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Reiteration theorems with extreme values of parameters

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Abstract. We consider real interpolation methods defined by means of slowly varying functions b and symmetric spaces E, for which we present extreme reiteration theorems. As an application we identify, for all possible values of $\theta\in[0,1]$, the interpolation spaces $(L_1, L\log L)_{\theta,b,E}$ and $(L_{\exp}, L_\infty)_{\theta,b,E}$.

1. Introduction

Many important results in analysis are based on the action of certain linear operators between L^p spaces. Quite often, optimal versions of these results require some knowledge about these operators in spaces close to L_1 or L_∞. The associated extremal spaces are not always part of the classical families, and more general Orlicz or Lorentz classes enter into play, see e.g. [3], [7], [9], [24], [32], [35] and [37]. Interpolation theory is a useful tool in all these problems, but it becomes necessary to extend the classical methods to include also these extreme situations (see e.g. [22] and [28]).

The real interpolation method admits not only numerical parameters, but also functional parameters, which fit better to these general situations, see [21] and [31]. A quite general family of functional parameters is given by $t^\theta b(t)$, where $0\le\theta\le1$ and $b(t)$, $t>0$, is in the family of *slowly varying functions* (see Definition 2.1 below). A further extension is possible by allowing the secondary parameter in the real interpolation method, an L_q space, to be any rearrangement-invariant space E.

The theory of real interpolation, which considers simultaneously these two extensions was developed in [15] and [17]. There, we considered spaces of the form

$$(X_0, X_1)_{\theta,b,E}$$

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with $0\le\theta\le 1$, a slowly varying function $b(t)$ and a general rearrangement-invariant space E. For this method we proved a number of reiteration theorems, by appropriately using Hardy-type inequalities, integral estimates and norm equivalences of slowly varying functions. Although these results cover and extend many known reiteration theorems, there are some *extreme cases*, which have not been studied, namely, the interpolation spaces

$$(X_0,(X_0,X_1)_{0,a,F})_{\theta,b,E} \quad \text{and} \quad ((X_0,X_1)_{1,a,F},X_1)_{\theta,b,E}, \tag{1}$$

where a and b are slowly varying functions, E and F are rearrangement-invariant spaces, and $\theta\in[0,1]$.

These extreme reiteration spaces are typically difficult to handle, and are only known in the literature in some special cases. In [20], Gómez and Milman studied (1) for the classical interpolation parameters (i.e. $a\equiv b\equiv 1$, and E and F within the L_q spaces), but only in the simpler case of ordered couples $X_1\subseteq X_0$. More recently, Cobos, Fernández-Cabrera, Kühn and Ullrich considered in [10] some particular cases of slowly varying functions a and b, still for ordered couples, and E and F within the L_q spaces. More general results are due to Evans, Opic and Pick [13] who obtained reiteration theorems for general Banach couples (X_0,X_1) when a and b are *broken-logarithmic functions*. See [1] and [11] for other recent results.

One of the difficulties of these extreme interpolation cases is the need of *limiting* Hardy-type inequalities, which in infinite-measure spaces must hold in the whole interval $(0,\infty)$. As explained in [13], such global inequalities cannot hold for functions like $b(t)=1+|\log t|$, which have the same behavior near 0 and near ∞. In [13] this problem is overcome for the case of broken logarithms by requiring powers of different signs at 0 and ∞.

For a general slowly varying function $b(t)$, $t>0$, explicit formulas are not available, but we develop a procedure to prove such inequalities when certain "indices", related with the behavior of $b(t)$ near 0 and ∞, have different signs. More precisely, we assume that $b(t^2)\sim b(t)$, and define the related functions B_0 and B_∞ by

$$B_0(u)=b(e^{1-1/u}) \text{ and } B_\infty(u)=b(e^{1/u-1}) \quad \text{for } 0<u<1.$$

Then, under appropriate conditions in the *extension indices* of B_0 and B_∞ we prove limiting Hardy-type inequalities in the whole line $(0,\infty)$; see Lemma 3.6. Moreover, these Hardy inequalities hold not only for L_q norms, but also in the richer family of rearrangement-invariant spaces E.

Suitable conditions on the extension indices of B_0, B_∞ and φ_E (the fundamental function of E) will lead to explicit equivalences of the form

$$\|b(s)\|_{\widetilde{E}(0,t)}\sim b(t)\varphi_E(\ell(t)) \quad \text{and} \quad \|b(s)\|_{\widetilde{E}(t,\infty)}\sim b(t)\varphi_E(\ell(t)), \quad t>0,$$

where $\ell(t)=1+|\log t|$, $t>0$, as well as to additional limiting estimates (see Lemmas 3.2 and 3.5). These simple formulas will play a crucial role in proving the main results of the paper, namely explicit identities for the *extreme reiteration spaces* in (1) for all possible values of $\theta\in[0,1]$; see Theorems 5.5 and 5.6.

As in our previous papers [15] and [17], we shall use a direct approach, which follows the classical methods to establish reiteration results. First, we obtain general Holmstedt-type formulas for the couples $(X_0,(X_0,X_1)_{0,a,F})$ and $((X_0,X_1)_{1,a,F},X_1)$. The second step in this process consists of establishing a formula of change of variables (see Lemma 5.1).

Putting together all the previous ingredients we shall see that the resulting spaces in the reiteration formulas (1) lie outside the original scale, and to describe them new interpolation functors are needed (see Definitions 5.2 and 5.3).

Finally, to illustrate our results, we include some applications to spaces of Lorentz–Karamata type $L_{\infty,b,E}$ and $L_{(1,b,E)}$. For example, if we work on a finite measure space, our results applied to the ordered Banach couple (L_1,L_∞) enable us to identify the interpolation spaces $(L_1,L\log L)_{\theta,b,E}$ and $(L_{\exp},L_\infty)_{\theta,b,E}$ for all possible values of $\theta\in[0,1]$, which seems to be a new result in the literature. We can also identify the interpolation spaces between the *Schatten ideals* S_1 and S_∞, and the *Macaev ideals* S_w and $S_{\mathcal{M}}$.

The paper is organized as follows. In Section 2 we review basic concepts about rearrangement-invariant spaces and slowly varying functions. In Section 3 we provide the essential lemmas (equivalence lemma, limiting estimates and limiting Hardy-type inequality). The description of the interpolation method $\overline{X}_{\theta,b,E}$ and general Holmstedt-type formulas for the K-functional of the couples $(X_0,(X_0,X_1)_{0,b,E})$ and $((X_0,X_1)_{1,b,E},X_1)$ can be found in Section 4. The extreme reiteration results appear in Section 5, and applications to interpolation of Lorentz–Karamata type spaces are presented in Section 6. Finally, in Section 7 we show interpolation formulas for the couples $(L_1,L\log L)$ and $(L_{\exp},L_\infty)$, and also for (S_∞,S_w) and $(S_{\mathcal{M}},S_1)$.

Throughout the paper we shall write $f\lesssim g$ instead of $f\le Cg$ for some constant $C>0$. The functions f and g are equivalents, $f\sim g$, if $f\lesssim g$ and $g\lesssim f$. We also say that a function f is *almost increasing* (*almost decreasing*) if it is equivalent to an increasing (decreasing) function.

2. Preliminaries

We refer to the monographs [4] and [14] for the main definitions and properties concerning rearrangement-invariant spaces and interpolation theory. Recall that a Banach function space E on $(0,\infty)$ is *rearrangement-invariant* (r.i.) if, for any two measurable functions f and g,

$$g \in E \text{ and } f^* \le g^* \quad \Longrightarrow \quad f \in E \text{ and } \|f\|_E \le \|g\|_E,$$

where f^* and g^* stand for the decreasing rearrangements of f and g. Following the approach of [4], we assume the *Fatou property* in Banach function spaces; then every r.i. space E is obtained by applying an *exact* interpolation method to the couple (L_1, L_∞).

2.1. Measures

Throughout this paper we will handle three different measures on $(0,\infty)$. The usual Lebesgue measure dt, the homogeneous measure dt/t and the measure $dt/t\ell(t)$, where $\ell(t):=(1+|\log t|)$, $t>0$. Following [12], we use letters with a tilde for spaces with the measure dt/t and with a hat for the measure $dt/t\ell(t)$. For example, the spaces $\tilde{L}_1$ and $\hat{L}_1$ are defined by the norms

$$\|f\|_{\tilde{L}_1} = \int_0^\infty |f(t)| \frac{dt}{t} \quad \text{and} \quad \|f\|_{\hat{L}_1} = \int_0^\infty |f(t)| \frac{dt}{t\ell(t)},$$

respectively, while $\tilde{L}_\infty$ and $\hat{L}_\infty$ coincide with L_∞. More precisely, if the space $E=E((0,\infty),dt)$ is obtained by the interpolation functor $\mathcal{F}$ from the basic couple (L_1, L_∞) as $E=\mathcal{F}(L_1,L_\infty)$, then

$$\widetilde{E} = \mathcal{F}(\tilde{L}_1, L_\infty) \quad \text{and} \quad \widehat{E} = \mathcal{F}(\hat{L}_1, L_\infty).$$

Sometimes we need to restrict the space to some partial interval $(a,b)\subset(0,\infty)$. Then we use the notation $E(a,b)$, $\widetilde{E}(a,b)$ and $\widehat{E}(a,b)$. Such spaces can be obtained by applying the functor $\mathcal{F}$ to the couples $(L_1(a,b), L_\infty(a,b))$, $(\tilde{L}_1(a,b), L_\infty(a,b))$ and $(\hat{L}_1(a,b), L_\infty(a,b))$, respectively. Moreover the norms in E and $E(a,b)$ can be related in the following way: $\|f\|_{E(a,b)} = \|f(t)\chi_{(a,b)}(t)\|_E$. Using this notation, we can write

$$\|f\|_E \sim \|f\|_{E(0,t)} + \|f\|_{E(t,\infty)} \quad \text{for all } t>0.$$

The same is true for $\widetilde{E}$ and $\widehat{E}$.

Since $f(s)\in\widetilde{E}(t,\infty)$ if and only if $f(1/s)\in\widetilde{E}(0,1/t)$, $t>0$, we shall often prove our assertions only for f in the space $\widetilde{E}(0,t)$, extending them to $\widetilde{E}(t,\infty)$ through the operator $s\mapsto f(1/s)$. We also observe that $\|f(s)\|_{\widetilde{E}} = \|f(1/s)\|_{\widetilde{E}}$ for all $f\in\widetilde{E}$.

