

# Nilpotent $p$ -local finite groups

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**Abstract.** We provide characterizations of  $p$ -nilpotency for fusion systems and  $p$ -local finite groups that are inspired by known result for finite groups. In particular, we generalize criteria by Atiyah, Brunetti, Frobenius, Quillen, Stambach and Tate.

## 1. Introduction

This article is concerned with the concept of  $p$ -nilpotency. A finite group  $G$  is  $p$ -nilpotent if its Sylow  $p$ -subgroup  $S$  has a normal complement  $N$  in  $G$ , that is, the composition  $S \rightarrow G \rightarrow G/N$  is an isomorphism.

In the literature, there are characterizations of  $p$ -nilpotency using group cohomology, Quillen categories, group theory, and fusion systems. For example, the restriction map in mod- $p$  cohomology  $H^*(G; \mathbb{F}_p) \rightarrow H^*(S; \mathbb{F}_p)$  is an isomorphism if and only if  $G$  is  $p$ -nilpotent [Q]. The Frobenius  $p$ -nilpotency criterion in group theory [R, Theorem 10.3.2] characterizes these groups as those for which  $N_G(P)/C_G(P)$  is a  $p$ -group for all  $p$ -subgroups  $P$  of  $S$ .

For our purposes, an interesting characterization of  $p$ -nilpotency is given by [Gla, Theorem 8.6]:  $G$  is  $p$ -nilpotent if and only if  $S$  controls  $G$ -fusion in  $S$ , or equivalently, in the terminology of fusion systems [BLO2],  $\mathcal{F}_S(G) = \mathcal{F}_S(S)$ . Here  $\mathcal{F}_S(G)$  is the category whose objects are subgroups of  $S$  and morphisms are maps induced by conjugations in  $G$ , and  $\mathcal{F}_S(S)$  is the category with the same objects but whose morphisms are maps induced by conjugations in  $S$ . This characterization seems appropriate as a definition of  $p$ -nilpotency for fusion systems. This has already been adopted by several authors, starting with Kessar and Linckelmann in [KL2].

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The objects known as  $p$ -local finite groups were introduced by Broto, Levi, and Oliver in [BLO2] as a mean to extract the essence of the  $p$ -local structure of a group of course, but also of a block, or of more exotic occurrences such as the Solomon 2-local groups [LO]. A  $p$ -local finite group is an enrichment of a saturated fusion system, a notion introduced by Puig [P] in 1990 under the name of “full Frobenius system”. For a finite group  $G$ , this  $p$ -local structure encodes the homotopical properties of the lattice of its  $p$ -subgroups under the conjugation action of  $G$ . It is natural then that most  $p$ -local properties enjoyed by finite groups also hold for  $p$ -local finite groups.

Following this motivation and the definition for nilpotency of fusion systems in [KL2], we say that a  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  (see Section 1 for background on  $p$ -local finite groups) is *nilpotent* when  $\mathcal{F} = \mathcal{F}_S(S)$ . This definition does not mention the centric linking system  $\mathcal{L}$ , but it follows from Oliver’s solution to the Martino–Priddy conjecture in [O1] and [O2], that there is a unique isomorphism class of centric linking systems associated with the fusion system of a finite group. Chermak proved recently the stronger result that every saturated fusion system has an associated centric linking system that is unique up to isomorphism (see [Ch] and [O3]). Both proofs rely on the classification of finite simple groups. In the case of the fusion system  $\mathcal{F}_S(S)$  it is straightforward to show that it has a unique centric linking system  $\mathcal{L}_S^c(S)$  up to isomorphism, see Proposition 3.1, and hence we do not need the heavy machinery used in the general case. In particular, a  $p$ -local finite group is nilpotent if and only if its fusion system is that of the group  $S$ . Sometimes it is more convenient to check a topological condition than to stay at the level of the algebra of the fusion data, and having a centric linking system allows for topological characterizations. Nevertheless, the reader might find it convenient to forget about  $\mathcal{L}$  where it is not needed.

Note that the prime  $p$  is implicitly given when we have a fusion system or a  $p$ -local finite group, so we will simply say nilpotent instead of  $p$ -nilpotent. In fact, considering that a  $p$ -local finite group is an object created to keep track of the  $p$ -local structure only, this naming convention agrees with the well known fact that a finite group is nilpotent if and only if it is  $p$ -nilpotent for all prime numbers  $p$ . According to the previous comments, if  $G$  is a finite group with Sylow  $p$ -subgroup  $S$ , the associated  $p$ -local group is nilpotent if and only if  $G$  is  $p$ -nilpotent.

We offer a list of characterizations of nilpotency in this context, inspired by the work on  $p$ -nilpotency of groups. Apart from the criteria we have already mentioned above, we also obtain analogues of the criteria of Tate [T], Stammbach [S], Atiyah [Q], and Quillen [Q].

