal of R.*

Proof. Under our hypotheses, it is well known that the idealization (see [13]) and the duplication (see [4] and [15]) produce a Gorenstein ring if and only if R is CM and I is a canonical ideal. Since a CM ring is Gorenstein if and only if its CM type is one, the thesis follows immediately by Theorem 1.3. □

We notice that, if I is a canonical ideal of R, we can apply a result of Eisenbud (stated and proved in [4]) to prove one direction of the above result, i.e. that $R(I)_{a,b}$ is Gorenstein for every $a, b \in R$.

Corollary 1.5. *Let R be a regular local ring. The ring $R(I)_{a,b}$ is CM if and only if it is Gorenstein.*

Proof. An ideal I of a regular local ring is a maximal CM module if and only if it is a principal ideal by the Auslander–Buchsbaum formula (cf. [3, Theorem 1.3.3]) if and only if it is a canonical ideal, because a regular local ring is Gorenstein. Therefore it is enough to apply Proposition 1.2 and Corollary 1.4. □

Notice that, if R is zero-dimensional, its canonical module is isomorphic to an ideal if and only if R is Gorenstein and, in this case, we have $\omega_R \cong R$; in any case, ω_R is never isomorphic to a proper ideal of R and therefore the next theorem is not surprising.

Theorem 1.6. *Let $(R, \mathfrak{m})$ be a local Artinian ring. Then $R(I)_{a,b}$ is not Gorenstein.*

Proof. Since $R(I)_{a,b}$ is an Artinian ring, it is Gorenstein if and only if its socle, $\operatorname{soc} R(I)_{a,b} = (0 :_{R(I)_{a,b}} M)$, is a k-vector space of dimension one, where $k = R(I)_{a,b}/M \cong R/\mathfrak{m}$. We have that $r + it \in \operatorname{soc} R(I)_{a,b}$ if and only if

$$\begin{cases} rm - ijb = 0, \\ rj + mi - aij = 0 \end{cases}$$

for any $m + jt \in M$ (i.e. for any $m \in \mathfrak{m}$ and any $j \in I$). In particular, if $j = 0$ we get $r \in \operatorname{soc} R$ and $i \in I \cap \operatorname{soc} R$; thus

$$\operatorname{soc} R(I)_{a,b} \subseteq \{r + it \mid r \in \operatorname{soc} R, i \in I \cap \operatorname{soc} R\}.$$

It is straightforward to check the opposite inclusion, so we have an equality. We claim that $I \cap \operatorname{soc} R \neq (0)$. Indeed, if $0 \neq x \in I$, we have $x\mathfrak{m}^n = (0)$ for some $n \in \mathbb{N}$, because $\mathfrak{m}$ is nilpotent by artinianity. We can assume that $x\mathfrak{m}^{n-1} \neq (0)$ and clearly $x\mathfrak{m}^{n-1} \subseteq I \cap \operatorname{soc} R$.

Consequently, if $0\neq i\in I\cap \operatorname{soc} R$, we have that i and it are elements of $\operatorname{soc} R(I)_{a,b}$ and they are linearly independent on k; hence $R(I)_{a,b}$ is not a Gorenstein ring. □

We end this section studying when $R(I)_{a,b}$ is a complete intersection (briefly, c.i.). We recall that, following [7, Section 18.5], a local ring R is a c.i. if its completion with respect to the $\mathfrak{m}$-adic topology $\widehat{R}$ can be written as a regular local ring modulo a regular sequence.

Remark 1.7. Assume that $(R,\mathfrak{m})$ is local; in this case we know that also $R(I)_{a,b}$ is local, with maximal ideal $M=\mathfrak{m}\oplus I$ (see Proposition 1.1(6)). Since the powers of M are, as R-modules, $M^n=\mathfrak{m}^n\oplus\mathfrak{m}^{n-1}I$ (see the proof of [1, Proposition 2.3]), it is straightforward that the M-adic topology on $R(I)_{a,b}$ coincides with the $\mathfrak{m}$-adic topology induced by the structure of $R(I)_{a,b}$ as R-module. Hence, as R-module, $\widehat{R(I)_{a,b}}\cong\widehat{R}\oplus\hat{I}$.

Since we are supposing R to be Noetherian, we can assume that $R\subset\widehat{R}$ and thus $a,b\in\widehat{R}$; now it is clear that $\widehat{R(I)_{a,b}}\cong\widehat{R}(\hat{I})_{a,b}$.

Proposition 1.8. *Let R be a local ring and let I be a regular ideal of R. The ring $R(I)_{a,b}$ is a c.i. if and only if R is a c.i. and I is a canonical ideal. In particular the property of being a c.i. is independent of the choice of a and b.*

Proof. Let I be minimally generated by $i_1,...,i_p$. By Cohen's structure theorem we have that $\widehat{R}\cong S/J$, where S is a complete regular local ring. It follows that the ring $\widehat{R(I)_{a,b}}$ can be presented as $S[[y_1,...,y_p]]/\ker\varphi$, where $\varphi: S[[y_1,...,y_p]]\to \widehat{R(I)_{a,b}}$ is defined by $\varphi(s)=s+J$ and $\varphi(y_h)=i_ht$, for every $h=i,...,p$. Notice that $S[[y_1,...,y_p]]$ is again a regular local ring.

