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$\mathcal{D}$-modules with finite support are semi-simple

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Abstract. Let $(R, \mathfrak{m}, k_R)$ be a regular local k-algebra satisfying the weak Jacobian criterion, and such that k_R/k is an algebraic field extension. Let $\mathcal{D}_R$ be the ring of k-linear differential operators of R. We give an explicit decomposition of the $\mathcal{D}_R$-module $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}_R^{n+1}$ as a direct sum of simple modules, all isomorphic to $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}$, where certain "Pochhammer" differential operators are used to describe generators of the simple components.

1. Introduction

The main purpose of this note is to give explicit semi-simple decompositions of $\mathcal{D}_R$-modules with a finite support, where $\mathcal{D}_R$ is the ring of differential operators associated with a rather general regular local ring R. Let first $A=k[x_1, ..., x_n]$ be the polynomial ring over a field k of characteristic 0 and $\mathcal{D}_A$ be its ring of k-linear differential operators. Let $\mathfrak{m}_A$ be a maximal ideal of A and $\mathrm{Mod}_{\mathfrak{m}_A}(\mathcal{D}_A)$ be the category of finitely generated $\mathcal{D}_A$-modules whose support is the single point $\mathfrak{m}_A \in \operatorname{Spec} A$. According to Kashiwara's theorem, $\mathrm{Mod}_{\mathfrak{m}_A}(\mathcal{D}_A)$ is equivalent to the category of finite-dimensional vector spaces over the residue field $k_A=A/\mathfrak{m}_A$, where the equivalence is $N \mapsto N^{\mathfrak{m}_A}=\{n\in N \mid \mathfrak{m}_A \cdot n=0\}$ (see [1, Sections V.3.1.2 and VI.7.3]); hence $\mathrm{Mod}_{\mathfrak{m}_A}(\mathcal{D}_A)$ is a semi-simple category. That the argument works fine also when k_A is not algebraically closed will follow from the discussion below, and is due to the fact that k_A/k is an algebraic field extension. Thus the point of departure for this paper, namely the question whether $N=\mathcal{D}_A/\mathcal{D}_A\mathfrak{m}_A$ is simple, posed to me by J.-E. Björk, has an affirmative answer, since $\dim_{k_A} N^{\mathfrak{m}_A}=1$ (see Lemma 2.4).

We shall work over a local regular noetherian k-algebra $(R, \mathfrak{m}, k_R)$ of characteristic 0, only requiring that the R-module of k-linear derivations $T_{R/k}$ is big enough. For this reason we recall the following result from Matsumura.

Theorem 1.1. ([4, Theorems 30.6 and 30.8]) *Let $(R, \mathfrak{m}_R)$ be a regular local ring of dimension n containing the rational numbers* $\mathbf{Q}$*. Let R^* be a completion* *of* $\mathbf{R}$, k_1 *be a quasi-coefficient field of* R, *and* K *be a coefficient field of* R^* *such that* $k_1 \subset K$. *The following conditions are equivalent*:

(1) *There exist* $\partial_1, ..., \partial_n \in T_{R/k_1}$ *and* $f_1, ..., f_n \in \mathfrak{m}_R$ *such that* $\det \partial_i(f_j) \notin \mathfrak{m}_R$;

(2) *If* $\{x_1, ..., x_n\}$ *is a regular system of parameters and* ∂_{x_i} *are the partial derivatives of* $R^* = K[[x_1, ..., x_n]]$, $\partial_{x_i}(x_j) = \delta_{ij}$, *then* $\partial_{x_i} \in T_{R/k_1}$;

(3) T_{R/k_1} *is free of rank* n.

Furthermore, if these conditions hold, then for any $P \in \operatorname{Spec} R$, *putting* $A = R/P$, *we have* $T_{A/k_1} = T_{R/k_1}(P)/PT_R$, *and* $\operatorname{rank} T_{A/k_1} = \dim A$. ($T_{R/k_1}(P) \subset T_{R/k_1}$ *denotes the submodule of derivations* ∂ *such that* $\partial(P) \subset P$.)

If the equivalent conditions in Theorem 1.1 hold, then we say that $(R, \mathfrak{m}, k_R)$ satisfies the weak Jacobian condition $(WJ)_{k_1}$. Note that if k_R/k is algebraic, then we can replace k_1 by k and write $(WJ)_k$. In this paper $(R, \mathfrak{m}, k_R)$ denotes a k-algebra of characteristic 0 satisfying $(WJ)_k$, where the field extension k_R/k is algebraic. For example, R could be the localization at a maximal ideal of a regular ring of finite type over k, a formal power series ring over k, or a ring of convergent power series when k is either the field of real or complex numbers.