In the last decade the concept of  $p$ -nilpotency has attracted a lot of attention. Let us mention for example the work of Kessar and Linckelmann, who proved the  $p$ -nilpotency theorem of Glaubermann and Thompson for blocks in [KL1] and later for fusion systems [KL2].

Even more recently, a result of Tate has been extended to fusion systems by the four authors in [DGPS], providing another proof of one part of our Theorem 7.1. In [DGMP] the Frobenius nilpotency criterion, already present in [L], is used to prove Thompson's result for saturated fusion systems. This criterion is also available in this article as part of Theorem 5.8.

The work carried out in this paper is also of importance because nilpotency is used to determine sparseness and extreme sparseness of fusion systems, as defined in [Gle], as well as to define quasicentric subgroups, a concept heavily used in [BCGLO1] and [BCGLO2].

The basic definitions of saturated fusion systems and their associated centric linking systems are given in Section 2. Section 3 is dedicated to generalities about nilpotent fusion systems and  $p$ -local finite groups. Section 4 deals with homological and cohomological characterizations of nilpotency in low degrees, and some global fusion criteria are studied in Section 5. A criterion in terms of elementary abelian subgroups is described in Section 6. Section 7 is then concerned with cohomological criteria in high degrees, as well as Morava  $K$ -theory. Finally, Section 8 contains criteria obtained by using Quillen categories.

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## 2. Fusion systems and associated centric linking systems

This first section is devoted to the basic definitions and properties we will use from the beautiful theory of  $p$ -local finite groups. Our main reference is the article [BLO2] by Broto, Levi, and Oliver. The expert can skip this section.

Given two finite groups  $P$  and  $Q$ , let  $\text{Hom}(P, Q)$  denote the set of group homomorphisms from  $P$  to  $Q$ , and let  $\text{Inj}(P, Q)$  denote the set of monomorphisms. If  $P$  and  $Q$  are subgroups of a larger group  $G$ , then  $\text{Hom}_G(P, Q) \subseteq \text{Inj}(P, Q)$  denotes the subset of homomorphisms induced by conjugation by elements of  $G$ , and  $\text{Aut}_G(P)$  the group of automorphisms of  $P$  induced by conjugation in  $G$ .

*Definition 2.1.* A fusion system  $\mathcal{F}$  over a finite  $p$ -group  $S$  is a category whose objects are the subgroups of  $S$ , and whose morphism sets  $\text{Hom}_{\mathcal{F}}(P, Q)$  satisfy the following conditions:

- (a)  $\text{Hom}_S(P, Q) \subseteq \text{Hom}_{\mathcal{F}}(P, Q) \subseteq \text{Inj}(P, Q)$  for all  $P, Q \leq S$ .
- (b) Every morphism in  $\mathcal{F}$  factors as an isomorphism in  $\mathcal{F}$  followed by an inclusion.

If  $\mathcal{F}$  is a fusion system over  $S$  and  $P, Q \leq S$ , then we write  $\text{Hom}_{\mathcal{F}}(P, Q)$  for the set of morphisms in  $\mathcal{F}$  to emphasize that all morphisms in the category  $\mathcal{F}$  are homomorphisms. If  $\text{Iso}_{\mathcal{F}}(P, Q)$  denotes the subset of isomorphisms in  $\mathcal{F}$ , we see that  $\text{Iso}_{\mathcal{F}}(P, Q) = \text{Hom}_{\mathcal{F}}(P, Q)$  if  $|P| = |Q|$ , and  $\text{Iso}_{\mathcal{F}}(P, Q) = \emptyset$  otherwise. We also write  $\text{Aut}_{\mathcal{F}}(P) = \text{Iso}_{\mathcal{F}}(P, P)$  and  $\text{Out}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{F}}(P) / \text{Inn}(P)$ . Two subgroups  $P, P' \leq S$  are called  $\mathcal{F}$ -conjugate if  $\text{Iso}_{\mathcal{F}}(P, P') \neq \emptyset$ .

Here, and throughout the paper, we write  $\text{Syl}_p(G)$  for the set of Sylow  $p$ -subgroups of  $G$ . Also, for any  $P \leq G$  and any  $g \in N_G(P)$ ,  $c_g \in \text{Aut}(P)$  denotes the automorphism  $c_g(x) = gxg^{-1}$ . The fusion systems we consider in this paper will all be saturated in the following sense [BLO2, Definition 1.2].

*Definition 2.2.* Let  $\mathcal{F}$  be a fusion system over a  $p$ -group  $S$ .