Since $(i_ht)^2=-ai_h^2t-bi_h^2$, with $ai_h,bi_h^2\in\widehat{R}$, if we choose $\alpha_h,\beta_h\in S$ such that $\varphi(\alpha_h)=ai_h$ and $\varphi(\beta_h)=bi_h^2$, then $\ker\varphi$ contains the elements of the form $F_h:=y_h^2+\alpha_hy_h+\beta_h$. Hence $\ker\varphi\supseteq J+(F_1,...,F_p)$. For every index h, an element of the form F_h is necessary as a generator of $\ker\varphi$, since it contains a pure power of y_h of the lowest possible degree. Moreover, $\ker\varphi\cap S=J$, since the restriction of φ to S gives the presentation of $\widehat{R}$. It follows that $\mu(\ker\varphi)$ (i.e. the cardinality of a minimal set of generators of $\ker\varphi$) is bigger than or equal to $\mu(J)+p$.

Assume that $R(I)_{a,b}$ is a c.i.; this means that $\dim S+p-\dim\widehat{R(I)_{a,b}}=\mu(\ker\varphi)$. Hence we have the following chain of inequalities:

$$\dim S+p-\dim\widehat{R(I)_{a,b}}=\mu(\ker\varphi)\geq\mu(J)+p\geq\dim S-\dim\widehat{R}+p.$$

Since $\dim\widehat{R}=\dim\widehat{R(I)_{a,b}}$, all the above inequalities are equalities and, in particular, $\mu(J)=\dim S-\dim\widehat{R}$, i.e. R is a c.i.

Moreover, since $R(I)_{a,b}$ is a c.i., it is Gorenstein and I has to be a canonical ideal of R by Corollary 1.4.

Conversely, assume that R is a c.i. and that I is a canonical ideal of R. We have that $\mu(I)=1$, since it equals the type of $\widehat{R}$, which is Gorenstein. Using the above notation we have $\ker\varphi\supseteq J+(F_1)$. The reverse inclusion is also true, since, if $g(y_1)\in\ker\varphi$, its class modulo $J+(F_1)$ is of the form $g_0+g_1y_1$ (with $g_0,g_1\in S$) and it belongs to $\ker\varphi$ if and only if $g_0\in J$ and $\varphi(g_1)i_1t=0$; the last equality, since i_1 is a non-zerodivisor, implies that also $g_1\in J$. This proves that $\mu(\ker\varphi)=\mu(J)+1$; since $\mu(J)=\dim S-\dim\widehat{R}$, also $R(I)_{a,b}$ is a c.i. □

2. Almost Gorenstein property for $R(I)_{a,b}$

Let $(R,\mathfrak{m})$ be a local one-dimensional Cohen–Macaulay ring. We say that R is an *almost Gorenstein* ring if it has a canonical module ω_R which is isomorphic to a fractional ideal of R such that

$$R\subseteq\omega_R\subseteq(\mathfrak{m}:\mathfrak{m}).$$

This definition generalizes the first one given in [2] for one-dimensional analytically unramified rings and it is equivalent to the definition given in [8] if $R/\mathfrak{m}$ is infinite, since, in this case, we can assume that $R\subseteq\omega_R\subseteq\overline{R}$ (see [8, Theorem 3.11]). Thus for a local one-dimensional almost Gorenstein ring we have an exact sequence of R-modules

$$0\longrightarrow R\longrightarrow\omega_R\longrightarrow\omega_R/R\longrightarrow 0$$

with $\mathfrak{m}\omega_R\subseteq\mathfrak{m}$ or, equivalently, $\mathfrak{m}\omega_R\subseteq R$, i.e. $\mathfrak{m}(\omega_R/R)=0$.

Following [9], a CM local ring $(R,\mathfrak{m})$ of any Krull dimension d possessing a canonical module ω_R is defined to be *almost Gorenstein* if there exists an exact sequence of R-modules

$$0\longrightarrow R\longrightarrow\omega_R\longrightarrow C\longrightarrow 0$$

such that $\mu_R(C)=e^0_{\mathfrak{m}}(C)$, where $\mu_R(C)=\lambda_R(C/\mathfrak{m}C)$ is the number of generators of C and $e^0_{\mathfrak{m}}(C)$ is the multiplicity of C with respect to $\mathfrak{m}$. It turns out that $\dim_R C=d-1$. Thus, in dimension d, the condition $\mathfrak{m}C=0$, given in dimension one, becomes $\mathfrak{m}C=(f_1,\dots,f_{d-1})C$, for some $f_1,\dots,f_{d-1}\in\mathfrak{m}$, i.e. $\mu_R(C)=\lambda_R(C/\mathfrak{m}C)=\lambda_R(C/(f_1,\dots,f_{d-1})C)=e^0_{\mathfrak{m}}(C)$.

Moreover, if R is one-dimensional and satisfies this general definition, we may assume that the canonical module ω_R is a fractional ideal of R, i.e. that ω_R is a canonical ideal of R, because the total ring of fractions of R turns out to be a Gorenstein ring (cf. [9, Lemma 3.1, (1) and Remark 3.2]). If we also assume that $R/\mathfrak{m}$ is infinite, the definition of one-dimensional almost Gorenstein ring given in the beginning of this section, which we adopt, is equivalent to that given in [9], as proved in [9, Proposition 3.4].

We finally recall that, if $d=0$, a ring is almost Gorenstein if and only if it is Gorenstein.

