Approximation numbers of composition operators on the Dirichlet space

Pascal Lefèvre, Daniel Li, Hervé Queffélec and Luis Rodríguez-Piazza

Abstract. We study the decay of approximation numbers of compact composition operators on the Dirichlet space. We give upper and lower bounds for these numbers. In particular, we improve on a result of El-Fallah, Kellay, Shabankhah and Youssfi, on the set of contact points with the unit circle of a compact symbolic composition operator acting on the Dirichlet space $\mathcal{D}$. We extend their results in two directions: first, the contact only takes place at the point 1. Moreover, the approximation numbers of the operator can be arbitrarily subexponentially small.

1. Introduction

1.1. Organization of the paper

The paper deals with composition operators. This area is widely studied nowadays, on various spaces of analytic functions (e.g. Hardy, Bergman and Dirichlet spaces): one may read for instance the monographs [15] or [4] to get an overview on the subject until the nineties, and [5] or [9] for some recent results in the framework of the Dirichlet space. It seems natural to try to apply again some of the techniques used in the framework of Hardy and Bergman spaces. Nevertheless it is far from being that simple. Actually it often turns out that the Dirichlet space is one of the most difficult "classical" spaces to handle. For instance, a first difficulty appears at the very beginning of the theory: the composition operators are not necessarily bounded when we only require the symbol to belong to the Dirichlet space (whereas all the composition operators are bounded on the Hardy and Bergman spaces).

The study of approximation numbers of composition operators acting on classical spaces of analytic functions (like the Hardy and Bergman spaces) was initiated in [10] and [11] by the three last named authors. In the present paper, we get in-

terested in the same kind of questions but for composition operators acting on the Dirichlet space. Some results already appeared in [9] (among other things), but we focus exclusively on this topic in the sequel.

The notation and definitions are made precise in the next subsection.

In Section 2, we show that some similar phenomena (as in Hardy and Bergman spaces) hold in the framework of Dirichlet spaces. More precisely, the approximation numbers of composition operators on the Dirichlet space cannot decay more rapidly than exponentially, and this speed of convergence can only be attained for symbols φ satisfying $\|\varphi\|_\infty < 1$ (see Theorems 2.1 and 2.2). On the other hand, we investigate the extremal case and it turns out that C_φ may have almost geometric decay (in particular it belongs to all Schatten classes) and $\varphi(\mathbb{D})$ may touch the boundary of $\mathbb{D}$ (see Theorem 2.8).

In Section 3, we focus on composition operators whose symbol is a cusp map. They play the same role in the theory as the lens maps do in the theory of Hardy spaces. The rate of decay of its approximation numbers is given in Theorem 3.1.

At last, in Section 4, we make Theorem 2.8 more precise and prove in Theorem 4.1 that the symbol (which will be the composition of a cusp map and a peak function) may belong to both the disk algebra and the Dirichlet space, and moreover meet the boundary precisely at 1 with a level set which is any compact set with zero logarithmic capacity.

1.2. Notation and background

Given two Banach spaces X and Y, and an integer $n \geq 1$, we recall that the nth *approximation number* of an operator $T\colon X \to Y$ is

$$a_n(T) = \inf\{\|T - R\| \mid R\colon X \to Y \text{ and } \operatorname{rank}(R) < n\}.$$

We point out that the sequence $\{a_n(T)\}_{n=1}^\infty$ is nonincreasing and bounded by $a_1(T) = \|T\|$. We refer to [2] for more on the subject. In the sequel, we shall work with separable Hilbert spaces. Hence, in our framework, $\{a_n(T)\}_{n=1}^\infty$ belongs to c_0 if and only if T is compact.

We denote by $\mathbb{D}$ the open unit disk of the complex plane and by A the normalized area measure $dx\,dy/\pi$ of $\mathbb{D}$. The unit circle is denoted by $\mathbb{T} = \partial\mathbb{D}$.

A *Schur function* is an analytic self-map of $\mathbb{D}$ and the associated composition operator is defined, formally, by $C_\varphi(f) = f \circ \varphi$. The function φ is called the symbol of C_φ.

The *Dirichlet space* $\mathcal{D}$ is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$(1) \qquad \|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty.$$

If $f(z)=\sum_{n=0}^{\infty} c_n z^n$, one has

$$
(2) \qquad \|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n|c_n|^2.
$$

Then $\|\cdot\|_{\mathcal{D}}$ is a norm on $\mathcal{D}$, making $\mathcal{D}$ a Hilbert space. We consider its subspace $\mathcal{D}_*$, consisting of functions $f \in \mathcal{D}$ such that $f(0)=0$. In this paper, we call $\mathcal{D}_*$ the *Dirichlet space*. For further information on the Dirichlet space, the reader may see [1] or [14].

Recall that, whereas every Schur function φ generates a bounded composition operator C_{φ} on the Hardy or Bergman spaces, this is no longer the case for the Dirichlet space (see [12], Proposition 3.12, for instance).

The *Bergman space* $\mathfrak{B}$ is the space of analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ such that

$$
\|f\|_{\mathfrak{B}}^2 := \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty.
$$

If $f(z)=\sum_{n=0}^{\infty} c_n z^n$, one has $\|f\|_{\mathfrak{B}}^2 = \sum_{n=0}^{\infty} |c_n|^2/(n+1)$.

We denote by S_n the truncation operator: if $f(z)=\sum_{k=1}^{\infty} c_k z^k$ is in $\mathcal{D}_*$, then

$$
(3) \qquad (S_n f)(z) = \sum_{k=1}^{n} c_k z^k.
$$

The *Carleson window* centered at $\xi \in \mathbb{T}$ and of size $h \in (0,1)$ is defined as

$$
(4) \qquad S(\xi, h) = \{z \in \mathbb{D} | \, |z-\xi| < h\}.
$$

The notation $A \lesssim B$ (equivalently $B \gtrsim A$) means that $A \leq CB$ for some constant $C>0$, and $A \approx B$ means that we have both $A \lesssim B$ and $B \lesssim A$.

2. Approximation numbers in the general case

2.1. Geometric decay of the approximation numbers

It was shown in [10], that the approximation numbers of composition operators on the Hardy space H^2 as well as on the (weighted) Bergman spaces $\mathfrak{B}_{\alpha}$, $\alpha > -1$, cannot decay more rapidly than exponentially, and that this speed of convergence can only be attained for symbols mapping the unit disk $\mathbb{D}$ into a smaller disk $r\mathbb{D}$, with $0<r<1$. In this section, we see that the same phenomenon holds for the Dirichlet space. The proofs will be adapted from those of [10].

