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On normal forms for Levi-flat hypersurfaces with an isolated line singularity

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Abstract. We prove the existence of normal forms for some local real-analytic Levi-flat hypersurfaces with an isolated line singularity. We also give sufficient conditions for a Levi-flat hypersurface with a complex line as singularity to be a pullback of a real-analytic curve in $\mathbb{C}$ via a holomorphic function.

1. Introduction

Let $M\subset U\subset\mathbb{C}^n$ be a real-analytic hypersurface, where U is an open set. Denote by M^* the regular part, that is, near each point $p\in M^*$, the variety M is a manifold of real codimension one. For each $p\in M^*$, there is a unique complex hyperplane L_p contained in the tangent space T_pM^*, and this consequently defines a real-analytic distribution $p\mapsto L_p$ of complex hyperplanes in T_pM^*, the so-called *Levi distribution*. We say that M is *Levi-flat*, if the Levi distribution is integrable in the sense of Frobenius. The foliation defined by this distribution is called the *Levi-foliation*. The local structure near regular points is very well understood, according to É. Cartan, around each $p\in M^*$ we can find local holomorphic coordinates $z_1, ..., z_n$ such that $M^*=\{(z_1, ..., z_n)|\mathrm{Re}(z_n)=0\}$, and consequently the leaves of the Levi-foliation are imaginary levels of z_n. They case was studied by Burns–Gong [3]. They classified singular Levi-flat hypersurfaces in $\mathbb{C}^n$ with quadratic singularities and also proved the existence of a normal form, in the case of generic (Morse) singularities. In [4], Cerveau–Lins Neto proved that a local real-analytic Levi-flat hypersurface M with a sufficiently small singular set is given by the zeros of the real part of a holomorphic function.

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The aim of this paper is to prove the existence of some normal forms for local real-analytic Levi-flat hypersurfaces defined by the vanishing of the real part of holomorphic functions with an *isolated line singularity* (for short: ILS). In particular, we establish an analogous result like in singularity theory for germs of holomorphic functions.

The main motivation for this work is a result due to D. Siersma, who introduced in [13] the class of germs of holomorphic functions with an ILS. More precisely, let $\mathcal{O}_{n+1}:=\{f\colon(\mathbb{C}^{n+1},0)\to\mathbb{C}\}$ be the ring of germs of holomorphic functions and let m be its maximal ideal. If $(x,y)=(x,y_1,...,y_n)$ denote the coordinates in $\mathbb{C}^{n+1}$, consider the line $L:=\{(x,y)|y_1=...=y_n=0\}$, let $I:=(y_1,...,y_n)\subset\mathcal{O}_{n+1}$ be its ideal and denote by $\mathcal{D}_I$ the group of local analytic isomorphisms $\varphi\colon(\mathbb{C}^{n+1},0)\to(\mathbb{C}^{n+1},0)$ for which $\varphi(L)=L$. Then $\mathcal{D}_I$ acts on I^2 and for $f\in I^2$, the tangent space of (the orbit of) f with respect to this action is the ideal defined by

$$\tau(f):=m.\frac{\partial f}{\partial x}+I.\frac{\partial f}{\partial y}$$

and the codimension of (the orbit) of f is $c(f):=\dim_{\mathbb{C}}(I^2/\tau(f))$.

A line singularity is a germ $f\in I^2$. An ILS is a line singularity f such that $c(f)<\infty$. Geometrically, $f\in I^2$ is an ILS if and only if the singular locus of f is L and for every $x\neq0$, the germ of (a representative of) f at $(x,0)\in L$ is equivalent to $y_1^2+...+y_n^2$. In a certain sense ILSs are the first generalization of isolated singularities. Siersma proved the following result. (The topology on $\mathcal{O}_{n+1}$ is introduced as in [5, p. 145].)

Theorem 1.1. *A germ $f\in I^2$ is D_I-simple (i.e. $c(f)<\infty$ and f has a neighborhood in I^2 which intersects only a finite number of D_I-orbits) if and only if f is D_I-equivalent to one of the germs in Table 1.*

The singularities in Theorem 1.1 are analogous of the *A*-*D*-*E* singularities due to Arnold [1]. A new characterization of simple ILSs have been proved by Zaharia [14]. We prove the existence of normal forms for Levi-flat hypersurfaces with an ILS.

Theorem 1.2. *Let $M=\{(x,y)|F(x,y)=0\}$ be a germ of an irreducible real-analytic hypersurface on $(\mathbb{C}^{n+1},0)$, $n\geq3$. Suppose that*

(1) *$F(x,y)=\mathrm{Re}(P(x,y))+H(x,y)$, where $P(x,y)$ is one of the germs of Table 1;*