The norms of the spaces E, $\widetilde{E}$ and $\widehat{E}$ can also be directly connected without the use of interpolation functors. For measurable functions $f\colon(0,\infty)\to(0,\infty)$ we have

$$\|f\|_{\widetilde{E}(0,t)} = \|f(e^{-u})\|_{E(\log t,\infty)}, \quad \text{if } t\le 1,$$
$$\|f\|_{\widetilde{E}(1,t)} = \|f(e^{u})\|_{E(0,\log t)}, \quad \text{if } t>1,$$

while

$$\|f\|_{\widehat{E}(0,t)} = \|f(e^{1-e^u})\|_{E(\log \ell(t),\infty)}, \quad \text{if } t \le 1,$$
$$\|f\|_{\widehat{E}(1,t)} = \|f(e^{e^u-1})\|_{E(0,\log \ell(t))}, \quad \text{if } t > 1.$$

2.2. Extension indices

Given an everywhere positive finite function φ on $(0,a)$, $0<a\le\infty$, we denote its associated *dilation function* by

$$m_\varphi(t) = \sup_{0<s<\min\{a,a/t\}} \frac{\varphi(ts)}{\varphi(s)}, \quad 0<t<\infty.$$

If $m_\varphi(t)$ is finite everywhere then the *lower* and *upper extension indices* of φ exist and they are defined as

$$\pi_\varphi = \lim_{t\to 0} \frac{\log m_\varphi(t)}{\log t} \quad \text{and} \quad \rho_\varphi = \lim_{t\to\infty} \frac{\log m_\varphi(t)}{\log t}.$$

In general, $-\infty<\pi_\varphi\le\rho_\varphi<\infty$, but if φ is increasing we have $0\le\pi_\varphi\le\rho_\varphi<\infty$, and if φ is quasi-concave then $0\le\pi_\varphi\le\rho_\varphi\le 1$. Moreover, if $0<\pi_\varphi\le\rho_\varphi<\infty$, then

$$\varphi(t) \sim \int_0^t \varphi(s)\frac{ds}{s}$$

and if $-\infty<\pi_\varphi\le\rho_\varphi<0$ the function $\varphi(t)$ is equivalent to $\int_t^\infty \varphi(s)\,ds/s$. Note also that both indices remain the same after replacing $\varphi(t)$ by arbitrary equivalent function. As an example, $\varphi(t)=t^\alpha \ell(t)^\beta$, $\alpha,\beta\in\mathbb{R}$, has $\pi_\varphi=\rho_\varphi=\alpha$.

The following properties of extension indices can be easily proved:

(i) If $\varphi(t)=t^\sigma\psi(t)$, for all $\sigma\in\mathbb{R}$, then $\pi_\varphi=\sigma+\pi_\psi$ and $\rho_\varphi=\sigma+\rho_\psi$.

(ii) If $\varphi(t)=\alpha(t)\psi(t)$ then $\pi_\varphi\ge\pi_\alpha+\pi_\psi$ and $\rho_\varphi\le\rho_\alpha+\rho_\psi$.

(iii) If $\varphi(t)=\psi(t^{-1})$ then $\pi_\varphi=-\rho_\psi$ and $\rho_\varphi=-\pi_\psi$.

(iv) If $\varphi(t)=1/\psi(t)$ then $\pi_\varphi=-\rho_\psi$ and $\rho_\varphi=-\pi_\psi$.

(v) If $\varphi(t)=\theta(\psi(t))$ then $\pi_\varphi\ge\pi_\theta\pi_\psi$ and $\rho_\varphi\le\rho_\theta\rho_\psi$.

From (i) it follows that the ratio $\varphi(t)/t^\sigma$ is almost increasing for any $\sigma<\pi_\varphi$ and almost decreasing for any $\sigma>\rho_\varphi$.

An important function in any r.i. space E is its fundamental function $\varphi_E(\lambda)=\|\chi_{(0,\lambda)}\|_E$, which is continuous and quasi-concave. Moreover, the space E always admits an equivalent renorming such that φ_E becomes concave and the derivative φ_E' exists a.e. and is decreasing. In particular, $0\le\pi_{\varphi_E}\le\rho_{\varphi_E}\le 1$.

2.3. Slowly varying functions

Following [18] we give the definition of *slowly varying functions.*

Definition 2.1. A positive and Lebesgue measurable function b is *slowly varying* on $(0,\infty)$ (notation $b\in\mathrm{SV}$) if, for every $\varepsilon>0$, the function $t^{\varepsilon}b(t)$ is almost increasing on $(0,\infty)$ while the function $t^{-\varepsilon}b(t)$ is almost decreasing on $(0,\infty)$.

Powers of logarithms, $\ell^{\alpha}(t)=(1+|\log t|)^{\alpha}$, $\alpha\in\mathbb{R}$, are slowly varying in $(0,\infty)$. More generally, broken logarithmic functions, defined as

$$\ell^{(\alpha,\beta)}(t)=\begin{cases}\ell^{\alpha}(t), & 0<t\le 1,\\ \ell^{\beta}(t), & t>1,\end{cases} \tag{2}$$

with $(\alpha,\beta)\in\mathbb{R}^2$ are also in SV. Two further examples are the iterated logarithms

$$b(t)=(\ell\circ\ldots\circ\ell)^{\alpha}(t),\quad \alpha\in\mathbb{R},\ t>0,$$

and the family of functions

$$b(t)=\exp(|\log t|^{\alpha}),\quad \alpha\in(0,1),\ t>0.$$

The latter have the special property of growing faster to infinity than any positive power of a logarithm.

For equivalent definitions and further examples of slowly varying functions see [6] or [25]. Some basic properties are summarized in the following lemma.

Lemma 2.2. *Assume* $b,b_1,b_2\in\mathrm{SV}$. *Then the following are true*:

(i) $b_1b_2\in\mathrm{SV}$, $b(1/t)\in\mathrm{SV}$ *and* $b^r\in\mathrm{SV}$ *for all* $r\in\mathbb{R}$.

(ii) $b(t^{\alpha}b_1(t))\in\mathrm{SV}$ *for any* $\alpha>0$.

(iii) *If* ε *and* s *are positive numbers, then there are positive constants* c_ε *and* C_ε *such that*

$$c_\varepsilon\min\{s^{-\varepsilon},s^{\varepsilon}\}b(t)\le b(st)\le C_\varepsilon\max\{s^{\varepsilon},s^{-\varepsilon}\}b(t).$$

In particular, $\pi_b=\rho_b=0$.

(iv) $b\circ f\sim b\circ g$ *if* f *and* g *are positive finite equivalent functions on* $(0,\infty)$.

(v) *Let* E *be an r.i. space and* $\alpha>0$. *Then, for* $t>0$,

$$\|s^{\alpha}b(s)\|_{\widetilde{E}(0,t)}\sim t^{\alpha}b(t)\quad\textit{and}\quad \|s^{-\alpha}b(s)\|_{\widetilde{E}(t,\infty)}\sim t^{-\alpha}b(t).$$

(vi) *The functions*

$$\Phi_0(t):=\|b(s)\|_{\widetilde{E}(0,t)}\quad\textit{and}\quad \Phi_\infty(t):=\|b(s)\|_{\widetilde{E}(t,\infty)},\quad t>0,$$

are slowly varying functions, for all r.i. spaces E.

(vii) *For any* $t>0$ *and any r.i. space* E,

$$b(t) \lesssim \|b(s)\|_{\widetilde{E}(0,t)} \quad \textit{and} \quad b(t) \lesssim \|b(s)\|_{\widetilde{E}(t,\infty)}.$$

We refer to [18] for the proof of (i)–(iv). Properties (v)–(vii) are proved in [15].

Remark 2.3. The property (iii) above implies that if $b \in \mathrm{SV}$ is such that $b(t_0)=0$ ($b(t_0)=\infty$) for some $t_0>0$, then $b\equiv 0$ ($b\equiv\infty$). In particular, if $\|b\|_{\widetilde{E}(0,1)}<\infty$ then $\|b\|_{\widetilde{E}(0,t)}<\infty$ for all $t>0$ and if $\|b\|_{\widetilde{E}(1,\infty)}<\infty$ then $\|b\|_{\widetilde{E}(t,\infty)}<\infty$ for all $t>0$. These facts appear implicitly in the theorems of Section 4.

3. Equivalences of norms and Hardy-type inequalities

In this section we consider slowly varying functions b such that $b(t^2)\sim b(t)$. All previous examples, except $b(t)=\exp(|\log t|^{\alpha})$, satisfy this condition. For every $b\in\mathrm{SV}$ as above, we define new functions B_0 and B_∞ by

$$B_0(u) = b(e^{1-1/u}) \quad \text{and} \quad B_\infty(u) = b(e^{1/u-1}), \quad \text{for } 0<u\le 1,$$

or equivalently by the formula

$$b(t) = \begin{cases} B_0(1/\ell(t)), & \text{if } t\in(0,1], \\ B_\infty(1/\ell(t)), & \text{if } t\in(1,\infty). \end{cases} \tag{3}$$

For example, if $b(t)=\ell^{(\alpha,\beta)}(t)$ then $B_0(u)=1/u^{\alpha}$ and $B_\infty(u)=1/u^{\beta}$, $u\in(0,1)$. The condition $b(t^2)\sim b(t)$ implies that B_0 and B_∞ satisfy the Δ_2 condition, that is $B_0(t)\sim B_0(2t)$ and $B_\infty(t)\sim B_\infty(2t)$. Hence the extension indices of B_0 and B_∞ exist and are both finite.

3.1. Equivalences of norms

Next we study the counterparts of the equivalences in Lemma 2.2(v),

$$\|s^{\alpha} b(s)\|_{\widetilde{E}(0,t)} \sim t^{\alpha} b(t) \quad \text{and} \quad \|s^{-\alpha} b(s)\|_{\widetilde{E}(t,\infty)} \sim t^{-\alpha} b(t) \tag{4}$$

in the limit case $\alpha=0$. First we study the simple case $\|\ell(s)^{-\sigma}\|_{\widetilde{E}(1,t)}$, $t>e$, and deduce afterwards the general case.

Lemma 3.1. *Let E be an r.i. space and choose $\sigma<\pi_{\varphi_E}$, then*

$$\|\ell(s)^{-\sigma}\|_{\widetilde{E}(1,t)} \sim \|\ell(s)^{-\sigma}\|_{\widetilde{E}(\sqrt{t},t)} \sim \ell(t)^{-\sigma}\varphi_E(\ell(t))$$

for all $t\in(e,\infty)$.

Proof. We shall distinguish two different cases: $\sigma\leq 0$ and $0<\sigma<\pi_{\varphi_E}$. Assume first $\sigma\leq 0$, then for all $t>1$,

$$\begin{aligned}\|\ell(s)^{-\sigma}\|_{\widetilde{E}(1,t)} &= \|\ell(s)^{-\sigma}\chi_{(1,t)}(s)\|_{\widetilde{E}(1,\infty)} \leq \ell(t)^{-\sigma}\|\chi_{(1,t)}(e^u)\|_E \\ &= \ell(t)^{-\sigma}\|\chi_{(0,\log t)}(u)\|_E = \ell(t)^{-\sigma}\varphi_E(\log t) \leq \ell(t)^{-\sigma}\varphi_E(\ell(t)).\end{aligned}$$

On the other hand

$$\begin{aligned}\|\ell(s)^{-\sigma}\|_{\widetilde{E}(1,t)} &\geq \left\|\ell(s)^{-\sigma}\right\|_{\widetilde{E}(\sqrt{t},t)} \geq \ell\big(\sqrt{t}\big)^{-\sigma}\|\chi_{(\sqrt{t},t)}(e^u)\|_E \\ &= \ell\big(\sqrt{t}\big)^{-\sigma}\|\chi_{(\frac{1}{2}\log t,\log t)}(u)\|_E \gtrsim \ell(t)^{-\sigma}\varphi_E(\ell(t)),\end{aligned} \tag{5}$$

where the last inequality uses the quasi-concavity of φ_E and the fact $t>e$. This proves the equivalence for $\sigma\leq 0$.

