Ark. Mat., **53** (2015), 203–236
DOI: 10.1007/s11512-014-0207-6

Extremal functions for real convex bodies

Daniel M. Burns, Norman Levenberg and Sione Maʻu

Abstract. We study the smoothness of the Siciak–Zaharjuta extremal function associated to a convex body in $\mathbb{R}^2$. We also prove a formula relating the complex equilibrium measure of a convex body in $\mathbb{R}^n$ $(n\geq 2)$ to that of its Robin indicatrix. The main tool we use is extremal ellipses.

1. Introduction

The Siciak–Zaharjuta extremal function for a compact set $K\subset\mathbb{C}^n$ is the plurisubharmonic (psh) function on $\mathbb{C}^n$ given by

$$V_K(z) := \sup\{u(z) : u\in L(\mathbb{C}^n) \text{ and } u\leq 0 \text{ on } K\},$$

where $L(\mathbb{C}^n)=\{u \text{ psh on } \mathbb{C}^n : \text{there exists } C\in\mathbb{R} \text{ such that } u(z)\leq \log^+|z|+C\}$ denotes the class of psh functions on $\mathbb{C}^n$ with logarithmic growth.

The upper semicontinuous regularization $V_K^*(z):=\limsup_{\zeta\to z} V_K(\zeta)$ is identically ∞ if K is pluripolar; otherwise $V_K^*\in L(\mathbb{C}^n)$. In fact, $V_K^*\in L^+(\mathbb{C}^n)$, where

$$L^+(\mathbb{C}^n) = \{u\in L(\mathbb{C}^n) : \text{there exists } C\in\mathbb{R} \text{ such that } u(z)\geq \log^+|z|+C\}.$$

The set K is *L-regular* if K is nonpluripolar and $V_K=V_K^*$; this is equivalent to V_K being continuous. In this paper, V_K will always have a continuous foliation structure that automatically gives L-regularity.

The complex Monge–Ampère operator applied to a function u of class C^2 on some domain in $\mathbb{C}^n$ is given by

$$(dd^c u)^n = i\partial\bar{\partial}u\wedge\ldots\wedge i\partial\bar{\partial}u, \quad n \text{ times}.$$

The third author was partially supported by University of Auckland grant 3704154.

Its action can be extended to certain nonsmooth classes of plurisubharmonic functions (cf., [3]). In particular, for a locally bounded psh function u, $(dd^c u)^n$ is well-defined as a positive measure.

If K is compact and nonpluripolar, we define the *complex equilibrium measure of* K as $(dd^c V_K^*)^n$. We also call it the (*complex*) *Monge–Ampère measure of* K. This is a positive measure supported on K.

For L-regular sets, the relationship between the higher order smoothness of V_K and geometric properties of K is not completely understood, except in a few special cases. It is not known whether V_K is smooth if K is the closure of a bounded domain and the boundary of K is smooth or even real-analytic. It is known that if K is the disjoint union of the closures of finitely many strictly pseudoconvex domains with smooth boundary, then V_K is $C^{1,1}$ [10].

However, the extremal function has particularly nice properties when K is the closure of a bounded, smoothly bounded, *strictly lineally convex* domain $D\subset\mathbb{C}^n$. Then V_K is smooth on $\mathbb{C}^n\setminus K$, as a consequence of Lempert's results ([12], [13] and [14]). He showed that there is a smooth foliation of $\mathbb{C}^n\setminus K$ by holomorphic disks on which V_K is harmonic (*extremal disks*).

If K is a convex body in $\mathbb{R}^n\subset\mathbb{C}^n$, it was shown in [8] that as long as ∂K in $\mathbb{R}^n$ does not contain parallel line segments, there is a continuous foliation of $\mathbb{C}^n\setminus K$ by extremal disks. For a *symmetric* convex body, the existence of such a foliation was proved earlier in [1] by different methods.

The existence of extremal disks through each point of $\mathbb{C}^n\setminus K$ (K being a real convex body) was obtained in [7] by an approximation argument using Lempert theory, and it was shown that these disks must be contained in complexified real ellipses (*extremal ellipses*). An important tool used in this study was a real geometric characterization of such ellipses, which was derived from a variational description of the extremal disks. The goal of this paper is to establish further properties of V_K by studying its foliation in more detail. We begin in the next section by recalling basic properties of V_K and its associated extremal ellipses that will be used in what follows.

In Section 3, we study the smoothness of V_K. Results are proved in $\mathbb{R}^2\subset\mathbb{C}^2$ as the geometric arguments work only in dimension 2. For a convex body $K\subset\mathbb{R}^2$ we first show that at certain points of $\mathbb{C}^2\setminus K$, V_K is pluriharmonic (and therefore smooth). At other points, we use the foliation structure of V_K by extremal ellipses to study its smoothness. We derive geometric conditions on extremal ellipses that ensure smoothness of the foliation, under the assumption that the real boundary ∂K (i.e., the boundary of K as a subset of $\mathbb{R}^2$) is sufficiently smooth. We also give simple examples to illustrate what happens when these conditions fail. Two types of ellipses are considered separately:

(1) extremal ellipses intersecting ∂K in exactly two points;

(2) extremal ellipses intersecting ∂K in exactly three points.

This accounts for most ellipses; those that remain are contained in a subset of $\mathbb{C}^2$ of real codimension 1.

Theorem 3.11. *Let $K\subset\mathbb{R}^2\subset\mathbb{C}^2$ be a convex body whose boundary ∂K is C^r-smooth ($r\in\{2,3,...\}\cup\{\infty,\omega\}$). Then V_K is C^r on $\mathbb{C}^2\setminus K$ except for a set of real dimension at most* 3.

Finally, in Section 4, we study the complex equilibrium measure of a convex body $K\subset\mathbb{R}^n\subset\mathbb{C}^n$, $n\geq 2$. If a compact set $K\subset\mathbb{C}^n$ has the foliation property, we can use a "transfer of mass" argument to relate its complex equilibrium measure to $(dd^c\rho_K^+)^n$, where ρ_K denotes the Robin function of K and $\rho_K^+=\max\{\rho_K,0\}$. The measure $(dd^c\rho_K^+)^n$ is in fact the equilibrium measure of the *Robin indicatrix of* K, $K_\rho:=\{z:\rho_K(z)\leq 0\}$, and the relation is given in terms of the *Robin exponential map*, first defined in [8].

Theorem 4.6. *Let $K\subset\mathbb{R}^n$ be a convex body with unique extremals. Then for any ϕ continuous on K,*

$$\int\phi(dd^cV_K)^n=\int(\phi\circ F)(dd^c\rho_K^+)^n.$$

Here F denotes the extension of the Robin exponential map as a continuous function from ∂K_ρ onto K. A preliminary step is to prove a version of this (Theorem 4.3) for $K=\overline{D}$, the closure of a smoothly bounded, strictly lineally convex domain D.

2. Background

In this section we recall essential properties of extremal functions and foliations associated to convex bodies.

The following properties of extremal functions are well-known.

Theorem 2.1. (1) *Suppose $K_1\subset\mathbb{C}^n$ and $K_2\subset\mathbb{C}^m$ are compact sets. Then for $(z,w)\in\mathbb{C}^{n+m}\setminus K_1\times K_2$ we have*

$$V_{K_1\times K_2}(z,w)=\max\{V_{K_1}(z),V_{K_2}(w)\}. \tag{2.1}$$

(2) *Let $K\subset\mathbb{C}^n$ be compact and let $P=(P_1,...,P_n)\colon\mathbb{C}^n\to\mathbb{C}^n$ be a polynomial map of degree d, $P_j=\widehat{P}_j+r_j$ with $\widehat{P}_j$ homogeneous of degree d and* $\deg(r_j)<d$.

Suppose $\widehat{P}^{-1}(0)=\{0\}$, *where* $\widehat{P}=(\widehat{P}_1,...,\widehat{P}_n)$. *Then for all* $z\in\mathbb{C}^n$,

$$V_{P^{-1}(K)}(z)=d\cdot V_K(P(z)). \tag{2.2}$$

Proof. See e.g. Chapter 5 of [11]. □

If L is an affine change of coordinates then (2.2) shows that $V_{L(K)}(L(z))=V_K(z)$.

Next, let K be a convex body. We summarize the essential properties of extremal curves for V_K.

Theorem 2.2. (1) *Through every point* $z\in\mathbb{C}^n\setminus K$ *there is either a complex ellipse* E *with* $z\in E$ *such that* V_K *restricted to* E *is harmonic on* $E\setminus K$, *or there is a complexified real line* L *with* $z\in L$ *such that* V_K *is harmonic on* $L\setminus K$.

(2) *For* E *as above,* $E\cap K$ *as above is a real ellipse inscribed in* K, *i.e., for its given eccentricity and orientation, it is the ellipse with largest area completely contained in* K; *if* L *is as above,* $L\cap K$ *is the longest line segment (for its given direction) completely contained in* K.

(3) *Conversely, suppose* $C_T\subset K$ *is a real inscribed ellipse (or line segment) with maximal area (or length) as above. Form* E *(or* L*) by complexification (i.e., find the unique complex algebraic curve of degree* ≤ 2 *containing* C_T*). Then* V_K *is harmonic on* $E\setminus C_T$ *(resp.* $L\setminus C_T$*).*

Proof. See Theorem 5.2 and Section 6 of [8]. □

The ellipses and lines discussed above have parametrizations of the form

$$F(\zeta)=a+c\zeta+\frac{\bar{c}}{\zeta}, \tag{2.3}$$

where $a\in\mathbb{R}^n$, $c\in\mathbb{C}^n$ and $\zeta\in\mathbb{C}\setminus\{0\}$ with $V_K(F(\zeta))=\big|\log|\zeta|\big|$. (As usual, $\bar{c}$ denotes the component-wise complex conjugate of c.) These are higher-dimensional analogs of the classical Joukowski function $\zeta\mapsto\frac{1}{2}(\zeta+1/\zeta)$.

A curve parametrized as in (2.3) is the image of the curve $\{(z_1,z_2)\in\mathbb{C}^2: z_1^2+z_2^2=1\}$ under the affine map

$$\mathbb{C}^2\ni(z_1,z_2)\longmapsto a+z_1 2\operatorname{Re}(c)-z_2 2\operatorname{Im}(c)\in\mathbb{C}^n.$$

Hence F parametrizes an ellipse in $\mathbb{C}^n$ if $\operatorname{Im}(c)\neq 0$, otherwise (for $c\in\mathbb{R}^2$) it gives the complex line $\{a+\lambda c:\lambda\in\mathbb{C}\}$. For convenience, we usually consider both cases together by regarding the complex lines to be degenerate ellipses with infinite eccentricity. These algebraic curves will be referred to as *extremal curves* or *extremal ellipses*, including the degenerate case.

From Theorem 2.2(3), one can see that an extremal curve for V_K may not be unique for a given eccentricity and orientation if K contains parallel line segments in its boundary ∂K (as a boundary in $\mathbb{R}^n$), as it may be possible to translate the curve and obtain another extremal. On the other hand, if no such line segments exist (e.g. if K is strictly convex) then extremal curves are unique.

The following was shown in [8].

Theorem 2.3. *If $K\subset\mathbb{R}^n$ is a convex body such that all its extremal curves are unique, then these curves give a continuous foliation of $\mathbb{CP}^n\setminus K$ by analytic disks such that the restriction of V_K to any leaf of the foliation is harmonic.*

In the above result we are considering $\mathbb{CP}^n=\mathbb{C}^n\cup H_\infty$ via the usual identification of homogeneous coordinates $[Z_0:Z_1:...:Z_n]$ with the affine coordinates $(z_1,...,z_n)$ given by $z_j=Z_j/Z_0$ when $Z_0\neq 0$, and $H_\infty=\{[Z_0:...:Z_n]:Z_0=0\}$. An analytic disk which is a leaf of the foliation is precisely 'half' of an extremal ellipse. Letting $\bar\Delta=\{\zeta:|\zeta|\leq 1\}$ denote the closed unit disk in $\mathbb{C}$ and $\widehat{\mathbb{C}}$ the Riemann sphere, a leaf of the foliation may be given by $F(\widehat{\mathbb{C}}\setminus\bar\Delta)$, with F as in (2.3) extended holomorphically to infinity via $F(\infty)=[0:c_1:...:c_n]=:[0:c]$.

Analytic disks through conjugate points $[0:c]$ and $[0:\bar c]$ at H_∞ (called *conjugate leaves* in [7]) are the two 'halves' of an extremal ellipse, and fit together along the corresponding real inscribed ellipse in K.

The bulk of the proof of Theorem 2.3 consisted in verifying that two extremal ellipses can only intersect in the set K, and hence they are disjoint in $\mathbb{C}^n\setminus K$. This was done on a case-by-case basis using the geometry of real convex bodies.