(2) *$M=\{(x,y)|F(x,y)=0\}$ is Levi-flat;*

(3) *$H(x,0)=0$ for all $x\in(\mathbb{C},0)$, and $j_0^k(H)=0$ for $k=\deg(P)$.*

Type	Normal form	Conditions
A_∞	$y_1^2+y_2^2+\ldots+y_n^2$	
D_∞	$xy_1^2+y_2^2+\ldots+y_n^2$	
$J_{k,\infty}$	$x^k y_1^2+y_1^3+y_2^2+\ldots+y_n^2$	$k\geq 2$
$T_{\infty,k,2}$	$x^2y_1^2+y_1^k+y_2^2+\ldots+y_n^2$	$k\geq 4$
$Z_{k,\infty}$	$xy_1^3+x^{k+2}y_1^2+y_2^2+\ldots+y_n^2$	$k\geq 1$
$W_{1,\infty}$	$x^3y_1^2+y_1^4+y_2^2+\ldots+y_n^2$	
$T_{\infty,q,r}$	$xy_1y_2+y_1^q+y_2^r+y_3^2+\ldots+y_n^2$	$q\geq r\geq 3$
$Q_{k,\infty}$	$x^ky_1^2+y_1^3+xy_2^2+y_3^2+\ldots+y_n^2$	$k\geq 2$
$S_{1,\infty}$	$x^2y_1^2+y_1^2y_2+y_3^2+\ldots+y_n^2$	

Table 1. Isolated Line singularities.

Then there exists a biholomorphism $\varphi\colon(\mathbb{C}^{n+1},0)\to(\mathbb{C}^{n+1},0)$ *preserving* L *such that*

$$\varphi(M)=\{(x,y)\,|\,\mathrm{Re}(P(x,y))=0\}.$$

This result is a Siersma-type theorem for singular Levi-flat hypersurfaces. We remark that the function H is of course restricted by the assumption that M is Levi flat. Now, if $\varphi(M)=\{(x,y)|\mathrm{Re}(P(x,y))=0\}$, where P is a germ with an ILS at L then $\mathrm{Sing}(M)=L$. In other words, M is a Levi-flat hypersurface with an ILS at L. If $P(x,y)$ is the germ A_∞, we prove that Theorem 1.2 is true in the case $n=2$.

Theorem 1.3. *Let* $M=\{(x,y)|F(x,y)=0\}$ *be a germ of an irreducible real-analytic Levi-flat hypersurface on* $(\mathbb{C}^3,0)$. *Suppose that* F *is defined by*

$$F(x,y)=\mathrm{Re}(y_1^2+y_2^2)+H(x,y),$$

where H *is a germ of a real-analytic function such that* $H(x,0)=0$ *and* $j_0^k(H)=0$ *for* $k=2$. *Then there exists a biholomorphism* $\varphi\colon(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ *preserving* L *such that* $\varphi(M)=\{(x,y)|\mathrm{Re}(y_1^2+y_2^2)=0\}$.

The above result should be compared to [3, Theorem 1.1]. This result can be viewed as a Morse lemma for Levi-flat hypersurfaces with an ILS at L. The problem of normal forms of Levi-flat hypersurfaces in $\mathbb{C}^3$ with an ILS seems difficult in the other cases. To prove these results we use techniques of holomorphic foliations developed in [4] and [6]. Similar normal forms of singular Levi-flat hypersurfaces have been obtained in [3], [7] and [9].

This paper is organized as follows: in Section 2, we recall some definitions and known results about Levi-flat hypersurfaces and holomorphic foliations. Section 3 is devoted to prove Theorem 1.2. In Section 4, we prove Theorem 1.3. Finally, in

Section 5, using holomorphic foliations, we give sufficient conditions for a Levi-flat hypersurface with a complex line as singularity to be a pullback of a real-analytic curve in $\mathbb{C}$ via a holomorphic function (see Theorem 5.7).

2. Levi-flat hypersurfaces and foliations

In this section we work with germs at $0\in\mathbb{C}^{n+1}$ of irreducible real-analytic hypersurfaces and of codimension-one holomorphic foliations. Let $M=\{(x,y)|F(x,y)=0\}$, where $F\colon(\mathbb{C}^{n+1},0)\to(\mathbb{R},0)$ is a germ of an irreducible real-analytic function, and $M^*:=\{(x,y)|F(x,y)=0\}\setminus\{(x,y)|dF(x,y)=0\}$. Let us define the singular set of M (or "set of critical points" of M) by

$$\mathrm{Sing}(M):=\{(x,y)\,|\,F(x,y)=0\}\cap\{(x,y)\,|\,dF(x,y)=0\}. \tag{1}$$

Note that $\mathrm{Sing}(M)$ contains all points $q\in M$ such that M is smooth at q, but the codimension of M at q is at least two. In general the singular set of a real-analytic subvariety M in a complex manifold is defined as the set of points near which M is not a real-analytic submanifold (of any dimension) and "in general" has structure of a semianalytic set; see for instance, [10]. In this paper, we work with $\mathrm{Sing}(M)$ as defined in (1). We recall that (in this case) the Levi distribution L on M^* is defined by

$$L_p:=\ker(\partial F(p))\subset T_pM^*=\ker(dF(p))\quad \text{for any } p\in M^*. \tag{2}$$

Let us suppose that M is *Levi-flat*. This implies that M^* is foliated by complex codimension-one holomorphic submanifolds immersed on M^*.

Note that the Levi distribution L on M^* can be defined by the real-analytic 1-form $\eta=i(\partial F-\bar\partial F)$, which is called the *Levi* 1-*form of* F. It is well known that the integrability condition of L is equivalent to the equation $(\partial F-\bar\partial F)\wedge\partial\bar\partial F|_{M^*}=0$.