For the case $0<\sigma<\pi_{\varphi_E}$, we may assume without loss of generality that the function φ_E is concave. Then the Lorentz space Λ_{φ_E} is included in E (see [14, p. 118]) and $\varphi_E'(s)\leq\varphi_E(s)/s$. So,

$$\begin{aligned}\|\ell(s)^{-\sigma}\|_{\widetilde{E}(1,t)} &= \|(1+u)^{-\sigma}\|_{E(0,\log t)} = \|u^{-\sigma}\|_{E(1,\ell(t))} \leq \|u^{-\sigma}\|_{\Lambda_{\varphi_E}(1,\ell(t))} \\ &= \int_1^{\ell(t)} u^{-\sigma}\varphi_E'(u)\,du \leq \int_1^{\ell(t)} u^{-\sigma}\varphi_E(u)\frac{du}{u}.\end{aligned}$$

The condition $\pi_{\varphi_E}>\sigma$ implies that $\int_1^{\ell(t)} u^{-\sigma}\varphi_E(u)\,du/u\lesssim\ell(t)^{-\sigma}\varphi_E(\ell(t))$, which gives the estimate from above. The estimate from below is similar to (5),

$$\|\ell(s)^{-\sigma}\|_{\widetilde{E}(1,t)} \geq \|\ell(s)^{-\sigma}\|_{\widetilde{E}(\sqrt{t},t)} \geq \ell(t)^{-\sigma}\|\chi_{(\sqrt{t},t)}(e^u)\|_E \gtrsim \ell(t)^{-\sigma}\varphi_E(\ell(t)). \tag{6}$$ □

We now establish the limiting case of (4) for $\alpha=0$, generalizing to slowly varying functions on r.i. spaces the results from Lemma 6.1 from [13].

Lemma 3.2. *Let E be an r.i. space, $b\in\mathrm{SV}$ be such that $b(t^2)\sim b(t)$ and let B_0 and B_∞ be its associated functions defined by (3).*

(i) *If $\rho_{B_\infty}<\pi_{\varphi_E}\leq\rho_{\varphi_E}<\pi_{B_0}$, then*

$$\|b(s)\|_{\widetilde{E}(0,t)} \sim b(t)\varphi_E(\ell(t)), \quad t>0.$$

(ii) *If* $\rho_{B_0}<\pi_{\varphi_E}\leq\rho_{\varphi_E}<\pi_{B_\infty}$, *then the equivalence is*

$$\|b(s)\|_{\widetilde{E}(t,\infty)}\sim b(t)\varphi_E(\ell(t)),\quad t>0.$$

Proof. We split $(0,\infty)=(0,1)\cup[1,e]\cup(e,\infty)$ and we shall establish (i) on each interval. For the first interval we may quote Lemma 2.1 of [33], which under the conditions $b(t^2)\sim b(t)$ and $\rho_{\varphi_E}<\pi_{B_0}$, gives that

$$\|b(t\tau)\|_{\widetilde{E}(0,1)}=\|b(te^{-u})\|_E\sim b(t)\varphi_E(\ell(t))\quad\text{for all } t\in(0,1).$$

Then the change of variables $s=t\tau$ yields (i) on the interval $(0,1)$.

When $t\in[1,e]$, the monotonicity of the functions $b(t)/t$ and $\varphi_E(\ell(t))$ implies that

$$\|b(s)\|_{\widetilde{E}(0,t)}\sim 1\sim b(t)\varphi_E(\ell(t)).$$

Finally let $t\in(e,\infty)$. Observe that

$$\|b(s)\|_{\widetilde{E}(0,t)}\sim\|b(s)\|_{\widetilde{E}(0,1)}+\|b(s)\|_{\widetilde{E}(1,t)}\sim 1+\|b(s)\|_{\widetilde{E}(1,t)}.$$

Then to establish (i) on the interval (e,∞) it suffices to show that for all $t>e$,

$$1\lesssim b(t)\varphi_E(\ell(t)) \tag{7}$$

and

$$\|b(s)\|_{\widetilde{E}(1,t)}\sim b(t)\varphi_E(\ell(t)). \tag{8}$$

Consider the function $f(t)=B_\infty(1/t)\varphi_E(t)$, $t>0$, whose indices satisfy

$$0<-\rho_{B_\infty}+\pi_{\varphi_E}\leq\pi_f\leq\rho_f\leq-\pi_{B_\infty}+\rho_{\varphi_E}<\infty.$$

Then f is almost increasing and therefore $f(\ell(t))=b(t)\varphi_E(\ell(t))$, for $t>e$, is almost increasing too. In particular we have (7) for all $t\in(e,\infty)$.

Take now $\rho_{B_\infty}<\sigma<\pi_{\varphi_E}$ so that the function $t^{-\sigma}B_\infty(t)$, $t>0$, is almost decreasing. Then, for all $t\in(e,\infty)$, we have

$$\|b(s)\|_{\widetilde{E}(1,t)}=\left\|B_\infty\left(\frac{1}{\ell(s)}\right)\right\|_{\widetilde{E}(1,t)}\lesssim B_\infty\left(\frac{1}{\ell(t)}\right)\ell(t)^\sigma\|\ell(s)^{-\sigma}\|_{\widetilde{E}(1,t)}\sim b(t)\varphi_E(\ell(t)),$$

where last inequality uses Lemma 3.1. A similar argument yields the reverse inequality

$$\|b(s)\|_{\widetilde{E}(1,t)}\gtrsim\|b(s)\|_{\widetilde{E}(\sqrt{t},t)}\gtrsim B_\infty\left(\frac{1}{\ell(\sqrt{t})}\right)\ell(\sqrt{t})^\sigma\|\ell(s)^{-\sigma}\|_{\widetilde{E}(\sqrt{t},t)}\sim b(t)\varphi_E(\ell(t)).$$

This establishes (8) and concludes the proof of (i).

Finally, (ii) follows from (i) by using the function $\bar{b}(t)=b(1/t)$, $t>0$, and recalling that $\|g(t)\|_{\widetilde{E}}=\|g(1/t)\|_{\widetilde{E}}$ for all $g\in\widetilde{E}$. □

Below, we shall also use the following lemmas.

Lemma 3.3. *Let $b\in\mathrm{SV}$ be such that $b(t^2)\sim b(t)$. If the associated function B_0 satisfies that $\pi_{B_0}>0$, then*

$$\int_0^t b(s)\frac{ds}{s\ell(s)}\sim b(t) \quad \text{for all } t\in(0,1).$$

Proof. Changing variables $\ell(s)=1/u$ we obtain

$$\int_0^t b(s)\frac{ds}{s\ell(s)}=\int_0^t B_0\left(\frac{1}{\ell(s)}\right)\frac{ds}{s\ell(s)}=\int_0^{1/\ell(t)} B_0(u)\frac{du}{u}.$$

Since $\pi_{B_0}>0$, we have (see [14, p. 57])

$$\int_0^{1/\ell(t)} B_0(u)\frac{du}{u}\sim B_0\left(\frac{1}{\ell(t)}\right)=b(t). \quad \square$$

Lemma 3.4. *Let $b\in\mathrm{SV}$ be such that $b(t^2)\sim b(t)$. Then*

$$\int_{\sqrt{t}}^t b(s)\frac{ds}{s\ell(s)}\gtrsim b(t) \quad \text{for all } t\in(e,\infty).$$

Proof. Since $\ell(t)$ is increasing in $(1,\infty)$, we get

$$\int_{\sqrt{t}}^t b(s)\frac{ds}{s\ell(s)}\geq\frac{1}{\ell(t)}\int_{\sqrt{t}}^t b(s)\frac{ds}{s}.$$

Hence it is enough to estimate the second integral from below by $b(t)\ell(t)$. Take $\alpha\in\mathbb{R}$ such that $\rho_{B_\infty}<\alpha$. Then the function $t^{-\alpha}B_\infty(t)$, $t>0$, is almost decreasing and we have, for all $t\in(e,\infty)$,

$$\begin{aligned}\int_{\sqrt{t}}^t b(s)\frac{ds}{s}&=\int_{\sqrt{t}}^t B_\infty\left(\frac{1}{\ell(s)}\right)\ell(s)^\alpha\ell(s)^{-\alpha}\frac{ds}{s}\\&\gtrsim B_\infty\left(\frac{1}{\ell(\sqrt{t}\,)}\right)\ell(\sqrt{t}\,)^\alpha\int_{\sqrt{t}}^t\ell(s)^{-\alpha}\frac{ds}{s}.\end{aligned}$$

Arguing as in (5) and (6) and using that $b(t)\sim b(\sqrt{t}\,)$, one obtains that

$$\int_{\sqrt{t}}^t b(s)\frac{ds}{s}\gtrsim b(t)\ell(t)$$

for all $t\in(e,\infty)$. □

3.2. Limiting estimates

Let E be an r.i. space, $b\in\mathrm{SV}$ and $\alpha\in\mathbb{R}$. Lemma 2.4 from [15] asserts that

$$\|s^\alpha b(s)\varphi(s)\|_{\widetilde{E}(0,t)} \lesssim \int_0^t s^\alpha b(s)\varphi(s)\,\frac{ds}{s}$$

and

$$\|s^\alpha b(s)\varphi(s)\|_{\widetilde{E}(t,\infty)} \lesssim \int_t^\infty s^\alpha b(s)\varphi(s)\,\frac{ds}{s},$$

for any quasi-concave function φ and any $t\in(0,\infty)$. Next we state better estimates, when $\alpha\in\{-1,0\}$, which will play a crucial role later on.

Lemma 3.5. *Let E be an r.i. space, φ be a quasi-concave function and let $b\in\mathrm{SV}$ be such that $b(t^2)\sim b(t)$.*

(i) *If the associated function B_0 satisfies $\pi_{B_0}>0$, then*

$$\|s^{-1}b(s)\varphi(s)\|_{\widetilde{E}(0,t)} \lesssim \int_0^t s^{-1}b(s)\varphi(s)\varphi_E(\ell(s))\frac{ds}{s\ell(s)}$$

for any $t\in(0,\infty)$.

(ii) *If B_∞ satisfies $\pi_{B_\infty}>0$, then*

$$\|b(s)\varphi(s)\|_{\widetilde{E}(t,\infty)} \lesssim \int_t^\infty b(s)\varphi(s)\varphi_E(\ell(s))\frac{ds}{s\ell(s)}$$

for any $t\in(0,\infty)$.