The goal of this section is to study when $R(I)_{a,b}$ is almost Gorenstein and to prove that this property is independent also of the choice of $a, b \in R$. We first study the one-dimensional case, giving an explicit description of the canonical ideal of $R(I)_{a,b}$ and some constructive methods to get almost Gorenstein rings; then, we study the case of dimension $d>1$.

Throughout this section we assume that $R(I)_{a,b}$ is a CM local ring; we recall that it is equivalent to require that R is CM and local and I is a maximal CM module (cf. Proposition 1.2); we will also assume that $R/\mathfrak{m}$ is infinite.

2.1. The one-dimensional case

Let $(R, \mathfrak{m})$ be a one-dimensional Cohen–Macaulay local ring and I be an $\mathfrak{m}$-primary ideal of R. We further assume throughout this subsection that R has a canonical ideal ω_R that is a fractional ideal such that $R \subseteq \omega_R \subseteq \overline{R}$.

Let H be a fractional ideal of R; since by definition there exists a regular element $y \in R$ such that $yH = J \subset R$, we can consider a minimal reduction xR of J and, with a slight abuse of terminology, we call $xy^{-1}R$ a minimal reduction of H, where now $xy^{-1} \in Q(R)$, the total ring of fractions of R.

Let zR (with $z \in Q(R)$) be a minimal reduction of $(\omega_R : I)$ and let us fix this notation for the whole current subsection; note that in this case z has to be an invertible element of $Q(R)$: in fact, if $z = xy^{-1}$ as above, $y \in y(\omega_R : I)$, so this is a regular ideal and a minimal reduction of a regular ideal has to be generated by a non-zerodivisor.

The inclusion $R \subseteq R(I)_{a,b}$ is a local homomorphism and $R(I)_{a,b}$ is a finite R-module, hence the canonical module of $R(I)_{a,b}$ is $\mathrm{Hom}_R(R(I)_{a,b}, \omega_R)$ (by [3, Theorem 3.3.7(b)]), where the structure of $R(I)_{a,b}$-module is given by $((r+it)\varphi)(s+jt) = \varphi((r+it)(s+jt))$, for each $\varphi \in \mathrm{Hom}_R(R(I)_{a,b}, \omega_R)$.

Our first goal is to give an explicit description of a canonical ideal K of $R(I)_{a,b}$ such that $R(I)_{a,b} \subseteq K \subseteq \overline{R(I)_{a,b}}$.

Clearly, as R-modules,

$$\begin{aligned}\omega_{R(I)_{a,b}} &\cong \mathrm{Hom}_R\big(R(I)_{a,b}, \omega_R\big) \cong \mathrm{Hom}_R(R \oplus I, \omega_R) \\ &\cong \mathrm{Hom}_R(R, \omega_R) \oplus \mathrm{Hom}_R(I, \omega_R) \cong \omega_R \oplus (\omega_R : I) \cong \frac{1}{z}(\omega_R : I) \oplus \frac{1}{z}\omega_R.\end{aligned}$$

We want to see that $\frac{1}{z}(\omega_R\!:\!I)\oplus\frac{1}{z}\omega_R$ is also an $R(I)_{a,b}$-module isomorphic to $\omega_{R(I)_{a,b}}$. More precisely, we define

$$K=\left\{\frac{x}{z}+\frac{y}{z}t|\ x\in(\omega_R:I), y\in\omega_R\right\}$$

and, given $(r+it)\in R(I)_{a,b}$ and $(\frac{x}{z}+\frac{y}{z}t)\in K$, we set

$$(r+it)\left(\frac{x}{z}+\frac{y}{z}t\right)=\left(\frac{rx}{z}-\frac{biy}{z}+\left(\frac{ry}{z}+\frac{ix}{z}-\frac{aiy}{z}\right)t\right)\in K;$$

it is easy to see that, in this way, we define an $R(I)_{a,b}$-module.

Proposition 2.1. *The $R(I)_{a,b}$-module K, defined above, is a canonical ideal of $R(I)_{a,b}$ such that $R(I)_{a,b}\subseteq K\subseteq\overline{R(I)_{a,b}}$.*

Proof. Consider the map $\varphi\!:\!K\to\operatorname{Hom}_R(R(I)_{a,b},\omega_R)$ that associates with $(\frac{x}{z}+\frac{y}{z}t)$ the homomorphism $f_{(\frac{x}{z}+\frac{y}{z}t)}\!:\!(s+jt)\mapsto(xj+y(s-ja))$. It is enough to prove that this is an isomorphism of $R(I)_{a,b}$-modules. Clearly φ is well defined. Let $(r+it),(s+jt)\in R(I)_{a,b}$ and $(\frac{x}{z}+\frac{y}{z}t)\in K$, one has

$$\begin{aligned}
&\left((r+it)\varphi\left(\frac{x}{z}+\frac{y}{z}t\right)\right)(s+jt)\\
&=(r+it)f_{(\frac{x}{z}+\frac{y}{z}t)}(s+jt)\\
&=f_{(\frac{x}{z}+\frac{y}{z}t)}\big((r+it)(s+jt)\big)=f_{(\frac{x}{z}+\frac{y}{z}t)}\big(rs-bij+(rj+is-aij)t\big)\\
&=xrj+xis-aijx+yrs-bijy-arjy-aisy+a^2ijy\\
&=f_{(\frac{rx-biy}{z}+\frac{ix+ry-aiy}{z}t)}(s+jt)=\varphi\left((r+it)\left(\frac{x}{z}+\frac{y}{z}t\right)\right)(s+jt).
\end{aligned}$$

This proves that φ is an homomorphism of $R(I)_{a,b}$-modules. Moreover, if $f_{(\frac{x}{z}+\frac{y}{z}t)}(s+jt)=0$ for any $(s+jt)\in R(I)_{a,b}$, chosen $\lambda\in I$ regular, one has

$$\begin{cases}y=f_{(\frac{x}{z}+\frac{y}{z}t)}(1)=0\\ \lambda x=f_{(\frac{x}{z}+\frac{y}{z}t)}(\lambda a+\lambda t)=0\end{cases}$$

then $(x,y)=(0,0)$ and therefore φ is injective.