Our first result is on the geometric decay.

Theorem 2.1. *Let $\varphi\colon \mathbb{D}\to\mathbb{D}$ be an analytic self-map inducing a bounded composition operator on $\mathcal{D}_*$. Then, there exist positive constants $c',c>0$ and $0<r<1$ such that the approximation numbers of the composition operator $C_\varphi\colon \mathcal{D}_*\to\mathcal{D}_*$ satisfy*

$$\text{(5)} \qquad c'\sqrt{n}\|\varphi\|_\infty^n \geq a_n(C_\varphi)\geq cr^n, \quad n=1,2,\ldots.$$

Proof. We introduce some notation. First, we set, for any operator T on some Hilbert space H,

$$\text{(6)} \qquad \beta(T)=\liminf_{n\to\infty} a_n(T)^{1/n}.$$

Next, let

$$\varphi^\#(z)=\frac{\varphi'(z)(1-|z|^2)}{1-|\varphi(z)|^2}$$

be the *pseudo-hyperbolic derivative* of φ; we set

$$\text{(7)} \qquad [\varphi]=\sup_{z\in\mathbb{D}}|\varphi^\#(z)|=\|\varphi^\#\|_\infty.$$

Note that $[\varphi]\leq 1$, by the Schwarz–Pick inequality.

The upper bound is easy. We may assume that $\|\varphi\|_\infty<1$. Note that, since C_φ is bounded on $\mathcal{D}_*$, we have $\varphi^k=C_\varphi(z^k)\in\mathcal{D}_*$. Then

$$\begin{aligned}
a_n(C_\varphi)^2 &\leq \|C_\varphi-C_\varphi S_{n-1}\|^2 \\
&\leq \|C_\varphi-C_\varphi S_{n-1}\|_{\mathrm{HS}}^2 \\
&= \sum_{k=n}^\infty \frac{\|\varphi^k\|_\mathcal{D}^2}{k} \\
&= \sum_{k=n}^\infty \int_\mathbb{D} k|\varphi^{k-1}(z)|^2|\varphi'(z)|^2\,dA(z) \\
&= \int_\mathbb{D} \sum_{k=n}^\infty k|\varphi^{k-1}(z)|^2|\varphi'(z)|^2\,dA(z) \\
&\leq \int_\mathbb{D} \sum_{k=n}^\infty k\|\varphi\|_\infty^{2k-2}|\varphi'(z)|^2\,dA(z) \\
&\leq K(\varphi)n\|\varphi\|_\infty^{2n}\|\varphi\|_\mathcal{D}^2
\end{aligned}$$

(we have used that

$$\sum_{k=n}^{\infty} k\rho^{k-1} = \frac{\rho^{n-1}[n-(n-1)\rho]}{(1-\rho)^2} \leq \frac{1}{(1-\rho)^2}n\rho^{n-1},$$

with $\rho = \|\varphi\|_\infty^2$), implying that

$$(8) \qquad a_n(C_\varphi) \lesssim \sqrt{n}\|\varphi\|_{\mathcal{D}}\|\varphi\|_\infty^n \approx \sqrt{n}\|\varphi\|_\infty^n$$

and $\beta(C_\varphi) \leq \|\varphi\|_\infty$.

For the lower bound in (5), we shall prove that

$$(9) \qquad [\varphi]^2 \leq \beta(C_\varphi) \leq \|\varphi\|_\infty,$$

which will give the result, since for each $\varkappa < [\varphi]^2$ there will be some constant $c_\varkappa > 0$ such that $a_n(C_\varphi) \geq c_\varkappa \varkappa^n$, $n \geq 1$.

The inequality (9) is obtained as in the Hardy and Bergman cases in [10]. We may assume that C_φ is compact on $\mathcal{D}_*$ (since otherwise $\beta(C_\varphi)=1$ and the result is trivial). Now, set $\phi_u(z)=u-z/(1-\bar{u}z)$, $u \in \mathbb{D}$. Then, if φ is a symbol with C_φ compact on $\mathcal{D}_*$ and $a \in \mathbb{D}$, let $\psi = \phi_{\varphi(a)} \circ \varphi \circ \phi_a$. Note that the compactness of C_φ on $\mathcal{D}_*$ implies its compactness on $\mathcal{D}$. Hence we can write $C_\psi = C_{\phi_a} \circ C_\varphi \circ C_{\phi_{\varphi(a)}}$. Now, the relations $\psi(0)=0$, $\psi'(0)=\varphi'(a)(1-|a|^2)/(1-|\varphi(a)|^2)=\varphi^\#(a)$ and the diagrams

$$\mathcal{D}_* \xrightarrow{C_{\phi_a}} \mathcal{D} \xrightarrow{C_\varphi} \mathcal{D} \xrightarrow{C_{\phi_{\varphi(a)}}} \mathcal{D}_*,$$

with $0 \xrightarrow{\phi_a} a \xrightarrow{\varphi} \varphi(a) \xrightarrow{\phi_{\varphi(a)}} 0$, show that $\psi \in \mathcal{D}_*$ and that C_ψ is also compact on $\mathcal{D}_*$. Now we notice that, for any compact composition operator C_τ on $\mathcal{D}_*$, the solution σ of the König equation

$$\sigma \circ \tau = \tau'(0)\sigma, \quad \sigma(0)=0 \text{ and } \sigma'(0)=1,$$

has to belong to $\mathcal{D}_*$ as this is the case for any Hilbert space of analytic functions on $\mathbb{D}$. Hence, if $\psi'(0)=\varphi^\#(a)\neq 0$, the sequence of eigenvalues of C_ψ is $\{[\psi'(0)]^n\}_{n=0}^\infty$. It follows from [10], Lemma 3.2 (which is an easy consequence of Weyl's inequality) that $\beta(C_\varphi)=\beta(C_\psi)\geq|\varphi^\#(a)|$. Since this remains trivially true when $\varphi^\#(a)=0$, Theorem 2.1 is proved. $\square$

Now, we shall see that the geometric decay can take place only for symbols φ such that $\|\varphi\|_\infty < 1$.

Theorem 2.2. *For each $r \in (0,1)$, there exists $s = s(r) \in (0,1)$, with $s(r) \to 1$ as $r \to 1$, such that, whenever C_φ a bounded composition operator on $\mathcal{D}_*$, we have*

$$(10) \qquad \|\varphi\|_\infty > r \quad \Longrightarrow \quad a_n(C_\varphi) \gtrsim \frac{s^n}{\sqrt{n}}.$$

We shall see in the proof that we can take $s = e^{-\varepsilon\pi}$, with $\varepsilon = 2\pi/[\log(1+r)/(1-r)]$ (see (13), where s is changed into s^2).