Proof. First we prove (i) for $t\in(0,1)$. Using Lemma 3.3 and that the function $s\mapsto\varphi(s)/s$ is almost decreasing, one get

$$s^{-1}b(s)\varphi(s)\sim s^{-1}\varphi(s)\int_0^s b(\tau)\frac{d\tau}{\tau\ell(\tau)} \lesssim \int_0^s \tau^{-1}b(\tau)\varphi(\tau)\frac{d\tau}{\tau\ell(\tau)}$$

for all $s\in(0,1)$. Hence

$$\|s^{-1}b(s)\varphi(s)\|_{\widetilde{E}(0,t)} \lesssim \left\|\int_0^s \tau^{-1}b(\tau)\varphi(\tau)\frac{d\tau}{\tau\ell(\tau)}\right\|_{\widetilde{E}(0,t)}$$

for all $0<t<1$. Therefore, in order to obtain (i) for $t\in(0,1)$, it is sufficient to verify that

$$\left\|\int_0^s \tau^{-1}b(\tau)\varphi(\tau)\frac{d\tau}{\tau\ell(\tau)}\right\|_{\widetilde{E}(0,t)} \lesssim \int_0^t \tau^{-1}b(\tau)\varphi(\tau)\varphi_E(\ell(\tau))\frac{d\tau}{\tau\ell(\tau)}$$

for all $0<t<1$. Observe that

$$\left\|\int_0^s \tau^{-1}b(\tau)\varphi(\tau)\frac{d\tau}{\tau\ell(\tau)}\right\|_{\widetilde{E}(0,t)} \leq \left\|\int_0^t \chi_{(0,s)}(\tau)\tau^{-1}b(\tau)\varphi(\tau)\frac{d\tau}{\tau\ell(\tau)}\right\|_{\widetilde{E}(0,1)}$$
$$= \left\|\int_0^t \chi_{(0,e^{-u})}(\tau)\tau^{-1}b(\tau)\varphi(\tau)\frac{d\tau}{\tau\ell(\tau)}\right\|_E$$

for all $0<t<1$. Then, using the general Minkowski inequality, see [14, p. 45], and the relation $\|\chi_{(0,e^{-u})}(\tau)\|_E=\|\chi_{(0,\log(1/\tau))}(u)\|_E=\varphi_E(\log(1/\tau))\leq\varphi_E(\ell(\tau))$, for all $\tau>0$, we obtain the result.

If t lays in the interval $[1,e]$ it is easy to check that

$$\|s^{-1}b(s)\varphi(s)\|_{\widetilde{E}(0,t)} \sim 1 \sim \int_0^t s^{-1}b(s)\varphi(s)\varphi_E(\ell(s))\frac{ds}{s\ell(s)}.$$

Finally, if $t\in(e,\infty)$ then

$$\|s^{-1}b(s)\varphi(s)\|_{\widetilde{E}(0,t)} \sim \|s^{-1}b(s)\varphi(s)\|_{\widetilde{E}(0,e)} + \|s^{-1}b(s)\varphi(s)\|_{\widetilde{E}(e,t)}$$
$$\sim \int_0^e s^{-1}b(s)\varphi(s)\varphi_E(\ell(s))\frac{ds}{s\ell(s)} + \|s^{-1}b(s)\varphi(s)\|_{\widetilde{E}(e,t)}.$$

Hence to finish the proof it is enough to verify that

$$\|s^{-1}b(s)\varphi(s)\|_{\widetilde{E}(e,t)} \leq \int_e^t \tau^{-1}b(\tau)\varphi(\tau)\varphi_E(\ell(\tau))\frac{d\tau}{\tau\ell(\tau)} \tag{9}$$

for all $t\in(e,\infty)$. Lemma 3.4 and the fact that $\varphi(s)/s$ is almost decreasing yield the inequalities

$$s^{-1}b(s)\varphi(s) \lesssim s^{-1}\varphi(s)\int_{\sqrt{s}}^s b(\tau)\frac{d\tau}{\tau\ell(\tau)} \lesssim \int_{\sqrt{s}}^s \tau^{-1}b(\tau)\varphi(\tau)\frac{d\tau}{\tau\ell(\tau)}$$

for all $s\in(e,\infty)$. Now taking norms on $\widetilde{E}(e,t)$, for all $t>e$, and using the connection between the norms of the spaces, we get

$$\|s^{-1}b(s)\varphi(s)\|_{\widetilde{E}(e,t)} \lesssim \left\|\int_{\sqrt{s}}^s \tau^{-1}b(\tau)\varphi(\tau)\frac{d\tau}{\tau\ell(\tau)}\right\|_{\widetilde{E}(e,t)}$$
$$\leq \left\|\int_e^t \tau^{-1}b(\tau)\varphi(\tau)\chi_{(\sqrt{s},s)}(\tau)\frac{d\tau}{\tau\ell(\tau)}\right\|_{\widetilde{E}(1,\infty)}$$
$$= \left\|\int_e^t \tau^{-1}b(\tau)\varphi(\tau)\chi_{(e^{u/2},e^u)}(\tau)\frac{d\tau}{\tau\ell(\tau)}\right\|_E$$

for all $t>e$. Using now the general Minkowski inequality, see [14, p. 45], and the estimate $\|\chi_{(e^{u/2},e^u)}(\tau)\|_E=\|\chi_{(\log\tau,2\log\tau)}(u)\|_E=\varphi_E(\log\tau)\le\varphi_E(\ell(\tau))$, for all $\tau>e$, we establish (9) for all $t>e$.

In order to prove (ii), we consider the slowly varying function $\bar{b}(t)=b(1/t)$, $t>0$, and the quasi-concave function $\overline{\varphi}(s)=s\varphi(1/s)$. We observe that

$$\|b(s)\varphi(s)\|_{\widetilde{E}(t,\infty)}=\left\|s\bar{b}\left(\frac{1}{s}\right)\overline{\varphi}\left(\frac{1}{s}\right)\right\|_{\widetilde{E}(t,\infty)}=\|s^{-1}\bar{b}(s)\overline{\varphi}(s)\|_{\widetilde{E}(0,1/t)}.$$

Then, by (i) and the change of variables $s=1/\tau$, we obtain (ii). In fact,

$$\begin{aligned}\|b(s)\varphi(s)\|_{\widetilde{E}(t,\infty)}&\lesssim\int_0^{1/t}s^{-1}\bar{b}(s)\overline{\varphi}(s)\varphi_E(\ell(s))\frac{ds}{s\ell(s)}\\&=\int_t^\infty b(\tau)\varphi(\tau)\varphi_E(\ell(\tau))\frac{d\tau}{\tau\ell(\tau)}.\qquad\square\end{aligned}$$

3.3. Limiting Hardy-type inequalities

When $\alpha>0$, general Hardy-type inequalities, involving $b\in\mathrm{SV}$ and an r.i. space E, were obtained in [15]. Namely

$$\left\|t^{-\alpha}b(t)\int_0^t f(s)\,ds\right\|_{\widetilde{E}}\lesssim\|t^{1-\alpha}b(t)f(t)\|_{\widetilde{E}},$$

$$\left\|t^{\alpha}b(t)\int_t^\infty f(s)\,ds\right\|_{\widetilde{E}}\lesssim\|t^{1+\alpha}b(t)f(t)\|_{\widetilde{E}}$$

hold for each measurable positive function f on $(0,\infty)$. These inequalities were a cornerstone for the reiteration results of [15] and [17]. Now we focus on the limit case $\alpha=0$ in which the power function disappears; then an additional logarithmic term comes out and the measure used is $dt/t\ell(t)$. Recall that $\widehat{E}=E(dt/t\ell(t))$.

Lemma 3.6. *Let E be an r.i. space, $b\in\mathrm{SV}$ be such that $b(t^2)\sim b(t)$, and let B_0 and B_∞ be its associated functions.*

(i) *If $\rho_{B_0}<0<\pi_{B_\infty}$, then*

$$\left\|b(t)\int_0^t f(s)\,ds\right\|_{\widehat{E}}\lesssim\|tb(t)f(t)\ell(t)\|_{\widehat{E}}$$

for any positive measurable function f on $(0,\infty)$.

(ii) *If* $\rho_{B_\infty}<0<\pi_{B_0}$, *then*

$$\left\| b(t)\int_t^\infty f(s)\,ds\right\|_{\widehat{E}} \lesssim \|tb(t)f(t)\ell(t)\|_{\widehat{E}}$$

for any positive measurable function f *on* $(0,\infty)$.

Proof. In order to prove (i) we consider the operator

$$Tg(t)=b(t)\int_0^t \frac{g(s)}{b(s)\ell(s)}\frac{ds}{s}.$$

We claim that T is bounded from $\hat{L}_1$ into $\hat{L}_1$, and from L_∞ into L_∞. Actually, let $g\in\hat{L}_1$, then

$$\begin{aligned}\|Tg\|_{\hat{L}_1} &= \int_0^\infty \left| b(t)\int_0^t \frac{g(s)}{b(s)\ell(s)}\frac{ds}{s}\right| \frac{dt}{t\ell(t)}\\ &\leq \int_0^\infty \frac{|g(s)|}{b(s)}\left(\int_s^\infty \frac{b(t)}{\ell(t)}\frac{dt}{t}\right)\frac{ds}{s\ell(s)} \sim \int_0^\infty |g(s)|\frac{ds}{s\ell(s)} = \|g\|_{\hat{L}_1},\end{aligned}$$

where last inequality follows from Lemma 3.2(ii) applied to the r.i. space $E=L_1$ and to the slowly varying function $\bar{b}(t)=b(t)/\ell(t)$. Observe that $\rho_{\overline{B}_0}=1+\rho_{B_0}$ and $\pi_{\overline{B}_\infty}=1+\pi_{B_\infty}$.

Let us estimate the norm $\|T\colon L_\infty\to L_\infty\|$. Choose $g\in L_\infty$. Then

$$\|Tg\|_{L_\infty} = \sup_{t>0}\left| b(t)\int_0^t \frac{g(s)}{b(s)\ell(s)}\frac{ds}{s}\right| \leq \|g\|_{L_\infty}\sup_{t>0} b(t)\int_0^t \frac{1}{b(s)\ell(s)}\frac{ds}{s} \sim \|g\|_{L_\infty}.$$

In order to prove last inequality take the slowly varying function $\tilde{b}(t)=1/b(t)\ell(t)$. Its associated functions are $\widetilde{B}_0(t)=t/B_0(t)$ and $\widetilde{B}_\infty(t)=t/B_\infty(t)$, with indices $\pi_{\widetilde{B}_0}=1-\rho_{B_0}$, $\rho_{\widetilde{B}_\infty}=1-\pi_{B_\infty}$. Then, we can apply Lemma 3.2(i) with $\tilde{b}$ and $E=L_1$ to obtain

$$\int_0^t \frac{1}{b(s)\ell(s)}\frac{ds}{s} \sim \frac{1}{b(t)}.$$

Now, since $\widehat{E}$ is an interpolation space for the couple $(\hat{L}_1, L_\infty)$ we have that the operator

$$T\colon \widehat{E} \longrightarrow \widehat{E}$$

is bounded. It suffices to choose $g(t)=tb(t)f(t)\ell(t)$, $t>0$, to complete the proof of (i).

In order to show (ii) we proceed similarly. First we prove that the operator

$$Tg(t) = b(t) \int_t^\infty \frac{g(s)}{b(s)\ell(s)} \frac{ds}{s}$$

is bounded from $\hat{L}_1$ to $\hat{L}_1$ and also bounded from L_∞ to L_∞. As a matter of fact,

$$\|Tg\|_{\hat{L}_1} \le \int_0^\infty \frac{|g(s)|}{b(s)} \left(\int_0^s \frac{b(t)}{\ell(t)} \frac{dt}{t} \right) \frac{ds}{s\ell(s)} \sim \int_0^\infty |g(s)| \frac{ds}{s\ell(s)} = \|g\|_{\hat{L}_1}.$$

The last inequality follows from Lemma 3.2(i) applied to the slowly varying function $\bar{b}$ and to the r.i. space $E = L_1$. Similarly using Lemma 3.2(ii), with $\tilde{b}$ and $E = L_1$, we get

$$\|Tg\|_{L_\infty} \le \|g\|_{L_\infty} \sup_{t>0} b(t) \int_t^\infty \frac{1}{b(s)\ell(s)} \frac{ds}{s} \sim \|g\|_{L_\infty}.$$

Now the same interpolation argument as above yield that

$$T\colon \widehat{E} \longrightarrow \widehat{E}$$

is a bounded operator and taking $g(t) = tb(t)f(t)\ell(t)$, $t>0$, we establish (ii). □

Remark 3.7. As a special case we recover various results stated in [13, Lemma 4.2], corresponding to a broken logarithm b and $E = L_q$. See also [30].