As for the surjectivity, consider $g\in\operatorname{Hom}_R(R(I)_{a,b},\omega_R)$. Let $\lambda\in I$ be a regular element and set

$$\begin{cases}x=\frac{g(\lambda t)}{\lambda}+g(a)\\ y=g(1)\end{cases}$$

Clearly $y\in\omega_R$ and we claim that $x\in(\omega_R:I)$; in fact, if $i\in I$,

$$ix=\frac{ig(\lambda t)}{\lambda}+ig(a)=\frac{\lambda g(it)}{\lambda}+g(ai)=g(ai+it)\in\omega_R.$$

Hence $\frac{x}{z}+\frac{y}{z}t\in K$. Finally, for any $s+jt\in R(I)_{a,b}$, one has

$$\begin{aligned} f_{(\frac{x}{z}+\frac{y}{z}t)}(s+jt) = xj+y(s-ja) &= \frac{g(\lambda t)}{\lambda}j+g(aj)+g(s)-g(aj) \\ &= \frac{\lambda g(jt)}{\lambda}+g(s)=g(s+jt) \end{aligned}$$

and consequently φ is surjective.

We recall that, by Corollary 1.8 of [1], the integral closure of $R(I)_{a,b}$ contains the ring $\overline{R}[t]/(t^2+at+b)=\{r_1+r_2t|\ r_1,r_2\in\overline{R}, t^2=-at-b\}$.

One has $R\subseteq\frac{1}{z}(\omega_R:I)$ and $I\subseteq\frac{1}{z}\omega_R$, because $z\in(\omega_R:I)$, thus $R(I)_{a,b}\subseteq K$. Moreover $\omega_R\subseteq(\omega_R:I)\subseteq z\overline{R}$, since z is a minimal reduction of $(\omega_R:I)$ (see e.g. [2, Proposition 16]). Hence

$$R(I)_{a,b}\subseteq K\subseteq\overline{R}[t]/\big(t^2+at+b\big)\subseteq\overline{R(I)_{a,b}}.$$

Finally K is a fractional ideal of $R(I)_{a,b}$. In fact we can choose two regular elements $i\in I$ and $r\in R$, such that $r\omega_R\subseteq R$; hence $riz\in R\subseteq R(I)_{a,b}$ is such that $riz\,K\subseteq R(I)_{a,b}$. □

The next lemma is proved, in a different way, in the proof of [8, Proposition 6.1]. If $\mathfrak{a}$ is a fractional ideal of R, we denote its dual, $(\omega_R:\mathfrak{a})$, by $\mathfrak{a}^\vee$.

Lemma 2.2. *Let $\mathfrak{a},\mathfrak{b}$, and $\mathfrak{c}$ be fractional ideals of R and let z be a reduction of $I^\vee$. The following statements hold.*

(1) $\mathfrak{a}\mathfrak{b}\subseteq\mathfrak{c}$ *if and only if* $\mathfrak{a}\mathfrak{c}^\vee\subseteq\mathfrak{b}^\vee$.

(2) $\mathfrak{m}I^\vee\subseteq zR$ *if and only if* $\mathfrak{m}\omega_R\subseteq zI$.

(3) $II^\vee=zI$ *if and only if* $zI^\vee=(I^\vee)^2$.

Proof. As for the first point we have that

$$\mathfrak{a}\mathfrak{b}\subseteq\mathfrak{c}\Longleftrightarrow\mathfrak{c}^\vee\subseteq(\mathfrak{a}\mathfrak{b})^\vee\Longleftrightarrow\mathfrak{c}^\vee\subseteq\big(\mathfrak{b}^\vee:\mathfrak{a}\big)\Longleftrightarrow\mathfrak{a}\mathfrak{c}^\vee\subseteq\mathfrak{b}^\vee.$$

The second point follows applying this to $\mathfrak{a}=\mathfrak{m}$, $\mathfrak{b}=I^\vee$, $\mathfrak{c}=zR$. In the same way we get the last point in the particular case $\mathfrak{a}=I^\vee$, $\mathfrak{b}=I$ and $\mathfrak{c}=zI$, because the other inclusions are trivial. □

We can see that Proposition 6.1 of [8], proved for the idealization $R\ltimes I\cong R(I)_{0,0}$, holds also for arbitrary a and b.