Let us consider the Taylor series of F at $0\in\mathbb{C}^{n+1}$,

$$F(x,y)=\sum_{j,\mu,k,\nu}F_{j\mu k\nu}x^jy^\mu\bar x^k\bar y^\nu,$$

where $\overline{F}_{j\mu k\nu}=F_{k\nu j\mu}$ for all $j,k\in\mathbb{N}$, $\mu=(\mu_1,...,\mu_n)$, $\nu=(\nu_1,...,\nu_n)$, $(x,y)\in\mathbb{C}\times\mathbb{C}^n$, $y^\mu=y_1^{\mu_1}...y_n^{\mu_n}$ and $\bar y^\nu=\bar y_1^{\nu_1}...\bar y_n^{\nu_n}$. The complexification $F_{\mathbb{C}}\in\mathcal{O}_{2n+2}$ of F is defined by the series

$$F_{\mathbb{C}}(x,y,z,w)=\sum_{j,\mu,k,\nu}F_{j\mu k\nu}x^jy^\mu z^kw^\nu,$$

where $z\in\mathbb{C}$, $w=(w_1,...,w_n)\in\mathbb{C}^n$ and $w^\nu=w_1^{\nu_1}...w_n^{\nu_n}$. Notice that

$$F(x,y)=F_{\mathbb{C}}(x,y,\bar x,\bar y).$$

The complexification $M_{\mathbb{C}}$ of M is defined as $M_{\mathbb{C}}:=\{(x,y,z,w)|F_{\mathbb{C}}(x,y,z,w)=0\}$ and defines a complex subvariety in $\mathbb{C}^{2n+2}$, its regular part is

$$M_{\mathbb{C}}^{*} := M_{\mathbb{C}} \setminus \{(x,y,z,w) \mid dF_{\mathbb{C}}(x,y,z,w) = 0\}.$$

Now, assume that M is Levi-flat. Then the integrability condition of

$$\eta = i(\partial F - \bar{\partial} F)|_{M^*}$$

implies that $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ is integrable, where

$$\eta_{\mathbb{C}} := i[(\partial_x F_{\mathbb{C}} + \partial_y F_{\mathbb{C}}) - (\partial_z F_{\mathbb{C}} + \partial_w F_{\mathbb{C}})].$$

Therefore $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ defines a codimension-one holomorphic foliation $\mathcal{L}_{\mathbb{C}}$ on $M_{\mathbb{C}}^*$ that will be called the *complexification of* $\mathcal{L}$.

Let $W := M_{\mathbb{C}}^* \setminus \mathrm{Sing}(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$ and denote by L_ζ the leaf of $\mathcal{L}_{\mathbb{C}}$ through ζ, where $\zeta \in W$. The next results will be used several times througout the paper.

Lemma 2.1. (Cerveau–Lins Neto [4]) *For any* $\zeta \in W$, *the leaf* L_ζ *of* $\mathcal{L}_{\mathbb{C}}$ *through* ζ *is closed in* $M_{\mathbb{C}}^*$.

Definition 2.2. The algebraic dimension of $\mathrm{Sing}(M)$ is the complex dimension of the singular set of $M_{\mathbb{C}}$.

The following result will be used enunciated in the context of Levi-flat hypersurfaces in $\mathbb{C}^{n+1}$.

Theorem 2.3. (Cerveau-Lins Neto [4]) *Let* $M = \{(x,y) \mid F(x,y) = 0\}$ *be a germ of an irreducible analytic Levi-flat hypersurface at* $0 \in \mathbb{C}^{n+1}$, $n \geq 2$, *with Levi* 1*-form* $\eta = i(\partial F - \bar{\partial} F)$. *Assume that the algebraic dimension of* $\mathrm{Sing}(M)$ *is* $\leq 2n-2$. *Then there exists a unique germ at* $0 \in \mathbb{C}^{n+1}$ *of the holomorphic codimension-one foliation* $\mathcal{F}_M$ *tangent to* M, *if one of the following conditions is fulfilled*:

(1) $n \geq 3$ *and* $\mathrm{cod}_{M_{\mathbb{C}}^*}(\mathrm{Sing}(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})) \geq 3$;

(2) $n \geq 2$, $\mathrm{cod}_{M_{\mathbb{C}}^*}(\mathrm{Sing}(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})) \geq 2$ *and* $\mathcal{L}_{\mathbb{C}}$ *admits a non-constant holomorphic first integral.*

Moreover, in both cases the foliation $\mathcal{F}_M$ *admits a non-constant holomorphic first integral* f *such that* $M = \{(x,y) \mid \mathrm{Re}(f(x,y)) = 0\}$.