Note that, in particular, with the notation (6), one has

$$\|\varphi\|_\infty = 1 \quad \Longrightarrow \quad \beta(C_\varphi) = 1.$$

The converse implication is true by (9).

The proof follows the same pattern as in [10], with the following additional argument.

Lemma 2.3. *Let ν be a probability measure, which is compactly carried by $\varphi(\mathbb{D})$, and let $R_\nu \colon \mathfrak{B} \to L^2(\nu)$ be the canonical inclusion. Then,*

$$a_n(C_\varphi) \gtrsim a_n(R_\nu).$$

To prove this lemma, we need another one. For $f \in \mathcal{H}(\mathbb{D})$ and $0 < r < 1$, we set as usual

$$M(r, f) = \sup_{|z|=r} |f(z)|.$$

We then have the following result.

Lemma 2.4. *Let $g \in \mathcal{H}(\mathbb{D})$, not identically zero, and $0 < r < 1$. Then, there exists $C > 0$, depending only on g and r, such that*

$$(11) \qquad M(r, f) \leq C \|fg\|_{\mathfrak{B}} \quad \text{for all } f \in \mathcal{H}(\mathbb{D}).$$

Therefore, for each compact subset $L \subseteq \mathbb{D}$, there exists a constant $C = C(L, g)$ such that, for any $f \in \mathcal{H}(\mathbb{D})$, one has

$$(12) \qquad \|f\|_{\mathcal{C}(L)} \leq C \|fg\|_{\mathfrak{B}}.$$

Proof. Since the zeros of g are at most countable, we can find $r \leq \rho < 1$ such that g does not vanish on the circle of radius ρ. Hence there is some $\mu_r > 0$ such that

$$|g(a)| \geq \mu_r > 0 \quad \text{for } |a| = \rho.$$

Let $\delta=1-\rho$, $f\in\mathcal{H}(\mathbb{D})$ and $|a|=\rho$. By the subharmonicity of $|fg|^2$, we have

$$\mu_r^2|f(a)|^2 \leq |f(a)g(a)|^2 \leq \frac{1}{\delta^2}\int_{D(a,\delta)}|fg|^2\,dA \leq \frac{1}{\delta^2}\int_{\mathbb{D}}|fg|^2\,dA,$$

whence $M(\rho,f)\leq C\|fg\|_{\mathfrak{B}}$ with $C=1/\delta\mu_r$. But $M(r,f)\leq M(\rho,f)$ by the maximum modulus principle, and we get (11). This ends the proof of Lemma 2.4, since if $L\subseteq\overline{D(0,r)}$ and $f\in\mathcal{H}(\mathbb{D})$, then $\|f\|_{\mathcal{C}(L)}\leq M(r,f)\leq C\|fg\|_{\mathfrak{B}}$ by the maximum modulus principle again. $\square$

Proof of Lemma 2.3. Let $D\colon\mathcal{D}_*\to\mathfrak{B}$ be the differentiation operator (which is a unitary operator by the definition of the norms of these spaces). We can write (see [10], proof of Lemma 3.5) $\nu=\varphi(\sigma)$, for some probability measure σ carried by a compact subset L of $\mathbb{D}$. We then have, for any $f\in\mathcal{D}_*$, with the help of Lemma 2.4 applied to the nonzero function $g=\varphi'$,

$$\|R_\nu Df\|_{L^2(\nu)}^2 = \int|f'|^2\,d\nu = \int_L|f'\circ\varphi|^2\,d\sigma \leq \|f'\circ\varphi\|_{\mathcal{C}(L)}^2$$

$$\leq C^2\int_{\mathbb{D}}|f'\circ\varphi|^2|\varphi'|^2\,dA = C^2\|C_\varphi f\|_{\mathcal{D}_*}^2.$$

This implies that $a_n(R_\nu D)\leq Ca_n(C_\varphi)$, or, equivalently, $a_n(R_\nu)\leq Ca_n(C_\varphi)$, since D is unitary. $\square$

Recall now the following lemmas from [10] (the first one will be used again later, in Lemma 3.6).

Lemma 2.5. ([10], Lemma 3.6) *For every* $r\in(0,1)$ *there exist* $s=s(r)<1$ *and* $f=f_r\in H^\infty$ *with the following properties*:
  (1) $\lim_{r\to1^-}s(r)=1$;
  (2) $\|f\|_\infty\leq1$;
  (3) $f((0,r])=s\partial\mathbb{D}$ *in a one-to-one way.*
Explicitly, one has

(13) $$s=e^{-\varepsilon\pi/2} \quad\text{with } \varepsilon=\frac{2\pi}{\log\dfrac{1+r}{1-r}}.$$

Note that in [10], ε was defined with the help of a parameter ρ, but

$$\frac{1+\rho}{1-\rho}=\sqrt{\frac{1+r}{1-r}}.$$

Lemma 2.6. (See [10], Lemma 3.7 and its proof) *Let $0<r<1$ and s be as in* (13). *Then, there exists a probability measure μ carried by $[0,r]$ such that, if $R_\mu\colon \mathfrak{B}\to L^2(\mu)$ is the canonical inclusion, one has, for every $n\geq 1$,*

$$a_n(R_\mu)\gtrsim \frac{s^n}{\sqrt{n}}.$$

Lemma 2.7. (See [10], Lemma 3.8 and its sequel) *Let $\varphi\colon \mathbb{D}\to\mathbb{D}$ be a Schur function and suppose that 0 and r belong to $\varphi(\mathbb{D})$. Then, for any probability measure μ carried by $[0,r]$, there exists a probability measure ν compactly carried by $\varphi(\mathbb{D})$ such that*

$$a_{2n}(R_\mu)\leq 2a_n(R_\nu).$$

Proof of Theorem 2.2. The three lemmas put together give the result. Indeed, assume that $\|\varphi\|_\infty>r$. By making a rotation, we may assume that $r\in\varphi(\mathbb{D})$. Let then μ be as in Lemma 2.6 and ν be as in Lemma 2.7 (which we may use since $0=\varphi(0)\in\varphi(\mathbb{D})$). Using Lemma 2.3, we obtain

$$a_n(C_\varphi)\gtrsim a_n(R_\nu)\gtrsim a_{2n}(R_\mu)\gtrsim \frac{s^{2n}}{\sqrt{2n}},$$

and, changing s into s^2, this ends the proof of Theorem 2.2. $\square$

2.2. Extremal behavior

In this subsection, we see that we may have "very compact" composition operators on $\mathcal{D}_*$ whose symbol has an image touching the boundary of $\mathbb{D}$.