Theorem 2.3. *The ring $R(I)_{a,b}$ is almost Gorenstein if and only if $II^\vee = zI$ and $z\mathfrak{m} = \mathfrak{m}I^\vee$. In particular, almost Gorensteinness does not depend on a and b.*

Proof. By Proposition 2.1, we have the canonical ideal K of $R(I)_{a,b}$ defined above. Let M be the maximal ideal of $R(I)_{a,b}$. $R(I)_{a,b}$ is almost Gorenstein if and only if $MK \subseteq M$ or, equivalently, $MzK \subseteq zM$. Given $(m+it) \in M$ and $(x+yt) \in zK$ (i.e. $m \in \mathfrak{m}$, $i \in I$, $x \in I^\vee$, $y \in \omega_R$), the latter condition means that $(m+it)(x+yt) = mx - biy + (my + ix - aiy)t \in zM$, that is

$$\begin{cases} mx - biy \in z\mathfrak{m} \\ my + ix - aiy \in zI. \end{cases}$$

Suppose now that $R(I)_{a,b}$ is almost Gorenstein. If we choose $i=0$, the first equation becomes $\mathfrak{m}I^\vee \subseteq z\mathfrak{m}$, i.e. $\mathfrak{m}I^\vee = z\mathfrak{m}$. Moreover, if in the second equation we set $y=0$, we get $II^\vee \subseteq zI$, i.e. $II^\vee = zI$.

Conversely, if the conditions of the statement hold, in light of the previous lemma we have

$$mx - biy \in \mathfrak{m}I^\vee + I\omega_R \subseteq z\mathfrak{m} + \mathfrak{m}\omega_R \subseteq z\mathfrak{m} + zI \subseteq z\mathfrak{m} + z\mathfrak{m} = z\mathfrak{m},$$
$$my + ix - aiy \in \mathfrak{m}\omega_R + II^\vee + I\omega_R \subseteq zI + zI + \mathfrak{m}\omega_R \subseteq zI. \qquad \square$$

Theorem 2.3 may have the following equivalent formulation:

Corollary 2.4. *The ring $R(I)_{a,b}$ is almost Gorenstein if and only if $z^{-1}I^\vee$ is a ring and $R \subseteq z^{-1}I^\vee \subseteq (\mathfrak{m}:\mathfrak{m})$.*

Proof. By Lemma 2.2, the condition $II^\vee = zI$ is equivalent to $zI^\vee = (I^\vee)^2$ and we note that this happens if and only if $z^{-1}I^\vee$ is a ring. Indeed $zI^\vee = (I^\vee)^2$ if and only if for any $x, y \in I^\vee$ one has $xy \in zI^\vee$, that is equivalent to $z^{-1}xz^{-1}y \in z^{-1}I^\vee$ for any $z^{-1}x, z^{-1}y \in z^{-1}I^\vee$, i.e. $z^{-1}I^\vee$ is a ring.

Furthermore, the condition $z\mathfrak{m} = \mathfrak{m}I^\vee$ is equivalent to $(\mathfrak{m}:\mathfrak{m}) \supseteq z^{-1}I^\vee$, i.e. $(z\mathfrak{m} : \mathfrak{m}) \supseteq I^\vee$, because we always have $\mathfrak{m}I^\vee \supseteq z\mathfrak{m}$. Finally, since $zI \subseteq \omega_R$, we have $R = (\omega_R : \omega_R) \subseteq (\omega_R : zI) = z^{-1}I^\vee$. $\square$

Corollary 2.4 allows us to construct a large class of one-dimensional almost Gorenstein rings. In fact, let A be an overring of R, $A \subseteq (\mathfrak{m}:\mathfrak{m})$. Then $A^\vee = (\omega_R : A)$ is a fractional ideal of R. Let $r \in R$ be a regular element such that $rA^\vee \subseteq R$ and set $I := rA^\vee$. It is easy to check that I satisfies the conditions of Corollary 2.4, in fact a minimal reduction of $I^\vee = r^{-1}A$ is $z = r^{-1}$ and $z^{-1}I^\vee = rr^{-1}A = A$.

If $A=R$, then $A^\vee=(\omega_R:A)=\omega_R$ and any integral ideal $I=rA^\vee$ is a canonical ideal, giving $R(I)_{a,b}$ Gorenstein.

If $A=(\mathfrak{m}:\mathfrak{m})$, then $A^\vee=(\omega_R:(\mathfrak{m}:\mathfrak{m}))$ and any integral ideal of the form $r(\omega_R:(\mathfrak{m}:\mathfrak{m}))$ gives $R(I)_{a,b}$ almost Gorenstein (cf. [8, Corollary 6.2]).

In particular, if R is Gorenstein then there are not proper overrings between R and $(\mathfrak{m}:\mathfrak{m})$. It follows that $R(I)_{a,b}$ is almost Gorenstein if and only if either $I=rR^\vee=rR$ or $I=r(R:(\mathfrak{m}:\mathfrak{m}))=r\mathfrak{m}$ (cf. [8, Corollary 6.4]).

Example 2.5. Consider $R:=k[[X^4,X^5,X^{11}]]$, where k is a field. In this case $(\mathfrak{m}:\mathfrak{m})=k[[X^4,X^5,X^6,X^7]]$. If we choose the overring $A=k[[X^4,X^5,X^7]]$ of R, then $A^\vee$ is the fractional ideal (X,X^4) of R and taking for example $r=X^4$, we get $I=X^4A^\vee=(X^5,X^8)$. Thus, for any choice of $a,b\in R$, we obtain that $R(I)_{a,b}$ is an almost Gorenstein ring.