Remark 3.8. See [26] and [27] for Hardy-type inequalities in weighted r.i. spaces.

4. Interpolation methods and generalized Holmstedt-type formulas

Let $\overline{X} = (X_0, X_1)$ be a compatible couple of Banach spaces, that is, X_0 and X_1 are Banach spaces continuously embedded in some common Hausdorff topological vector space. We equip $X_0 + X_1$ with the norm $K(1, \cdot)$, where

$$K(t, f) = K(t, f; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1, f_i \in X_i, i = 0, 1\}$$

is the *Peetre K-functional*. We refer to the well known texts [4], [5], [8] and [14] for basic concepts on interpolation theory.

The following interpolation methods were introduced in [15] and constitute an extension of the well known real interpolation method with a functional parameter (see, e.g. [21] and [31]).

Definition 4.1. Let $\overline{X}=(X_0,X_1)$ be a compatible Banach couple, E be an r.i. space, $b\in\mathrm{SV}$ and $0\le\theta\le1$. The real interpolation space $\overline{X}_{\theta,b,E}\equiv(X_0,X_1)_{\theta,b,E}$ consists of all f in X_0+X_1 for which

$$\|f\|_{\theta,b,E}=\|t^{-\theta}b(t)K(t,f)\|_{\widetilde{E}}<\infty.$$

The space $\overline{X}_{\theta,b,E}$ is a Banach space. It is also an interpolation space provided that

$$0<\theta<1,\quad \text{or}\quad \theta=0 \text{ and } \|b\|_{\widetilde{E}(1,\infty)}<\infty,\quad \text{or}\quad \theta=1 \text{ and } \|b\|_{\widetilde{E}(0,1)}<\infty.$$

Moreover, if none of the previous conditions hold, then $\overline{X}_{\theta,b,E}=\{0\}$.

In [15] and [17] we identified the reiteration spaces

$$(\overline{X}_{\theta_0,b_0,E_0},\overline{X}_{\theta_1,b_1,E_1})_{\theta,b,E}$$

for almost all possible values of θ_0, θ_1 and θ in $[0,1]$. Our interest now is to study the extreme cases

$$(X_0,\overline{X}_{0,b_1,E_1})_{\theta,b,E}\quad\text{and}\quad(\overline{X}_{1,b_0,E_0},X_1)_{\theta,b,E},$$

that were not covered in [15] and [17]; notice that both spaces of the couple are now very close to X_0 or to X_1, respectively. In order to identify these interpolation spaces we relate the K-functional of the underlying couples through generalized Holmstedt-type formulas collected in the following theorems.

Theorem 4.2. *Let $\overline{X}=(X_0,X_1)$ be a compatible Banach couple and E be an r.i. space. Let $b\in\mathrm{SV}$ be such that $\|b\|_{\widetilde{E}(1,\infty)}<\infty$ and choose $\rho(t)=\|b\|^{-1}_{\widetilde{E}(t,\infty)}$, $t>0$. Then*

$$K(\rho(t),f;X_0,\overline{X}_{0,b,E})\sim\rho(t)\|b(s)K(s,f)\|_{\widetilde{E}(t,\infty)}$$

for all $f\in X_0+\overline{X}_{0,b,E}$ and $t\in(0,\infty)$.

Proof. Given $f\in X_0+X_1$ define

$$(Pf)(t)=\|b(s)K(s,f)\|_{\widetilde{E}(0,t)},\qquad t>0,$$

$$(Qf)(t)=\|b(s)K(s,f)\|_{\widetilde{E}(t,\infty)},\qquad t>0.$$

In the first stage we prove the inequality

$$K(\rho(t),f;X_0,\overline{X}_{0,b,E})\lesssim\rho(t)(Qf)(t). \tag{10}$$

Let $f\in X_0+\overline{X}_{0,b,E}$ and $t>0$. Choose a decomposition $f=g+h$ in X_0+X_1 such that

$$\|g\|_{X_0}+t\|h\|_{X_1}\le 2K(t,f).$$

It is easy to check that for all $s\in(0,\infty)$,

$$K(s,g)\le 2K(t,f) \quad\text{and}\quad \frac{K(s,h)}{s}\le 2\frac{K(t,f)}{t}. \tag{11}$$

Then

$$\|g\|_{X_0}\le 2K(t,f)\sim K(t,f)\rho(t)\|b\|_{\widetilde{E}(t,\infty)}\lesssim \rho(t)(Qf)(t). \tag{12}$$

Besides, for the term $\|h\|_{\overline{X}_{0,b,E}}$ we have the estimate

$$\|h\|_{\overline{X}_{0,b,E}}\le (Ph)(t)+(Qf)(t)+(Qg)(t).$$

Here $(Ph)(t)$ can be estimated by $(Qf)(t)$ using (11), Lemma 2.2(v) and (vii), and the monotonicity of $K(t,f)$,

$$\begin{aligned}(Ph)(t)&=\|b(s)K(s,h)\|_{\widetilde{E}(0,t)}\le t^{-1}K(t,f)\|sb(s)\|_{\widetilde{E}(0,t)}\\&\sim K(t,f)b(t)\lesssim K(t,f)\|b\|_{\widetilde{E}(t,\infty)}\lesssim (Qf)(t).\end{aligned}$$

Using the monotonicity of the K-functional it is easy to prove that $(Qg)(t)\lesssim(Qf)(t)$ and so $\|h\|_{\overline{X}_{0,b,E}}\lesssim(Qf)(t)$. This, combined with (12), proves (10).

The second part of the proof deals with the reverse inequality

$$\rho(t)(Qf(t))\lesssim K(\rho(t),f;X_0,\overline{X}_{0,b,E}). \tag{13}$$

Fix $t>0$ and let $f=g+h$ be any decomposition of f in $X_0+\overline{X}_{0,b,E}$. Then

$$(Qf)(t)\le (Qg)(t)+(Qh)(t)\le (Qg)(t)+\|h\|_{\overline{X}_{0,b,E}}.$$

Since

$$(Qg)(t)=\|b(s)K(s,g)\|_{\widetilde{E}(t,\infty)}\le \|b\|_{\widetilde{E}(t,\infty)}\|g\|_{X_0}=\rho(t)^{-1}\|g\|_{X_0},$$

we have that $\rho(t)(Qf)(t)\le\|g\|_{X_0}+\rho(t)\|h\|_{\overline{X}_{0,b,E}}$. Taking infimum over all possible representations of f we establish (13). □

Next we take care of the K-functional for the couple $((X_0,X_1)_{1,b,E},X_1)$. Although it can be proved in a similar way as Theorem 4.2, we shall use a symmetry argument.

Theorem 4.3. *Let $\overline{X}=(X_0,X_1)$ be a compatible Banach couple and E be an r.i. space. Let $b\in\mathrm{SV}$ be such that $\|b\|_{\widetilde{E}(0,1)}<\infty$ and consider the function $\rho(t)=\|b\|_{\widetilde{E}(0,t)}$, $t>0$. Then*

$$K(\rho(t),f;\overline{X}_{1,b,E},X_1)\sim\|s^{-1}b(s)K(s,f)\|_{\widetilde{E}(0,t)} \tag{14}$$

for all $f\in\overline{X}_{1,b,E}+X_1$ and $t\in(0,\infty)$.

Proof. Consider the function $\bar{b}(t)=b(1/t)$. Then $\rho(t)=\|\bar{b}(s)\|_{\widetilde{E}(1/t,\infty)}$. Using the equalities (see [4, Proposition 5.1.2] and [15, Lemma 3.4])

$$K(t,f;X_0,X_1)=tK(t^{-1},f;X_1,X_0)\quad\text{and}\quad(X_0,X_1)_{1,b,E}=(X_1,X_0)_{0,\bar{b},E},$$

together with Theorem 4.2 we have

$$\begin{aligned}K(\rho(t),f;(X_0,X_1)_{1,b,E},X_1)&=\rho(t)K\left(\frac{1}{\rho(t)},f;X_1,(X_1,X_0)_{0,\bar{b},E}\right)\\&\sim\|\bar{b}(s)K(s,f;X_1,X_0)\|_{\widetilde{E}(1/t,\infty)}.\end{aligned}$$

Hence (14) follows using that $\|f(s)\|_{\widetilde{E}(0,t)}=\|f(1/s)\|_{\widetilde{E}(1/t,\infty)}$ and the relation between the K-functionals again. □

Remark 4.4. The Holmstedt-type formulas in Theorems 4.2 and 4.3 hold for more general parameters b than slowly varying functions. In fact it is only needed that

$$\|sb(s)\|_{\widetilde{E}(0,t)}\lesssim t\|b(s)\|_{\widetilde{E}(t,\infty)}\quad\text{for all }t>0.$$

5. Reiteration theorems

In the present section we identify the interpolation spaces

$$(X_0,\overline{X}_{0,b_1,E_1})_{\theta,b,E}\quad\text{and}\quad(\overline{X}_{1,b_0,E_0},X_1)_{\theta,b,E}$$

for all possible values of $\theta\in[0,1]$. To do so we will need a formula for a change of variables, which is stated in the next lemma. Recall again that $\widehat{E}=E(dt/t\ell(t))$.

We say that a function f belongs to the family $\mathcal{F}$ if it has the following property:

$$\text{if }\varphi\sim\psi,\quad\text{then }f\circ\varphi\sim f\circ\psi.$$

For example, potential functions, slowly varying functions and the K-functional belong to $\mathcal{F}$. In particular, if $b\in\mathrm{SV}$ and $0\le\theta\le1$, then $f(t)=t^{-\theta}b(t)K(t,\cdot)$ belongs to $\mathcal{F}$.

Lemma 5.1. *Let E be an r.i. space, $b\in\mathrm{SV}$ be such that $b(t^2)\sim b(t)$ with associated functions B_0 and B_∞.*

(i) *If $\rho_{B_0} < \pi_{\varphi_E} \le \rho_{\varphi_E} < \pi_{B_\infty}$ and $\rho(t)=1/b(t)\varphi_E(\ell(t))$, $t>0$, then*

$$\|f\circ\rho\|_{\widehat{E}} \sim \|f\|_{\widetilde{E}}$$

for all $f\in\widetilde{E}$ belonging to $\mathcal{F}$, with equivalence constant independent of f.

(ii) *If $\rho_{B_\infty} < \pi_{\varphi_E} \le \rho_{\varphi_E} < \pi_{B_0}$ and $\rho(t)=b(t)\varphi_E(\ell(t))$, $t>0$, then*

$$\|f\circ\rho\|_{\widehat{E}} \sim \|f\|_{\widetilde{E}}$$

for all $f\in\widetilde{E}$ belonging to $\mathcal{F}$, with equivalence constant independent of f.