Theorem 2.8. *For every vanishing sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive numbers, there exists a symbol φ with $\|\varphi\|_\infty=1$ such that C_φ is compact on $\mathcal{D}_*$, but*

$$(14) \qquad\qquad\qquad a_n(C_\varphi)\lesssim e^{-n\varepsilon_n}.$$

In particular, C_φ may be in all Schatten classes $S_p(\mathcal{D}_*)$, $p>0$, of the Dirichlet space.

The result will follow from Theorem 2.9, which is the analogue of Theorem 5.1 in [10]. In the sequel, we shall consider the counting function $n_\varphi\colon \mathbb{D}\to\mathbb{N}$ of φ, namely, for $w\in\mathbb{D}$,

$$n_\varphi(w)=\sum_{\varphi(z)=w} 1$$

with each z occurring as many times as its multiplicity.

Of course, when $w\notin\varphi(\mathbb{D})$, we have $n_\varphi(w)=0$.

Theorem 2.9. *Let φ be a Schur function inducing a bounded composition operator on $\mathcal{D}_*$. Set*

$$(15) \qquad m(t) = \sup_{\substack{|\xi|=1 \\ 0<h\leq t}} \frac{1}{h^2} \int_{S(\xi,h)} n_\varphi \, dA.$$

Then

$$(16) \qquad a_n(C_\varphi) \lesssim \inf_{0<t<1} \left[\sqrt{n}(1-t)^n + \sqrt{m(t)} \right].$$

Proof. We set $\mu = n_\varphi \, dA$ and denote by I_μ the natural embedding from $\mathfrak{B}$ to $L^2(\mu)$. We know from [9], proof of Theorem 3.1, that $C_\varphi^* C_\varphi$ is unitarily equivalent to the Toeplitz operator $T_\mu = I_\mu^* I_\mu$ and therefore that

$$(17) \qquad a_n(C_\varphi) = a_n(I_\mu).$$

Let now $R\colon \mathfrak{B} \to L^2(\mu)$ be the operator of rank $\leq n$ defined by

$$R(f)(z) = \sum_{k=0}^{n-1} \hat{f}(k) z^k, \quad f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k,$$

so that $(I_\mu - R)(f) = I_\mu(u)$ with

$$u(z) = \sum_{k=n}^{\infty} \hat{f}(k) z^k =: z^n v(z)$$

and $v \in \mathfrak{B}$, with $\|v\|_{\mathfrak{B}} \leq \sqrt{n+1}\|u\|_{\mathfrak{B}}$ as is obvious by inspection of the coefficients. Assume that $\|f\|_{\mathfrak{B}} \leq 1$. Then $\|v\|_{\mathfrak{B}} \leq \sqrt{n+1}\|u\|_{\mathfrak{B}} \leq \sqrt{n+1}\|f\|_{\mathfrak{B}} \leq \sqrt{n+1}$.

Fix $0<t<1$. We have

$$\|(I_\mu - R)(f)\|_{L^2(\mu)}^2 = \|I_\mu(u)\|_{L^2(\mu)}^2$$

$$= \int_{\mathbb{D}} |u(z)|^2 \, d\mu(z)$$

$$= \int_{|z|\leq 1-t} |u(z)|^2 \, d\mu(z) + \int_{1-t\leq|z|<1} |u(z)|^2 \, d\mu(z)$$

$$\leq (1-t)^{2n} \int_{|z|\leq 1-t} |v(z)|^2 \, d\mu(z) + \int_{1-t\leq|z|<1} |u(z)|^2 \, d\mu(z)$$

$$=: I_1 + I_2.$$

Clearly, using that $\|v\|_{\mathfrak{B}} \leq \sqrt{n+1}$, we have

$$I_1 \leq (1-t)^{2n} \|I_\mu(v)\|^2_{L^2(\mu)} \lesssim n(1-t)^{2n}.$$

For I_2, we use the Carleson–Hastings embedding theorem [6] for the natural embedding $I_\nu \colon \mathfrak{B} \to L^2(\nu)$, with the measure $\nu = \mu_t$ which is the restriction of μ to the annulus $1-t < |z| < 1$. This theorem tells us that

$$\|I_{\mu_t}\|^2 \approx \sup_{\substack{|\xi|=1 \\ 0<h<1}} \frac{1}{h^2} \int_{S(\xi,h)} d\mu_t(z).$$

Since μ_t is supported by the annulus $1-t < |z| < 1$, we can restrict ourselves to $0 < h \leq t$ to get from the previous inequality,

$$I_2 = \|I_{\mu_t}(u)\|^2_{L^2(\mu_t)} \lesssim m(t) \|u\|^2_{\mathfrak{B}} \lesssim m(t).$$

It follows that

$$\|(I_\mu - R)(f)\|^2_{L^2(\mu)} \lesssim [n(1-t)^{2n} + m(t)].$$

Taking the supremum over f, and then square roots, we get

$$a_{n+1}(I_\mu) \leq \|I_\mu - R\| \lesssim \left[\sqrt{n}(1-t)^n + \sqrt{m(t)} \, \right].$$

Finally, taking the infimum on t and changing n into $n-1$, we end the proof of Theorem 2.9. $\square$

Remark. In [10], Theorem 4.1, it was proved in the opposite direction, following [3], that a composition operator on the weighted Bergman space $\mathfrak{B}_\alpha$ may be compact, but not much more. It is likely that the same occurs in $\mathcal{D}_*$, namely that for every vanishing sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive numbers, there exists a symbol φ such that C_φ is compact on $\mathcal{D}_*$ and for which $\liminf_{n\to\infty} a_n(C_\varphi)/\varepsilon_n > 0$ (in particular, if it happens to be true, we might have C_φ compact and in no Schatten class $S_p(\mathcal{D}_*)$, $p < \infty$, of the Dirichlet space). But we have not succeed in proving that.

3. Approximation numbers of the cusp map

In [11], it is shown that there is a composition operator $C_\chi \colon H^2 \to H^2$, whose symbol is called the cusp map, defined on the Hardy space, such that, for some constants $c_1 > c_2 > 0$, one has

$$(18) \qquad e^{-c_1 n/\log n} \lesssim a_n(C_\chi \colon H^2 \to H^2) \lesssim e^{-c_2 n/\log n}, \quad n = 2, 3, \dots .$$

In [9], it was shown that every composition operator which is compact on the Dirichlet space is in all Schatten classes $S_p(H^2)$, $p>0$, on the Hardy space. Therefore the approximation numbers of $C_\varphi\colon H^2\to H^2$ must be (much) smaller than those of $C_\varphi\colon \mathcal{D}_*\to\mathcal{D}_*$. The next theorem gives, for the cusp map, this order of smallness.