3. Proof of Theorem 1.2

We write

$$F(x,y) = \mathrm{Re}(P(x, y_1, \ldots, y_n)) + H(x, y_1, \ldots, y_n),$$

where $P(x, y_1, ..., y_n)$ is one of the polynomials of Table 1, $H\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{R}, 0)$ is a germ of a real-analytic function such that $H(x, 0) = 0$ for all $x \in (\mathbb{C}, 0)$, and $j_0^k(H) = 0$ for $k = \deg(P)$. The complexification of F is given by

$$F_{\mathbb{C}}(x, y, z, w) = \tfrac{1}{2} P(x, y) + \tfrac{1}{2} P(z, w) + H_{\mathbb{C}}(x, y, z, w),$$

and therefore $M_{\mathbb{C}} = \{(x, y, z, w) \mid F_{\mathbb{C}}(x, y, z, w) = 0\} \subset (\mathbb{C}^{2n+2}, 0)$, where $z \in \mathbb{C}$ and $w = (w_1, ..., w_n) \in \mathbb{C}^n$.

Since $P(x, y)$ has an ILS at L, we get $\mathrm{Sing}(M_{\mathbb{C}}) = \{(x, y, z, \omega) \mid y = w = 0\} \simeq \mathbb{C}^2$. In particular, the algebraic dimension of $\mathrm{Sing}(M)$ is 2. On the other hand, the complexification of $\eta = i(\partial F - \bar{\partial} F)$ is

$$\eta_{\mathbb{C}} := i[(\partial_x F_{\mathbb{C}} + \partial_y F_{\mathbb{C}}) - (\partial_z F_{\mathbb{C}} + \partial_w F_{\mathbb{C}})].$$

Recall that $\eta|_{M^*}$ and $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ define $\mathcal{L}$ and $\mathcal{L}_{\mathbb{C}}$ respectively. Now we compute $\mathrm{Sing}(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$. We can write $dF_{\mathbb{C}} = \alpha + \beta$, with

$$\alpha := \frac{\partial F_{\mathbb{C}}}{\partial x}\, dx + \sum_{j=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial y_j}\, dy_j = \frac{1}{2} \frac{\partial P}{\partial x}(x, y)\, dx + \frac{1}{2} \sum_{j=1}^{n} \frac{\partial P}{\partial y_j}(x, y)\, dy_j + \theta_1$$

and

$$\beta := \frac{\partial F_{\mathbb{C}}}{\partial z}\, dz + \sum_{j=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial w_j}\, dw_j = \frac{1}{2} \frac{\partial P}{\partial z}(z, w)\, dz + \frac{1}{2} \sum_{j=1}^{n} \frac{\partial P}{\partial w_j}(z, w)\, dw_j + \theta_2,$$

where

$$\theta_1 = \frac{\partial H_{\mathbb{C}}}{\partial x}\, dx + \sum_{j=1}^{n} \frac{\partial H_{\mathbb{C}}}{\partial z_j}\, dz_j \quad \text{and} \quad \theta_2 = \frac{\partial H_{\mathbb{C}}}{\partial z}\, dz + \sum_{j=1}^{n} \frac{\partial H_{\mathbb{C}}}{\partial w_j}\, dw_j.$$

Note that $\eta_{\mathbb{C}} = i(\alpha - \beta)$, and so

$$\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*} = (\eta_{\mathbb{C}} + i\, dF_{\mathbb{C}})|_{M_{\mathbb{C}}^*} = 2i\alpha|_{M_{\mathbb{C}}^*} = -2i\beta|_{M_{\mathbb{C}}^*}. \tag{3}$$

In particular, $\alpha|_{M_{\mathbb{C}}^*}$ and $\beta|_{M_{\mathbb{C}}^*}$ define $\mathcal{L}_{\mathbb{C}}$. Therefore $\mathrm{Sing}(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$ can be split in two parts. In fact, let

$$M_1 := \left\{ (x, y, z, w) \in M_{\mathbb{C}} \;\middle|\; \frac{\partial F_{\mathbb{C}}}{\partial z} \neq 0 \text{ or } \frac{\partial F_{\mathbb{C}}}{\partial w_j} \neq 0 \text{ for some } j = 1, ..., n \right\},$$

$$M_2 := \left\{ (x, y, z, w) \in M_{\mathbb{C}} \;\middle|\; \frac{\partial F_{\mathbb{C}}}{\partial x} \neq 0 \text{ or } \frac{\partial F_{\mathbb{C}}}{\partial z_j} \neq 0 \text{ for some } j = 1, ..., n \right\}.$$

Then $M_{\mathbb{C}} = M_1 \cup M_2$. If we let $A_0 = \partial H_{\mathbb{C}} / \partial x$, $A_j = \partial H_{\mathbb{C}} / \partial z_j$, $B_0 = \partial H_{\mathbb{C}} / \partial z$ and $B_j = \partial H_{\mathbb{C}} / \partial w_j$ for all $1 \leq j \leq n$, we obtain that $\mathrm{Sing}(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}) = X_1 \cup X_2$, where

$$X_1 := M_1 \cap \left\{ (x,y,z,w) \,\middle|\, \frac{\partial P}{\partial x}(x,y) + A_0 = \frac{\partial P}{\partial y_1}(x,y) + A_1 = \dots = \frac{\partial P}{\partial y_n}(x,y) + A_n = 0 \right\}$$

and

$$X_2 := M_2 \cap \left\{ (x,y,z,w) \,\middle|\, \frac{\partial P}{\partial z}(z,w) + B_0 = \frac{\partial P}{\partial w_1}(z,w) + B_1 = \dots = \frac{\partial P}{\partial w_n}(z,w) + B_n = 0 \right\}.$$