Let us recall that the ring of (k-linear) differential operators $\mathcal{D}_R \subset \operatorname{End}_k(R)$ of R is defined inductively as

$$\mathcal{D}_R = \bigcup_{m=0}^{\infty} \mathcal{D}_R^m, \quad \mathcal{D}^0 = \operatorname{End}_R(R) = R \quad \text{and} \quad \mathcal{D}_R^{m+1} = \{P \in \operatorname{End}_k(R) | [P, R] \subset \mathcal{D}_R^m\},$$

where $[P, R] = PR - RP \subset \operatorname{End}_k(R)$. It is easy to see that $T_R \subset \mathcal{D}_R^1 \subset \mathcal{D}_R$, and conversely, if $P \in \mathcal{D}_R^1$, then $P - P(1) \in T_R$; hence

$$\mathcal{D}_R^1 = R + T_R.$$

The following companion to Theorem 1.1 should be well known; see [3].

Proposition 1.2. *Let* R/k *be a regular local* k*-algebra satisfying* $(WJ)_k$ *and such that* k_R/k *is algebraic. Then* $\mathcal{D}_R^1$ *generates the algebra* $\mathcal{D}_R$.

Select x_i and ∂_{x_i} as in Theorem 1.1. Given a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, $n = \dim R$, we put $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} \in R$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_n}^{\alpha_n} \in \mathcal{D}_R$, $|\alpha| = \sum_{i=1}^n \alpha_i$, and $\alpha! = \alpha_1! ... \alpha_n!$.

We recall some important well-known facts for the algebra $\mathcal{D}_R$.

Fact 1.3. *The* R*-module* $\mathcal{D}_R$ *is free with basis* $\{\partial^\alpha\}_{\alpha \in \mathbf{N}^d}$, *where* $\mathcal{D}_R$ *is either regarded as a left or right module.*

Proof. That the ∂^α generate $\mathcal{D}_R$ both as a left and a right R-algebra follows from Proposition 1.2. First consider $\mathcal{D}_R$ as a left R-module. Assume that $P=\sum_{\alpha\in\Omega} a_\alpha\partial^\alpha=0$, where Ω is a finite set of multi-indices. If one of the indices has minimal $|\alpha|$ in the set Ω, then $P(X^\alpha)=a_\alpha\alpha!=0$. This implies that $a_\alpha=0$ for all α. Now take the right module structure, and assume that $\sum_{\alpha\in\Omega}\partial^\alpha a_\alpha=0$. Then $\sum_{\alpha\in\Omega}(a_\alpha\partial^\alpha+[\partial^\alpha,a_\alpha])=0$, where $[\partial^\alpha,a_\alpha]\in\mathcal{D}_R^{|\alpha|-1}$ and $a_\alpha\partial^\alpha\in\mathcal{D}_R^{|\alpha|}$. Since the ∂^α are free generators as a left module, it follows that if $\alpha\in\Omega$ has maximal $|\alpha|$, then $a_\alpha=0$. This implies that all $a_\alpha=0$. $\square$

Fact 1.4. *R is a simple $\mathcal{D}_R$-module.*

Proof. Let $I\subset R$ be a non-zero $\mathcal{D}_R$-module. If $I\neq R$ there exists a non-zero element $f\in I\cap\mathfrak{m}^l$ with smallest $l\geq 1$. But then there exists a derivation ∂ such that $\partial(f)\in\mathfrak{m}^{l-1}$, $\partial(f)\neq 0$, which gives a contradiction. $\square$

Fact 1.5. *$\mathcal{D}_R$ is a simple ring.*

Proof. If $P\in\mathcal{D}_R^n$ belongs to a 2-sided ideal J, then $P_r=[r,P]\in J\cap\mathcal{D}_R^{n-1}$ for all $r\in R$. Unless $P\notin\mathcal{D}_R^0=R$ there exists an element r such that $P_r\neq 0$. Iterating, it follows that $J\cap R\neq 0$. By Fact 1.4, $R\subset J$; hence $J=\mathcal{D}_R$. $\square$

2. $\mathcal{D}$-modules with finite support

Let $\mathcal{D}_X=\mathcal{D}_{X/k}$ denote the sheaf of differential operators on a scheme X/k; we refer to [2] for the basic definitions. Instead of schemes we could in a similar way consider sheaves on complex or real-analytic manifolds (or even ringed spaces where the local rings are regular and satisfy $(WJ)_k$ at all closed points), but the reader will have little problems in transcribing the theorem below to such a situation.