Theorem 3.1. *Let χ be the cusp map. There exist two constants $0<c'<c$ such that the approximation numbers $a_n(C_\chi)$ of the associated composition operator $C_\chi\colon \mathcal{D}_*\to\mathcal{D}_*$ satisfy*

$$(19) \qquad e^{-c\sqrt{n}} \lesssim a_n(C_\chi\colon \mathcal{D}_*\to\mathcal{D}_*) \lesssim e^{-c'\sqrt{n}}, \quad n=1,2,\dots.$$

Recall the definition of the *cusp map* χ, introduced in [7], and later used, with a slightly different definition in [11]. Actually, as in [9], we have to modify it slightly again in order to have $\chi(0)=0$. We first define:

$$\chi_0(z) = \frac{\left(\dfrac{z-i}{iz-1}\right)^{1/2}-i}{-i\left(\dfrac{z-i}{iz-1}\right)^{1/2}+1};$$

we note that $\chi_0(1)=0$, $\chi_0(-1)=1$, $\chi_0(i)=-i$, $\chi_0(-i)=i$, and $\chi_0(0)=\sqrt{2}-1$. Then we set

$$\chi_1(z)=\log\chi_0(z), \quad \chi_2(z)=-\frac{2}{\pi}\chi_1(z)+1 \quad \text{and} \quad \chi_3(z)=\frac{a}{\chi_2(z)},$$

and finally

$$\chi(z)=1-\chi_3(z),$$

where

$$(20) \qquad a=1-\frac{2}{\pi}\log\left(\sqrt{2}-1\right)\in(1,2)$$

is chosen in order that $\chi(0)=0$. The image Ω of the (univalent) cusp map is formed by the intersection of the inside of the disk $D\left(1-\frac{1}{2}a,\frac{1}{2}a\right)$ and the outside of the two disks $D\left(1+\frac{1}{2}ia,\frac{1}{2}a\right)$ and $D\left(1-\frac{1}{2}ia,\frac{1}{2}a\right)$.

3.1. Proof of the upper bound of Theorem 3.1

We need some lemmas.

Lemma 3.2. *We have*

$$\|\chi^n\|_{\mathcal{D}} \leq C n^{-\delta}, \tag{21}$$

for every $n \geq 1$, where C and δ are positive numerical constants.

Proof. Since χ is univalent, we have, for every $0 < h < 1$,

$$\|\chi^n\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} n^2 |w|^{2n-2} n_\chi(w)\, dA(w) \leq n^2 (1-h)^{2n-2} + n^2 \int_{|w| \geq 1-h} n_\chi(w)\, dA(w).$$

But $\int_{|w| \geq 1-h} n_\chi(w)\, dA(w)$ is the area of $\chi(\mathbb{D}) \cap \{w \,|\, |w| \geq 1-h\}$; since $\chi(\mathbb{D})$ is delimited at the cuspidal point 1 by two circular arcs, this area is $\approx h^3$. We hence get that

$$\|\chi^n\|_{\mathcal{D}} \lesssim n[e^{-nh} + h^{3/2}].$$

The choice $h = 2 \log n / n$ gives $\|\chi^n\|_{\mathcal{D}} \lesssim n^{-1/2} (\log n)^{3/2}$ and hence the lemma, with any $\delta < \frac{1}{2}$. $\quad\square$

An immediate corollary, in which S_N denotes the operator of Nth partial sum, as defined in (3), is the following.

Corollary 3.3. *We have*

$$\|C_\chi - C_\chi S_N\| \lesssim N^{-\delta}.$$

Proof. Using the Hilbert–Schmidt norm, we get

$$\|C_\chi - C_\chi S_N\|^2 \leq \|C_\chi - C_\chi S_N\|_{\mathrm{HS}}^2 = \sum_{n > N} \frac{\|\chi^n\|^2}{n} \lesssim \sum_{n > N} n^{-1-2\delta} \lesssim N^{-2\delta}. \quad\square$$

Now, the idea for majorizing $a_n(C_\chi)$ is to write, for every operator R with rank $< n$,

$$\|C_\chi - R\| \leq \|C_\chi - C_\chi S_N\| + \|C_\chi S_N - R\|,$$

which gives, taking the infimum over all such R,

$$a_n(C_\chi) \leq \|C_\chi - C_\chi S_N\| + a_n(C_\chi S_N). \tag{22}$$

Using the corollary, we get

$$a_n(C_\chi) \lesssim N^{-\delta} + a_n(C_\chi S_N) \tag{23}$$

and our goal is to give a good upper bound of $a_n(C_\chi S_N)$.

Lemma 3.4. *For some numerical constant $\varepsilon > 0$, we have*

$$(24) \qquad a_n(C_\chi S_N) \lesssim \sqrt{N} e^{-\varepsilon\sqrt{n}}.$$

With this estimation, we get

$$a_n(C_\chi) \lesssim \left[N^{-\delta} + \sqrt{N} e^{-\varepsilon\sqrt{n}} \right]$$

and, by adjusting $N = [e^{\varepsilon\sqrt{n}}]$, we obtain the upper bound in (19).

Proof. To prove (24), we shall replace $C_\chi S_N$ by a "dominating" operator.

We begin with observing that, if $f(z) = \sum_{j=1}^{\infty} c_j z^j \in \mathcal{D}_*$, we have by a change of variables, setting $d\mu = n_\chi \, dA = \mathbb{1}_{\chi(\mathbb{D})} \, dA$,

$$(25) \qquad \|C_\chi S_N f\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} \left| \sum_{j=1}^{N} j c_j w^{j-1} \right|^2 n_\chi(w) \, dA(w) = \int_{\mathbb{D}} \left| \sum_{j=1}^{N} j c_j w^{j-1} \right|^2 d\mu(w).$$

Now, denote by $\Delta_N \colon \mathcal{D}_* \to H^2$ the map defined by

$$\Delta_N f(w) = \sum_{j=1}^{N} j c_j w^{j-1}.$$

Observe that

$$\|\Delta_N f\|_{H^2}^2 = \sum_{j=1}^{N} j^2 |c_j|^2 \leq N \sum_{j=1}^{N} j |c_j|^2 \leq N \|f\|_{\mathcal{D}}^2,$$

so that $\|\Delta_N\| \leq \sqrt{N}$.