Since P is a polynomial with an ILS at $L=\{(x,y) | y=0\}$, we conclude that

$$\operatorname{cod}_{M_{\mathbb{C}}^*} \operatorname{Sing}(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}) = n.$$

By hypothesis $n \geq 3$. Then it follows from Theorem 2.3(1) that there exists a germ $f \in \mathcal{O}_{n+1}$ such that the holomorphic foliation $\mathcal{F}$ defined by $df=0$ is tangent to M. Moreover $M=\{(x,y) | \operatorname{Re}(f(x,y))=0\}$. Note that if

$$M = \{(x,y) \,|\, \operatorname{Re}(f(x,y)) = 0\} = \{(x,y) \,|\, F(x,y) = 0\},$$

with F being an irreducible germ, we must have $\operatorname{Re}(f)=UF$, where U is a germ of a real-analytic function with $U(0)\neq 0$. Without loss of generality, we can assume that $U(0)=1$. In particular, $\operatorname{Re}(f)=UF$ implies that

$$f = P + \text{higher order terms}.$$

According to Theorem 1.1, there exists a biholomorphism $\varphi\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C}^{n+1},0)$ preserving L such that $f \circ \varphi^{-1} = P$, (f is D_I-equivalent to P, because f is a germ with ILS at L). Therefore, $\varphi(M)=\{(x,y) | \operatorname{Re}(P(x,y))=0\}$ and the proof is complete.

4. Proof of Theorem 1.3

The idea is to use Theorem 2.3(2). In order to prove our result in the case $n=2$, we are going to prove that $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.

We begin by a blow-up along $C := \{(x,y,z,w) | y_1 = y_2 = w_1 = w_2 = 0\} \simeq \mathbb{C}^2 \subset \mathbb{C}^6$. Let

$$F(x,y_1,y_2) = \operatorname{Re}(y_1^2 + y_2^2) + H$$

and assume that $M=\{(x,y) | F(x,y)=0\}$ is Levi-flat. Its complexification can be written as

$$F_{\mathbb{C}}(x,y_1,y_2,z,w_1,w_2) = \tfrac{1}{2}(y_1^2+y_2^2) + \tfrac{1}{2}(w_1^2+w_2^2) + H_{\mathbb{C}}(x,y_1,y_2,z,w_1,w_2).$$

Note that

$$\operatorname{Sing}(M_{\mathbb{C}}) = \{(x,y,z,w) \,|\, y = w = 0\} = C.$$

Let E be the exceptional divisor of the blow-up $\pi\colon \widetilde{\mathbb{C}}^6 \to \mathbb{C}^6$ along C. Denote by $\widetilde{M}_{\mathbb{C}} := \overline{\pi^{-1}(M_{\mathbb{C}} \setminus \{C\})} \subset \widetilde{\mathbb{C}}^6$ the strict transform of $M_{\mathbb{C}}$ via π and by $\widetilde{\mathcal{F}} := \pi^*(\mathcal{L}_{\mathbb{C}})$ the foliation on $\widetilde{M}_{\mathbb{C}}$.

Now, we consider a special situation. Suppose that $\widetilde{M}_{\mathbb{C}}$ is smooth and set $\widetilde{C} := \widetilde{M}_{\mathbb{C}} \cap E$. Moreover, assume that $\widetilde{C}$ is invariant by $\widetilde{\mathcal{F}}$. Take $S = \widetilde{C} \setminus \operatorname{Sing}(\widetilde{\mathcal{F}})$. Then S is a smooth leaf of $\widetilde{\mathcal{F}}$. Pick $p_0 \in S$ and a transverse section Σ through p_0. Let $G \subset \operatorname{Diff}(\Sigma, p_0)$ be the holonomy group of the leaf S of $\widetilde{\mathcal{F}}$. Since $\dim \Sigma = 1$, we can assume that $G \subset \operatorname{Diff}(\Sigma, 0)$. We state a fundamental lemma.

Lemma 4.1. (Fernández-Pérez [9]) *In the above situation, suppose that the following properties are satisfied*:

(1) *For any* $p \in S \setminus \operatorname{Sing}(\widetilde{\mathcal{F}})$ *the leaf* L_p *of* $\widetilde{\mathcal{F}}$ *through* p *is closed in* S;

(2) $g'(0)$ *is a primitive root of unity for all* $g \in G \setminus \{\mathrm{id}\}$.

Then $\mathcal{L}_{\mathbb{C}}$ *admits a non-constant holomorphic first integral.*

Proof. Let $G' = \{g'(0) \mid g \in G\}$ and consider the homomorphism $\phi\colon G \to G'$ defined by $\phi(g) = g'(0)$. We claim that ϕ is injective. In fact, assume that $\phi(g) = 1$ and suppose by contradiction that $g \neq \mathrm{id}$. In this case $g(z) = z + az^{r+1} + \dots$, where $a \neq 0$. According to [11], the pseudo-orbits of this transformation accumulate at $0 \in (\Sigma, 0)$, contradicting the fact that the leaves of $\widetilde{\mathcal{F}}$ are closed and so the assertion is proved. Now, it suffices to prove that any element $g \in G$ has finite order (cf. [12]). In fact, $\phi(g) = g'(0)$ is a root of unity, and thus g has finite order because ϕ is injective. Hence, all transformations of G have finite order and G is linearizable.