That these ellipses foliate $\mathbb{CP}^n\setminus K$ continuously was obtained as a by-product of the approximation techniques used to prove their existence. This was to approximate K by a decreasing sequence $K_j\searrow K$ of strictly convex, conjugation invariant bodies in $\mathbb{C}^n$ with real-analytic boundary. For such sets K_j, Lempert theory gives the existence of a smooth foliation of $\mathbb{C}^n\setminus K_j$ by analytic disks such that the restriction of V_{K_j} to each disk is harmonic. It was also verified in [8] that these foliations extend smoothly across H_∞ in local coordinates. In the limit as $j\to\infty$, they converge to a continuous foliation parametrized by H_∞.

We remark that H_∞ is a natural parameter space for leaves of the foliation by recalling its real geometric interpretation. Two ellipses

$$\zeta\longmapsto a+b\zeta+\frac{\bar b}{\zeta}\quad\text{and}\quad\zeta\longmapsto a'+b'\zeta+\frac{\bar b'}{\zeta}$$

intersect the same point $c=[0:b]=[0:b']\in H_\infty$ if and only if $b=\lambda b'$ for some $\lambda\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$. Writing $\lambda=re^{i\psi}$, $r>0$, and putting $\zeta=e^{i\theta}$, the parametrizations become

$$e^{i\theta}\longmapsto a+2(\mathrm{Re}(b)\cos\theta-\mathrm{Im}(b)\sin\theta)\quad\text{and}$$
$$e^{i\theta}\longmapsto a'+2r(\mathrm{Re}(b)\cos(\theta+\psi)-\mathrm{Im}(b)\sin(\theta+\psi)).$$

As θ runs through $\mathbb{R}$ these parametrizations trace real ellipses in $\mathbb{R}^n$ related by the translation $a-a'$ and the scale factor r, but with the same eccentricity and orientation. If $a=a'$ and $|\lambda|=1$ we get a reparametrization of the same ellipse.

Given a parameter $c\in H_\infty$, write $a=a(c)$ and $b=b(c)$, where $\zeta\mapsto a+b\zeta+\bar{b}/\zeta$, $b=(b_1,...,b_n)$, is an extremal ellipse for the eccentricity and orientation given by $c\in H_\infty$. When $b_1\neq 0$, we may reparametrize the ellipse so that $b_1\in(0,\infty)$. Put $c_j=b_j/b_1$ and $\rho(c)=b_1$. We then write an extremal as

$$\zeta\longmapsto a(c)+\rho(c)\left((1,c_2,...,c_n)\zeta+\frac{(1,\bar{c}_2,...,\bar{c}_n)}{\zeta}\right)=a(c)+\rho(c)\left(c\zeta+\frac{\bar{c}}{\zeta}\right)\tag{2.4}$$

(slightly abusing notation in the last expression). When extremals are unique, $a(c)$ and $\rho(c)$ are uniquely determined by c, so by Theorem 2.3 they are continuous functions. (Note that this is only valid locally, i.e., when $b_1\neq 0$.)

3. Smoothness of V_K in $\mathbb{C}^2$

We specialize now to a compact convex body $K\subset\mathbb{R}^2\subset\mathbb{C}^2$, and ∂K will then denote the boundary in $\mathbb{R}^2$. Denote coordinates in $\mathbb{C}^2$ by $z=(z_1,z_2)$, and use $x=(x_1,x_2)$ when restricting to $\mathbb{R}^2$. In analyzing the extremal ellipses associated to K, we will employ elementary geometric arguments, which do not directly generalize to higher dimensions.

3.1. Points at which V_K is pluriharmonic

From classical potential theory in one complex variable we have the well-known formula

$$V_{[-1,1]}(\zeta)=\log|h(\zeta)|,\quad \zeta\notin[-1,1],$$

where $h(\zeta)=\zeta+\sqrt{\zeta^2-1}$ is the inverse of the Joukowski function (cf., [18]). Hence if $S=[-1,1]\times[-1,1]\subset\mathbb{R}^2\subset\mathbb{C}^2$ is the square centered at the origin, then by (2.1),

$$V_S(z)=\max\{\log|h(z_1)|,\log|h(z_2)|\}.\tag{3.1}$$

On $\mathbb{C}^2\setminus S$ this is the maximum of two pluriharmonic functions. A continuous foliation for V_S is given by extremal ellipses for S centered at the origin [1].

Lemma 3.1. *Suppose C is an extremal curve for the square S, and $z=(z_1,z_2)\in C$. Let $j=1$ or $j=2$. Then C intersects both of the lines $z_j=\pm 1$ if and only if $V_S(z)=\log|h(z_j)|$.*

Proof. Take $j=1$; the proof when $j=2$ is identical. We have a parametrization $z=(a_1,a_2)+\rho((1,c_2)t+(1,\bar{c}_2)/t)\in C$. Since C is extremal, if it intersects the lines $z_1=\pm 1$, then it intersects these lines tangentially. By symmetry the midpoint of the ellipse lies on the line $z_1=0$; so $a_1=0$.

We verify that $\rho=\frac{1}{2}$. Since C intersects the lines $z_1=\pm 1$ there exist $\phi_1,\phi_2\in[-\pi,\pi]$ such that

$$1=\rho(e^{i\phi_1}+e^{-i\phi_1})=2\rho\cos\phi_1,$$
$$-1=\rho(e^{i\phi_2}+e^{-i\phi_2})=2\rho\cos\phi_2.$$

Either equation immediately implies that $\rho\geq\frac{1}{2}$, and the reverse inequality follows since the real points of C lie in S. Hence z_1 is given by the Joukowski function: $z_1=\frac{1}{2}(t+1/t)=h^{-1}(t)$ for $t\in\mathbb{C}$ and

$$V_S(z)=\log|t|=\log|h\circ h^{-1}(t)|=\log|h(z_1)|.$$

Conversely, suppose C does not intersect, say, $z_1=1$. We want to show that $V_S(z)>\log|h(z_1)|$. Now the real points of C, which are contained in S, tangentially intersect the line $z_1=1-\varepsilon$ for some $\varepsilon>0$. At a point $(z_1,z_2)\in C$, the parametrization of C yields $z_1=a_1+\rho(t+1/t)$. The z_1-coordinates of real points of C, given by $t=e^{i\theta}$, $\theta\in\mathbb{R}$, must satisfy

$$a_1+2\rho\cos\theta\in[-1,1-\varepsilon]\quad\text{for all }\theta.$$

We show that $\rho\geq\frac{1}{2}$ cannot hold. Suppose it does; then we must have $\cos\theta\in I$, where $I:=[-1-a_1,1-\varepsilon-a_1]$. If $a_1<0$, then $\cos\pi=-1\notin I$, a contradiction. But on the other hand, if $a_1\geq 0$ then $\cos 0=1\notin I$, which is also a contradiction. Hence $\rho<\frac{1}{2}$. Finally, taking $a_1=0$ for simplicity, a calculation yields

$$\log|h(z_1)|=\log\left|h\left(\rho\left(t+\frac{1}{t}\right)\right)\right|=\log|2\rho t|<\log|t|=V_S(z_1).\qquad\square$$

The above lemma shows that if $z\in\mathbb{C}^2\setminus S$ lies on an extremal ellipse for S that does not intersect all four sides, then V_S is pluriharmonic in a neighborhood of z.

Theorem 3.2. *Let $K\subset\mathbb{R}^2$ be a convex body, and suppose that ∂K contains a pair of parallel line segments. Suppose C is an extremal curve of K that intersects ∂K in the interior of these two line segments and in no other points. Then for any $z\in C\setminus K$, V_K is pluriharmonic in a neighborhood of z.*

Proof. First, we simplify the situation using Theorem 2.1 and the fact that pluriharmonicity is unaffected by linear transformations. Hence we may assume that the parallel line segments lie on the lines $x_2=1$ and $x_2=-1$ and that C is centered at the origin. By rescaling and translating the x_1-axis, we may further assume that $K\subset S$, $S=[-1,1]\times[-1,1]$ as above, and that C intersects ∂K in the two points $(\alpha,1)$ and $(-\alpha,-1)$, where $0<\alpha<1$.

Write $C_T=\{F(e^{i\theta}):\theta\in\mathbb{R}\}$ for the real ellipse contained in K, where $F(t)=bt+\bar{b}/t$ is the parametrization of C. By elementary topology in $\mathbb{R}^2$ there exists $\varepsilon>0$ such that for any $s\in[-\varepsilon,\varepsilon]$, the translated sets $(s,0)+C_T$ are contained in K.

By construction, C is extremal for S as well as for K, so for any $z\in C\setminus\mathbb{R}^2$, we have $V_K(z)=V_S(z)$. We show that for any sufficiently close point z' we also have $V_K(z')=V_S(z')$.

Consider an extremal ellipse C' for S containing a point z' that is given by the parametrization $t\mapsto b't+\overline{b'}/t$. By the continuity of the foliation for V_S given by extremal ellipses centered at the origin, then given $\varepsilon>0$ there is $\delta>0$ such that $|z-z'|<\delta$ implies $|b-b'|<\varepsilon/2$ and $|t_0-t_0'|<\varepsilon/2$, where

$$z=ct_0+\frac{\bar{c}}{t_0} \quad \text{and} \quad z'=c't_0'+\frac{\overline{c'}}{t_0'}.$$

Since $K\subset S$, to show that C' is extremal for K we only need to verify that $C_T'=C'\cap S\subset K$. Let $x=(x_1,x_2)\in C_T'$ and define y_1 by the condition that $y=(y_1,x_2)\in C_T$ is the closest point to x with the same second coordinate. We have

$$\begin{aligned} x_1=b_1'e^{i\theta}+\overline{b_1'}e^{-i\theta}&=b_1e^{i\theta}+\bar{b}_1e^{-i\theta}+(b_1'-b_1)e^{i\theta}+(\overline{b_1'}-\bar{b}_1)e^{-i\theta}\\ &=y_1+(b_1'-b_1)e^{i\theta}+(\overline{b_1'}-\bar{b}_1)e^{-i\theta}, \end{aligned}$$

where $y=(y_1,y_2)$ for some $y\in C_T$. Hence $|x_1-y_1|\le 2|b_1'-b_1|<\varepsilon$. So C_T' is contained in the convex hull of $C_T-(\varepsilon,0)$ and $C_T+(\varepsilon,0)$, which in turn is contained in K. (See Figure 1.) Therefore C_T' is an extremal curve through z' for both S and K, and $V_K(z')=V_S(z')$ follows.

Applying the previous lemma, $V_K(z')=V_S(z')=\log|h(z_2')|$ for all $z'=(z_1',z_2')$ with $|z-z'|<\delta$. So V_K is pluriharmonic in a neighborhood of z. □

Remark 3.3. If C is an extremal ellipse that does not intersect ∂K in a pair of parallel line segments, then it is unique for its value of $c\in H_\infty$. If K contains parallel line segments elsewhere, we may get rid of this parallelism by modifying K slightly (e.g. shaving off a thin wedge along one of the line segments). This can be done without affecting C and nearby extremals. Hence in studying the local behavior of extremal ellipses near a unique extremal C, we may assume that uniqueness

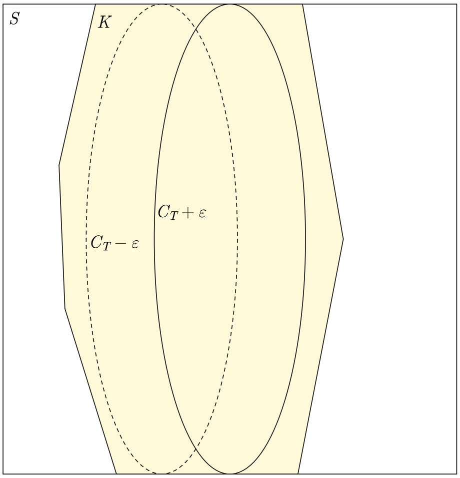

Figure 1. Proof of Theorem 3.2.

holds globally, and that extremal ellipses give a continuous foliation of $\mathbb{C}^n \setminus K$ (by Theorem 2.3).

We now turn to study the smoothness of the foliation at points on unique extremals.

3.2. Extremal ellipses meeting ∂K in two points

As before, write C to denote an extremal ellipse for V_K, F its parametrization, and $C_T = \{F(e^{i\theta}) : \theta \in \mathbb{R}\}$ its trace on K.

Note. From now until the end of Section 3, coordinates in $\mathbb{R}^2 \subset \mathbb{C}^2$ will be denoted by (x, y).

We will assume in what follows that ∂K is at least C^2. For a point $a \in \partial K$, denote by $T_a(\partial K)$ the tangent line to ∂K that passes through a.