- A subgroup  $P \leq S$  is *fully centralized* in  $\mathcal{F}$  if  $|C_S(P)| \geq |C_S(P')|$  for all  $P' \leq S$  that are  $\mathcal{F}$ -conjugate to  $P$ .
- A subgroup  $P \leq S$  is *fully normalized* in  $\mathcal{F}$  if  $|N_S(P)| \geq |N_S(P')|$  for all  $P' \leq S$  that are  $\mathcal{F}$ -conjugate to  $P$ .
- $\mathcal{F}$  is a *saturated fusion system* if the following two conditions hold:
  - (I) Any  $P \leq S$  which is fully normalized in  $\mathcal{F}$  is fully centralized in  $\mathcal{F}$ , and  $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(P))$ .
  - (II) If  $P \leq S$  and  $\phi \in \text{Hom}_{\mathcal{F}}(P, S)$  are such that  $\phi P$  is fully centralized, and if we set

$$N_{\phi} = \{g \in N_S(P) \mid \phi c_g \phi^{-1} \in \text{Aut}_S(\phi P)\},$$

then there is  $\bar{\phi} \in \text{Hom}_{\mathcal{F}}(N_{\phi}, S)$  such that  $\bar{\phi}|_P = \phi$ .

The motivating example for this definition is the fusion system of a finite group  $G$ . For any  $S \in \text{Syl}_p(G)$ , we let  $\mathcal{F}_S(G)$  be the fusion system over  $S$  defined by setting  $\text{Hom}_{\mathcal{F}_S(G)}(P, Q) = \text{Hom}_G(P, Q)$  for all  $P, Q \leq S$ .

**Proposition 2.3.** ([BLO2, Proposition 1.3]) *Let  $G$  be a finite group, and let  $S$  be a Sylow  $p$ -subgroup of  $G$ . Then, the fusion system  $\mathcal{F}_S(G)$  over  $S$  is saturated. Furthermore, a subgroup  $P \leq S$  is fully centralized in  $\mathcal{F}_S(G)$  if and only if  $C_S(P) \in \text{Syl}_p(C_G(P))$ , while  $P$  is fully normalized in  $\mathcal{F}_S(G)$  if and only if  $N_S(P) \in \text{Syl}_p(N_G(P))$ .*

In order to help motivate the next constructions, we recall some definitions. If  $G$  is a finite group and  $p$  is a prime, then a  $p$ -subgroup  $P \leq G$  is  $p$ -centric if  $C_G(P) = Z(P) \times C'_G(P)$ , where  $C'_G(P)$  has order prime to  $p$ . For any  $P, Q \leq G$ , let  $N_G(P, Q)$  denote the transporter, that is, the set of all  $g \in G$  such that  $gPg^{-1} \leq Q$ . For any  $S \in \text{Syl}_p(G)$ ,  $\mathcal{L}_S^c(G)$  denotes the category whose objects are the  $p$ -centric subgroups of  $S$ , and where  $\text{Mor}_{\mathcal{L}_S^c(G)}(P, Q) = N_G(P, Q)/C'_G(P)$ . By comparison,  $\text{Hom}_G(P, Q) \cong N_G(P, Q)/C_G(P)$ . Hence there is a functor from  $\mathcal{L}_S^c(G)$  to  $\mathcal{F}_S(G)$  which is the inclusion on objects, and which sends the morphism corresponding to  $g \in N_G(P, Q)$  to  $c_g \in \text{Hom}_G(P, Q)$ .

*Definition 2.4.* Let  $\mathcal{F}$  be any fusion system over a  $p$ -group  $S$ . A subgroup  $P \leq S$  is  $\mathcal{F}$ -centric if  $P$  and all of its  $\mathcal{F}$ -conjugates contain their  $S$ -centralizers. We denote by  $\mathcal{F}^c$  the full subcategory of  $\mathcal{F}$  whose objects are the  $\mathcal{F}$ -centric subgroups of  $S$ .

*Definition 2.5.* Let  $\mathcal{F}$  be any fusion system over a finite  $p$ -group  $S$ . A subgroup  $P \leq S$  is  $\mathcal{F}$ -radical if  $\text{Out}_{\mathcal{F}}(P)$  is  $p$ -reduced, that is, if the maximal normal  $p$ -subgroup of  $\text{Out}_{\mathcal{F}}(P)$  is  $\{1\}$ .

**Theorem 2.6.** (Alperin’s fusion theorem [BLO2, Theorem A.10]) *Let  $\mathcal{F}$  be a saturated fusion system over a finite  $p$ -group  $S$ . Then for each  $\phi \in \text{Iso}_{\mathcal{F}}(P, P')$ , there exist sequences of subgroups of  $S$ ,*

$$P = P_0, P_1, \dots, P_k = P' \quad \text{and} \quad Q_1, Q_2, \dots, Q_k,$$

and elements  $\phi_i \in \text{Aut}_{\mathcal{F}}(Q_i)$ , such that

- (a)  $Q_i$  is fully normalized in  $\mathcal{F}$ ,  $\mathcal{F}$ -radical, and  $\mathcal{F}$ -centric for each  $i$ ;
- (b)  $P_{i-1}, P_i \leq Q_i$  and  $\phi_i(P_{i-1}) = P_i$  for each  $i$ ;
- (c)  $\phi = \phi_k \circ \phi_{k-1} \circ \dots \circ \phi_1$ .