The theorem below can be regarded as a version of Kashiwara's embedding theorem.

Theorem 2.1. *Let X/k be a scheme of characteristic 0 such that the local rings at all closed points are regular and satisfy $(WJ)_k$, and that all closed points are rational over k. Let M be a coherent $\mathcal{D}_{X/k}$-module whose support* $\operatorname{supp} M\subset X$ *is a finite set of closed points. Let n_x be the length of the maximal submodule of M with support at the point x, and let $\mathfrak{m}_x$ be the sheaf of ideals of x. Then M is a semi-simple module of the form*

$$M=\bigoplus_{x\in\operatorname{supp} M}\bigoplus^{n_x}\frac{\mathcal{D}_X}{\mathcal{D}_X\mathfrak{m}_x},$$

and $n_x=\dim_{k_x} M^{\mathfrak{m}_x}$.

We remark that by Fact 1.3, $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}=k_R[\partial_{x_1},...,\partial_{x_d}]$, where the action of $k_R[x_1,...,x_d]$ on the right-hand side is determined by $X^\alpha\partial^\beta=-(\beta!/\alpha!)\partial^{\beta-\alpha}$, when $\beta_i\geq\alpha_i$, $i=1,...,n$, and otherwise $X^\alpha\partial^\beta=0$.

Our goal is to give a concrete decomposition when we are given a presentation in terms of cyclic modules. First recall that a ring is simple if it has no non-trivial 2-sided ideals, and that we have the following well-known lemma.

Lemma 2.2. ([5]) *Let $\mathcal{D}$ be a simple ring and M be a $\mathcal{D}$-module of finite length. Assume that for any element m in M there exists a non-zero element P in $\mathcal{D}$ such that $Pm=0$ (i.e. M is a torsion module). Then M is cyclic.*

We remark that artinian $\mathcal{D}$-modules are torsion in the above sense if the ring $\mathcal{D}$ is not artinian. For example, $\mathcal{D}_R$ is not artinian as soon as $\dim R\geq 1$, so in particular any $\mathcal{D}_R$-module of finite length is cyclic, by Fact. 1.5.

If now M is a $\mathcal{D}_R$-module of finite type with support at the maximal ideal $\mathfrak{m}$, then $\dim_{k_R} M^{\mathfrak{m}}<\infty$, and any set of generators of M will belong to $M^{\mathfrak{m}^n}$ for sufficiently high n. Therefore M cannot have an infinite composition series, i.e. M is of finite length, and since any element in M is killed by $\mathfrak{m}^n$ for sufficiently high n, it is clearly a torsion module (which we thus can see without using the fact that $\mathcal{D}_R$ is non-artinian); hence M is cyclic by Lemma 2.2. If m is a cyclic generator we have a surjective homomorphism $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1}\to\mathcal{D}_Rm=M$, so that after iteration we have a finite resolution

$$0\longrightarrow\frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}^{n_r}}\longrightarrow...\longrightarrow\frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}^{n_i}}\longrightarrow...\longrightarrow\frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}^{n+1}}\longrightarrow M\longrightarrow 0. \tag{2.1}$$

Proof of Lemma 2.2. We prove this by induction over the length of the $\mathcal{D}$-module M. If $l(M)=1$, then M is simple so any non-zero vector is a cyclic generator. Now assume $l(M)\geq 2$ and that the assertion holds for all modules of length $<l(M)$. If $L\subset M$ is a non-zero simple submodule, we have the exact sequence

$$0\longrightarrow L\longrightarrow M\longrightarrow M/L\longrightarrow 0,$$

where $l(M/L)<l(M)$. By assumption there exists an element m in M that maps to a cyclic generator in M/L. Choose a non-zero vector $m_0\in L$. Since M is a torsion module, $\mathrm{Ann}_{\mathcal{D}}(m)\neq 0$, and since $\mathcal{D}$ is simple, the 2-sided ideal $\mathrm{Ann}_{\mathcal{D}}(m)\mathcal{D}$ contains the identity 1; hence there exists $Q\in\mathrm{Ann}_{\mathcal{D}}(m)$ and $P\in\mathcal{D}$ such that $QPm_0\neq 0$. Putting $m_1=m+Pm_0$, we have $Qm_1=QPm_0\in L$, and since L is simple, both Pm_0 and m_0 belong to $\mathcal{D}m_1$; hence also $m\in\mathcal{D}m_1$. By assumption any element m' in M can be written as $m'=P_0m_0+P_1m$; since $m,m_0\in\mathcal{D}m_1$ this shows that m_1 is a cyclic generator of M. □

Recall the Pochhammer symbol

$$(a)_n = a(a+1)...(a+n-1),$$

and we also put $(a)_0=1$. In the theorem below we use the notation in Theorem 1.1.