This implies that there is a coordinate system w on $(\Sigma, 0)$ such that

$$G = \langle w \mapsto \lambda w \rangle,$$

where λ is a dth-primitive root of unity (cf. [12]). In particular, $\psi(w) = w^d$ is a first integral of G, that is $\psi \circ g = \psi$ for any $g \in G$.

Let Γ be the union of the separatrices of $\mathcal{L}_{\mathbb{C}}$ through $0 \in \mathbb{C}^6$ and $\widetilde{\Gamma}$ be its strict transform under π. The first integral ψ can be extended to a first integral $\varphi\colon \widetilde{M}_{\mathbb{C}} \setminus \widetilde{\Gamma} \to \mathbb{C}$ by setting

$$\varphi(q) = \psi(\tilde{L}_q \cap \Sigma),$$

where $\tilde{L}_p$ denotes the leaf of $\widetilde{\mathcal{F}}$ through q. Since ψ is bounded (in a compact neighborhood of $0 \in \Sigma$), so is φ. It follows from Riemann's extension theorem that φ can be extended holomorphically to $\widetilde{\Gamma}$ with $\varphi(\widetilde{\Gamma}) = 0$. This provides the first integral of $\mathcal{L}_{\mathbb{C}}$. $\square$

The rest of the proof is devoted to prove that we are indeed in the conditions of Lemma 4.1. It follows from Lemma 2.1 that the leaves of $\mathcal{L}_{\mathbb{C}}$ are closed. Therefore, we need to prove that each generator of the holonomy group G of $\widetilde{\mathcal{F}}$ with respect to S has finite order.

Consider for instance the chart $(U_1,(x,t,s,z,u,v))$ of $\widetilde{\mathbb{C}}^6$, where

$$\pi(x,t,s,z,u,v)=(x,tu,su,z,u,vu)=(x,y_1,y_2,z,w_1,w_2).$$

We have

$$\widetilde{M}_{\mathbb{C}}\cap U_1=\{(x,t,s,z,u,v)\in U_1 \mid 1+t^2+s^2+v^2+uH_1(x,t,s,z,u,v)=0\},$$

where $H_1=H(x,ut,us,z,u,uv)/u^3$ and this fact imply that

$$E\cap\widetilde{M}_{\mathbb{C}}\cap U_1=\{(x,t,s,z,u,v)\in U_1 \mid 1+t^2+s^2+v^2=u=0\}.$$

It is not difficult to see that these complex subvarieties are smooth. Now, let us describe the foliation $\widetilde{\mathcal{F}}$ on U_1. In fact, note that the foliation $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M^*_{\mathbb{C}}}=0$, where

$$\alpha=\frac{1}{2}\frac{\partial P}{\partial x}\,dx+\frac{1}{2}\frac{\partial P}{\partial y_1}\,dy_1+\frac{1}{2}\frac{\partial P}{\partial y_2}\,dy_2+\frac{\partial H_{\mathbb{C}}}{\partial x}\,dx+\sum_{j=1}^{2}\frac{\partial H_{\mathbb{C}}}{\partial y_j}\,dy_j.$$

It follows that $\alpha=y_1\,dy_1+y_2\,dy_2+(\partial H_{\mathbb{C}}/\partial x)\,dx+\sum_{j=1}^{2}(\partial H_{\mathbb{C}}/\partial y_j)\,dy_j$, and then $\widetilde{\mathcal{F}}|_{U_1}$ is defined by $\widetilde{\alpha}|_{\tilde{M}_{\mathbb{C}}\cap U_1}=0$, where

$$\widetilde{\alpha}=(t^2+s^2)\,du+ut\,dt+us\,ds+u\tilde{\theta}, \tag{4}$$

and

$$\tilde{\theta}=\frac{\pi^*\left(\dfrac{\partial H_{\mathbb{C}}}{\partial x}\,dx+\displaystyle\sum_{j=1}^{2}\frac{\partial H_{\mathbb{C}}}{\partial y_j}\,dy_j\right)}{u^2}.$$

Therefore, the singular set of $\widetilde{\mathcal{F}}|_{U_1}$ is given by

$$\operatorname{Sing}(\widetilde{\mathcal{F}}|_{U_1})=\{(x,t,s,z,u,v)\mid u=t+is=0 \text{ or } u=t-is=0\}.$$

On the other hand, note that the exceptional divisor E is invariant by $\widetilde{\mathcal{F}}$ and the intersection with $\operatorname{Sing}(\widetilde{\mathcal{F}})$ is

$$\operatorname{Sing}(\widetilde{\mathcal{F}}|_{U_1})\cap E=\{(x,t,s,z,u,v)\mid u=t+is=v^2+1=0 \text{ or } u=t-is=v^2+1=0\}.$$

In particular, $S:=(E\cap\widetilde{M}_{\mathbb{C}})\setminus\operatorname{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}})$ is a leaf of $\widetilde{\mathcal{F}}$. We calculate the generators of the holonomy group G of the leaf S. We work in the chart U_1, because of the symmetry of the variables in the definition of the variety $\widetilde{M}_{\mathbb{C}}$.