Proposition 3.4. *Let $C_T \cap \partial K = \{a, b\}$. If ∂K is smooth at a and b, then the tangent lines $T_a(\partial K)$ and $T_b(\partial K)$ are parallel.*

Proof. Let $\mathbf{v}_a$ and $\mathbf{v}_b$ be unit vectors parallel to $T_a(\partial K)$ and $T_b(\partial K)$ respectively. We may assume they are oriented so that $\mathbf{v}_a \cdot \mathbf{v}_b \geq 0$. Suppose $\mathbf{v}_a \neq \mathbf{v}_b$. Then take any unit vector $\mathbf{v}$ for which

$$\mathbf{v}_a \cdot \mathbf{v}_b < \mathbf{v} \cdot \mathbf{v}_b < 1 = \mathbf{v}_b \cdot \mathbf{v}_b.$$

For $t>0$ sufficiently small, the translated ellipse $C_{T,t}:=C_T+t\mathbf{v}$ is then contained in the interior of K so that $C_{T,t}$ can be expanded to an ellipse with the same orientation and eccentricity as C_T. This contradicts the fact that C_T is extremal. $\square$

Let us start now by fixing an extremal curve C, with $C_T \subset K$, corresponding to a fixed value $c=c_0 \in H_\infty$ and $\rho(c_0)=\rho_0$. The parametrization of C may be written as

$$\zeta \longmapsto (x_0, y_0) + \rho_0 \left((1, c_0)\zeta + \frac{(1, \bar{c}_0)}{\zeta} \right). \tag{3.2}$$

(In the above equation, we identify c_0 with its representation in local coordinates, i.e., as a complex number $c_0 \in \mathbb{C}$.)

For convenience, we will use a more natural parametrization of C_T from the point of view of real geometry, i.e., $C_T = F(\theta) = (F_1(\theta), F_2(\theta))$, where

$$F_1(\theta) = \rho_0[\alpha \cos\theta \cos\psi - \beta \sin\theta \sin\psi] + x_0, \tag{3.3}$$

$$F_2(\theta) = \rho_0[\alpha \cos\theta \sin\psi + \beta \sin\theta \cos\psi] + y_0. \tag{3.4}$$

Here α, β and ψ incorporate the parameter $c \in H_\infty$; in local coordinates, they are real-analytic functions of c. Precisely, c determines ψ and $\gamma:=\beta/\alpha$, which are scale-invariant parameters. (See Figure 2 for the explicit geometry.) By rotating coordinates, it is no loss of generality to assume that $\alpha \sin\psi \neq 0$, which we will assume in what follows.

Differentiating (3.3) and (3.4), we have

$$F_1'(\theta) = \rho_0[-\alpha \sin\theta \cos\psi - \beta \cos\theta \sin\psi], \tag{3.5}$$

$$F_2'(\theta) = \rho_0[-\alpha \sin\theta \sin\psi + \beta \cos\theta \cos\psi]. \tag{3.6}$$

Let $a=F(\theta_0)$ and $b=F(\theta_1)$ be the points of intersection with ∂K. Write $K=\{(x,y)\in\mathbb{R}^2 : r(x,y)\le 0\}$ with $\nabla r(a), \nabla r(b) \neq (0,0)$. By Proposition 3.4, $\theta_1=\theta_0+\pi$. We rotate coordinates and normalize r so that $\nabla r(a)=(0,1)$ and $\nabla r(b)=(0,-\lambda)$ with $\lambda>0$. Since

$$\nabla r(a)\cdot F'(\theta_0) = \nabla r(b)\cdot F'(\theta_1) = 0,$$

we have

$$F_2'(\theta_0) = F_2'(\theta_1) = 0.$$

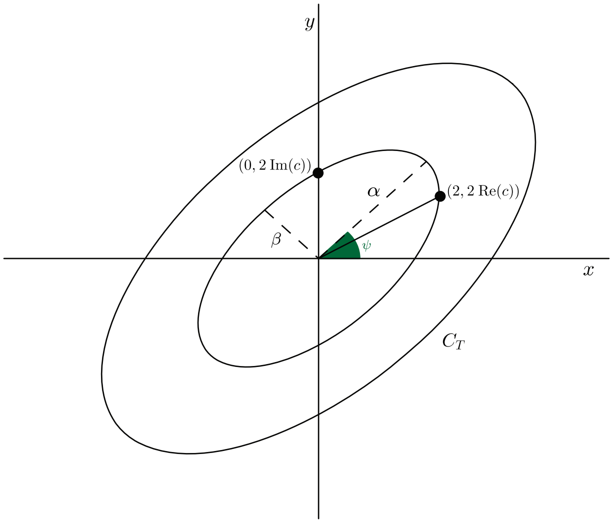

Figure 2. The parameters α, β and ψ and their relation to c in local coordinates. The smaller ellipse is the reference ellipse given by setting $(x_0, y_0)=(0,0)$ and $\rho_0=1$ in (3.2). Then α and β are the lengths of its axes and ψ is the angle between the axis of length α and the horizontal axis. Scaling by ρ_0 (and translating to (x_0, y_0)) yields the extremal ellipse C_T. Note that in local coordinates, a reference ellipse is the unique ellipse, for a given eccentricity and orientation, that is centered at the origin and tangent to the vertical line $x=2$.

Take $a=(0,0)$ and write ∂K near a as $\partial K=\{(s,\eta(s)):|s|<\varepsilon\}$ with $\eta(0)=\eta'(0)=0$. Now we consider variations in s, so that we consider the point $a(s):=(s,\eta(s))$ on ∂K. This determines the normal $\nabla r(a(s))$ and an "antipodal" point $b(s)=(x(s),y(s))\in\partial K$ such that $\nabla r(b(s))=\lambda\nabla r(a(s))$ for some $\lambda=\lambda(s)<0$. We define the parameter $\theta_0(s)$ via the defining relation

$$r_x(s,\eta(s))\frac{\partial F_1}{\partial\theta}(\theta_0(s))+r_y(s,\eta(s))\frac{\partial F_2}{\partial\theta}(\theta_0(s))=0$$

and this defines $\theta_1(s):=\theta_0(s)+\pi$. We also write the center as

$$(x_0(s),y_0(s))=\frac{a(s)+b(s)}{2}=\left(\frac{s}{2}+\frac{x(s)}{2},\frac{\eta(s)}{2}+\frac{y(s)}{2}\right).$$

Allowing ρ and c (i.e., ρ, γ and ψ) to vary, we now consider F as a function

$$F(\theta)=F(\theta_0(s),x_0(s),y_0(s),\rho,\alpha,\beta,\psi).$$

Consider the equations

$$A(s,\rho,\gamma,\psi):=s-F_1(\theta_0(s),x_0(s),y_0(s),\rho,\gamma,\psi),$$
$$B(s,\rho,\gamma,\psi):=\eta(s)-F_2(\theta_0(s),x_0(s),y_0(s),\rho,\gamma,\psi).$$

We get a mapping $(s,\rho,\gamma,\psi)\mapsto(A,B)$ near $(0,\rho_0,\gamma_0,\psi_0)$, where $\rho_0=\rho(c_0)$ for the parameter $c=c_0$ corresponding to γ_0 and ψ_0; that is at $s=0$. Therefore we have $A(0,\rho_0,\gamma_0,\psi_0)=B(0,\rho_0,\gamma_0,\psi_0)=0$,

$$r_x(s,\eta(s))\frac{\partial F_1}{\partial\theta}(\theta_0(s))+r_y(s,\eta(s))\frac{\partial F_2}{\partial\theta}(\theta_0(s))=0,$$

and

$$r_x(x(s),y(s))\frac{\partial F_1}{\partial\theta}(\theta_1(s))+r_y(x(s),y(s))\frac{\partial F_2}{\partial\theta}(\theta_1(s))=0.$$

We want to find conditions for which

$$\det\begin{pmatrix}\dfrac{\partial A}{\partial s} & \dfrac{\partial A}{\partial\rho}\\[2ex] \dfrac{\partial B}{\partial s} & \dfrac{\partial B}{\partial\rho}\end{pmatrix}\Bigg|_{\substack{s=0\\ \rho=\rho_0}}\neq 0. \tag{3.7}$$

Then by the implicit function theorem we can solve for s,ρ near $0,\rho_0$ in terms of γ,ψ (i.e., c) near γ_0,ψ_0.

We write, for simplicity, $\theta_0=\theta_0(0)$ and $\theta_1=\theta_1(0)$ so that

$$\frac{\partial F_2}{\partial\theta}(\theta_0)=\frac{\partial F_2}{\partial\theta}(\theta_1)=0.$$

From (3.6), $(\partial F_2/\partial\theta)(\theta_0)=0$ says that

$$\alpha\sin\theta_0\sin\psi=\beta\cos\theta_0\cos\psi. \tag{3.8}$$

We compute the entries of the matrix in (3.7). Below, prime ($'$) denotes differentiation with respect to s,

$$\frac{\partial A}{\partial s}=1-\frac{\partial F_1}{\partial\theta}\theta'-\frac{\partial F_1}{\partial x_0}x_0'-\frac{\partial F_1}{\partial y_0}y_0'.$$

From (3.3), $\partial F_1/\partial x_0=1$ and $\partial F_1/\partial y_0=0$. Thus

$$\frac{\partial A}{\partial s}=1-\frac{\partial F_1}{\partial\theta}\theta'-x_0'. \tag{3.9}$$

Next,

$$\frac{\partial A}{\partial\rho}=-\frac{\partial F_1}{\partial\rho}=-(\alpha\cos\theta\cos\psi-\beta\sin\theta\sin\psi).$$

Then

$$\frac{\partial B}{\partial s}=\eta'-\frac{\partial F_2}{\partial\theta}\theta'-\frac{\partial F_2}{\partial x_0}x_0'-\frac{\partial F_2}{\partial y_0}y_0'=\eta'-\frac{\partial F_2}{\partial\theta}\theta'-y_0'$$

since (3.4) implies $\partial F_2/\partial x_0=0$ and $\partial F_2/\partial y_0=1$. Moreover, at $s=0$, we have $\eta'(0)=0$ and $(\partial F_2/\partial\theta)(\theta_0)=0$ so that

$$\frac{\partial B}{\partial s}\bigg|_{s=0}=y_0'(0).$$

But $y_0(s)=\frac{1}{2}(\eta(s)+y(s))$ so that

$$y_0'(0)=\tfrac{1}{2}(\eta'(0)+y'(0))=0.$$

Thus $(\partial B/\partial s)|_{s=0}=0$.

On the other hand, we claim that if $\alpha\sin\psi\neq 0$, then $(\partial B/\partial\rho)|_{s=0}\neq 0$. By (3.4),

$$\frac{\partial B}{\partial\rho}=-\frac{\partial F_2}{\partial\rho}=-(\alpha\cos\theta\sin\psi+\beta\sin\theta\cos\psi).$$

If on the contrary, $(\partial B/\partial\rho)|_{s=0}=0$, then using (3.8),

$$\begin{aligned}\alpha\sin\theta_0\sin\psi&=\beta\cos\theta_0\cos\psi,\\ \alpha\cos\theta_0\sin\psi&=-\beta\sin\theta_0\cos\psi.\end{aligned}$$

Multiplying the top equation by $\sin\theta_0$ and the bottom one by $\cos\theta_0$ and adding, we obtain $\alpha\sin\psi=0$, a contradiction. Thus (3.7) holds precisely when $(\partial A/\partial s)|_{s=0}\neq 0$.