Recall the following definition from [BLO2, Definition A.3].

*Definition 2.7.* The normalizer fusion system  $N_{\mathcal{F}}(Q)$  of a fully normalized subgroup  $Q \leq S$  in  $\mathcal{F}$  is the fusion system defined over  $N_S(Q)$  whose morphisms are given by

$$\begin{aligned} \text{Hom}_{N_{\mathcal{F}}(Q)}(P, P') &= \{ \phi \in \text{Hom}_{\mathcal{F}}(P, P') \mid \text{there is } \psi \in \text{Hom}_{\mathcal{F}}(PQ, P'Q) \\ &\quad \text{with } \psi|_P = \phi \text{ and } \psi(Q) \leq Q \}. \end{aligned}$$

Proposition A.6 in [BLO2] shows that this fusion system is saturated. Moreover, it has an associated centric linking system, see [BLO2, Definition 6.1]. When a fusion system coincides with the normalizer fusion system of a normal subgroup, one can define a quotient fusion system. We recall the definition from [L, Definition 6.1].

*Definition 2.8.* Let  $\mathcal{F}$  be a fusion system over a finite  $p$ -group  $S$  such that  $\mathcal{F} = N_{\mathcal{F}}(Q)$  for some normal subgroup  $Q$  of  $S$ . We define the category  $\mathcal{F}/Q$  as follows. Its objects are subgroups of  $S/Q$  and for any two subgroups  $H$  and  $K$  of  $S$  containing  $Q$ , a group homomorphism  $\psi: H/Q \rightarrow K/Q$  is a morphism in the category  $\mathcal{F}/Q$  if there exists a morphism  $\phi: H \rightarrow K$  in  $\mathcal{F}$  satisfying  $\phi(u)Q = \psi(uQ)$  for all  $u$  in  $H$ .

**Theorem 2.9.** ([L, Theorem 6.2]) *Let  $\mathcal{F}$  be a fusion system over a finite  $p$ -group  $S$  such that  $\mathcal{F} = N_{\mathcal{F}}(Q)$  for some normal subgroup  $Q$  of  $S$ . Then the category  $\mathcal{F}/Q$  is a saturated fusion system over  $S/Q$ .*

*Definition 2.10.* Let  $\mathcal{F}$  be a fusion system over the  $p$ -group  $S$ . A *centric linking system associated with  $\mathcal{F}$*  is a category  $\mathcal{L}$  whose objects are the  $\mathcal{F}$ -centric subgroups of  $S$ , together with a functor

$$\pi: \mathcal{L} \longrightarrow \mathcal{F}^c$$

and “distinguished” monomorphisms  $\delta_P: P \rightarrow \text{Aut}_{\mathcal{L}}(P)$  for each  $\mathcal{F}$ -centric subgroup  $P \leq S$ , which satisfy the following conditions:

(A)  $\pi$  is the identity on objects and is surjective on morphisms. More precisely, for each pair of objects  $P, Q \in \mathcal{L}$ ,  $Z(P)$  acts freely on  $\text{Mor}_{\mathcal{L}}(P, Q)$  by composition (upon identifying  $Z(P)$  with  $\delta_P(Z(P)) \leq \text{Aut}_{\mathcal{L}}(P)$ ), and  $\pi$  induces a bijection

$$\text{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \text{Hom}_{\mathcal{F}}(P, Q);$$

(B) For each  $\mathcal{F}$ -centric subgroup  $P \leq S$  and each  $g \in P$ ,  $\pi$  sends  $\delta_P(g) \in \text{Aut}_{\mathcal{L}}(P)$  to  $c_g \in \text{Aut}_{\mathcal{F}}(P)$ ;

(C) For each  $f \in \text{Mor}_{\mathcal{L}}(P, Q)$  and each  $g \in P$ , the following square commutes in  $\mathcal{L}$ :

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \delta_P(g) \downarrow & & \downarrow \delta_Q(\pi(f)(g)) \\ P & \xrightarrow{f} & Q. \end{array}$$

One easily checks that for any finite group  $G$  and any  $S \in \text{Syl}_p(G)$ ,  $\mathcal{L}_S^c(G)$  is a centric linking system associated with the fusion system  $\mathcal{F}_S(G)$ . We are now ready to give the definition of a  $p$ -local finite group, as in [BLO2, Definition 1.8].