Theorem 2.3. *Let $(R, \mathfrak{m}, k_R)$ be an allowed regular local k-algebra R of dimension d, such that the residue field $k_R=R/\mathfrak{m}$ is algebraic over k, and let $\mathcal{D}_R$ be the ring of differential operators of R. Define the derivations ∂_{x_i} by $\partial_{x_i}(x_j)=\delta_{ij}$ and the "Pochhammer" differential operators*

$$Q_{n,d}(x_1, ..., x_d) = \prod_{i=1}^{d} (1+\partial_{x_i} x_i)_n \in \mathcal{D}_R.$$

(1) *The module $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}$ is a simple $\mathcal{D}_R$-module and $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1}$ is a semi-simple $\mathcal{D}_R$-module for each positive integer n.*

(2) *There is an isomorphism of $\mathcal{D}_R$-modules*

$$\psi\colon \bigoplus_{j=0}^{n} \bigoplus_{|\alpha|=j} \frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}} \longrightarrow \frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}^{n+1}},$$

$$P_{\alpha,j} \bmod \mathcal{D}_R\mathfrak{m} \longmapsto P_{\alpha,j} Q_{n-j,d}(x_1, ..., x_d) X^{\alpha} \bmod \mathcal{D}_R\mathfrak{m}^{n+1}.$$

Lemma 2.4. *Let M be a $\mathcal{D}_R$-module which is generated by its $\mathfrak{m}$-invariant subspace $M^{\mathfrak{m}}=\{m\in M \mid \mathfrak{m}\cdot m=0\}$. Then M is semi-simple. More precisely, if S is a basis of the k_R-vector space $M^{\mathfrak{m}}$, we have*

$$M = \bigoplus_{v\in S} \mathcal{D}_R v,$$

where all the modules $\mathcal{D}_R v$ are isomorphic to the simple module $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}$.

Proof. We first note that if L is a $\mathcal{D}_R$-module of finite type which is generated by the invariant space $L^{\mathfrak{m}}$, and this space is one-dimensional over k_R, then L is simple. This follows since any element in L is killed by a sufficiently high power of $\mathfrak{m}$, so if L_1 is a non-zero submodule we have $L_1^{\mathfrak{m}}\neq 0$. Hence $L_1^{\mathfrak{m}}=L^{\mathfrak{m}}$, which gives $L_1=L$.

To see that the module $N=\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}$ is simple, by the previous paragraph it suffices to prove that $N^{\mathfrak{m}}$ is one-dimensional over k_R. So if $P\in\mathcal{D}_R$ and $P \bmod \mathcal{D}_R\mathfrak{m}\in N^{\mathfrak{m}}$, i.e. $\mathfrak{m}P\subset\mathcal{D}_R\mathfrak{m}$, we need to see that $P\in R+\mathcal{D}_R\mathfrak{m}$. Expressed in a regular system of parameters $P=\sum_\alpha \partial^\alpha a_\alpha$ we have $(x_1, ..., x_d)\cdot\sum_\alpha \partial^\alpha a_\alpha \subset \mathcal{D}_R\cdot(x_1, ..., x_d)$. This implies, from the fact that the differential operators ∂^α form a basis of the right R-module $\mathcal{D}_R$, that $a_\alpha \in (x_1, \ldots, x_d)$ when $|\alpha|>0$, and therefore $P \in R + \mathcal{D}_R \mathfrak{m}$.