Let also J be the canonical inclusion $J \colon H^2 \to L^2(\mu)$. The equality (25) reads $\|C_\chi S_N f\|_{\mathcal{D}}^2 = \|J \Delta_N f\|_{L^2(\mu)}^2$; therefore there is a contraction $C_N \colon \mathcal{D}_* \to \mathcal{D}_*$ such that

$$(26) \qquad C_\chi S_N = C_N J \Delta_N.$$

The ideal property of approximation numbers now implies that

$$a_n(C_\chi S_N) = a_n(C_N J \Delta_N) \leq \|C_N\| a_n(J) \|\Delta_N\| \leq \sqrt{N} a_n(J),$$

and we are left with the task of majorizing $a_n(J)$. To this end, we use the Gelfand numbers c_n ([13] or [2]). Recall that if $T \colon X \to X$ is an operator on some Banach space X, then $c_n(T) = \inf\{\|T|_Z\| \mid Z \subseteq X \text{ and } \operatorname{codim} Z < n\}$, and if $X = H$ is a Hilbert space, then $c_n(T) = a_n(T)$.

Let B be a Blaschke product of length $< n$, and let $E = B H^2$ which is a subspace of H^2 of codimension $< n$. We have

$$a_n(J) = c_n(J) \leq \|J|_E\|.$$

The majorization is then made using the Carleson embedding theorem. Let r be the greatest integer $<\sqrt{n}$, and B_0 be a Blaschke product with r zeros well distributed on the interval $(0,1)$. More precisely, B_0 has its zeros at the points

$$z_j = 1 - 2^{-j}, \quad 1 \leq j \leq r.$$

Set $\Omega = \chi(\mathbb{D})$ and observe that

(27) $$z \in \Omega \text{ and } \operatorname{Re} z \geq 1-h \implies |\operatorname{Im} z| \lesssim h^2,$$

(28) $$A[S(\xi,h) \cap \Omega] \lesssim h^3 \quad \text{for every } \xi \in \mathbb{T}.$$

Let now $B = B_0^r$. This is a Blaschke product of length $r^2 < n$. Using the Carleson embedding theorem (for the measure $d\mu = n_\chi \, dA$), as in [10] and [11], and the univalence of χ, we get

(29) $$\|J|_E\|^2 \lesssim \sup_{\substack{0<h<1 \\ \|\xi\|=1}} \frac{1}{h} \int_{S(\xi,h) \cap \Omega} |B|^2 \, dA.$$

To estimate the supremum in the right-hand side of (29), we may assume that $h = 2^{-l}$ and we separate two cases.

• $l \geq r$. Then, using (28) and the fact that $|B| \leq 1$, we have

(30) $$\frac{1}{h} \int_{S(\xi,h) \cap \Omega} |B|^2 \, dA \lesssim \frac{1}{h} h^3 = h^2 = 2^{-2l} \leq 2^{-2r}.$$

• $l < r$. Then, we have

$$\frac{1}{h} \int_{S(\xi,h) \cap \Omega} |B|^2 \, dA \leq \frac{1}{h} \int_{\{z \mid |z-1| \leq 2^{-r}\} \cap \Omega} |B(z)|^2 \, dA(z)$$

$$+ \sum_{j=l+1}^{r} \frac{1}{h} \int_{C_j \cap \Omega} |B|^2 \, dA,$$

where C_j is the annulus

$$C_j = \{z \in \mathbb{D} \mid 2^{-j} \leq |z-1| \leq 2^{-j+1}\}.$$

The first term is handled as before. Now, since Ω is contained in some sector $1-|z| \geq \delta|1-z|$, we have, for $z \in C_j \cap \Omega$,

$$1-|z| \geq \delta 2^{-j} \quad \text{and} \quad 1-|z_j| = 2^{-j},$$

whereas

$$|z-z_j| = |z-1+2^{-j}| \le |z-1|+2^{-j} \le 3\cdot 2^{-j}.$$

This implies that, for some absolute constant $M>0$,

$$|z-z_j| \le M \min(1-|z|, 1-|z_j|)$$

and, by [8], Lemma 2.3, the jth factor of B_0 is, in modulus, less than $\varkappa = M/\sqrt{M^2+1}<1$. Therefore $|B| = |B_0^r| \le \varkappa^r$ on all sets $C_j \cap \Omega$, so that

$$\sum_{j=l+1}^{r} \frac{1}{h} \int_{C_j \cap \Omega} |B|^2 \, dA \le \sum_{j=l+1}^{r} \frac{1}{h} \int_{C_j \cap \Omega} \varkappa^{2r} \, dA$$

$$\lesssim 2^l \varkappa^{2r} A[S(\xi, 2^{-l})] \lesssim \varkappa^{2r} 2^l 2^{-2l} \lesssim \varkappa^{2r}.$$

This finally shows, due to (29) and (30), that $\|J|_E\| \lesssim \varkappa^r$, or, after setting $\varkappa = e^{-\varepsilon}$, that (recall that r is the greatest integer $<\sqrt{n}$, and hence $r \approx \sqrt{n}$)

$$\|J|_E\| \lesssim e^{-\varepsilon\sqrt{n}}.$$

This proves (24) and ends the proof of the upper bound in Theorem 3.1. $\square$

3.2. Proof of the lower bound of Theorem 3.1

Recall that μ is the measure $d\mu = n_\chi \, dA$ and that $\Omega = \chi(\mathbb{D})$.
Consider the diagram

$$H^2 \xrightarrow{P} \mathcal{D}_* \xrightarrow{C_\chi} \mathcal{D}_* \xrightarrow{D} L^2(\mu),$$

in which

$$P\left(\sum_{n=0}^{\infty} c_n z^n\right) = \sum_{n=0}^{\infty} c_n \frac{z^{n+1}}{n+1}$$

is the "primitivation" operator and D is the differentiation operator. We have

$$DC_\chi Pf = (f\circ\chi)\chi'.$$

We note that, by the definition of the norms, $\|P\| \le 1$. For $0<h<1$ fixed, let also

$$R\colon H^2 \longrightarrow L^\infty([0, 1-h])$$

be the canonical injection.

The rest of the proof consists of two steps, the first of which consists of showing that $a_n(C_\chi)$ is not much smaller than $a_n(R)$.