Proof. We prove (i). A similar argument proves (ii). We observe that by an interpolation argument, it suffices to show the equivalence of the norm for $E=L_1$ and $E=L_\infty$.

Let $\Phi_0(u)=(B_0(u)\varphi_E(1/u))^{-1}$, $0<u\le 1$. By properties of the indices

$$0<\pi_{\varphi_E}-\rho_{B_0}\le \pi_{\Phi_0}\le \rho_{\Phi_0}\le \rho_{\varphi_E}-\pi_{B_0}<\infty.$$

Then there exists a smooth function $\Psi_0\sim\Phi_0$ such that $u\Psi_0'(u)\sim\Psi_0(u)$ for all $0<u\le 1$, $\Psi_0(1)=\Phi_0(1)$ and $\lim_{u\to 0}\Psi_0(u)=0$ (see [34, Lemma 2.1]). Hence,

$$\int_0^{\Psi_0(1)} |f(s)|\frac{ds}{s} = \int_0^1 |f(\Psi_0(u))|\frac{\Psi_0'(u)}{\Psi_0(u)}\,du \sim \int_0^1 |f(\Psi_0(u))|\frac{du}{u}.$$

Since $f\in\mathcal{F}$,

$$\int_0^{\Psi_0(1)} |f(s)|\frac{ds}{s} \sim \int_0^1 |f(\Phi_0(u))|\frac{du}{u}.$$

Now using the change of variables $u=1/\ell(t)$ and the fact that $\Phi_0(1/\ell(t))=\rho(t)$, for all $0<t\le 1$, we obtain that

$$\int_0^{\Psi_0(1)} |f(s)|\frac{ds}{s} \sim \int_0^1 |f(\Phi_0(u))|\frac{du}{u} = \int_0^1 |f(\rho(t))|\frac{dt}{t\ell(t)}. \tag{15}$$

On the other hand, we consider the function $\Phi_\infty(u)=(B_\infty(1/u)\varphi_E(u))^{-1}$ for $1\le u<\infty$. By hypothesis

$$0<\pi_{B_\infty}-\rho_{\varphi_E}\le \pi_{\Phi_\infty}\le \rho_{\Phi_\infty}\le \rho_{B_\infty}-\pi_{\varphi_E}<\infty.$$

As before, there exists a smooth function $\Psi_\infty\sim\Phi_\infty$ such that $u\Psi_\infty'(u)\sim\Psi_\infty(u)$ for all $1\le u<\infty$, $\Psi_\infty(1)=\Phi_\infty(1)$ and $\lim_{u\to\infty}\Psi_\infty(u)=\infty$. Then

$$\int_{\Psi_\infty(1)}^\infty |f(s)|\frac{ds}{s} = \int_1^\infty |f(\Psi_\infty(u))|\frac{\Psi_\infty'(u)}{\Psi_\infty(u)}\,du \sim \int_1^\infty |f(\Psi_\infty(u))|\frac{du}{u}.$$

Using that $f\in\mathcal{F}$, the change of variables $u=\ell(t)$, $1\le t<\infty$, and that $\Phi_\infty(\ell(t))=\rho(t)$, for all $1\le t<\infty$, we have

$$\int_{\Psi_\infty(1)}^{\infty}|f(s)|\frac{ds}{s}\sim\int_1^\infty|f(\Phi_\infty(u))|\frac{du}{u}=\int_1^\infty|f(\rho(t))|\frac{dt}{t\ell(t)}. \tag{16}$$

Summing up (15) and (16) we get

$$\|f\|_{\tilde{L}_1}\sim\|f\circ\rho\|_{\hat{L}_1}$$

(observe that $\Psi_0(1)=\Psi_\infty(1)$). The same is true in the case $E=L_\infty$, that is $\|f\|_{L_\infty}\sim\|f\circ\rho\|_{L_\infty}$. Hence, using the interpolation properties of the space E, we obtain the inequality

$$\|f\circ\rho\|_{\widehat{E}}\lesssim\|f\|_{\widetilde{E}}.$$

The reverse inequality can be proved using the same techniques with inverse functions. □

We can now consider the extreme reiteration problem. To this end, we need to introduce the space $\overline{X}_{\theta,b,\widehat{E}}$ and the spaces $\mathcal{L}$ and $\mathcal{R}$ with respect to the measure $dt/t\ell(t)$.

Definition 5.2. Let $\overline{X}=(X_0,X_1)$ be a compatible Banach couple, E be an r.i. space, $b\in\mathrm{SV}$ and $0\le\theta\le1$. The space $\overline{X}_{\theta,b,\widehat{E}}$ consists of all f in X_0+X_1 for which the norm

$$\|f\|_{\theta,b,\widehat{E}}=\|t^{-\theta}b(t)K(t,f)\|_{\widehat{E}}$$

is finite.

When $E=L_q$, $1\le q<\infty$, the space $\overline{X}_{\theta,b,\widehat{E}}$ coincides with the interpolation space $\overline{X}_{\theta,b(t)/\ell^{1/q}(t),E}$ while $\overline{X}_{\theta,b,\hat{L}_\infty}=\overline{X}_{\theta,b,L_\infty}$.

Definition 5.3. Given a compatible Banach couple $\overline{X}=(X_0,X_1)$, a real parameter $0\le\theta\le1$, two r.i. spaces E and F, and $a,b\in\mathrm{SV}$, we define the space $\overline{X}^{\mathcal{L}}_{\theta,b,\widehat{E},a,F}$ as the collection of all those elements $f\in X_0+X_1$ for which

$$\|f\|_{\mathcal{L};\theta,b,\widehat{E},a,F}=\big\|b(t)\|s^{-\theta}a(s)K(s,f)\|_{\widetilde{F}(0,t)}\big\|_{\widehat{E}}<\infty.$$

Analogously, the space $\overline{X}^{\mathcal{R}}_{\theta,b,\widehat{E},a,F}$ consists of all those elements in X_0+X_1 for which the norm

$$\|f\|_{\mathcal{R};\theta,b,\widehat{E},a,F}=\big\|b(t)\|s^{-\theta}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\big\|_{\widehat{E}}$$

is finite.

These spaces can also be identified when $E=F=L_q$.

Lemma 5.4. *Let* $\overline{X}=(X_0,X_1)$ *be a compatible Banach couple,* $a,b\in\mathrm{SV}$, $1\le q\le\infty$ *and* $0\le\theta\le 1$. *Then*

$$\overline{X}^{\mathcal{L}}_{\theta,b,\hat{L}_q,a,L_q}=\overline{X}_{\theta,a\Phi_\infty,L_q} \quad \text{and} \quad \overline{X}^{\mathcal{R}}_{\theta,b,\hat{L}_q,a,L_q}=\overline{X}_{\theta,a\Phi_0,L_q},$$

where $\Phi_0(t)=\|b(s)/\ell^{1/q}(s)\|_{\tilde{L}_q(0,t)}$ *and* $\Phi_\infty(t)=\|b(s)/\ell^{1/q}(s)\|_{\tilde{L}_q(t,\infty)}$.

The proof is omitted since it is an easy consequence of Fubini's theorem (see [15, Lemma 6.7]).

The first reiteration result is described in the following theorem.

Theorem 5.5. *Let* $\overline{X}=(X_0,X_1)$ *be a compatible couple,* E *and* E_1 *be r.i. spaces, and* $b,b_1\in\mathrm{SV}$ *with* b_1 *satisfying* $b_1(t)\sim b_1(t^2)$. *Assume that* E_1 *and the associated functions of* b_1, $B_{1,0}$ *and* $B_{1,\infty}$ *satisfy*

$$\rho_{B_{1,0}}<\pi_{\varphi_{E_1}}\le\rho_{\varphi_{E_1}}<\pi_{B_{1,\infty}}.$$

Then, for any $0\le\theta<1$, *we have the equality*

$$(X_0,\overline{X}_{0,b_1,E_1})_{\theta,b,E}=\overline{X}_{0,\tilde{b},\widehat{E}}, \tag{17}$$

where $\tilde{b}(t)=(b_1(t)\varphi_{E_1}(\ell(t)))^\theta b(1/b_1(t)\varphi_{E_1}(\ell(t)))$, $t>0$. *In case* $\theta=1$, *we have that*

$$(X_0,\overline{X}_{0,b_1,E_1})_{1,b,E}=\overline{X}^{\mathcal{R}}_{0,b\circ\rho,\widehat{E},b_1,E_1}, \tag{18}$$

where $\rho(t)=(b_1(t)\varphi_{E_1}(\ell(t)))^{-1}$, $t>0$.

Proof. By Lemma 3.2(ii) we know that $\|b_1\|_{\widetilde{E}_1(t,\infty)}\sim b_1(t)\varphi_{E_1}(\ell(t))$, $t>0$. Hence we take

$$\rho(t)=\frac{1}{b_1(t)\varphi_{E_1}(\ell(t))},\quad t>0.$$

Choose $f\in(X_0,\overline{X}_{0,b_1,E_1})_{\theta,b,E}$. Then by the generalized Holmstedt-type formula, Theorem 4.2, we know that

$$\overline{K}(\rho(t),f):=K(\rho(t),f;X_0,\overline{X}_{0,b_1,E_1})\sim\rho(t)\|b_1(s)K(s,f)\|_{\widetilde{E}_1(t,\infty)}.$$

Thus, using Lemma 5.1(i) with $F(t)=t^\theta b(t)\overline{K}(t,f)\in\mathcal{F}$, we establish the equivalence

$$\begin{aligned}\|f\|_{(X_0,\overline{X}_{0,b_1,E_1})_{\theta,b,E}}&=\|s^{-\theta}b(s)\overline{K}(s,f)\|_{\widetilde{E}}\sim\|\rho(t)^{-\theta}b(\rho(t))\overline{K}(\rho(t),f)\|_{\widehat{E}}\\&\sim\|\rho(t)^{1-\theta}b(\rho(t))\|b_1(s)K(s,f)\|_{\widetilde{E}_1(t,\infty)}\|_{\widehat{E}}.\end{aligned}$$

Consequently, for $\theta=1$ we obtain (18). On the other hand, to get (17) for $0\le\theta<1$ it suffices to prove that

$$I := \|\rho(t)^{1-\theta} b(\rho(t)) \|b_1(s)K(s,f)\|_{\widetilde{E}_1(t,\infty)}\|_{\widehat{E}} \sim \|\tilde{b}(t)K(t,f)\|_{\widehat{E}}.$$

First we prove the inequality $\lesssim$. Observe that the function $\Theta(t)=\rho(t)^{1-\theta}b(\rho(t))$, $t>0$, satisfies $\Theta(t)\sim\Theta(t^2)$, and has as associated functions

$$B_{\Theta,0}(u) = \Theta(e^{1-1/u}) = (B_{1,0}(u)\varphi_{E_1}(1/u))^{\theta-1} b(\rho(e^{1-1/u})),$$

$$B_{\Theta,\infty}(u) = \Theta(e^{1/u-1}) = (B_{1,\infty}(u)\varphi_{E_1}(1/u))^{\theta-1} b(\rho(e^{1/u-1})),$$

for $0<u<1$. Moreover, the indices of $B_{\Theta,0}$ and $B_{\Theta,\infty}$ satisfy

$$\rho_{B_{\Theta,\infty}} \le (\theta-1)(\pi_{B_{1,\infty}} - \rho_{\varphi_{E_1}}) < 0 < (\theta-1)(\rho_{B_{1,0}} - \pi_{\varphi_{E_1}}) \le \pi_{B_{\Theta,0}}.$$

Hence Lemma 3.5(ii) and the Hardy-type Lemma 3.6(ii) yield

$$\begin{aligned} I &\lesssim \left\| \rho(t)^{1-\theta} b(\rho(t)) \int_t^\infty b_1(s) \frac{\varphi_{E_1(\ell(s))}}{\ell(s)} K(s,f) \frac{ds}{s} \right\|_{\widehat{E}} \\ &\lesssim \|\rho(t)^{-\theta} b(\rho(t)) K(t,f)\|_{\widehat{E}} = \|\tilde{b}(t)K(t,f)\|_{\widehat{E}}. \end{aligned}$$

For the reverse inequality we use the monotonicity of the K-functional and the equivalence $\|b_1\|_{\widetilde{E}_1(t,\infty)} \sim \rho(t)^{-1}$, $t>0$, to get

$$I \gtrsim \|\rho(t)^{1-\theta} b(\rho(t)) K(t,f) \|b_1(u)\|_{\widetilde{E}_1(t,\infty)}\|_{\widehat{E}} \sim \|\tilde{b}(t)K(t,f)\|_{\widehat{E}}. \qquad \square$$

The remaining case is studied in the following theorem. Although it can be proved using a symmetry argument, we shall follow similar techniques to those used in Theorem 5.5.