We now show that

$$\frac{\partial A}{\partial s}\bigg|_{s=0}=\frac{1}{2}+\left(\frac{\partial F_1}{\partial\theta}(\theta_0)\right)^2\frac{r_{xx}(0,0)}{y_0}-\frac{1}{2}\frac{r_{xx}(0,0)r_y(x(0),y(0))}{r_{xx}(x(0),y(0))r_y(0,0)}. \tag{3.10}$$

To see this, recall first that $r_x(0,0)=0$ and $r_y(0,0)=1$. Moreover,

$$r_x(s,\eta(s))\frac{\partial F_1}{\partial\theta}(\theta(s))+r_y(s,\eta(s))\frac{\partial F_2}{\partial\theta}(\theta(s))=0;$$

differentiating this equation with respect to s we get

$$(r_{xx}+r_{yx}\eta')\frac{\partial F_1}{\partial\theta}+r_x\frac{\partial^2F_1}{\partial\theta^2}\theta'+(r_{xy}+r_{yy}\eta')\frac{\partial F_2}{\partial\theta}+r_y\frac{\partial^2F_2}{\partial\theta^2}\theta'=0.$$

Now at $s=0$, $\eta'(0)=(\partial F_2/\partial\theta)(\theta_0)=r_x(0,0)=0$; moreover, writing $F_2:=J+y_0$ we see that $\partial^2F_2/\partial\theta^2=-J$ so that

$$\frac{\partial^2F_2}{\partial\theta^2}(\theta_0)=-J(\theta_0)=y_0-F_2(\theta_0)=y_0.$$

Hence

$$r_{xx}(0,0)\frac{\partial F_1}{\partial\theta}(\theta_0)+y_0\theta'(0)=0$$

and so

$$\theta'(0) = \frac{-r_{xx}(0,0)\dfrac{\partial F_1}{\partial\theta}(\theta_0)}{y_0}. \tag{3.11}$$

To compute/rewrite $x_0'(0)$, we use the fact that $\nabla r(s,\eta(s))=\lambda\nabla r(x(s),y(s))$. This implies the relation

$$r_x(x(s),y(s))\cdot r_y(s,\eta(s))-r_y(x(s),y(s))\cdot r_x(s,\eta(s))=0.$$

Differentiate this with respect to s, and set $s=0$ gives

$$\begin{aligned}(r_{xx}(x(0),y(0))\cdot x'(0)+r_{xy}(x(0),y(0))\cdot y'(0))r_y(0,0)&\\ -(r_{xx}(0,0)+r_{xy}(0,0)\cdot\eta'(0))r_y(x(0),y(0))&=0.\end{aligned}$$

Here we have used the fact(s) that $r_x(0,0)=r_x(x(0),y(0))=0$. But we also have $\eta'(0)=y'(0)=0$, $r_y(0,0)=1$, and $r_y(x(0),y(0))=1/\lambda$; and so

$$r_{xx}(x(0),y(0))\cdot x'(0)-\frac{1}{\lambda}r_{xx}(0,0)=0;$$

i.e.,

$$x'(0)=\frac{r_{xx}(0,0)}{\lambda r_{xx}(x(0),y(0))}.$$

Now $x_0(s)=\frac{1}{2}(s+x(s))$, so $x_0'(0)=\frac{1}{2}(1+x'(0))$, and so

$$x_0'(0)=\frac{1}{2}+\frac{1}{2}\frac{r_{xx}(0,0)}{\lambda r_{xx}(x(0),y(0))}. \tag{3.12}$$

Plugging (3.11) and (3.12) into (3.9) yields (3.10).

We now analyze the situation when $(\partial A/\partial s)|_{s=0}=0$. From (3.10), this occurs precisely when

$$\begin{aligned}\frac{\left(\dfrac{\partial F_1}{\partial\theta}(\theta_0)\right)^2}{y_0}&=\frac{1}{2}\left(\frac{-1}{r_{xx}(0,0)}+\frac{r_y(x(0),y(0))}{r_y(0,0)r_{xx}(x(0),y(0))}\right)\\ &=\frac{1}{2}\left(\frac{-1}{r_{xx}(0,0)}+\frac{r_y(x(0),y(0)}{r_{xx}(x(0),y(0))}\right).\end{aligned} \tag{3.13}$$

Since $r_x(0,0)=r_x(x(0),y(0))=0$, $r_y(0,0)=1$ and $r_y(x(0),y(0))<0$, the right-hand side of (3.13) is minus the average of the radii of the osculating circles of ∂K at $a=(0,0)$ and $b=(x(0),y(0))$.

We claim that the left-hand side of (3.13) is minus the average of the radii of the osculating circles of C_T at a and b. To see this, note from (3.4) and (3.5) (and $(\partial^2 F_2/\partial\theta^2)(\theta_0)=y_0$), that

$$\frac{\left(\dfrac{\partial F_1}{\partial\theta}(\theta_0)\right)^2}{y_0}=\frac{\rho(-\alpha\sin\theta_0\cos\psi-\beta\cos\theta_0\sin\psi)^2}{-(\alpha\cos\theta_0\sin\psi+\beta\sin\theta_0\cos\psi)}.$$

Let us rotate coordinates so that $\sin\psi=1$; i.e., $\psi=\pi/2$, and assume $\theta_0=0$. Then

$$\frac{\left(\dfrac{\partial F_1}{\partial\theta}(\theta_0)\right)^2}{y_0}=\frac{\rho\beta^2}{-\alpha}.$$

On the other hand, C_T now has the parametrization

$$F_1(\theta)=-\rho\beta\sin\theta+x_0 \quad\text{and}\quad F_2(\theta)=\rho\alpha\cos\theta+y_0,$$

and the curvature of C_T as a function of θ is

$$\varkappa(\theta)=\frac{1}{\rho}\frac{\alpha\beta}{(\beta^2\cos^2(\theta)+\alpha^2\sin^2\theta)^{3/2}}.$$

At $\theta=0,\pi$ we get

$$\varkappa(0)=\varkappa(\pi)=\frac{\alpha}{\beta^2\rho}$$

as claimed (precisely, $-\frac{1}{2}(1/\varkappa(0)+1/\varkappa(\pi))=-\frac{1}{2}(2\beta^2\rho/\alpha)=-\rho\beta^2/\alpha$).

Remark 3.5. (i) Note that the radii of the osculating circles for ∂K at the points a and b are at least as large as those for C_T since C_T is inscribed in ∂K. Thus the condition $(\partial A/\partial s)|_{s=0}=0$ fails if the curvature of C_T is strictly less than that of ∂K at either a or b.

(ii) A degenerate ellipse (i.e. line segment) occurs when $\beta=0$. A careful examination of the preceding calculations shows that (3.7) always holds in this case.

When ∂K is C^r, the implicit function theorem shows that s and ρ can be solved in terms of γ and ψ (equivalently, $c\in H_\infty$) as C^r functions. This implies that locally, the center $(x_0(s),y_0(s))$ is a C^r function of c, and must therefore coincide with $a(c)$ given in (2.4). Similarly, the scale factor $\rho(c)$ is also a C^r function of c. Altogether, this shows that the foliation of extremal ellipses near C_T is C^r.

The smoothness of the foliation at a point $z\in\mathbb{C}^n\setminus K$ in turn implies the smoothness of V_K at z, as the partial derivatives $\partial/\partial z_j$ may be computed explicitly in terms of foliation parameters using the chain rule. We summarize this in the following theorem.

Theorem 3.6. *Suppose* $z \in \mathbb{C}^2 \setminus K$ *lies on an extremal ellipse* C *for* K *with the following properties*:

(1) *The intersection* $\partial K \cap C$ *is exactly two points.*

(2) ∂K *is* C^r, $r \geq 2$, *in a neighborhood of* $\partial K \cap C$.

(3) *For at least one of these intersection points, the curvature of* C_T *is strictly greater than the curvature of* ∂K *at this point.*

Then V_K *is* C^r *in a neighborhood of* z.

Remark 3.7. Parameters $c \in H_\infty$ corresponding to extremal curves for V_K that intersect ∂K in two points but do not satisfy the curvature condition (3) form a set of real dimension at most 1. One way to see this is to consider the collection of all extremal ellipses $\mathcal{C}_a$ that intersect ∂K at a point a. Then $\mathcal{C}_a$ is parametrized by a subset of (real) dimension 1 in H_∞, and $\bigcup_{a \in \partial K} \mathcal{C}_a = \mathbb{C}^2$. For each a, however, there is at most one ellipse parameter c_a for which the curvature of the extremal ellipse coincides with the curvature of ∂K. Now as a varies smoothly over the curve ∂K, c_a varies smoothly over a one-dimensional subset of H_∞. Hence $\bigcup_{a \in \partial K} \{c_a\}$ is at most 1-dimensional.

Example 3.8. Consider K to be the real unit disk, $K = \{(x, y) \in \mathbb{R}^2 \subset \mathbb{C}^2 : x^2 + y^2 \leq 1\}$. Then by Lundin's formula [16],

$$V_K(z) = \tfrac{1}{2} \log^+(|z_1|^2 + |z_2|^2 + |z_1^2 + z_2^2 - 1|).$$

On $\mathbb{C}^2 \setminus K$ this function is nonsmooth precisely on the complex ellipse C given by $z_1^2 + z_2^2 = 1$. In this case $C_T = \partial K$, so trivially the curvatures are equal.

3.3. Extremal ellipses meeting ∂K in three points

Suppose an extremal ellipse C meets ∂K in three points:

$$C \cap \partial K = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}.$$

In this case, the ellipse is necessarily nondegenerate.

As before, we parametrize C_T via $F(\theta) = (F_1(\theta), F_2(\theta))$ as in (3.3) and (3.4). For $j = 1, 2, 3$ define $\theta_j \in [0, 2\pi)$ via $F(\theta_j) = (x_j, y_j)$. We may assume that $\alpha = 1$, otherwise replace β with β/α and ρ with $\rho\alpha$.

We want to analyze this setup under variations of β and ψ (i.e., c). We will assume that coordinates have been chosen so that ∂K can be represented as a graph over either x or y near each point (x_j, y_j); in particular, we will assume that for each $j = 1, 3$ there exist smooth functions η_j with $\partial K = \{(x, y) : y = \eta_j(x)\}$ near (x_j, y_j) and $\partial K = \{(x, y) : x = \eta_2(y)\}$ near (x_2, y_2) (see Figure 3).

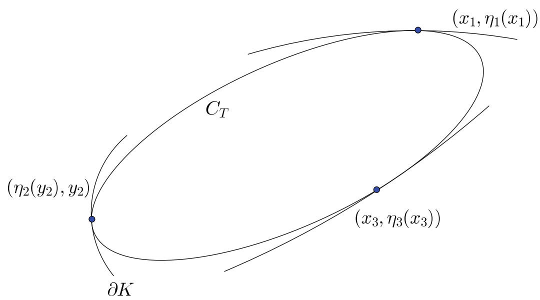

Figure 3. Local parametrizations of ∂K.

Now consider variations in β and ψ, and consider the variables ρ, x_0, y_0 and x_j, y_j, θ_j, $j=1,2,3$, to be dependent on these variations. In total, there are 12 dependent variables.

We will eliminate eight of these variables. Using the functions η_j, we immediately eliminate y_1, x_2 and y_3. We now proceed to eliminate θ_j, $j=1,2,3$. We use the fact that the intersection $C_T\cap\partial K$ is tangential at each point (x_j, y_j). For $j=1$, this says that

$$(F_1'(\theta_1), F_2'(\theta_1)) = \lambda(1, \eta_1'(x_1));$$

i.e., $F_1'(\theta_1)\eta_1'(x_1)=F_2'(\theta_1)$. Explicitly, using (3.5) and (3.6) we obtain

$$\tan\theta_1 = \frac{\beta(\cos\psi+\eta_1'(x_1)\sin\psi)}{\sin\psi-\eta_1'(x_1)\cos\psi}. \tag{3.14}$$

Locally, we may take the principal branch of arctan (that gives angles in $[0,2\pi)$) to obtain the function $\theta_1(x_1,\beta,\psi)$, and hence eliminate θ_1 as a dependent variable. Similarly we can do the same for θ_2 and θ_3.

The last two variables we will eliminate are x_0 and y_0. First, define the following functions (for notational convenience we suppress their dependence on the variables β, ψ, ρ, x_0 and y_0):

$$\begin{aligned} A_1(x_1)&=F_1(\theta_1)-x_1, & B_1(x_1)&=F_2(\theta_1)-\eta_1(x_1),\\ A_2(y_2)&=F_1(\theta_2)-\eta_2(y_2), & B_2(y_2)&=F_2(\theta_2)-y_2,\\ A_3(x_3)&=F_1(\theta_3)-x_3, & B_3(x_3)&=F_2(\theta_3)-\eta_3(x_3). \end{aligned}$$

The geometric condition that the points (x_j, y_j) are intersections of C_T with ∂K says that $A_j=B_j=0$, $j=1,2,3$.

Define $S_1(\theta):=F_1(\theta)-x_0$ and $S_2(\theta):=F_2(\theta)-y_0$; then $A_3=B_3=0$ says that

$$x_0=x_3-S_1(\theta_3) \quad \text{and} \quad y_0=\eta_3(x_3)-S_2(\theta_3).$$

Using this to eliminate x_0 and y_0, the system of equations reduces to

$$\begin{aligned} A_1&=S_1(\theta_1)-S_1(\theta_3)+x_3-x_1, & B_1&=S_2(\theta_1)-S_2(\theta_3)+\eta_3(x_3)-\eta_1(x_1),\\ A_2&=S_1(\theta_2)-S_1(\theta_3)+x_3-\eta_2(y_2), & B_2&=S_2(\theta_2)-S_2(\theta_3)+\eta_3(x_3)-y_2. \end{aligned}$$

In summary, we have a map $M\colon (x_1,y_2,x_3,\rho,\beta,\psi)\mapsto(A_1,A_2,B_1,B_2)$, where the geometric condition that C_T is inscribed in ∂K implies that $M=0$.