If $v \in M^{\mathfrak{m}}$, then there is a canonical non-zero homomorphism $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m} \to \mathcal{D}_R v$, which is injective and has a simple image by the previous paragraph. The canonical surjective homomorphism

$$\bigoplus_{v \in S} \mathcal{D}_R v \longrightarrow M$$

is an isomorphism since the left hand side is semi-simple and the restriction to any of its simple terms is non-zero. □

Proof of Theorems 2.1 and 2.3. The decomposition of M over the support is obvious, so one can assume that $\operatorname{supp} M = \{x\}$ is a single closed point in X, and thus M can be regarded as a $\mathcal{D}_R$-module, where R is the local ring $\mathcal{O}_{X,x}$. Let M^0 be an R-submodule of finite type that generates M. Hence $\mathfrak{m}^{n+1} M^0 = 0$ for high n; let n be the highest integer such that $\mathfrak{m}^n M^0 \neq 0$. Put $M_i^0 = \mathfrak{m}^i M^0$ and let M_i be the $\mathcal{D}_R$-module it generates, so we have a filtration by $\mathcal{D}_R$-modules $M_n \subset M_{n-1} \subset \ldots \subset M_0 = M$, and exact sequences

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow M_i/M_{i+1} \longrightarrow 0.$$

Then M_n and each quotient M_i/M_{i+1} is generated by its $\mathfrak{m}$-invariants, so at any rate M is a successive extension of $\mathcal{D}_R$-modules that are generated by $\mathfrak{m}$-invariants. All these modules can be decomposed into a direct sum of simple modules that are isomorphic to the module $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}$. It remains to see that M is a direct sum of such modules. To see this, first note that M is a quotient of a direct sum of modules of the form $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1}$ for different non-negative integers n, so to see that M is semi-simple it suffices to see that $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1}$ is semi-simple, and this follows if we prove (2) in Theorem 2.3.

We have $x_i^k(k+\partial_{x_i} x_i) = \partial_{x_i} x_i^{k+1}$, and therefore $x_i(1+\partial_{x_i} x_i)_{n-j} = \partial_{x_i}^{n-j} x_i^{n+1-j}$. Hence $x_i Q_{n-j,d}(x_1, \ldots, x_d) X^\alpha = Q_{n-j,d-1}(x_1, \ldots, \hat{x}_i, \ldots, x_d) x_i^{n+1-j} X^\alpha$, so if $|\alpha| = j$, then

$$(*) \qquad \mathfrak{m} Q_{n-j,d}(x_1, \ldots, x_d) X^\alpha \subset \mathcal{D}_R \mathfrak{m}^{n+1}.$$

Therefore there exists a homomorphism of $\mathcal{D}_R$-modules

$$\psi_\alpha\colon \frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}} \longrightarrow \frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}^{n+1}}$$
$$P \longmapsto P Q_{n-j,d}(x_1, \ldots, x_d) X^\alpha \bmod \mathcal{D}_R\mathfrak{m}^{n+1},$$

and we put $\psi_\alpha(1_\alpha) = m_\alpha \in (\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1})^{\mathfrak{m}}$, where 1_α is the cyclic generator of the term with index α in the right-hand side of (2).

The fact that any differential operator $P \in \mathcal{D}_R$ has a unique expansion $P = \sum_\alpha \partial^\alpha a_\alpha$, $a_\alpha \in R$, implies that $Q_{n-j,d}(x_1, \ldots, x_d)X^\alpha \notin \mathcal{D}_R \mathfrak{m}^{n+1}$, when $|\alpha| \le j$; hence $\psi_\alpha \neq 0$.

Lemma 2.4 implies that ψ is injective if we first prove that the vectors m_α are linearly independent in the k_R-vector space $(\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1})^{\mathfrak{m}}$. Assume that we have a linear relation

$$\sum_{|\alpha| \le n} \lambda_\alpha m_\alpha = 0, \quad \lambda_\alpha \in k_A,$$

which means that

$$\sum_{|\alpha| \le n} \hat{\lambda}_\alpha Q_{n-j} X^\alpha \in \mathcal{D}\mathfrak{m}^{n+1}, \quad \hat{\lambda}_\alpha \in R \text{ and } \hat{\lambda}_\alpha \bmod \mathfrak{m} = \lambda_\alpha.$$

Defining the Euler operator $\nabla = \sum_{i=1}^d x_i \partial_{x_i}$ we have $[\nabla, Q_{n-j}] = 0$, $[\nabla, X^\alpha] = |\alpha| X^\alpha$, and

$$\hat{\lambda}_\alpha Q_{n-j} X^\alpha \nabla = (d - |\alpha|)\hat{\lambda}_\alpha Q_{n-j} X^\alpha - \nabla(\hat{\lambda}_\alpha) Q_{n-j} X^\alpha + (\nabla - d)\hat{\lambda}_\alpha Q_{n-j} X^\alpha.$$

Here the two last terms on the right belong to $\mathcal{D}_R\mathfrak{m}^{n+1}$ due to $(*)$, after noting that $\nabla(\hat{\lambda}_\alpha) \in \mathfrak{m}$ and $\nabla - d = \sum_{i=1}^d \partial_{x_i} x_i$. Therefore we can define a k_R-linear action E on the linear space $\sum_{|\alpha| \le n} k_R m_\alpha \subset (\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1})^{\mathfrak{m}}$ such that $E m_\alpha = (d - |\alpha|) m_\alpha$. A standard weight argument now implies that all the coefficients $\lambda_\alpha = 0$.