Theorem 5.6. *Let $\overline{X}=(X_0,X_1)$ be a compatible couple, E and E_0 be r.i. spaces, and $b, b_0\in \mathrm{SV}$ with b_0 satisfying $b_0(t)\sim b_0(t^2)$. Assume that E_0 and the associated functions of b_0, $B_{0,0}$ and $B_{0,\infty}$, satisfy*

$$\rho_{B_{0,\infty}} < \pi_{\varphi_{E_0}} \le \rho_{\varphi_{E_0}} < \pi_{B_{0,0}}.$$

Then, for any $0<\theta\le 1$, we have the equality

$$(\overline{X}_{1,b_0,E_0}, X_1)_{\theta,b,E} = \overline{X}_{1,\tilde{b},\widehat{E}}, \tag{19}$$

where $\tilde{b}(t)=(b_0(t)\varphi_{E_0}(\ell(t)))^{1-\theta} b(b_0(t)\varphi_{E_0}(\ell(t)))$, $t>0$. In the case $\theta=0$ we have

$$(\overline{X}_{1,b_0,E_0}, X_1)_{0,b,E} = \overline{X}^{\mathcal{L}}_{1,b\circ\rho,\widehat{E},b_0,E_0}, \tag{20}$$

where $\rho(t)=b_0(t)\varphi_{E_0}(\ell(t))$, $t>0$.

Proof. Let $\rho(t)=b_0(t)\varphi_{E_0}(\ell(t))$, $t>0$, and choose $f\in(\overline{X}_{1,b_0,E_0},X_1)_{\theta,b,E}$. Since $\rho(t)\sim\|b_0\|_{\widetilde{E}(0,t)}$ (Lemma 3.2(i)), the generalized Holmstedt-type formula, Theorem 4.3, assures that

$$\overline{K}(\rho(t),f):=K(\rho(t),f;\overline{X}_{1,b_0,E_0},X_1)\sim\|s^{-1}b_0(s)K(s,f)\|_{\widetilde{E}_0(0,t)}.$$

Now, using Lemma 5.1(ii) with $F(t)=t^\theta b(t)\overline{K}(t,f)\in\mathcal{F}$, we obtain the equivalence

$$\|f\|_{(\overline{X}_{1,b_0,E_0},X_1)_{\theta,b,E}}=\|s^{-\theta}b(s)\overline{K}(s,f)\|_{\widehat{E}}\sim\|\rho(t)^{-\theta}b(\rho(t))\overline{K}(\rho(t),f)\|_{\widehat{E}}$$
$$\sim\big\|\rho(t)^{-\theta}b(\rho(t))\|s^{-1}b_0(s)K(s,f)\|_{\widetilde{E}_0(0,t)}\big\|_{\widehat{E}}.$$

This establishes (20). In other to prove (19) for $0\le\theta<1$, it suffices to show that

$$I:=\big\|\rho(t)^{-\theta}b(\rho(t))\|s^{-1}b_0(s)K(s,f)\|_{\widetilde{E}_0(0,t)}\big\|_{\widehat{E}}\sim\|\tilde{b}(t)K(t,f)\|_{\widehat{E}}.$$

The proof of the inequality $\gtrsim$ follows easily from the monotonicity of the K-functional and the equivalence $\|b_0\|_{\widetilde{E}_0(0,t)}\sim\rho(t)$, $t>0$,

$$I\gtrsim\big\|t^{-1}\rho(t)^{-\theta}b(\rho(t))K(s,f)\|b_0(s)\|_{\widetilde{E}_0(0,t)}\big\|_{\widehat{E}}\sim\|t^{-1}\tilde{b}(t)K(t,f)\|_{\widehat{E}}.$$

For the reverse inequality we take the function $\Theta(t)=\rho(t)^{-\theta}b(\rho(t))$, $t>0$, which satisfies $\Theta(t)\sim\Theta(t^2)$, and has as associated functions

$$B_{\Theta,0}(u)=\Theta(e^{1-1/u})=\left(B_{0,0}(u)\varphi_{E_0}\left(\frac{1}{u}\right)\right)^{-\theta}b(\rho(e^{1-1/u})),$$
$$B_{\Theta,\infty}(u)=\Theta(e^{1/u-1})=\left(B_{0,\infty}(u)\varphi_{E_0}\left(\frac{1}{u}\right)\right)^{-\theta}b(\rho(e^{1/u-1})),$$

for $0<u<1$. These functions satisfy the inequalities

$$\rho_{B_{\Theta,0}}\le-\theta(\pi_{B_{0,0}}-\rho_{\varphi_{E_0}})<0<-\theta(\rho_{B_{0,\infty}}-\pi_{\varphi_{E_0}})\le\pi_{B_{\Theta,\infty}}.$$

We are now in a position to apply Lemma 3.5(i) and the Hardy-type inequality, Lemma 3.6(i), to get

$$I\lesssim\left\|\rho(t)^{-\theta}b(\rho(t))\int_0^t s^{-1}b_0(s)\frac{\varphi_{E_0(\ell(s))}}{\ell(s)}K(s,f)\frac{ds}{s}\right\|_{\widehat{E}}$$
$$\lesssim\|t^{-1}\rho(t)^{1-\theta}b(\rho(t))K(t,f)\|_{\widehat{E}}=\|t^{-1}\tilde{b}(t)K(t,f)\|_{\widehat{E}},$$

and the proof is complete. □

Remark 5.7. If we choose b, b_0 and b_1 to be broken logarithmic functions, E, E_0 and E_1 to be L_q-spaces and $0<\theta<1$ in (17) and (19) we recover Theorems 7.6 and 7.9 of [13].

6. Applications to spaces of Lorentz–Karamata type

Let (Ω,μ) denote a σ-finite measure space with a non-atomic measure μ. Let E be an r.i. space and $b\in\mathrm{SV}$. We introduce the *Lorentz–Karamata type* space $L_{\infty,b,E}$, as the set of all measurable functions such that

$$\|f\|_{L_{\infty,b,E}} = \|b(t)f^*(t)\|_{\widetilde{E}} < \infty.$$

The space $L_{\infty,b,E}$ is non-trivial if and only if $\|b\|_{\widetilde{E}(0,1)}<\infty$. See [16] for more information. Similarly, the *Lorentz–Karamata type* space $L_{(1,b,E)}$ consists of all μ-measurable functions such that the norm

$$\|f\|_{L_{(1,b,E)}} = \|tb(t)f^{**}(t)\|_{\widetilde{E}}$$

is finite. See e.g. [17]. In this case the condition $\|b\|_{\widetilde{E}(1,\infty)}<\infty$ assures that the space is non-trivial.

Since

$$K(t,f;L_1,L_\infty) = \int_0^t f^*(s)\,ds = tf^{**}(t), \quad t>0,$$

and

$$\|b(t)f^{**}(t)\|_{\widetilde{E}} \sim \|b(t)f^*(t)\|_{\widetilde{E}}, \quad t>0,$$

(see e.g., [12, Lemma 2.16]), we have the equalities

$$(L_1,L_\infty)_{1,b,E} = L_{\infty,b,E} \quad \text{and} \quad (L_1,L_\infty)_{0,b,E} = L_{(1,b,E)}.$$

Therefore, we may use the reiteration theorems of the previous section to establish interpolation formulas for Lorentz–Karamata type spaces.

Corollary 6.1. *Let E and E_1 be r.i. spaces, and $b,b_1\in\mathrm{SV}$ with b_1 satisfying $b_1(t)\sim b_1(t^2)$. Assume that E_1 and the associated functions of b_1, $B_{1,0}$ and $B_{1,\infty}$ satisfy*

$$\rho_{B_{1,0}} < \pi_{\varphi_{E_1}} \le \rho_{\varphi_{E_1}} < \pi_{B_{1,\infty}}.$$

Then, for any $0\le\theta<1$,

$$(L_1,L_{(1,b_1,E_1)})_{\theta,b,E} = (L_1,L_\infty)_{0,\tilde{b},\widehat{E}},$$

where $\tilde{b}(t)=(b_1(t)\varphi_{E_1}(\ell(t)))^\theta b(1/b_1(t)\varphi_{E_1}(\ell(t)))$, $t>0$.

For the limit case $\theta=1$ we have the equality

$$(L_1,L_{(1,b_1,E_1)})_{1,b,E} = (L_1,L_\infty)^{\mathcal{R}}_{0,b\circ\rho,\widehat{E},b_1,E_1},$$

where $\rho(t)=(b_1(t)\varphi_{E_1}(\ell(t)))^{-1}$, $t>0$.

When the r.i. space E is equal to a Lebesgue space L_q, $1\leq q\leq\infty$, and under the conditions of Corollary 6.1, for any $0\leq\theta<1$, one obtains the following interpolation equality for classical Lorentz–Karamata spaces $L_{(1,q;b)}:=L_{(1,b,L_q)}$ (see [18] and [29])

$$(L_1, L_{(1,b_1,E_1)})_{\theta,b,L_q} = L_{(1,q;b^{\#})},$$

where $b^{\#}(t)=(b_1(t)\varphi_{E_1}(\ell(t)))^{\theta}b(1/b_1(t)\varphi_{E_1}(\ell(t)))\ell^{-1/q}(t)$, $t>0$. And if $\theta=1$ and $E_1=L_q$, then

$$(L_1, L_{(1,q;b_1)})_{1,b,L_q} = L_{(1,q;b_1\Phi_0)},$$

where $\Phi_0(t)=\|b(1/b_1(t)\ell^{1/q}(t))/\ell^{1/q}(t)\|_{\tilde{L}_q(0,t)}$, $t>0$.