We can solve for x_1, y_2, x_3, and ρ in terms of β and ψ provided the Jacobian matrix

$$JM=\begin{pmatrix} \frac{\partial A_1}{\partial x_1} & \frac{\partial A_1}{\partial y_2} & \frac{\partial A_1}{\partial x_3} & \frac{\partial A_1}{\partial \rho}\\ \frac{\partial B_1}{\partial x_1} & \frac{\partial B_1}{\partial y_2} & \frac{\partial B_1}{\partial x_3} & \frac{\partial B_1}{\partial \rho}\\ \frac{\partial A_2}{\partial x_1} & \frac{\partial A_2}{\partial y_2} & \frac{\partial A_2}{\partial x_3} & \frac{\partial A_2}{\partial \rho}\\ \frac{\partial B_2}{\partial x_1} & \frac{\partial B_2}{\partial y_2} & \frac{\partial B_2}{\partial x_3} & \frac{\partial B_2}{\partial \rho} \end{pmatrix}$$

has nonzero determinant. Note that $\partial A_1/\partial y_2=\partial B_1/\partial y_2=\partial A_2/\partial x_1=\partial B_2/\partial x_1=0$.

Fix an initial inscribed ellipse, and denote its parameters by x_{j0},y_{j0}, $j=1,2,3$, and ρ_0. To simplify the computations, without loss of generality we may assume that

$$\eta_1'(x_1)|_{x_1=x_{10}}=0, \quad \eta_2'(y_2)|_{y_2=y_{20}}=0 \quad \text{and} \quad \eta_3'(x_3)|_{x_3=x_{30}}>0$$

by applying a linear change of coordinates (see Figure 3). In these coordinates, the tangency of C_T at its intersections with ∂K says that $S_2'(\theta_1)|_{\theta_1=\theta_1(x_{10})}=0$, and so

$$\text{(3.15)} \qquad \frac{\partial B_1}{\partial x_1}\bigg|_{x_1=x_{10}}=\left[S_2'(\theta_1)\frac{\partial\theta_1}{\partial x_1}-\eta_1'(x_1)\right]_{x_1=x_{10}}=0;$$

a similar argument also gives $(\partial A_2/\partial y_2)|_{y_2=y_{20}}=0$. Hence

$$\det(JM) = \det \begin{pmatrix} \dfrac{\partial A_1}{\partial x_1} & 0 & \dfrac{\partial A_1}{\partial x_3} & \dfrac{\partial A_1}{\partial \rho} \\ 0 & 0 & \dfrac{\partial B_1}{\partial x_3} & \dfrac{\partial B_1}{\partial \rho} \\ 0 & 0 & \dfrac{\partial A_2}{\partial x_3} & \dfrac{\partial A_2}{\partial \rho} \\ 0 & \dfrac{\partial B_2}{\partial y_2} & \dfrac{\partial B_2}{\partial x_3} & \dfrac{\partial B_2}{\partial \rho} \end{pmatrix} = \frac{\partial A_1}{\partial x_1}\frac{\partial B_2}{\partial y_2}\det(M_1), \tag{3.16}$$

where

$$M_1 = \begin{pmatrix} \dfrac{\partial B_1}{\partial x_3} & \dfrac{\partial B_1}{\partial \rho} \\ \dfrac{\partial A_2}{\partial x_3} & \dfrac{\partial A_2}{\partial \rho} \end{pmatrix}.$$

We derive conditions under which each factor on the right-hand side of (3.16) is nonzero. First,

$$\frac{\partial A_1}{\partial x_1} = S_1'(\theta_1)\frac{\partial \theta_1}{\partial x_1} - 1, \tag{3.17}$$

$$\frac{\partial B_2}{\partial y_2} = S_2'(\theta_2)\frac{\partial \theta_2}{\partial y_2} - 1. \tag{3.18}$$

We analyze $\det(M_1)$. For convenience, translate coordinates to the origin, i.e., put $(x_{30}, y_{30}) = (0,0)$. We then rotate coordinates as follows. Let $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ denote the standard basis in $\mathbb{R}^2$; let

$$R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},$$

where $\alpha \in (0, \pi/2)$ is given by $\tan\alpha = \eta_3'(0)$; now let $\mathbf{e}_1' = R_\alpha^{-1}\mathbf{e}_1$ and $\mathbf{e}_2' = R_\alpha^{-1}\mathbf{e}_2$. Let $(\,\cdot\,)_{R_\alpha}$ denote coordinates and matrices written with respect to this new basis, e.g.,

$$(\tilde{x}, \tilde{y})_{R_\alpha} = \tilde{x}\mathbf{e}_1' + \tilde{y}\mathbf{e}_2'.$$

In these coordinates, the common tangent to ∂K and C_T at $0 = (0,0)_{R_\alpha}$ has no second (i.e., $\mathbf{e}_2'$) component. This says that $\tilde{\eta}_3'(0) = 0$, where $(\tilde{x}, \tilde{\eta}_3(\tilde{x}))_{R_\alpha} = (x, \eta_3(x))$.

In what follows, tilded quantities (e.g., $\widetilde{S}_1$, $\widetilde{S}_2$) denote quantities expressed with respect to rotated coordinates. We calculate that

$$\begin{aligned}
R_\alpha \begin{pmatrix} \dfrac{\partial A_2}{\partial x_3} \\ \dfrac{\partial B_1}{\partial x_3} \end{pmatrix} &= R_\alpha \begin{pmatrix} -1+S_1'(\theta_3)\dfrac{\partial\theta_3}{\partial x_3} \\ -\eta_3'(x_3)+S_2'(\theta_3)\dfrac{\partial\theta_3}{\partial x_3} \end{pmatrix} \\
&= R_\alpha \begin{pmatrix} -1+\widetilde{S}_1'(\theta_3)\dfrac{\partial\theta_3}{\partial \tilde{x}_3} \\ -\tilde{\eta}_3'(\tilde{x}_3)+\widetilde{S}_2'(\theta_3)\dfrac{\partial\theta_3}{\partial \tilde{x}_3} \end{pmatrix}_{R_\alpha} \dfrac{\partial \tilde{x}_3}{\partial x_3} \\
&= R_\alpha R_\alpha^{-1} \begin{pmatrix} -1+\widetilde{S}_1'(\theta_3)\dfrac{\partial\theta_3}{\partial \tilde{x}_3} \\ 0 \end{pmatrix} \cos\alpha,
\end{aligned}$$

where zero in the bottom component above follows from $\tilde{\eta}_3'(0)=0$, by the same argument that gave (3.15) earlier.

Define B_{R_α} and A_{R_α} by

$$\begin{pmatrix} A_{R_\alpha} \\ B_{R_\alpha} \end{pmatrix} := R_\alpha \begin{pmatrix} \dfrac{\partial A_2}{\partial \rho} \\ \dfrac{\partial B_1}{\partial \rho} \end{pmatrix}.$$

With $E=\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we have

$$\begin{aligned}
\det(M_1) = -\det(ER_\alpha M_1) &= \det \begin{pmatrix} 0 & B_{R_\alpha} \\ \left(1-\widetilde{S}_1'(\theta_3)\dfrac{\partial\theta_3}{\partial\tilde{x}_3}\right)\cos\alpha & A_{R_\alpha} \end{pmatrix} \\
&= \left(\widetilde{S}_1'(\theta_3)\dfrac{\partial\theta_3}{\partial\tilde{x}_3}-1\right)(\cos\alpha) B_{R_\alpha}.
\end{aligned}$$

Therefore, $\det(JM)\neq 0$ holds if and only if none of

$$\widetilde{S}_1'(\theta_3)\frac{\partial\theta_3}{\partial\tilde{x}_3} = 1 \quad \text{and} \tag{3.19}$$

$$B_{R_\alpha} = 0 \tag{3.20}$$

hold, and the right-hand sides of (3.17) and (3.18) are nonzero.

We analyze (3.17) when its right-hand side is zero. Differentiating (3.3) and (3.14) (the latter implicitly), we obtain

$$1=S_1'(\theta_1)\frac{\partial\theta_1}{\partial x_1}=-\rho\beta\eta_1''(x_1)\frac{\cos^2\theta_1(\sin\theta_1\cos\psi+\beta\cos\theta_1\sin\psi)}{(\sin\psi-\eta_1'(x_1)\cos\psi)^2}.$$

Using the fact that $\eta_1'(x_1)=0$, we can simplify this to

$$\frac{1}{\eta_1''(x_1)}=-\rho\beta\left(\frac{\sin\theta_1\cos^2\theta_1}{\tan\psi\sin\psi}+\frac{\beta\cos^3\theta_1}{\sin\psi}\right),$$

and also simplify (3.14) to $\tan\psi=\beta/\tan\theta_1$; hence

$$\sin\psi=\frac{\beta\cos\theta_1}{\sqrt{\beta^2\cos^2\theta_1+\sin^2\theta_1}}.$$

Eliminating ψ, we obtain

$$\begin{aligned}\frac{1}{\eta_1''(x_1)}&=-\frac{\rho}{\beta}\sqrt{\beta^2\cos^2\theta_1+\beta^2\cos^2\theta_1(\sin^2\theta_1+\beta^2\cos^2\theta_1)}\\&=-\frac{\rho}{\beta}(\sin^2\theta_1+\beta^2\cos^2\theta_1)^{3/2}.\end{aligned}$$

Now, note that $\varkappa_{\partial K}(x_1)=-\eta_1''(x_1)$ and $\varkappa_{C_T}(x_1)=(\beta/\rho)(\sin^2\theta_1+\beta^2\cos^2\theta_1)^{-3/2}$, where $\varkappa_{\partial K}(x_1)$ (resp., $\varkappa_{C_T}(x_1)$) denotes the curvature of ∂K (resp., C_T) at the point x_1. Hence

$$S_1'(\theta_1)\frac{\partial\theta_1}{\partial x_1}=1\quad\Longleftrightarrow\quad\varkappa_{C_T}(x_1)=\varkappa_{\partial K}(x_1).$$

By (3.17), the above condition in turn is equivalent to $\partial A_1/\partial x_1=0$.

Similar calculations as above show that

$$\begin{aligned}S_2'(\theta_2)\frac{\partial\theta_2}{\partial y_2}=1\quad&\Longleftrightarrow\quad\varkappa_{C_T}(x_2)=\varkappa_{\partial K}(x_2),\\ \widetilde{S}_1'(\theta_3)\frac{\partial\theta_3}{\partial\tilde{x}_3}=1\quad&\Longleftrightarrow\quad\varkappa_{C_T}(x_3)=\varkappa_{\partial K}(x_3).\end{aligned}$$

Using (3.17), (3.18) and (3.19), we obtain the following geometric criterion:

($\star$) *If* $\det JM\neq0$ *then the curvature of* C_T *is strictly greater than the curvature of* ∂K *at each of the three intersection points* (x_j,y_j), $j=1,2,3$.

It remains to show that (3.20) always fails. For if not, then

$$0=B_{R_\alpha}=(S_2(\theta_1)-S_2(\theta_3))\cos\alpha-(S_1(\theta_2)-S_1(\theta_3))\sin\alpha,$$

i.e.,

$$\tan\alpha=\frac{S_2(\theta_1)-S_2(\theta_3)}{S_1(\theta_2)-S_1(\theta_3)}.$$

However, interpreting each side of the above equation geometrically, $\tan\alpha=\eta_3'(x_3)$ is the slope of ∂K at (x_3,y_3), while the right-hand-side of the equation is minus the slope of the line connecting $\mathbf{v}_1$ and $\mathbf{v}_2$, where for $j=1,2$, $\mathbf{v}_j$ denotes the closest point to (x_3,y_3) that lies on the tangent line to ∂K through (x_j,y_j). Hence $\tan\alpha>0>(S_2(\theta_1)-S_2(\theta_3))/(S_1(\theta_2)-S_1(\theta_3))$, a contradiction. So (3.20) fails.

This shows that if condition $(\star)$ holds, then $\det(JM)\neq 0$ and locally we may solve for (x_1,y_2,x_3,ρ) as functions of (β,ψ), and hence for ρ and $a_0=(x_0,y_0)$ as functions of c. If ∂K is C^r, $r\geq 2$, then the foliation for V_K is locally C^r at points on C, and hence so is V_K.

We have proved the following result.

Theorem 3.9. *Suppose $z\in\mathbb{C}^2\setminus K$ lies on an extremal ellipse C for K with the following properties*:

(1) *The intersection $\partial K\cap C$ is exactly three points.*

(2) *∂K is C^r, $r\geq 2$, in a neighborhood of $\partial K\cap C$.*

(3) *The curvature of C_T is strictly greater than the curvature of ∂K at each of the three intersection points.*

Then V_K is C^r in a neighborhood of z.