Pick $p_0=(0,1,0,0,0,0)\in S\cap U_1$ and a transversal $\Sigma=\{(0,1,0,0,\lambda,0)\,|\,\lambda\in\mathbb{C}\}$ parameterized by λ at p_0. We have that

$$\mathrm{Sing}(\widetilde{\mathcal{F}}|_{U_1})\cap E=\{(x,t,s,z,u,v)\,|\,u=t+is=v^2+1=0 \text{ or } u=t-is=v^2+1=0\}.$$

For $j=1,2$, let ρ_j be a 2nd-primitive root of -1. The fundamental group $\pi_1(S,p_0)$ can be written in terms of generators as

$$\pi_1(S,p_0)=\langle\gamma_j,\delta_j\rangle_{j=1,2},$$

where for $j=1,2$, γ_j are loops that turn around $\{(x,t,s,z,u,v)|u=t+is=v-\rho_j=0\}$ and δ_j are loops that turns around $\{(x,t,s,z,u,v)|u=t-is=v-\rho_j=0\}$. Therefore, $G=\langle f_j,g_j\rangle_{j=1,2}$, where f_j and g_j correspond to $[\gamma_j]$ and $[\delta_j]$, respectively. We get from (4) that $f_j'(0)=e^{-\pi i}$ and $g_j'(0)=e^{-\pi i}$ for $j=1,2$. The proof of the theorem is complete.

5. Levi-flat hypersurfaces with a complex line as singularity

In this section, we work with the system of coordinates $z=(z_1,...,z_n)\in\mathbb{C}^n$. The canonical local model examples of Levi-flat hypersurfaces M in $\mathbb{C}^3$ such that $\mathrm{Sing}(M)=L=\{z|z_1=z_2=0\}$ are $\{z|\mathrm{Re}(z_1^2+z_2^2)=0\}$ and $\{z|z_1\bar{z}_2-\bar{z}_1z_2=0\}$.

Recently, Burns and Gong [3] classified, up to local biholomorphism, all germs of quadratic Levi-flat hypersurfaces. Namely, up to biholomorphism, there are only five models as given in Table 2.

We address the problem of providing conditions to characterize singular Levi-flat hypersurfaces with a complex line as singularity. Using the classification due to Burns and Gong [3], it is not hard to prove the following proposition.

Proposition 5.1. *Suppose that M is a quadratic real-analytic Levi-flat hypersurface in $\mathbb{C}^n$, $n\geq3$, such that* $\mathrm{Sing}(M)=\{z|z_1=...=z_{n-1}=0\}$. *Then*

(1) *if $n=3$, M is biholomorphically equivalent to $Q_{0,2}$ or $Q_{2,4}$;*

(2) *if $n\geq4$, M is biholomorphically equivalent to $Q_{0,2(n-1)}$.*

Proof. To prove part (1), observe that there only are two models of M that admits $\mathrm{Sing}(M)=\{z|z_1=z_2=0\}$ as singularity, viz. $Q_{0,2}$ and $Q_{2,4}$. Now to prove part (2), note that if $n\geq4$, the real hypersurface $\{z|z_1\bar{z}_2-\bar{z}_1z_2=0\}$ has a complex subvariety of dimension $n-2$ as singularity. It follows that M is biholomorphically equivalent to $Q_{0,2(n-1)}$. □

In order to obtain a characterization, we define the Segre varieties associated with real-analytic hypersurfaces. Let M be a real-analytic hypersurface defined by

Type	Normal form	Singular set
$Q_{0,2k}$	$\mathrm{Re}(z_1^2+z_2^2+...+z_k^2)$	$\mathbb{C}^{n-k}$
$Q_{1,1}$	$z_1^2+2z_1^2\bar{z}_1+z_1^2$	empty
$Q_{1,2}^{\lambda}$	$z_1^2+2\lambda z_1^2\bar{z}_1+z_1^2$	$\mathbb{C}^{n-1}$
$Q_{2,2}$	$(z_1+\bar{z}_1)(z_2+\bar{z}_2)$	$\mathbb{R}^2\times\mathbb{C}^{n-2}$
$Q_{2,4}$	$z_1\bar{z}_2-\bar{z}_1z_2$	$\mathbb{C}^{n-2}$

Table 2. Levi-flat quadrics.

$\{z|F(z)=0\}$. Fix $p\in M$. The Segre variety associated with M at p is the complex variety in $(\mathbb{C}^n,p)$ defined by

(5) $$Q_p := \{z\in(\mathbb{C}^n,p)\,|\,F_{\mathbb{C}}(z,\bar{p})=0\}.$$

Now assume that M is Levi-flat and denote by L_p the leaf of $\mathcal{L}$ through $p\in M^*$. We denote by Q_p' the union of all branches of Q_p which are contained in M. Observe that Q_p' could be the empty set when $p\in\mathrm{Sing}(M)$. Otherwise, it is a complex variety of pure dimension $n-1$.

The following result is classical, we prove it here for completeness.