We point out that, if I and J are two isomorphic ideals of R, $R(I)_{a,b}$ and $R(J)_{a,b}$ are not necessarily isomorphic. For example, if we choose the ideal I above and $J=X^7A^\vee=(X^8,X^{11})$, with $a=0$, $b=-X^5$, we get $R(I)_{0,-X^5}=k[[T_1]]$ and $R(J)_{0,-X^5}=k[[T_2]]$, with $T_1=\langle 8,10,15,21,22\rangle$ and $T_2=\langle 8,10,21,22,27\rangle$ (cf. [1, Theorem 3.4]). However, if $J=xI$, $R(I)_{a,b}$ is almost Gorenstein if and only if $R(J)_{a,b}$ is almost Gorenstein: in fact, if z is a reduction of $I^\vee$, then $x^{-1}z$ is a reduction of $J^\vee$; moreover $(x^{-1}z)^{-1}J^\vee=xz^{-1}J^\vee=z^{-1}I^\vee$, so, by previous corollary, the conditions required to be almost Gorenstein coincide for both rings.

If R is a numerical semigroup ring or an algebroid branch, it is possible to get information about $R(I)_{a,b}$ by studying a numerical semigroup, called *numerical duplication* (see [1, Theorems 3.4 and 3.6]). In numerical semigroup theory, the corresponding concept of almost Gorenstein ring is the notion of almost symmetric semigroup and, in this context, Corollary 2.4 generalizes Theorem 4.3 of [6]. Moreover, in this case a simple formula is known for the type of the numerical duplication (see [6, Proposition 4.8]). The next proposition generalizes this result, giving a formula for $t(R(I)_{a,b})$, the CM type of $R(I)_{a,b}$.

Proposition 2.6. *Suppose that $R(I)_{a,b}$ is almost Gorenstein, then*

$$t(R(I)_{a,b})=2\lambda_R\left(\frac{z^{-1}I^\vee}{R}\right)+1=2\lambda_R\left(\frac{\omega_R}{zI}\right)+1.$$

Proof. First recall that the CM type of $R(I)_{a,b}$ is

$$t(R(I)_{a,b})=\lambda_R\left(\frac{(I:I)\cap(R:\mathfrak{m})}{R}\right)+\lambda_R\left(\frac{(I:\mathfrak{m})}{I}\right)$$

by [1, Theorem 3.2].

We note that $(I:I) \subseteq (R:\mathfrak{m})$. Indeed it is easy to see that $z(I:I) \subseteq (\omega_R:I) \subseteq z(R:\mathfrak{m})$, because $\mathfrak{m}I^\vee = z\mathfrak{m}$ by Theorem 2.3.

Moreover $(I:I) = ((\omega_R:(\omega_R:I)):I) = ((\omega_R:I^\vee):I) = (\omega_R:II^\vee) = z^{-1}I^\vee$, because $zI = II^\vee$ again by Theorem 2.3.

Finally, to conclude the proof, it is enough to show that $\lambda_R((I:\mathfrak{m})/I) = \lambda_R(z^{-1}I^\vee/\mathfrak{m})$. This holds because

$$\lambda_R\left(\frac{(I:\mathfrak{m})}{I}\right) = \lambda_R\left(\frac{((\omega_R:(\omega_R:I)):\mathfrak{m})}{I}\right) = \lambda_R\left(\frac{(\omega_R:\mathfrak{m}I^\vee)}{I}\right)$$

$$= \lambda_R\left(\frac{(\omega_R:I)}{\mathfrak{m}I^\vee}\right) = \lambda_R\left(\frac{I^\vee}{z\mathfrak{m}}\right) = \lambda_R\left(\frac{z^{-1}I^\vee}{\mathfrak{m}}\right),$$

where, since $R(I)_{a,b}$ is almost Gorenstein, we used $\mathfrak{m}I^\vee = z\mathfrak{m}$ (cf. Theorem 2.3). □

By Corollary 1.5, if R is a DVR and $R(I)_{a,b}$ is almost Gorenstein, it follows that $R(I)_{a,b}$ is Gorenstein, i.e. has type 1. On the other hand if we assume that R is not a DVR, the type of R is the length of $(\mathfrak{m}:\mathfrak{m})/R$ and then, since in the almost Gorenstein case $z^{-1}I^\vee \subseteq (\mathfrak{m}:\mathfrak{m})$ (cf. Corollary 2.4), the previous proposition implies the following.

Corollary 2.7. *If $R(I)_{a,b}$ is almost Gorenstein, its type is odd and $1 \leq t(R(I)_{a,b}) \leq 2t(R)+1$.*

Example 2.8. In Example 2.5 we get $t(R(I)_{a,b}) = 2\lambda_R(A/R)+1 = 3$. Observe that in this example $t(R)=2$ and all the odd values t, $1 \leq t \leq 5 = 2t(R)+1$ can be realized for $t(R(I)_{a,b})$. In fact for example for $I=(X^4, X^5)$, which is a canonical ideal of R, we get $t(R(I)_{a,b})=1$ and for $I=(X^4,X^5,X^6)=X^4(\omega_R:(\mathfrak{m}:\mathfrak{m}))$ we get $t(R(I)_{a,b})=5$.

Since the almost Gorensteinness does not depend on a and b, from [8, Theorem 6.5] we get the following proposition, of which we include also a simple proof.