Similar reasoning as in Remark 3.7 shows that the parameters for which condition (1) holds but condition (3) fails form a subset of H_∞ of at most one real dimension.

Extremal ellipses meeting ∂K in more than three points

The same reasoning also shows that the parameters for extremal ellipses meeting ∂K in at least four points form a lower-dimensional subset. Note that V_K may not be smooth across such ellipses.

Example 3.10. For the unit square $S=[-1,1]\times[-1,1]$, we have by Lemma 3.1 that $V_S(z)=\log|h(z_1)|=\log|h(z_2)|$ if $z=(z_1,z_2)$ lies on an extremal ellipse that intersects all four sides. In $\mathbb{C}^2\setminus S$, V_S is not smooth precisely on the set where $|h(z_1)|=|h(z_2)|$, which is a submanifold of real dimension 3.

The results of this section may be summarized in the following theorem.

Theorem 3.11. *Let $K \subset \mathbb{R}^2 \subset \mathbb{C}^2$ be a convex body whose boundary ∂K is C^r-smooth ($r \in \{2, 3, ...\} \cup \{\infty, \omega\}$). Then V_K is C^r on $\mathbb{C}^2 \setminus K$ except for a set of real dimension at most* 3.

Proof. A point at which V_K is not smooth must lie on an extremal ellipse C that satisfies one of the following conditions:

- C satisfies property (1) but not (3) in Theorem 3.6;
- C satisfies property (1) but not (3) in Theorem 3.9;
- C meets ∂K in at least 4 points.

A collection of ellipses that satisfies one of the above conditions forms at most a (real) one-parameter family; so the union of these ellipses is at most a real 3-dimensional set. □

That V_S in Example 3.10 is a maximum of smooth functions is an instance of a more general phenomenon. Given a symmetric convex set K, suppose $z \in C$, where C is an extremal ellipse that intersects K in four points, and suppose that the curvature of C_T is strictly greater than that of ∂K at these points. We enlarge K in two different ways to obtain convex sets K_1 and K_2 with the following properties:

- $K = K_1 \cap K_2$;
- C is an extremal ellipse for each of K_1 and K_2;
- on some neighborhood of $C \setminus K$, V_{K_j} is smooth for each $j = 1, 2$ and

$$V_K = \max\{V_{K_1}, V_{K_2}\}. \tag{3.21}$$

Theorem 3.11 gives the smoothness of the V_{K_j}'s in a neighborhood of $C \setminus \mathbb{R}^2$. Figure 4 illustrates this method when ∂K is given by $x^4 + y^4 = 1$. Since V_K is the maximum of two functions, it is not a priori smooth across C (and we expect nonsmoothness in general). For ∂K given by $x^{2n} + y^{2n} = 1$, where $n > 1$, the ellipses that intersect ∂K in four points form a real 3-dimensional set in $\mathbb{C}^2$ in which we expect V_K to be nonsmooth.

Note that the same sort of argument works in other cases, e.g. C_T intersects ∂K in more than four points and/or K is not symmetric. One can show that V_K is locally a maximum of smooth functions by considering enlargements of K to convex sets whose boundaries each meet C_T in three points, and then taking their extremal functions.

The same argument can also be used to get local smooth approximations: given $\varepsilon > 0$ and $z_0 \in \mathbb{C}^2 \setminus K$, one can construct a convex set K_ε with the property that for all z in some neighborhood of z_0, V_{K_ε} is smooth and $V_{K_\varepsilon}(z) \le V_K(z) \le V_{K_\varepsilon}(z) + \varepsilon$.

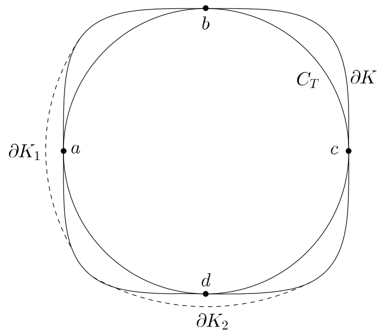

Figure 4. The construction of K_1 and K_2 by locally modifying ∂K. The real ellipse $C_T = C \cap K$ intersects ∂K_1 in $\{b, c, d\}$ and ∂K_2 in $\{a, b, c\}$. Equation (3.21) holds near C because an extremal ellipse for K whose parameters are sufficiently close to those of C is also an extremal ellipse for one of the K_j's.

For any compact convex set K, it seems plausible that one can make a finite number of local boundary modifications as in Figure 4 to remove the 'bad' conditions listed in the proof of Theorem 3.11, at least q.e.(1) This motivates the following conjecture.

Conjecture. *Let $K \subset \mathbb{R}^2$ be a convex body with smooth boundary.*

(1) *There is a finite collection $\{K_j\}_j$ of convex bodies with the property that V_{K_j} is smooth q.e. on $\mathbb{C}^2 \setminus K_j$ for each j, and $V_K = \max_j V_{K_j}$.*

(2) *Given $\varepsilon > 0$ there is a convex body K_ε such that V_{K_ε} is smooth q.e. on $\mathbb{C}^2 \setminus K_\varepsilon$ and $|V_K(z) - V_{K_\varepsilon}(z)| < \varepsilon$ for all $z \in \mathbb{C}^2$.*

Remark 3.12. The above conjecture is (trivially) true for the real disk, whose extremal function is smooth away from $z_1^2 + z_2^2 = 1$. It is not known if there is a real convex set whose extremal function is smooth everywhere on its complement.

4. The complex equilibrium measure and the Robin exponential map

In this section, we relate the complex equilibrium (or Monge–Ampère) measure of a convex set K to that of its Robin indicatrix, defined below.

Given a compact set $K \subset \mathbb{C}^n$, the Robin function ρ_K of K is the logarithmically homogeneous, psh function given by $\rho_K(z) = \limsup_{|\lambda| \to \infty} [V_K^*(\lambda z) - \log |\lambda|]$. The

(1) Recall that a property holds q.e. = *quasi-everywhere* if it holds everywhere outside a (possibly nonempty) pluripolar set.

Robin indicatrix of K is the set given by

$$K_\rho = \{z \in \mathbb{C}^n : \rho_K(z) \le 0\}. \tag{4.1}$$

Let D be a bounded, strictly lineally convex domain with smooth boundary. We recall some basic facts concerning Lempert extremal curves for $V_{\overline{D}}$; i.e., holomorphic curves which foliate $\mathbb{C}^n \setminus \overline{D}$ on which $V_{\overline{D}}$ is harmonic, and the Robin indicatrix of $\overline{D}$ (cf., [8], [15] and [17]). Recall that $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ is the open unit disk.

Proposition 4.1. *Let* $K = \overline{D}$, *where* D *is a bounded, strictly lineally convex body in* $\mathbb{C}^n$ *with smooth boundary. Then the following are true*:

(1) *A Lempert extremal curve may be represented as* $f : \mathbb{C} \setminus \Delta \to \mathbb{C}^n \setminus K$, *with Laurent expansion*

$$f(\zeta) = a_1\zeta + \sum_{j \le 0} a_j \zeta^j, \quad a_j \in \mathbb{C}^n, \ a_1 \ne 0, \tag{4.2}$$

and $V_K(f(\zeta)) = \log|\zeta|$.

(2) *A Lempert extremal curve may be extended continuously to a map on* $\partial\Delta$ *with* $f(\partial\Delta) \subset \partial D$.

(3) *A Lempert extremal curve is orthogonal to the level sets of* V_K. *Precisely, if* $z = f(\zeta) \in \mathbb{C}^n \setminus K$, *then the complex hyperplane* H_z *given by*

$$H_z = \{z + w : w \cdot t f'(\zeta) = 0 \text{ for all } t \in \mathbb{C}\}$$

is tangent to the level set of V_K *at* z.

(4) *If* $v \in \partial K_\rho$ *then* $v = \lim_{|\zeta| \to \infty} f(\zeta)/\zeta$ *for some* f *that parametrizes an extremal curve for* V_K. *We obtain the same extremal curve for* $w \in \partial K_\rho$ *if and only if* $w = ve^{i\theta}$: *in this case*, $w = \lim_{|\zeta| \to \infty} g(\zeta)/\zeta$, *where* $g(\zeta) = f(\zeta e^{-i\theta})$.

(5) *There exists a smooth diffeomorphism* $F : \mathbb{C}^n \setminus K_\rho \to \mathbb{C}^n \setminus K$ *such* $\rho_K(z) = V_K(F(z))$, *and for any Lempert extremal disk parametrized as in* (4.2), *we have* $F(a_1\zeta) = f(\zeta)$.

The smooth diffeomorphism F in (5) is called the *Robin exponential map*. The set K_ρ together with the complex lines through the origin may be regarded as a linearized model of K and the associated foliation for V_K.

By part (2), we can extend the Robin exponential map to ∂K_ρ via $F(a_1 e^{i\theta}) = f(e^{i\theta})$. Part (4) ensures that the map is well-defined.

Given a bounded, strictly lineally convex domain D with smooth boundary, let $V = V_K$ be the extremal function of its closure $K = \overline{D}$. For $\lambda \in (1, \infty)$, write

$$D_\lambda = \{z \in \mathbb{C}^n : V(z) < \log\lambda\}$$

for the sublevel sets of V.

The main ingredient to relate the equilibrium measure of K with that of its Robin indicatrix K_ρ is the following "transfer of mass" formula.

Lemma 4.2. (Transfer to a level set) *Let D be as above, and suppose ψ is a continuous function on $\mathbb{C}^n\setminus D$ with the property that $\psi(f(\zeta))=\psi(f(\zeta/|\zeta|))$ for all Lempert extremal disks $f\colon\Delta\to\mathbb{C}^n\setminus D$. Then for all $1<\lambda_1<\lambda_2<\infty$,*

$$\int_{\partial D_{\lambda_1}} \psi\, d^cV\wedge(dd^cV)^{n-1} = \int_{\partial D_{\lambda_2}} \psi\, d^cV\wedge(dd^cV)^{n-1}. \tag{4.3}$$

Proof. Let us first carry out the proof under the assumption that ψ is smooth. By Stokes' theorem,

$$\begin{aligned}\int_{-\partial D_{\lambda_1}\cup\partial D_{\lambda_2}} \psi\, d^cV\wedge(dd^cV)^{n-1} &= \int_{D_{\lambda_2}\setminus\bar{D}_{\lambda_1}} \psi(dd^cV)^n \\ &\quad+\int_{D_{\lambda_2}\setminus\bar{D}_{\lambda_1}} d\psi\wedge d^cV\wedge(dd^cV)^{n-1} \\ &:= I+II.\end{aligned}$$

Here $-\partial D_{\lambda_1}$ means that we use the opposite orientation on ∂D_{λ_1} (i.e., the boundary orientation induced by the complement of D).

Proving the lemma is equivalent to showing that $I+II=0$. Now $I=0$ since $(dd^cV)^n=0$ on $\mathbb{C}^n\setminus D$.

Therefore, we must show that $II=0$. First note that with respect to polar coordinates $z=re^{i\theta}$ in one variable, we have $d\log|z|=dr/r$ and $d^c\log|z|=d\theta$. Let $f\colon\mathbb{C}\setminus\Delta\to\mathbb{C}^n\setminus D$ parametrize a Lempert disk, with $V(f(t))=\log|t|$, $t=re^{i\theta}$. Then $f^*d^cV=d\theta$. Also, since $\psi\circ f(re^{i\theta})=\psi\circ f(e^{i\theta})$, we have $f^*\,d\psi=\gamma\, d\theta$ for some function γ; thus $f^*(d\psi\wedge d^cV)=0$. This says that $d\psi\wedge d^cV$ annihilates any pair of vectors tangent to the curve parametrized by f, so it can act nontrivially only on the components that are normal to the curve. Hence by Proposition 4.1(2), $d\psi\wedge d^cV$ can act nontrivially only on pairs of vectors with components in the complex tangent space of the level sets of V, which is of complex dimension $n-1$.

On the other hand, $f^*dd^cV=0$ since V is harmonic along the extremal curve; so dd^cV is a $(1,1)$-form that also can act nontrivially only on pairs of vectors with components in the complex tangent space of the level sets of V. Since $d\psi\wedge d^cV\wedge(dd^cV)^{n-1}$ is a smooth (n,n)-form that acts only on vectors spanning a space of complex dimension $n-1$, it must be identically zero. So $II=0$, which proves the lemma when ψ is smooth.