It remains to prove that ψ is an isomorphism. Let $N_n \subset \mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1}$ be the submodule that is generated by the canonical projection of $\mathfrak{m}^n$ in $\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1}$. Then N_n is generated by its $\mathfrak{m}$-invariants and $\dim_{k_R} N_n^{\mathfrak{m}}$ equals the number of monomials of degree n in d variables, which is thus equal to the length of N_n by Lemma 2.4. Since $(\mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^{n+1})/N_n = \mathcal{D}_R/\mathcal{D}_R\mathfrak{m}^n$, an induction over n gives that the lengths of both sides in (2) are equal. Since ψ is injective this implies that ψ is an isomorphism. □

Remark 2.5. 1. The proof gives that

$$\left(\frac{\mathcal{D}_R}{\mathcal{D}_R\mathfrak{m}^{n+1}}\right)^{\mathfrak{m}} = \sum_{|\alpha| \le n} k_R Q_{n-j,d}(x_1, \ldots, x_d) X^\alpha \bmod \mathcal{D}_R\mathfrak{m}^{n+1}.$$

2. Given a resolution as in (2.1), then (2) in Theorem 2.3 will give a decomposition of M.

Example 2.6. Let $A_1(k)$ be the Weyl algebra in one variable over a field k of characteristic 0, and consider a maximal ideal $\mathfrak{m}$ in the polynomial ring $k[x] \subset A_1(k)$; let ∂_x be the k-linear derivation of $k[x]$ such that $\partial_x(x) = 1$. By

Theorem 2.1, $M_l=A_1(k)/A_1(k)\mathfrak{m}^{l+1}=A_1(k)m_l$ is a semi-simple $A_1(k)$-module (here $m_l=1 \bmod A_1(k)\mathfrak{m}^{l+1}$). The localization $R=k[x]_\mathfrak{m}$ has a regular system of parameters formed by a generator x_1 of the principal ideal $\mathfrak{m}$, and given x_1 there exists a unique derivation $\partial_{x_1}\in T_{R/k}$ such that $\partial_{x_1}(x_1)=1$. The $\mathcal{D}_{R/k}$-module $R\otimes_{k[x]}M_l$ can be decomposed as

$$\bigoplus_{i=0}^{l} R\otimes_{k[x]}M_0 \longrightarrow R\otimes_{k[x]}M_l,$$

$$(P_i m_{0,i}) \longmapsto P_0\cdot x_1^l m_l+P_1\cdot(1+\partial_{x_1}\cdot x_1)x_1^{l-1}m_l+\dots$$
$$+P_l\cdot(1+\partial_{x_1}\cdot x_1)(2+\partial_{x_1}\cdot x_1)\dots(l+\partial_{x_1}\cdot x_1)m_l.$$

The isomorphism $M_0\oplus\dots\oplus M_0\to M_l$, where there are $l+1$ terms on the left, is defined similarly. We notice that although $\partial_{x_1}x_1$ in general does not act on $k[x]$, it has a well-defined action on M_l. Note also that the invariant space $M_0^\mathfrak{m}$ of the simple module M_0 is 1-dimensional over the residue field k_R and thus its dimension over k equals the degree of the field extension k_R/k.

One can reverse the roles of ∂_x and x in the Weyl algebra $A_1(k)$, and instead decompose modules according to their support in $\operatorname{Spec} k[\partial_x]$. Thus if $P\in k[\partial_x]\subset A_1(k)$ is a differential operator with constant coefficients, then

$$\frac{A_1(k)}{A_1(k)P}\cong\frac{A_1(k)}{A_1(k)P_1}\oplus\dots\oplus\frac{A_1(k)}{A_1(k)P_r},$$

where $P=P_1\dots P_r$ is a factorization into irreducible polynomials, where repetitions may occur. It is a good exercise to write down an isomorphism for some concrete polynomial P using "Pochhammer" operators $Q_{j,1}(P_i)$ when P has multiple factors.

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