Proposition 2.9. *The ring R is almost Gorenstein if and only if $R(\mathfrak{m})_{a,b}$ is almost Gorenstein. In this case, if R is not a DVR, the type of $R(\mathfrak{m})_{a,b}$ is $2t(R)+1$.*

Proof. We have that R is a DVR if and only if $R(\mathfrak{m})_{a,b}$ is Gorenstein (cf. Corollary 1.4). Thus we can exclude this case. If R is almost Gorenstein and not a DVR, then $(\omega_R:\mathfrak{m}) = (\mathfrak{m}:\mathfrak{m})$: in fact, $\lambda_R((\omega_R:\mathfrak{m})/\omega_R) = \lambda_R(R/\mathfrak{m}) = 1$; moreover, denoting by $t(R)$ the CM type of R, we obtain the following chain of equalities $t(R)+1 = \lambda_R((\mathfrak{m}:\mathfrak{m})\,/\,\mathfrak{m}) = \lambda_R((\mathfrak{m}:\mathfrak{m})\,/\,\omega_R) + \lambda_R(\omega_R\,/\mathfrak{m}) = \lambda_R((\mathfrak{m}:\mathfrak{m})/\omega_R) + \lambda_R(\omega_R/\mathfrak{m}\omega_R) =$

$\lambda_R((\mathfrak{m}{:}\mathfrak{m})/\omega_R)+t(R)$, that implies $\lambda_R((\mathfrak{m}{:}\mathfrak{m})/\omega_R)=1$ (cf. [2, Definition/Proposition 20]). So if $I=\mathfrak{m}$, then $z=1$, $z^{-1}I^\vee=(\mathfrak{m}{:}\mathfrak{m})$ and, by Corollary 2.4, $R(\mathfrak{m})_{a,b}$ is almost Gorenstein.

Conversely, if $R(\mathfrak{m})_{a,b}$ is almost Gorenstein but not Gorenstein, then $z^{-1}(\omega_R{:}\mathfrak{m})\subseteq(\mathfrak{m}{:}\mathfrak{m})$ (Corollary 2.4). Moreover it is well known that in dimension one ω_R is an irreducible fractional ideal and $\lambda_R((\omega_R{:}\mathfrak{m})/\omega_R)=1$. Thus $(\omega_R{:}\mathfrak{m})\subset\overline{R}$ (otherwise, if $x\in\overline{R}\setminus\omega_R$, we have $\omega_R=(\omega_R{:}\mathfrak{m})\cap(\omega_R,x)$, a contradiction) and by [2, Proposition 16] $z=1$ is a minimal reduction of $(\omega_R{:}\mathfrak{m})$. So $\omega_R\subseteq(\omega_R{:}\mathfrak{m})\subseteq(\mathfrak{m}{:}\mathfrak{m})$ and R is almost Gorenstein.

As for the last part of the statement, we have already proved that in this case $z^{-1}(\omega_R{:}\mathfrak{m})=(\mathfrak{m}{:}\mathfrak{m})$; then it is enough to apply the formula of Proposition 2.6. □

2.2. The general case

Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, with canonical module ω_R. The goal of this subsection is to prove that the property of being almost Gorenstein for $R(I)_{a,b}$ is independent of the choice of a and b also in the case $d>1$. We recall we are assuming that $R/\mathfrak{m}$ is infinite. The next lemma is proved e.g. in the proof of [3, Proposition 3.3.18], but we include the short proof for the sake of completeness.

Lemma 2.10. *Let I be a regular ideal and a maximal CM R-module. Then I has height one and R/I is a Cohen–Macaulay ring of dimension $d-1$.*

Proof. Since I is a maximal CM R-module, $\operatorname{depth} I=\operatorname{depth} R$; moreover I has positive height, because is regular. Hence the depth lemma [3, Proposition 1.2.9] implies that

$$\begin{aligned}\dim R-1\geq\dim R-\operatorname{height} I=\dim R/I\geq\operatorname{depth} R/I\\ \geq\min\{\operatorname{depth} I-1,\operatorname{depth} R\}=\operatorname{depth} R-1=\dim R-1.\end{aligned}$$

Therefore all inequalities above are equalities and the thesis follows immediately. □

The following lemma allows us to reduce to the one-dimensional case.

Lemma 2.11. *Let x be an element of the ring R, that determines a nonzerodivisor on R/I (i.e. $(I{:}x)=I$); then*

$$\frac{R(I)_{a,b}}{xR(I)_{a,b}}\cong\frac{R}{xR}\left(\frac{I+xR}{xR}\right)_{\overline{a},\overline{b}}$$

where $\overline{a}$ and $\overline{b}$ are the images of a and b in R/xR.

Proof. It is not difficult to check that there is a surjective ring homomorphism $\alpha: R(I)_{a,b} \to \frac{R}{xR}(\frac{I+xR}{xR})_{\overline{a},\overline{b}}$ defined by $r+it \mapsto (r+xR)+(i+xR)t$. The assumption on x implies that $I \cap xR = xI$; hence $i \in xR$ if and only if $i=xj$ with $j \in I$; therefore $\mathrm{Ker}(\alpha) = xR(I)_{a,b}$ and we obtain the thesis. □

In the next proposition we will use some results about filtrations and superficial elements that can be found, e.g, in [14, Chap. 1].

Proposition 2.12. *Let $d \geq 1$ and let I be a regular ideal of R. Then the almost Gorensteinness of $R(I)_{a,b}$ does not depend on the choice of a and b.*

Proof. By Theorem 2.3 it is enough to consider the case $d>1$. Assume that there exist two elements $a', b' \in R$ for which $R(I)_{a',b'}$ is almost Gorenstein. We have to show that $R(I)_{a,b}$ is almost Gorenstein for any $a, b \in R$. By Corollary 1.4, we can also assume that $R(I)_{a',b'}$ is not Gorenstein.