Corollary 6.2. *Let E and E_0 be r.i. spaces, and $b, b_0\in\mathrm{SV}$ with b_0 satisfying $b_0(t)\sim b_0(t^2)$. Assume that E_0 and the associated functions of b_0, $B_{0,0}$ and $B_{0,\infty}$ satisfy*

$$\rho_{B_{0,\infty}} < \pi_{\varphi_{E_0}} \leq \rho_{\varphi_{E_0}} < \pi_{B_{0,0}}.$$

Then, for any $0<\theta\leq 1$,

$$(L_{\infty,b_0,E_0}, L_\infty)_{\theta,b,E} = (L_1, L_\infty)_{1,\tilde{b},\widehat{E}},$$

where $\tilde{b}(t)=(b_0(t)\varphi_{E_0}(\ell(t)))^{1-\theta}b(b_0(t)\varphi_{E_0}(\ell(t)))$, $t>0$.

For the limit case $\theta=0$ we have the equality

$$(L_{\infty,b_0,E_0}, L_\infty)_{0,b,E} = (L_1, L_\infty)^{\mathcal{L}}_{1,b\circ\rho,\widehat{E},b_0,E_0},$$

where $\rho(t)=b_0(t)\varphi_{E_0}(\ell(t))$, $t>0$.

Again if we consider the Lebesgue space L_q, $1\leq q\leq\infty$, under the conditions of Corollary 6.2, for any $0<\theta\leq 1$, we have the following interpolation equality for classical Lorentz–Karamata spaces $L_{\infty,q;b}:=L_{\infty,b,L_q}$ in the extreme cases

$$(L_{\infty,b_0,E_0}, L_\infty)_{\theta,b,L_q} = L_{\infty,q;b^{\#}},$$

where $b^{\#}(t)=(b_0(t)\varphi_{E_0}(\ell(t)))^{1-\theta}b(b_0(t)\varphi_{E_0}(\ell(t)))\ell^{-1/q}(t)$, $t>0$. If $\theta=0$ and $E_0=L_q$, then we have the equality

$$(L_{\infty,q;b_0}, L_\infty)_{0,b,L_q} = L_{\infty,q;b_0\Phi_\infty},$$

where $\Phi_\infty(t)=\|b(b_0(t)\ell^{1/q}(t))/\ell^{1/q}(t)\|_{\tilde{L}_q(t,\infty)}$, $t>0$.

7. Ordered couples

In this section we assume additionally that the couple $\overline{X}=(X_0,X_1)$ is ordered in the sense that $X_1\hookrightarrow X_0$. A typical example is $X_0=L_1(\Omega)$ and $X_1=L_\infty(\Omega)$, where (Ω,μ) is a finite measure space.

Following [17, Section 7], we define the space $(X_0,X_1)_{\theta,b,E}$ in this case.

Definition 7.1. Let $\overline{X}=(X_0,X_1)$ be a compatible couple with $X_1\hookrightarrow X_0$, E be an r.i. space, b be a slowly varying function on $(0,1)$, and $0\le\theta\le1$. The real interpolation space $\overline{X}_{\theta,b,E}$ consists of all f in X_0 for which the norm

$$\|f\|_{\theta,b,E}=\|t^{-\theta}b(t)K(t,f)\|_{\widetilde{E}(0,1)}$$

is finite.

Similarly, replacing the interval $(0,\infty)$ by $(0,1)$, one defines the spaces $\overline{X}_{\theta,b,\widehat{E}}$, $\overline{X}^{\mathcal{L}}_{\theta,b,\widehat{E},a,F}$ and $\overline{X}^{\mathcal{R}}_{\theta,b,\widehat{E},a,F}$ as in Definitions 5.2 and 5.3, respectively.

For the definition of slowly varying functions on $(0,1)$ it is enough to replace the interval $(0,\infty)$ by $(0,1)$ in Definition 2.1. We will use the notation $\mathrm{SV}(0,1)$.

Observe that if we take $b\in\mathrm{SV}(0,1)$, a space $E(0,1)$ and an ordered couple (X_0,X_1), $X_1\hookrightarrow X_0$, it is possible to show that all the previous results of this article remain true if we omit all assumptions concerning the interval $(1,\infty)$.

Let $X_0=L_1(\Omega,\mu)$, $X_1=L_\infty(\Omega,\mu)$ and let (Ω,μ) be a finite measure space with $\mu(\Omega)=1$. Then,

$$(L_1,L_\infty)_{0,1,L_1}=L\log L \quad\text{and}\quad (L_1,L_\infty)_{1,\ell(t)^{-1},L_\infty}=L_{\exp}.$$

Remember that $L_\infty\hookrightarrow L_{\exp}\hookrightarrow L_p\hookrightarrow L\log L\hookrightarrow L_1$. Thus the couples $(L_1,L\log L)$ and $(L_{\exp},L_\infty)$ are also ordered couples. Moreover, the function $b_1(t)\equiv1$, $t\in(0,1)$, and the space L_1 satisfy the conditions of Theorem 5.5, and the function $b_0(t)=\ell(t)^{-1}$, $t\in(0,1)$, and the space L_∞ fulfill the hypotheses of Theorem 5.6. So, we can apply the previous extreme reiteration results to get the following corollaries.

Corollary 7.2. *Let E be an r.i. space and let $b\in\mathrm{SV}(0,1)$. Then, for $0\le\theta<1$, we have the equality*

$$(L_1,L\log L)_{\theta,b,E}=(L_1,L_\infty)_{0,\tilde{b},\widehat{E}},$$

where $\tilde{b}(t)=\ell^\theta(t)b(1/\ell(t))$, $0<t<1$. Moreover, in the limiting case $\theta=1$, we have

$$(L_1,L\log L)_{1,b,E}=(L_1,L_\infty)^{\mathcal{R}}_{0,\tilde{b},\widehat{E},1,L_1},$$

where $\tilde{b}(t)=b(1/\ell(t))$, $0<t<1$.

In particular, for $0\le\theta<1$ and $1\le q\le\infty$, we get the classical Lorentz–Karamata spaces

$$(L_1, L\log L)_{\theta,b,L_q} = L_{(1,q;b^{\#})},$$

where $b^{\#}(t)=\ell^{\theta-1/q}(t)b(1/\ell(t))$, $0<t<1$, and

$$(L_1, L\log L)_{1,b,L_1} = L_{(1,1;\Phi_0)}$$

for $\Phi_0(t)=\|b(1/\ell(s))/\ell(s)\|_{\tilde{L}_1(0,t)}$, $0<t<1$.

The intermediate space $(L_1, L\log L)_{\theta,1,L_q}$, when $0<\theta<1$ and $1\le q\le\infty$, was identified by Bennett in [2].

Corollary 7.3. *Let E be an r.i. space and let $b\in\mathrm{SV}(0,1)$. Then, for $0<\theta\le 1$, we have the equality*

$$(L_{\exp}, L_\infty)_{\theta,b,E} = (L_1, L_\infty)_{1,\tilde{b},\widehat{E}},$$

where $\tilde{b}(t)=\ell^{\theta-1}(t)b(1/\ell(t))$, $0<t<1$. Moreover, in the limit case $\theta=0$, we have

$$(L_{\exp}, L_\infty)_{0,b,E} = (L_1, L_\infty)^{\mathcal{L}}_{1,\tilde{b},\widehat{E},\ell(t)^{-1},L_\infty},$$

where $\tilde{b}(t)=b(1/\ell(t))$, $0<t<1$.

Again, for $0\le\theta<1$ and $1\le q\le\infty$, we obtain the classical Lorentz–Karamata spaces

$$(L_{\exp}, L_\infty)_{\theta,b,L_q} = L_{\infty,q;b^{\#}},$$

where $b^{\#}(t)=\ell^{\theta-1-1/q}(t)b(1/\ell(t))$, $0<t<1$, and

$$(L_{\exp}, L_\infty)_{0,b,L_\infty} = L_{\infty,\infty;\ell(t)^{-1}\Phi_\infty},$$

where $\Phi_\infty(t)=\|b(1/\ell(s))\|_{\tilde{L}_\infty(t,1)}$, $0<t<1$.

Another classical example of an ordered couple is that formed by the operator ideals S_∞ and S_1. Let H be a Hilbert space and let S_∞ be the Banach space of all bounded linear operators acting from H into H. For $T\in S_\infty$, the *singular numbers* of T are

$$s_n(T) = \inf\{\|T-R\|_H : R\in S_\infty \text{ with rank } R<n\},$$

$n\in\mathbb{N}$. For $1\le q\le\infty$, the *Schatten ideal* S_q is formed by all those $T\in S_\infty$ having finite norm

$$\|T\|_{S_q} = \left(\sum_{n=1}^{\infty} s_n(T)^q\right)^{1/q}.$$

See [19] and [23]. Other ideals that appear in the literature as suitable end point ideals for the scale of Schatten ideals S_p are the *Macaev ideals* defined as (see [19] and [28])

$$S_w=\left\{T\in S_\infty(H):\|T\|_{S_w}=\sum_{n=1}^{\infty}\frac{s_n(T)}{n}<\infty\right\}$$

and

$$S_{\mathcal{M}}=\left\{T\in S_\infty(H):\|T\|_{S_{\mathcal{M}}}=\sup_{n\in\mathbb{N}}\left\{\ell(n)^{-1}\sum_{k=1}^{n}s_k(T)\right\}<\infty\right\}.$$

It is well known that

$$K(t,T;S_1,S_\infty)\sim\sum_{n=1}^{[t]}s_n(T),$$

where $[t]$ is the integer part of t, see [23]. Therefore

$$(S_\infty,S_1)_{0,1,L_1}=S_w \quad\text{and}\quad (S_\infty,S_1)_{1,\ell(t)^{-1},L_\infty}=S_{\mathcal{M}},$$

see [28, p. 68] and [36]. Thus, the Macaev ideals S_w and $S_{\mathcal{M}}$ play the role in the theory of ideals of the spaces $L\log L$ and $L_{\exp}$. The following corollaries are consequences of the previous results.

Corollary 7.4. *Let E be an r.i. space and let $b\in\mathrm{SV}(0,1)$. Then, for $0\le\theta<1$, we have the equality*

$$(S_\infty,S_w)_{\theta,b,E}=(S_\infty,S_1)_{0,\tilde{b},\widehat{E}},$$

where $\tilde{b}(t)=\ell^{\theta}(t)b(1/\ell(t))$, $0<t<1$. Moreover, in the limit case $\theta=1$, we have

$$(S_\infty,S_w)_{1,b,E}=(S_\infty,S_1)^{\mathcal{R}}_{0,\tilde{b},\widehat{E},1,L_1},$$

where $\tilde{b}(t)=b(1/\ell(t))$, $0<t<1$.

Corollary 7.5. *Let E be an r.i. space and let $b\in\mathrm{SV}(0,1)$. Then, for $0<\theta\le1$, we have the equality*

$$(S_{\mathcal{M}},S_1)_{\theta,b,E}=(S_\infty,S_1)_{1,\tilde{b},\widehat{E}},$$

where $\tilde{b}(t)=\ell^{\theta-1}(t)b(1/\ell(t))$, $0<t<1$. Moreover, in the limit case $\theta=0$, we have

$$(S_{\mathcal{M}},S_1)_{0,b,E}=(S_\infty,S_1)^{\mathcal{L}}_{1,\tilde{b},\widehat{E},\ell(t)^{-1},L_\infty},$$

where $\tilde{b}(t)=b(1/\ell(t))$, $0<t<1$.

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