Proposition 5.2. *In the above situation, L_p is an irreducible component of (Q_p,p) and $Q_p'=L_p$.*

Proof. Since $p\in M^*$, É. Cartan's theorem assures that there exists a holomorphic coordinate system such that near p, M is given by $\{z|\mathrm{Re}(z_n)=0\}$ and p is the origin. In this coordinates system the foliation $\mathcal{L}$ is defined by $dz_n|_{M^*}=0$. In particular, $L_0=\{z|z_n=0\}$ and obviously $\{z|z_n=0\}$ is a branch of Q_0. Furthermore, L_0 is the unique germ of the complex variety of pure dimension $n-1$ at 0 which is contained in M. Hence $Q_0'=L_0$. □

Let $p\in\mathrm{Sing}(M)$, we say that p is a *Segre degenerate singularity* if Q_p has dimension n, that is, $Q_p=(\mathbb{C}^n,p)$. Otherwise, we say that p is a *Segre non-degenerate singularity.*

Suppose that M is defined by $\{z|F(z)=0\}$ in a neighborhood of p, observe that p is a degenerate singularity of M if $z\mapsto F_{\mathbb{C}}(z,\bar{p})$ is identically zero.

Remark 5.3. If V is a germ of a complex variety of dimension $n-1$ contained in M, then for $p\in V$ we have $(V,p)\subset(Q_p,p)$. In particular, if there exists infinitely many distinct complex varieties of dimension $n-1$ through $p\in M$ then p is a Segre degenerate singularity.

To continuation, we consider a germ at $0 \in \mathbb{C}^n$ of a codimension-one singular holomorphic foliation $\mathcal{F}$.

Definition 5.4. We say that $\mathcal{F}$ and M are *tangent*, if the leaves of the Levi-foliation $\mathcal{L}$ on M are also leaves of $\mathcal{F}$.

Definition 5.5. A meromorphic (holomorphic) function h is called a *meromorphic* (*holomorphic*) *first integral for* $\mathcal{F}$ if its indeterminacy (zeros) set is contained in $\mathrm{Sing}(\mathcal{F})$ and its level hypersurfaces contain the leaves of $\mathcal{F}$.

Recently, Cerveau and Lins Neto proved the following result.

Theorem 5.6. (Cerveau–Lins Neto [4]) *Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$, of a holomorphic codimension-one foliation tangent to a germ of an irreducible real-analytic hypersurface M. Then $\mathcal{F}$ has a non-constant meromorphic first integral.*

In our context, we prove the following result.

Theorem 5.7. *Let M be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$, of an irreducible real-analytic Levi-flat hypersurface such that $\mathrm{Sing}(M) = L := \{z \mid z_1 = \ldots = z_{n-1} = 0\}$. Suppose that*

(1) *every point in $\mathrm{Sing}(M)$ is a Segre non-degenerate singularity*;

(2) *the Levi-foliation $\mathcal{L}$ on M^* extends to a holomorphic foliation $\mathcal{F}$ in some neighborhood of M.*

Then there exists $f \in \mathcal{O}_n$ and a real-analytic curve $\gamma \subset \mathbb{C}$ such that $M = f^{-1}(\gamma)$.

Proof. Since the Levi-foliation $\mathcal{L}$ on M^* extends to a holomorphic foliation $\mathcal{F}$, we can apply directly Theorem 5.6, and thus $\mathcal{F}$ has a non-constant meromorphic first integral $f = g/h$, where g and h are relatively prime. We assert that f is holomorphic. In fact, if f is purely meromorphic, we have that for all $\zeta \in \mathbb{C}$, the complex hypersurfaces $V_\zeta = \{z \mid g(z) - \zeta h(z) = 0\}$ contains leaves of $\mathcal{F}$. In particular, M contains infinitely many hypersurfaces V_ζ, because M is closed and $\mathcal{F}$ is tangent to M. Set $\Lambda := \{\zeta \in \mathbb{C} \mid V_\zeta \subset M\}$. Note also that the foliation $\mathcal{F}$ is singular at L, so that $\mathcal{I}_f := \{z \mid h(z) = g(z) = 0\}$, the indeterminacy set of f, intersect L. Therefore, we have a point q in $\mathcal{I}_f \cap L$, which would be a Segre degenerate singularity, because $q \in V_\zeta$, for all $\zeta \in \Lambda$. This is a contradiction and the assertion is proved.

The foliation $\mathcal{F}$ is defined by $df = 0$, $f \in \mathcal{O}_n$, and is tangent to M. Without loss of generality, we can assume that f is an irreducible germ in $\mathcal{O}_n$. According to a remark of Brunella [2, p. 8], there exists a real-analytic curve $\gamma \subset \mathbb{C}$ through the origin such that $M = f^{-1}(\gamma)$. $\square$

Remark 5.8. In [10], Lebl gave conditions for the Levi-foliation on M^* to extend to a holomorphic foliation. One could consider these hypothesis and establish a more refined theorem. Note also that if Sing(M) is a germ of a smooth complex curve, it is possible to adapt the proof of Theorem 5.7. In general, the holomorphic extension problem for the Levi-foliation of a Levi-flat real-analytic hypersurface remains open and is of independent interest, for more details see [8].

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