*Definition 2.11.* A  *$p$ -local finite group* is a triple  $(S, \mathcal{F}, \mathcal{L})$ , where  $\mathcal{F}$  is a saturated fusion system over the  $p$ -group  $S$  and  $\mathcal{L}$  is a centric linking system associated with  $\mathcal{F}$ . The classifying space of the  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  is the space  $|\mathcal{L}|_p^\wedge$ .

Thus, for any finite group  $G$  and any  $S \in \text{Syl}_p(G)$ , the triple  $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$  is a  $p$ -local finite group. Its classifying space  $|\mathcal{L}_S^c(P)|_p^\wedge$  is homotopy equivalent to  $BG'_p^\wedge$  by [BLO1]. In general, the fact that  $S$  itself is an object of the centric linking system yields a “canonical inclusion”  $Bi: BS \rightarrow |\mathcal{L}|$  of the Sylow  $p$ -subgroup  $S$  into the  $p$ -local finite group, which allows us to compare their mod- $p$  cohomologies. We will denote by  $Bi_p^\wedge$  the composition of  $Bi$  with the  $p$ -completion map  $|\mathcal{L}| \rightarrow |\mathcal{L}|_p^\wedge$ .

### 3. Nilpotent fusion systems and $p$ -local finite groups

In this short section we give the definition of nilpotency in terms of fusion data and obtain different global characterizations, at the level of linking systems and their mod- $p$  cohomology. We start by giving an elementary proof of the fact that the fusion system of a finite  $p$ -group has a unique associated centric linking system (up to isomorphism). This also follows from [BCGLO1, Proposition 4.2], since a nilpotent fusion system is constrained, or from the much stronger [Ch] and [O3].

**Proposition 3.1.** *If  $S$  is a finite  $p$ -group, the fusion system  $\mathcal{F}_S(S)$  has a unique associated centric linking system up to isomorphism of categories.*

*Proof.* Note that  $\mathcal{L}_S^c(S)$  is a centric linking system associated with  $\mathcal{F}_S(S)$ . If  $\mathcal{L}$  is another centric linking system associated with  $\mathcal{F}_S(S)$ , consider the functor  $\delta: \mathcal{L}_S^c(S) \rightarrow \mathcal{L}$  constructed in [BLO2, Proposition 1.11]. It is clear that this functor is an equivalence of categories.  $\square$

Following Kessar and Linckelmann’s definition for fusion systems in [KL2] and the characterization of  $p$ -nilpotency for groups in terms of fusion stated in the introduction, we define a  $p$ -local finite group to be nilpotent when its fusion system is that of a  $p$ -group.

*Definition 3.2.* A  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  is *nilpotent* if  $\mathcal{F} = \mathcal{F}_S(S)$ .

Even though this definition seems to be stated at the level of fusion systems only, it says that the  $p$ -local finite group is the one induced by a Sylow  $p$ -subgroup. As it is sometimes more convenient to check a topological condition (using the centric linking system) than to stay at the purely algebraic level of the fusion data, we propose the following characterizations.

**Proposition 3.3.** *A  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if one of the following equivalent conditions is satisfied:*

- (1)  $|\mathcal{L}|_p^\wedge \simeq BS$ ;
- (2) the canonical inclusion induces an isomorphism  $H^*(|\mathcal{L}|; \mathbb{F}_p) \rightarrow H^*(BS; \mathbb{F}_p)$ ;
- (3)  $Bi_p^\wedge$  has a retraction  $r: |\mathcal{L}|_p^\wedge \rightarrow BS$ .

*Proof.* The first condition is equivalent to the nilpotency of the  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  by [BLO2, Theorem 7.4].

Since homotopy equivalences of  $p$ -completed spaces are  $\mathbb{F}_p$ -homology isomorphisms, and the groups  $H^n(|\mathcal{L}|; \mathbb{F}_p) \cong H_n(|\mathcal{L}|; \mathbb{F}_p)$  are finitely generated  $\mathbb{F}_p$ -modules by [BLO2, Proposition 5.2 and Theorem 5.8], it follows from [BS, Theorem 2.48] that conditions (1) and (2) are equivalent.