If ψ is only continuous, we first restrict it to ∂D and approximate it by a sequence of smooth functions ψ_n, with $\psi_n\to\psi$ uniformly. This can be done as follows. Since ∂D is smooth, locally we have a smooth diffeomorphism $\chi\colon U\subset\mathbb{R}^{2n-1}\to\partial D$, and if ψ is supported in $\chi(U)$, take $\psi_n=(\psi\circ\chi)_n\circ\chi^{-1}$, where $(\psi\circ\chi)_n$ are standard mollifications of $\psi\circ\chi$ in $\mathbb{R}^{2n-1}$ with $(\psi\circ\chi)_n\to\psi\circ\chi$ uniformly on U. As ∂D is compact, a general ψ continuous on ∂D can be mollified as above using a partition of unity, with $\psi_n\to\psi$ uniformly.

Next, extend ψ_n from ∂D to $\mathbb{C}^n\setminus D$ via $\psi_n(f(\zeta))=\psi_n(f(\zeta/|\zeta|))$, where f parametrizes a foliation disk. The extended functions ψ_n converge uniformly to ψ on $\mathbb{C}^n\setminus D$, and moreover are smooth on $\mathbb{C}^n\setminus\partial D$ since the Lempert foliation is smooth. Therefore (4.3) holds with ψ replaced by ψ_n. Taking the limit as $n\to\infty$, and using the uniform convergence $\psi_n\to\psi$, yields the result for ψ. $\square$

We now use the above lemma and a limiting procedure to transfer the Monge–Ampère measure of K to the boundary of K_ρ, the Robin indicatrix. In the calculations that follow we will use a standard Monge–Ampère formula (see e.g. [2]): given a smooth psh function u such that the boundary of the set $\{z:u(z)>0\}$ is a (real) smooth hypersurface S, and $u^+=\max\{u,0\}$, then for any continuous function φ,

$$\int\varphi(dd^cu^+)^n=\int_{\{u>0\}}\varphi(dd^cu)^n+\int_S\varphi\, d^cu\wedge(dd^cu)^{n-1}. \tag{4.4}$$

Note that by (4.4), the Monge–Ampère measures of $\overline{D}_\lambda$ ($\lambda\in(1,\infty)$) and K_ρ are given (with φ being an arbitrary continuous function) by the formulas

$$\int\varphi(dd^cV_{\overline{D}_\lambda})^n=\int_{\partial D_\lambda}\varphi\, d^cV_K\wedge(dd^cV_K)^{n-1} \tag{4.5}$$

and

$$\int\varphi(dd^c\rho_K^+)^n=\int_{\partial K_\rho}\varphi\, d^c\rho_K\wedge(dd^c\rho_K)^{n-1}. \tag{4.6}$$

We will also use standard convergence properties of Monge–Ampère measures (see e.g. [4]). Recall that for a sequence $\{u_j\}_{j=1}^\infty$ of locally bounded psh functions on a domain D we have the weak-$*$ convergence of measures $(dd^cu_j)^n\to(dd^cu)^n$ for some locally bounded psh function u on D whenever:

(1) $u_j\to u$ monotonically as $j\to\infty$ (i.e., $u_j\nearrow u$ a.e. or $u_j\searrow u$); or

(2) $u_j\to u$ locally uniformly as $j\to\infty$.

Theorem 4.3. *Let $K=\overline{D}\subset\mathbb{C}^n$ be the closure of a smoothly bounded, strongly lineally convex domain D; let $V=V_K$ be its Siciak–Zaharjuta extremal function, and let ρ be its Robin function. Then for any continuous ϕ on ∂K,*

$$\int\phi(dd^cV)^n=\int_{\partial K_\rho}(\phi\circ F)\, d^c\rho\wedge(dd^c\rho)^{n-1}, \tag{4.7}$$

where $K_\rho=\{z:\rho(z)\leq 0\}$ is the Robin indicatrix, and $F\colon \mathbb{C}^n\setminus K_\rho\to\mathbb{C}^n\setminus K$ is the Robin exponential map.

Proof. Using the Lempert foliation, we extend ϕ continuously to $\mathbb{C}^n\setminus D$ by the formula $\phi(f(\zeta)):=\phi(f(\zeta/|\zeta|))$, for all Lempert disks $f\colon \Delta\to\mathbb{C}^n\setminus D$. Next, we have $V_{\overline{D}_\lambda}\nearrow V$, so $(dd^cV_{\overline{D}_\lambda})^n\to(dd^cV)^n$ in the weak-$*$ convergence of measures. But for this particular ϕ, $\int\phi(dd^cV_{\overline{D}_\lambda})^n=\int_{\partial D_\lambda}\phi\, d^cV\wedge(dd^cV)^{n-1}$ is constant in λ by Lemma 4.2, and hence the equality

$$\int\phi(dd^cV)^n=\int_{\partial D_\lambda}\phi\, d^cV\wedge(dd^cV)^{n-1}$$

holds for all $\lambda>1$.

To prove the theorem, it suffices to show that the right-hand side of the above equation converges to the right-hand side of (4.7) as $\lambda\to\infty$. To see this, let $\lambda t=z$; then

$$\int_{z\in\partial D_\lambda}\phi(z)\, d^c_zV\wedge(dd^c_zV)^{n-1}$$
$$=\int_{t\in\lambda^{-1}(\partial D_\lambda)}\phi(t\lambda)\, d^c_t(V(t\lambda)-\log|\lambda|)\wedge(dd^c_t(V(t\lambda)-\log|\lambda|))^{n-1}.$$

For clarity, in the above lines the dependence (of derivatives and integrals) with respect to the variable z or t has been made explicit.

Away from the origin, the convergence $V_\lambda(t):=V(t\lambda)-\log|\lambda|\longrightarrow\rho(t)$ is uniform as $\lambda\to\infty$. Therefore we get the weak-* convergence $(dd^c(\max V_\lambda,0))^n\to(dd^c(\max\rho,0))^n$, i.e.,

$$\int_{\lambda^{-1}(\partial D_\lambda)}\varphi\, d^cV_\lambda\wedge(dd^cV_\lambda)^{n-1}\to\int_{\partial K_\rho}\varphi\, d^c\rho\wedge(dd^c\rho)^{n-1}$$

for any continuous function φ.

In particular, let ψ be the continuous function given by

$$\psi(\zeta a)=(\phi\circ F)(a)\quad\text{for all } a\in\partial K_\rho \text{ and } \zeta\in(1,\infty). \tag{4.8}$$

Then

$$\int_{\lambda^{-1}(\partial D_\lambda)}\psi\, d^cV_\lambda\wedge(dd^cV_\lambda)^{n-1}\to\int_{\partial K_\rho}\psi\, d^c\rho\wedge(dd^c\rho)^{n-1}\quad\text{as } \lambda\to\infty. \tag{4.9}$$

We want to replace $\psi(t)$ on the left-hand side by $\phi(t\lambda)$ for λ sufficiently large. To do this, we will prove a couple of lemmas before returning to the main proof.

Lemma 4.4. *There is a constant* $C>0$ *such that* $|a\xi - f_a(\xi)|<C$ *for all* $a\in\partial K_\rho$ *and* $|\xi|>1$.

Proof. Let f_a denote the parametrization of the Lempert extremal curve given by $\xi\mapsto a\xi+\sum_{j\le 0} a_j\xi^j$. The family of maps $g_a(\zeta)=a/\zeta-f_a(1/\zeta)$, $a\in\partial K_\rho$, are holomorphic in the unit disk Δ and uniformly bounded there by the maximum principle, since $f_a(\partial\Delta)\subset K$. In other words, there is a uniform bound $|a/\zeta-f_a(1/\zeta)|<C$ independent of a and $\zeta\in\Delta$; now identify $\xi=1/\zeta$. $\square$

Lemma 4.5. *Let* $V=V_K$ *be the Siciak–Zaharjuta extremal function for a compact set* K. *Suppose* V *is continuous, and that* $\mathbb{C}^n\setminus K$ *is foliated continuously by Lempert extremal disks. Suppose* φ *is a continuous function on* $\mathbb{C}^n\setminus K$ *such that for any leaf* $f\colon\mathbb{C}\setminus\Delta\to\mathbb{C}^n\setminus K$ *of the foliation,* $\varphi(f(\lambda))=\varphi(f(r\lambda))$ *for all* $r\in(1/|\lambda|,\infty)$.

Then given $C>0$ *and* $\varepsilon>0$, *there is* $R=R(C)>0$ *such that* $|\varphi(z)-\varphi(z')|<\varepsilon$ *whenever* $|z|,|z'|>R$ *and* $|z-z'|<C$.

Proof. Without loss of generality, $K\subset B(0,1)$, the unit ball. Then $|V(z)-V(z')|\le\omega(|z-z'|/|z|)$, where ω is the modulus of continuity of V on $B(0,2)$ (see e.g., Lemma 4.5 of [6]).

Hence given η such that $\omega(\eta)<\delta$ for some prescribed $\delta>0$, then by choosing $R_0>C/\eta$, we have that if $|z|,|z'|>R_0$ and $|z-z'|<C$, then

$$|V(z)-V(z')|<\delta. \tag{4.10}$$

We also have

$$z=f_a(\xi)=a\xi+\sum_{k=0}^{\infty} b_k\xi^{-k} \quad\text{and}\quad z'=f_{a'}(\xi')=a'\xi'+\sum_{k=0}^{\infty} b'_k\xi'^{-k}, \tag{4.11}$$

with $V(z)=\log|\xi|$, $V(z')=\log|\xi'|$, and $a,a'\in D_\rho$.

Without loss of generality, we assume that $\xi=|\xi|$ (by reparametrization). If z and z' satisfy (4.10), then $e^{-\delta}<|\xi'|/|\xi|<e^{\delta}$. Hence $\xi'/\xi=e^{\alpha+i\theta}$ for some $|\alpha|<\delta$ and $\theta\in[0,2\pi)$.

Using the series expansions in (4.11) to estimate $z-z'$, we have

$$C\ge|z-z'|\ge|a\xi-a'\xi'|-M=|\xi|\,|a-a'e^{\alpha+i\theta}|-M$$

for some constant M (which may be obtained using the uniform bound in the previous lemma). Hence $|a-a'e^{\alpha+i\theta}|\le(C+M)/|\xi|$. The right-hand side can be made smaller than $\delta>0$ by taking $|\xi|>(C+M)/\delta$.

Now take $\varepsilon>0$ as given by the hypothesis, and choose $\tilde\delta>0$ such that

$$\text{if } t,t'\in\partial K \text{ and } |t-t'|<\tilde\delta, \quad \text{then } |\varphi(t)-\varphi(t')|<\varepsilon. \tag{4.12}$$

By continuity of the foliation for V_K, we can choose $\delta>0$ such that

$$|f_c(1)-f_{c'}(1)|<\tilde{\delta} \quad \text{whenever } |c-c'|<\delta \text{ and } c,c'\in\partial K_\rho. \tag{4.13}$$

Choose $R>R_0$ sufficiently large so that if $|z|>R$, then $V_K(z)>(C+M)/\delta+1$. Now given $|z|,|z'|>R$, we have $z=f_a(\xi)$ and $z'=f_{a'}(\xi')$ with $V_K(z)=\log|\xi|$ and $V_K(z')=\log|\xi'|$; and $|z-z'|<C$ implies $|a-a'e^{\alpha+i\theta}|<\delta$. Finally,

$$\begin{aligned}|\varphi(z)-\varphi(z')| &= |\varphi(f_a(\xi))-\varphi(f_{a'}(\xi'))| = |\varphi(f_a(|\xi|))-\varphi(f_a(e^{-i\theta}|\xi'|))| \\ &= |\varphi(f_a(1))-\varphi(f_a(e^{-i\theta}))| = |\varphi(f_a(1))-\varphi(f_{e^{i\theta}a}(1))|<\varepsilon,\end{aligned}$$

where we apply (4.12) and (4.13) in the last line. This concludes the proof. $\square$

We now continue with the proof of Theorem 4.3. By Lemma 4.4, there is $C>0$ such that

$$|a\xi-f_a(\xi)|<C \quad \text{for all } |\xi|>1,\ a\in\partial K_\rho.$$

Given t, choose $s\in\mathbb{C}$ such that $t=sa$ for some $a\in\partial K_\rho$. Then for $\lambda>1$ sufficiently large (chosen so that $|s\lambda a|, |f_a(s\lambda)|>R$ for all $a\in\partial K_\rho$, where $R=R(C)$ is chosen as in Lemma 4.5), we have from (4.8),

$$|\psi(t)-\phi(t\lambda)| = |\psi(t\lambda)-\phi(t\lambda)| = |\psi(s\lambda a)-\phi(s\lambda a)| = |\phi(f_a(s\lambda))-\phi(s\lambda a)|<\varepsilon.$$