Lemma 3.5. *We have*

$$a_n(C_\chi) \geq \frac{h^2}{8} a_n(R).$$

Proof. We first notice that, if $f \in H^2$, we have

(31) $\|R(f)\|_{L^\infty([0,1-h])} \leq \frac{8}{h^2} \|f\|_{L^2(\mu)}.$

To that effect, we observe that (recall that $a \in (1,2)$ was defined in (20))

(32) $0 < h \leq a-1$ and $0 \leq x \leq 1-h \implies D(x, h^2/4a) \subseteq \Omega.$

Indeed, if $z = u+iv \in D(x, h^2/4a)$ and $0 \leq x \leq 1-h$, we have $1-u \geq h - h^2/4a \geq h/2$, as well as $|v| < h^2/4a$, and

$$\left| z - \left(1 + \frac{ia}{2} \right) \right|^2 = (1-u)^2 + \left(v - \frac{a}{2} \right)^2 \geq \frac{h^2}{4} + v^2 - a|v| + \frac{a^2}{4} > \frac{a^2}{4}.$$

Similarly, $\left| z - \left(1 - \frac{1}{2} ia \right) \right| > \frac{1}{2} a$. Moreover, since $1 - \frac{1}{2} a \leq \frac{1}{2} a - h$, we have

$$\left| z - \left(1 - \frac{a}{2} \right) \right| \leq |z - x| + \left| x - \left(1 - \frac{a}{2} \right) \right| \leq \frac{h^2}{4a} + \frac{a}{2} - h < \frac{a}{2}.$$

Hence $z \in \Omega$.

Therefore, by the subharmonicity of the function $|f|^2$,

$$|f(x)|^2 \leq \frac{16a^2}{h^4} \int_{D(x, h^2/4a)} |f|^2 \, dA \leq \frac{16a^2}{h^4} \int_\Omega |f|^2 \, dA$$

$$= \frac{16a^2}{h^4} \int_\mathbb{D} |f|^2 n_\chi \, dA = \frac{16a^2}{h^4} \int_\mathbb{D} |f|^2 \, d\mu,$$

which proves (31).

Let now $f \in H^2$ and $g = Pf \in \mathcal{D}_*$, so that $f = Dg$. As follows from (31) and from a change of variables, we have

$$\|Rf\|_\infty^2 \leq \frac{64}{h^4} \int_\mathbb{D} |f(w)|^2 n_\chi(w) \, dA(w) = \frac{64}{h^4} \int_\mathbb{D} |Dg(w)|^2 n_\chi(w) \, dA(w)$$

$$= \frac{64}{h^4} \int_\mathbb{D} |g'(\chi(z))|^2 |\chi'(z)|^2 \, dA(z) = \frac{64}{h^4} \|DC_\chi g\|_{L^2(\mathbb{D})}^2 = \frac{64}{h^4} \|C_\chi Pf\|_\mathcal{D}^2.$$

Therefore, there exists $C \colon \mathcal{D}_* \to L^\infty([0, 1-h])$ such that

$$R = CC_\chi P \quad \text{and} \quad \|C\| \leq \frac{8}{h^2}.$$

All this implies, by the ideal property of approximation numbers, that

$$a_n(R) \leq \|C\| a_n(C_\chi) \|P\| \leq \frac{8}{h^2} a_n(C_\chi),$$

which ends the proof of Lemma 3.5. $\square$

The second step consists of a minoration of $a_n(R)$, which uses the comparison with Bernstein numbers and a good choice of an n-dimensional space E.

Lemma 3.6. *For* $0 < h < 1$, *let* $r = 1 - h$ *and* s *be as in* (13). *We have*

$$a_n(R) \geq \frac{s^n}{\sqrt{n}}.$$

Recall (see [13] for example) that, if X and Y are two Banach spaces, and $T \colon X \to Y$ is a compact operator, the nth *Bernstein number* of T is

$$b_n(T) = \sup_{\dim E = n} \inf_{f \in S_E} \|Tf\|,$$

where S_E denotes the unit sphere of E, and we have

(33) $$a_n(T) \geq b_n(T).$$

Proof. Let $f = f_r$ be as in Lemma 2.5. Consider the n-dimensional space

$$E = [1, f, ..., f^{n-1}],$$

and let $g = \sum_{j=0}^{n-1} \alpha_j f^j \in E$ with $\|g\|_\infty = 1$. By Lemma 2.5 and the Cauchy–Schwarz inequality, we have

$$1 \leq \sum_{j=0}^{n-1} |\alpha_j| \, \|f^j\|_\infty \leq \sum_{j=0}^{n-1} |\alpha_j| \leq \sqrt{n} \left(\sum_{j=0}^{n-1} |\alpha_j|^2 \right)^{1/2}.$$

On the other hand, Lemma 2.5 again gives us that

$$\|R(g)\|_\infty \geq \left\| \sum_{j=0}^{n-1} \alpha_j s^j e^{ij\theta} \right\|_{L^\infty(\mathbb{T})} \geq \left\| \sum_{j=0}^{n-1} \alpha_j s^j e^{ij\theta} \right\|_{L^2(\mathbb{T})}$$

$$= \left(\sum_{j=0}^{n-1} |\alpha_j|^2 s^{2j} \right)^{1/2} \geq s^n \left(\sum_{j=0}^{n-1} |\alpha_j|^2 \right)^{1/2} \geq \frac{s^n}{\sqrt{n}}.$$

Therefore, $b_n(R) \geq s^n / \sqrt{n}$. Using (33), we get $a_n(R) \geq s^n / \sqrt{n}$ as well. $\square$

Let us now indicate how Lemmas 3.5 and 3.6 allow us to finish the proof. Write $h = e^{-A}$, where $A > 0$. Then, with the notation (13), we have

$$\varepsilon = \frac{2\pi}{\log \frac{1+r}{1-r}} \lesssim \frac{1}{\log \frac{1}{1-r}} = \frac{1}{\log \frac{1}{h}} = \frac{1}{A}$$

and

$$s \gtrsim e^{-c/A},$$

for some constant $c > 0$. Therefore Lemmas 3.5 and 3.6 give that

$$a_n(C_\chi) \gtrsim h^2 a_n(R) \gtrsim h^2 \frac{s^n}{\sqrt{n}} \gtrsim \frac{1}{\sqrt{n}} e^{-c'(A+n/A)}.$$

The optimal choice $A = \sqrt{n}$ gives the lower bound in Theorem 3.1, up to a change of the constant c'.