If  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent,  $\mathcal{F} = \mathcal{F}_S(S)$  and we have seen in the proof of Proposition 3.1 that  $\mathcal{L}_S^c(S)$  is the associated centric linking system. Thus  $Bi_p^\wedge: BS \rightarrow |\mathcal{L}|_p^\wedge$  is an equivalence. On the other hand, the existence of a retraction of  $Bi_p^\wedge$  implies that the cohomology of the Sylow  $p$ -subgroup is a retract of the cohomology of  $|\mathcal{L}|_p^\wedge$ . But the mod- $p$  cohomology of  $|\mathcal{L}|$  can be computed by stable elements by [BLO2, Theorem 5.8], and moreover it is a subalgebra of  $H^*(S; \mathbb{F}_p)$ . Therefore  $Bi_p^\wedge$  must be a homotopy equivalence.  $\square$

*Remark 3.4.* The result [BLO2, Proposition 5.5] on which the proof of the stable element formula relies is stronger than the use we make of it in the above proof. It actually tells us that the suspension spectrum of  $|\mathcal{L}|_p^\wedge$  appears as a stable summand in the suspension spectrum  $\Sigma^\infty BS$ . In particular  $Bi_p^\wedge$  always induces an epimorphism in homology and a monomorphism in cohomology with arbitrary constant coefficients (constant functors).

#### 4. Cohomological characterizations in low degrees

We show in this section that a nilpotent  $p$ -local finite group can be recognized by looking at (co)homological information in low degree. For finite groups the criteria we obtain are those of Tate [T], Stammbach [S], and some variations. The following result, which establishes an extension property for certain maps  $BS \rightarrow X$ , will be used in combination with Proposition 3.3 to prove these criteria.

**Proposition 4.1.** *Let  $S$  be a  $p$ -group and  $f: BS \rightarrow X$  be a map from the classifying space of  $S$  to a space  $X$ . Assume that  $f$  induces an epimorphism  $H^1(X; \mathbb{F}_p) \twoheadrightarrow H^1(BS; \mathbb{F}_p)$  and a monomorphism  $H^2(X; \mathbb{F}_p) \hookrightarrow H^2(BS; \mathbb{F}_p)$ . Then any map  $BS \rightarrow BP$  to the classifying space of a  $p$ -group  $P$  factors through  $f$  up to homotopy. In particular  $BS$  is a retract of  $X$ .*



*Proof.* We proceed by induction on the order of  $P$ . If  $P \cong C_p$ , the cyclic group of order  $p$ , the conclusion is a direct consequence of the surjectivity of  $H^1(f; \mathbb{F}_p)$ . Let us thus consider a  $p$ -group  $P$  of order  $p^n$  with  $n \geq 2$  and a group homomorphism  $\phi: S \rightarrow P$ .

The center of a  $p$ -group is never trivial. Therefore there is a central extension  $C_p \hookrightarrow P \xrightarrow{\pi} Q$  and by induction there exists a map  $g: X \rightarrow BQ$  such that the following square commutes up to homotopy

$$\begin{array}{ccc} BS & \xrightarrow{B\phi} & BP \\ f \downarrow & & \downarrow B\pi \\ X & \xrightarrow{g} & BQ. \end{array}$$

Now we have to lift  $g$  to a map  $h: X \rightarrow BP$ . The central extension gives rise to a fibration  $BP \rightarrow BQ \xrightarrow{k} K(\mathbb{Z}/p, 2)$ . The composite map  $k \circ B\pi \circ B\phi \simeq k \circ g \circ f$  is null-homotopic, and hence so is  $k \circ g: X \rightarrow K(\mathbb{Z}/p, 2)$  because of the injectivity of  $H^2(f; \mathbb{F}_p)$ . Therefore there exists a map  $h: X \rightarrow BP$  such that  $B\pi \circ h \simeq g$ .

In general,  $h \circ f \neq B\phi$  because they can differ by the action of the central  $C_p$ . This means, [BK, IX.4.1], that there exists a map  $\alpha: BS \rightarrow BC_p$  such that the composite map

$$BS \xrightarrow{\Delta} BS \times BS \xrightarrow{(h \circ f) \times \alpha} BP \times BC_p \longrightarrow BP$$

is homotopic to  $B\phi$ . By the induction hypothesis, there exists a map  $\beta: X \rightarrow BC_p$  such that  $\beta \circ f \simeq \alpha$ . The composite map

$$X \xrightarrow{\Delta} X \times X \xrightarrow{h \times \beta} BP \times BC_p \longrightarrow BP$$

is the map we seek.  $\square$

We emphasize that the factorization through  $f$  in the statement of the previous proposition is not unique. If it were so, the classifying space of the  $p$ -group  $P$  would be local with respect to  $f$  (in the sense of [F]). This property is the key ingredient in the proof of the  $p$ -local version of Stammbach’s criterion [S] about the second (co)homology group and the Huppert–Thompson–Tate criterion [T] about the first one. Our proof goes along the same lines as in Stammbach’s note.