Then

$$\left|\int_{\lambda^{-1}(\partial D_\lambda)}\psi(t)\,d^cV_\lambda\wedge(dd^cV_\lambda)^{n-1}-\int_{\lambda^{-1}(\partial D_\lambda)}\phi(t\lambda)\,d^cV_\lambda\wedge(dd^cV_\lambda)^{n-1}\right|<(2\pi)^n\varepsilon.$$

(The factor $(2\pi)^n$ is due to the fact that $V_\lambda\in L^+(\mathbb{C}^n)$ implies $\int(dd^cV_\lambda)^n=(2\pi)^n$; see e.g. [5].) Since ε is arbitrary, we obtain

$$\lim_{\lambda\to\infty}\int_{\lambda^{-1}(\partial D_\lambda)}\psi(t)\,d^cV_\lambda\wedge(dd^cV_\lambda)^{n-1}=\lim_{\lambda\to\infty}\int_{\lambda^{-1}(\partial D_\lambda)}\phi(t\lambda)\,d^cV_\lambda\wedge(dd^cV_\lambda)^{n-1}. \tag{4.14}$$

The expression inside the limit on the right-hand side is actually constant in λ: changing back to the variable z, where $\lambda t=z$, we have

$$\int_{\lambda^{-1}(\partial D_\lambda)}\phi(t\lambda)\,d^cV_\lambda\wedge(dd^cV_\lambda)^{n-1}=\int_{\partial D_\lambda}\phi(z)\,d^cV\wedge(dd^cV)^{n-1}=\int\phi(dd^cV_\lambda)^n,$$

which is independent of $\lambda>1$ by Lemma 4.2. Then

$$\lim_{\lambda\to\infty}\int_{\partial D_\lambda}\phi\,d^cV\wedge(dd^cV)^{n-1}=\lim_{\lambda\to1^+}\int\phi(dd^cV_\lambda)^n=\int\phi(dd^cV_K)^n,$$

where the last equality follows by Monge–Ampère convergence. Finally, putting the above together with (4.9) and (4.14) finishes the proof. □

We use Theorem 4.3 to get a similar result for convex bodies $K\subset\mathbb{R}^n$ with unique extremals. The Robin exponential map in this case is of the form

$$F(b\zeta)=a+b\zeta+\frac{\bar{b}}{\zeta},\quad b\in\partial K_\rho,\ \zeta\in\mathbb{C}\setminus\Delta.$$

(Recall that a depends on b; more precisely, on $c=[0:b_1:\ldots:b_n]\in H_\infty$.)

The Robin exponential map extends continuously to ∂K_ρ via $F(be^{i\theta})=a+be^{i\theta}+\bar{b}e^{-i\theta}$. Given $x\in K$ and $v\in\mathbb{R}^n$, there is always an extremal ellipse through x with tangent in the direction of v (see Section 3 of [9]). Hence $F\colon\partial K_\rho\to K$ is onto but not injective.

Theorem 4.6. *Let $K\subset\mathbb{R}^n$ be a convex body with unique extremals. Then for any ϕ continuous on K,*

$$\int\phi(dd^cV_K)^n=\int(\phi\circ F)(dd^c\rho_K^+)^n.$$

We will prove this using Theorem 4.3 and an approximation argument. Let $\{K_k\}_{k=1}^\infty$ be a strictly decreasing sequence of compact sets with the following properties:

(1) For each k, K_k is the closure of a smoothly bounded, strongly lineally convex domain D_k.

(2) $K_{k+1}\subset K_k$ with $\bigcap_{k=1}^\infty K_k=K$.

For convenience, let us denote the Siciak–Zaharjuta extremal functions by $V_k=V_{K_k}$ and $V=V_K$, and also write $\rho=\rho_K^+=\max\{\rho_K,0\}$ and $\rho_k=\rho_{K_k}^+$. Write $(K_k)_\rho=\{z:\rho_{K_k}(z)\leq 0\}$ and $K_\rho=\{z:\rho_K(z)\leq 0\}$ for the Robin indicatrices, and write $F_k\colon\mathbb{C}^n\setminus(K_k)_\rho\to\mathbb{C}^n\setminus K_k$ and $F\colon\mathbb{C}^n\setminus K_\rho\to\mathbb{C}^n\setminus K$ for the Robin exponential maps.

A key ingredient in the approximation will be the following result, stated without proof (cf., Corollary 7.2 of [8]).

Proposition 4.7. *On any compact subset of $\mathbb{C}^n\setminus K_\rho$ we have the uniform convergence $F_k\to F$.*

Given $r>1$, we put

$$\begin{aligned}V_{k,r}&=\max\{0,V_k-\log r\},\quad &\rho_{k,r}&=\max\{0,\rho_k-\log r\},\\ V_r&=\max\{0,V-\log r\},\quad &\rho_r&=\max\{0,\rho-\log r\}.\end{aligned}$$

It is easy to see that $V_{k,r}$ is the extremal function for the set $K_{k,r}=\{z:V_k(z)\leq\log r\}$, $\rho_{k,r}$ is the Robin function for $K_{k,r}$, and that $V_{k,r}\nearrow V_r$ and $\rho_{k,r}\nearrow\rho_r$ as $k\to\infty$.

Remark 4.8. By Lempert theory, the set $K_{k,r}$ is smoothly bounded and strongly lineally convex. Also, if $F_{k,r}$ denotes the corresponding Robin exponential map, then $F_{k,r}=F_k$ on the domain of $F_{k,r}$. This follows from the fact that the images of Lempert extremal disks for $V_{k,r}$ are contained in those of V_k. Precisely, if $f(\zeta)$ parametrizes a Lempert extremal for V_k, then $f_r(\zeta):=f(r\zeta)$ parametrizes a Lempert extremal disk for $V_{k,r}$.

We will also make use of the following lemma whose proof is straightforward.

Lemma 4.9. *Suppose we have the uniform convergence $\varphi_k\to\varphi$ of continuous functions on a domain D and the weak-$*$ convergence $\mu_k\to\mu$ of measures on D, and the total masses $\mu_k(D)$ and $\mu(D)$ are uniformly bounded above.*

Then $\int_D \varphi_k\, d\mu_k\to\int_D \varphi\, d\mu$.

We will apply this to Monge–Ampère measures of functions in $L^+(\mathbb{C}^n)$, which have total mass $(2\pi)^n$.

We can now prove Theorem 4.6.

Proof of Theorem 4.6. Let ϕ be a real-valued continuous function on K. Form the continuous function

$$\tilde{\phi}(z)=\begin{cases}\phi(z), & \text{if } z\in K,\\ \phi(f(\zeta/|\zeta|)), & \text{if } z=f(\zeta)\in\mathbb{C}^n\setminus K,\end{cases}$$

where f parametrizes an extremal disk that goes through z. Note that $\tilde{\phi}$ is continuous since the foliation of extremals for V_K is continuous and each leaf extends holomorphically across K (as a complex ellipse).

Fix $r>1$. By Theorem 4.3, we have

$$\int \tilde{\phi}(dd^cV_{k,r})^n=\int(\tilde{\phi}\circ F_{k,r})(dd^c\rho_{k,r})^n=\int(\tilde{\phi}\circ F_k)(dd^c\rho_{k,r})^n, \tag{4.15}$$

where we use the observation in Remark 4.8 to get the second equality.

The Monge–Ampère formula (4.6) applied to $\rho_{k,r}$ and ρ_r shows that $(dd^c\rho_{k,r})^n$ and $(dd^c\rho_r)^n$ are supported on the sets $\{z:\rho_k(z)=\log r\}$ and $\{z:\rho(z)=\log r\}$, which are contained in $\mathbb{C}^n\setminus K_\rho$. Here we have $F_k\to F$ uniformly; hence $\tilde{\phi}\circ F_k\to\tilde{\phi}\circ F$ uniformly on a compact set $S\subset\mathbb{C}^n\setminus K_\rho$ that contains a neighborhood of $\{z:\rho(z)=\log r\}$ and hence contains $\{z:\rho_k(z)=\log r\}$ for all sufficiently large k. Applying Lemma 4.9, we have

$$\int(\tilde{\phi}\circ F_k)(dd^c\rho_{k,r})^n\to\int(\tilde{\phi}\circ F)(dd^c\rho_r)^n \quad \text{as } k\to\infty.$$

The standard Monge–Ampère convergence also gives

$$\int \tilde{\phi}(dd^cV_{k,r})^n \to \int \tilde{\phi}(dd^cV_r)^n \quad \text{as } k\to\infty.$$

As (4.15) is true for all k, $\int \tilde{\phi}(dd^cV_r)^n=\int(\tilde{\phi}\circ F)(dd^c\rho_r)^n$ follows by taking the limit as $k\to\infty$. This latter formula is true for all $r>1$.

Since $V_r\nearrow V$ and $\rho_r\nearrow\rho$ as $r\to 1^-$, we also have the weak-$*$ convergences $(dd^cV_r)^n\to(dd^cV)^n$ and $(dd^c\rho_r)^n\to(dd^c\rho)^n$. Taking the limit as $r\to 1^-$ yields

$$\int \tilde{\phi}(dd^cV)^n = \int (\tilde{\phi}\circ F)(dd^c\rho)^n.$$

Finally, note that on K, where $(dd^cV)^n$ is supported, we have $\tilde{\phi}=\phi$. Similarly, it is easy to check that $\tilde{\phi}\circ F=\phi\circ F$ on the support of $(dd^c\rho)^n$. The theorem is proved. $\square$

Remark 4.10. Note that the formula in Theorem 4.6 exhibits $(dd^cV_K)^n$ as the push-forward of the measure $(dd^c\rho_K^+)^n$ under the Robin exponential map:

$$F_*((dd^c\rho_K^+)^n) = (dd^cV_K)^n.$$

References

1. Baran, M., Plurisubharmonic extremal functions and complex foliations for the complement of convex sets in $\mathbb{R}^n$, *Michigan Math. J.* **39** (1992), 395–404.
2. Bedford, E. and Ma'u, S., Complex Monge–Ampère of a maximum, *Proc. Amer. Math. Soc.* **136** (2008), 95–101.
3. Bedford, E. and Taylor, B. A., The Dirichlet problem for a complex Monge–Ampère equation, *Invent. Math.* **37** (1976), 1–44.
4. Bedford, E. and Taylor, B. A., A new capacity for plurisubharmonic functions, *Acta Math.* **149** (1982), 1–40.
5. Bedford, E. and Taylor, B. A., Uniqueness for the complex Monge–Ampère equation for functions of logarithmic growth, *Indiana Univ. Math. J.* **38** (1989), 455–469.
6. Bloom, T., Levenberg, N. and Ma'u, S., Robin functions and extremal functions, *Ann. Polon. Math.* **80** (2003), 55–84.
7. Burns, D., Levenberg, N. and Ma'u, S., Pluripotential theory for convex bodies in $\mathbb{R}^n$, *Math. Z.* **250** (2005), 91–111.
8. Burns, D., Levenberg, N. and Ma'u, S., Exterior Monge–Ampère solutions, *Adv. Math.* **222** (2009), 331–358.
9. Burns, D., Levenberg, N., Ma'u, S. and Revesz, Sz., Monge–Ampère measures for convex bodies and Bernstein–Markov type inequalities, *Trans. Amer. Math. Soc.* **362** (2010), 6325–6340.

10. Guan, B., On the regularity of the pluricomplex Green functions, *Int. Math. Res. Not.* **22** (2007), 19 pp.
11. Klimek, M., *Pluripotential Theory*, Oxford University Press, New York, 1991.
12. Lempert, L., La métrique de Kobayashi et la représentation des domaines sur la boule, *Bull. Soc. Math. France* **109** (1981), 427–474.
13. Lempert, L., Intrinsic distances and holomorphic retracts, in *Complex Analysis and Applications* '81 (*Varna*, *1981*), pp. 341–364, Publ. House Bulgar. Acad. Sci., Sofia, 1984.
14. Lempert, L., Symmetries and other transformations of the complex Monge–Ampère equation, *Duke Math. J.* **52** (1985), 869–885.
15. Lempert, L., Holomorphic invariants, normal forms, and the moduli space of convex domains, *Ann. of Math.* **128** (1988), 43–78.
16. Lundin, M., The extremal PSH for the complement of convex, symmetric subsets of $\mathbb{R}^n$, *Michigan Math. J.* **32** (1985), 197–201.
17. Momm, S., An extremal plurisubharmonic function associated to a convex pluricomplex Green function with pole at infinity, *J. Reine Angew. Math.* **471** (1996), 139–163.
18. Ransford, T., *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

Daniel M. Burns
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1043
U.S.A.
dburns@umich.edu

Norman Levenberg
Department of Mathematics
Indiana University
Bloomington, IN 47405
U.S.A.
nlevenbe@indiana.edu

Sione Ma'u
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland 1142
New Zealand
s.mau@auckland.ac.nz

Received December 11, 2013
published online January 23, 2015