Remark. One sees that the approximation numbers of C_χ behave quite differently on the Hardy space H^2 (like $e^{-cn/\log n}$, see [11]) and on the Dirichlet space (like $e^{-c\sqrt{n}}$). This seems to be due to the following: On the Hardy space, the important fact is the parametrization $t \mapsto \chi(e^{it})$ where logarithms are involved. On the Dirichlet space, we only need to know the geometry of $\chi(\mathbb{D})$, a domain limited by three circles, where logarithms are no longer involved.

4. Capacity of the set of contact points

Here is now the improvement of a theorem in [5] in terms of approximation numbers (see also [9], Theorem 4.1). This improvement is definitely optimal in view of our previous Theorem 2.2, stating that, for every bounded composition operator C_φ on $\mathcal{D}_*$ of symbol φ, one has

$$\|\varphi\|_\infty = 1 \implies \beta(C_\varphi) := \liminf_{n \to \infty} a_n(C_\varphi)^{1/n} = 1.$$

Recall the following notation, where φ belongs to the disk algebra $A(\mathbb{D})$, i.e. the space of continuous functions $f \colon \overline{\mathbb{D}} \to \mathbb{C}$ which are analytic in $\mathbb{D}$,

$$E_\varphi = \{e^{it} \in \mathbb{T} \mid |\varphi(e^{it})| = 1\}.$$

Theorem 4.1. *Let K be a compact set of the circle $\mathbb{T}$ with logarithmic capacity* Cap $K = 0$, *and $\{\varepsilon_n\}_{n=1}^\infty$ be a sequence of positive numbers with limit 0. Then, there exists a Schur function φ generating a composition operator C_φ bounded on $\mathcal{D}$ and with the following properties*:

(1) $\varphi \in A(\mathbb{D}) \cap \mathcal{D}$, the "Dirichlet algebra";

(2) $E_\varphi = K$ and $E_\varphi = \{e^{it} \in \mathbb{T} | \varphi(e^{it}) = 1\}$;

(3) $a_n(C_\varphi) \lesssim e^{-n\varepsilon_n}$ for all $n \geq 1$.

Before proving this theorem, we need two results from [9]. The first one is the existence of a peculiar peaking function. Recall that a function $q \in A(\mathbb{D})$ is said to *peak* on a compact subset $K \subseteq \partial \mathbb{D}$, and is called a *peaking function*, if $q(z) = 1$ for $z \in K$ and $|q(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus K$.

Theorem 4.2. ([9], Theorem 4.2) *For every compact set $K \subseteq \partial \mathbb{D}$ of logarithmic capacity* $\operatorname{Cap} K = 0$, *there exists a Schur function* $q \in A(\mathbb{D}) \cap \mathcal{D}_*$ *which peaks on K and is such that the composition operator* $C_q \colon \mathcal{D}_* \to \mathcal{D}_*$ *is bounded* (*and even Hilbert–Schmidt*).

The other one is a lemma borrowed from the proof of Theorem 3.3 in [9].

Lemma 4.3. *Let* $\delta \colon (0,1) \to (0,\infty)$ *be a positive function with* $\lim_{h \to 0} \delta(h) = 0$. *Then, there exists a univalent Schur function* $\gamma \in A(\mathbb{D})$ *such that* $\gamma(1) = 1$ *and*

$$(34) \qquad \int_{|w| \geq 1-h} n_\gamma(w) \, dA(w) = A[\gamma(\mathbb{D}) \cap \{w | 1-h \leq |w| < 1\}] \leq \delta(h).$$

Proof of Theorem 4.1. It suffices to use Lemma 4.3 to construct a generalized cusp map γ in order to have $a_n(C_\gamma) \lesssim e^{-n\varepsilon_n}$. Indeed, we then compose this generalized cusp map γ with a symbol q peaking on K, as given by Theorem 4.2; namely consider $\varphi = \gamma \circ q$. Then, we know that $E_\varphi = \{e^{it} | \varphi(e^{it}) = 1\} = K$. Moreover, $C_\varphi = C_q \circ C_\gamma$, so that, using the fact that $\|C_q\| < \infty$,

$$a_n(C_\varphi) \leq \|C_q\| a_n(C_\gamma) \lesssim e^{-n\varepsilon_n}.$$

It remains to find such a generalized cusp map γ. Set

$$\delta_n = \varepsilon_n + \frac{\log n}{2n}.$$

Let Φ be a positive continuous concave increasing function $\Phi \colon [0,1] \to [0,1]$ such that $\Phi(0) = 0$ and $\Phi(1/n) \geq \delta_n$. Let $\Psi = \Phi^{-1}$. By Lemma 4.3, we can adjust γ so as to have, using the notation (15),

$$m(h) \leq \rho^2(h),$$

where

$$\rho(h) = \exp\left(-\frac{h}{\Psi(h)}\right).$$

Plugging this into (16), we get

$$a_n(C_\gamma) \lesssim \inf_{0<h<1} \left[\sqrt{n}(1-h)^n + \rho(h) \right] \leq \inf_{0<h<1} \left[\sqrt{n} \exp(-nh) + \exp\left(-\frac{h}{\Psi(h)} \right) \right].$$

In particular, if we choose $h = \Phi(1/n)$, we obtain

$$a_n(C_\gamma) \lesssim \sqrt{n} e^{-n\Phi(1/n)} \leq \sqrt{n} e^{-n\delta_n} = e^{-n\varepsilon_n}.$$

In view of the initial observation, this ends the proof of Theorem 4.1. □

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Pascal Lefèvre
Laboratoire de Mathématiques de Lens
EA 2462, Fédération CNRS
Nord-Pas-de-Calais FR 2956
Faculté des Sciences Jean Perrin
Université Lille-Nord-de-France UArtois
Rue Jean Souvraz, S.P. 18
FR-62300 Lens
France
pascal.lefevre@univ-artois.fr

Daniel Li
Laboratoire de Mathématiques de Lens
EA 2462, Fédération CNRS
Nord-Pas-de-Calais FR 2956
Faculté des Sciences Jean Perrin
Université Lille-Nord-de-France UArtois
Rue Jean Souvraz, S.P. 18
FR-62300 Lens
France
daniel.li@euler.univ-artois.fr

Hervé Queffélec
Laboratoire Paul Painlevé U.M.R. CNRS
8524
Université Lille-Nord-de-France USTL
FR-59655 Villeneuve D'ascq Cedex
France
queff@math.univ-lille1.fr

Luis Rodríguez-Piazza
Facultad de Matemáticas, Departamento
de Análisis Matemático & IMUS
Universidad de Sevilla
Apartado de Correos 1160
ES-41080 Sevilla
Spain
piazza@us.es