**Theorem 4.2.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and let  $Bi: BS \rightarrow |\mathcal{L}|$  be the standard inclusion. Then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if one of the following equivalent four conditions is satisfied:*

- (1)  $Bi^* : H^1(|\mathcal{L}|; \mathbb{F}_p) \rightarrow H^1(BS; \mathbb{F}_p)$  is an isomorphism;
- (2)  $Bi^* : H^2(|\mathcal{L}|; \mathbb{F}_p) \rightarrow H^2(BS; \mathbb{F}_p)$  is an isomorphism;
- (3)  $Bi_* : H_1(BS; \mathbb{F}_p) \rightarrow H_1(|\mathcal{L}|; \mathbb{F}_p)$  is an isomorphism;
- (4)  $Bi_* : H_2(BS; \mathbb{F}_p) \rightarrow H_2(|\mathcal{L}|; \mathbb{F}_p)$  is an isomorphism.

*Proof.* The universal coefficient formula clearly implies the equivalence of the homological conditions and the cohomological ones. We follow Stambach’s strategy [S] and assume that (4) holds. Consider the universal coefficients exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_2(S; \mathbb{Z}) \otimes \mathbb{Z}/p & \longrightarrow & H_2(S; \mathbb{F}_p) & \longrightarrow & \text{Tor}(H_1(S; \mathbb{Z}); \mathbb{Z}/p) \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \cong & & \downarrow \beta \\
 0 & \longrightarrow & H_2(|\mathcal{L}|_p^\wedge; \mathbb{Z}) \otimes \mathbb{Z}/p & \longrightarrow & H_2(|\mathcal{L}|_p^\wedge; \mathbb{F}_p) & \longrightarrow & \text{Tor}(H_1(|\mathcal{L}|_p^\wedge; \mathbb{Z}); \mathbb{Z}/p) \longrightarrow 0.
 \end{array}$$

We see that  $\alpha$  must be a monomorphism and  $\beta$  an epimorphism. By Remark 3.4 we know that  $\alpha$  is an epimorphism, so that  $\beta$  is actually an isomorphism.

Now the short exact sequence  $K = \text{Ker}(H_1(Bi; \mathbb{Z})) \hookrightarrow H_1(S; \mathbb{Z}) \twoheadrightarrow H_1(|\mathcal{L}|_p^\wedge; \mathbb{Z})$  induces an exact sequence

$$0 \longrightarrow \text{Tor}(K; \mathbb{Z}/p) \longrightarrow \text{Tor}(H_1(S; \mathbb{Z}); \mathbb{Z}/p) \xrightarrow{\beta} \text{Tor}(H_1(|\mathcal{L}|_p^\wedge; \mathbb{Z}); \mathbb{Z}/p),$$

so that  $\text{Tor}(K; \mathbb{Z}/p) = 0$ . Since the homology of  $S$  is  $p$ -torsion,  $K$  must be trivial. Therefore  $H_1(Bi; \mathbb{Z})$  is an isomorphism, and in particular (3) holds.

It remains to prove that condition (3)—or equivalently condition (1)—implies that  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent. We assume therefore that  $H^1(Bi; \mathbb{F}_p)$  is an isomorphism. Since  $H^2(Bi; \mathbb{F}_p)$  is always a monomorphism, Proposition 4.1 applies and we deduce that  $BS$  is a retract of  $|\mathcal{L}|_p^\wedge$ . This means by Proposition 3.3 that the  $p$ -local finite group is nilpotent.  $\square$

Using the universal coefficients theorems for the  $\mathbb{Z}_p^\wedge$ -module  $\mathbb{F}_p$ , one can replace the  $\mathbb{F}_p$  coefficients by the  $p$ -adic integers.

**Corollary 4.3.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and let  $Bi : BS \rightarrow |\mathcal{L}|$  be the standard inclusion. Then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if one of the following two equivalent conditions is satisfied:*

- (1)  $Bi_* : H_1(BS; \mathbb{Z}_p^\wedge) \rightarrow H_1(|\mathcal{L}|; \mathbb{Z}_p^\wedge)$  is an isomorphism;
- (2)  $Bi^* : H^2(|\mathcal{L}|; \mathbb{Z}_p^\wedge) \rightarrow H^2(BS; \mathbb{Z}_p^\wedge)$  is an isomorphism.

### 5. Global fusion criteria

This section is devoted to criteria which allow us to recognize a nilpotent  $p$ -local finite group from its global fusion characteristics. We first need some facts about the fundamental group of a  $p$ -local finite group. The following definition is due to Puig [P]. The notation  $O^p(-)$  stands for the subgroup generated by elements of order prime to  $p$ .

*Definition 5.1.* Let  $\mathcal{F}$  be any saturated fusion system over a  $p$ -group  $S$ . The normal subgroup  $\langle g^{-1}\alpha(g) \mid g \in P \leq S \text{ and } \alpha \in O^p(\text{Aut}_{\mathcal{F}}(P)) \rangle$  is called the *hyperfocal subgroup of  $\mathcal{F}$*  and is denoted by  $\text{Hyp}(\mathcal{F})$ . The *focal subgroup*  $\text{Foc}(\mathcal{F})$  is the normal subgroup  $\langle g^{-1}\alpha(g) \mid g \in P \leq S \text{ and } \alpha \in \text{Aut}_{\mathcal{F}}(P) \rangle$ .