Our assumption means that there exists an exact sequence of $R(I)_{a',b'}$-modules

$$0 \longrightarrow R(I)_{a',b'} \longrightarrow \omega_{R(I)_{a',b'}} \longrightarrow C \longrightarrow 0,$$

where the number of elements of a minimal system of generators of C equals its multiplicity. Let M be the maximal ideal of $R(I)_{a',b'}$ and consider the filtration of C induced by M:

$$C \supseteq MC \supseteq M^2C \supseteq ... \supseteq M^iC \supseteq ... ;$$

this is an M-filtration of the $R(I)_{a',b'}$-module C, but, if we consider C as an R-module, it is also an $\mathfrak{m}$-filtration. Therefore, we know that in R there exists a C-superficial sequence for $\mathfrak{m}$ of length $d-1$; by definition it is clear that it is also a C-superficial sequence for M. Moreover, we can choose a sequence $\mathbf{f} = f_1, ..., f_{d-1}$ that is also R-regular and, since I has height one, such that $I + (\mathbf{f})$ is $\mathfrak{m}$-primary (see [10, Corollary 8.5.9]). Consequently $\mathbf{f}$ is an $R(I)_{a,b}$-regular sequence for any $a, b \in R$ and the ideal of R/I generated by the classes $\overline{f}_1, ..., \overline{f}_{d-1}$ is $\mathfrak{m}/I$-primary; therefore $\overline{\mathbf{f}}$ is a regular sequence, because R/I is a CM ring of dimension $d-1$ by Lemma 2.10. Hence we can use the previous lemma and, from [9, Theorem 3.7(2)], it follows that

$$\frac{R(I)_{a',b'}}{\mathbf{f}R(I)_{a',b'}} \cong \frac{R}{\mathbf{f}R}\left(\frac{I+\mathbf{f}R}{\mathbf{f}R}\right)_{\overline{a'},\overline{b'}}$$

is almost Gorenstein of dimension 1; by Theorem 2.3,

$$\frac{R}{\mathbf{f}R}\left(\frac{I+\mathbf{f}R}{\mathbf{f}R}\right)_{\overline{a},\overline{b}}$$

is an almost Gorenstein ring for any $\overline{a},\overline{b}\in R/\mathbf{f}R$. Observe also that, as above, the ideal $(I+\mathbf{f}R)/\mathbf{f}R$ is $\mathfrak{m}/\mathbf{f}R$-primary.

Finally, since $\mathbf{f}$ is an $R(I)_{a,b}$-regular sequence, this implies that $R(I)_{a,b}$ is almost Gorenstein for any $a,b\in R$, by [9, Theorem 3.7(1)]. □

By Proposition 2.12, $R(I)_{a,b}$ is almost Gorenstein if and only if $R(I)_{0,0}\cong R\ltimes I$ is almost Gorenstein. We have already observed that if I and J are two isomorphic ideals, then $R(I)_{a,b}$ and $R(J)_{a,b}$ do not need to be isomorphic (cf. Example 2.5). Anyway, it is easy to see that this happens for idealization, i.e. if N_1 and N_2 are two isomorphic R-modules then $R\ltimes N_1\cong R\ltimes N_2$. Thus applying Proposition 2.12 we obtain the following corollary.

Corollary 2.13. *If I, J are two isomorphic regular ideals of R, then $R(I)_{a,b}$ is almost Gorenstein if and only if $R(J)_{a,b}$ is almost Gorenstein.*

In [9] the authors study when the idealization is almost Gorenstein. Proposition 2.12 implies the following generalization of [9, Theorem 6.1].

Corollary 2.14. *Let I be a regular ideal of R and assume that $I^\vee$ is isomorphic to a regular ideal of R. Then the following are equivalent:*

(1) *$R(I)_{a,b}$ is almost Gorenstein for some $a,b\in R$;*

(2) *$R(I)_{a,b}$ is almost Gorenstein for all $a,b\in R$;*

(3) *I is a maximal CM R-module and any proper ideal J of R isomorphic to $I^\vee$ is such that $f_1\in J$, $\mathfrak{m}(J+Q)=\mathfrak{m}Q$, and $(J+Q)^2=Q(J+Q)$, for some parameter ideal $Q=(f_1,...,f_d)$ of R.*

Proof. The equivalence between (1) and (2) follows from Proposition 2.12.

(2) ⇒ (3): I is a maximal CM R-module by Proposition 1.2, therefore $J\cong I^\vee$ is a maximal CM R-module by [3, Theorem 3.3.10] and it is also isomorphic to a regular ideal; moreover $J^\vee\cong I$, thus the idealizations $R\ltimes I$ and $R\ltimes J^\vee$ are isomorphic. Furthermore, by Lemma 2.10, it follows that R/J is a CM ring of dimension $d-1$. Thus it follows from [9, Theorem 6.1, (1) ⇒ (2)] that (3) holds because $R(I)_{0,0}\cong R\ltimes I\cong R\ltimes J^\vee$ is almost Gorenstein.

(3) ⇒ (1): By [9, Theorem 6.1, (2) ⇒ (1)] we have that $R\ltimes J^\vee\cong R(I)_{0,0}$ is almost Gorenstein. □

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