**Theorem 5.2.** ([BCGLO2, Theorem B]) *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group. Then  $\pi_1(|\mathcal{L}|_p^\wedge) \cong S/\text{Hyp}(\mathcal{F})$ .*

We have seen in the previous section that the group  $H_1(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge) \cong \pi_1(|\mathcal{L}|_p^\wedge)_{ab}$  plays an important role in establishing the nilpotency of a given  $p$ -local finite group. The computation in the next proposition (compare with [Go, Theorem 7.3.4]) explains the relation between the focal and the hyperfocal subgroups.

**Proposition 5.3.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group. Then  $H_1(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge) \cong S/\text{Foc}(\mathcal{F})$ .*

*Proof.* The group  $\text{Foc}(\mathcal{F})$  is generated by  $\text{Hyp}(\mathcal{F})$  and elements  $g^{-1}\alpha(g)$  where  $\alpha \in \text{Aut}_{\mathcal{F}}(P) \setminus O^p(\text{Aut}_{\mathcal{F}}(P))$  for some  $P \leq S$ . By Alperin’s fusion theorem, Theorem 2.6, it is enough to consider those subgroups  $P$  that are fully normalized in  $\mathcal{F}$ .

Assume then that  $P$  is fully normalized in  $\mathcal{F}$ . Since  $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(P))$ , any elements of  $p$ -power order in  $\text{Aut}_{\mathcal{F}}(P)$  can be conjugated via an element of  $O^p(\text{Aut}_{\mathcal{F}}(P))$  to an element of  $\text{Aut}_S(P)$ . Therefore  $\text{Foc}(\mathcal{F})$  is generated by  $\text{Hyp}(\mathcal{F})$  and commutators in  $[S, S]$ . By Theorem 5.2, we have that  $\pi_1(|\mathcal{L}|_p^\wedge) \cong S/\text{Hyp}(\mathcal{F})$ , so its abelianization is  $S/\text{Foc}(\mathcal{F})$ .  $\square$

Recall the following definitions from [BLO2, Definition A.3].

*Definition 5.4.* The *normalizer fusion system*  $N_{\mathcal{F}}(Q)$  of a fully normalized subgroup  $Q \leq S$  in  $\mathcal{F}$  is the fusion system defined over  $N_S(Q)$  whose morphisms are given by

$$\begin{aligned} \text{Hom}_{N_{\mathcal{F}}(Q)}(P, P') &= \{\phi \in \text{Hom}_{\mathcal{F}}(P, P') \mid \text{there is } \psi \in \text{Hom}_{\mathcal{F}}(PQ, P'Q) \\ &\quad \text{with } \psi|_P = \phi \text{ and } \psi(Q) \leq Q\}. \end{aligned}$$

The *centralizer fusion system*  $C_{\mathcal{F}}(Q)$  of a fully centralized subgroup  $Q \leq S$  in  $\mathcal{F}$  is the fusion system defined over  $C_S(Q)$  whose morphisms are given by

$$\begin{aligned} \text{Hom}_{C_{\mathcal{F}}(Q)}(P, P') &= \{\phi \in \text{Hom}_{\mathcal{F}}(P, P') \mid \text{there is } \psi \in \text{Hom}_{\mathcal{F}}(PQ, P'Q) \\ &\quad \text{with } \psi|_P = \phi \text{ and } \psi|_Q = \text{id}\}. \end{aligned}$$

Proposition A.6 in [BLO2] shows that these fusion systems are saturated. Moreover, they have associated centric linking systems, see [BLO2, Definition 2.4] and [BLO2, Definition 6.1].

*Definition 5.5.* ([BCGLO1, Definition 1.5]) A subgroup of  $S$  is  $\mathcal{F}$ -weakly closed if no other subgroup is  $\mathcal{F}$ -conjugate to it.

**Lemma 5.6.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and let  $V \leq Z(S)$  be an  $\mathcal{F}$ -weakly closed subgroup of  $S$ . Then  $V$  is normal in  $(S, \mathcal{F}, \mathcal{L})$ , that is,  $(S, \mathcal{F}, \mathcal{L}) = (N_S(V), N_{\mathcal{F}}(V), N_{\mathcal{L}}(V))$ .*

*Proof.* By Alperin's fusion theorem for saturated fusion systems, Theorem 2.6, every morphism in  $\mathcal{F}$  is the composition of automorphisms of  $\mathcal{F}$ -centric subgroups. More precisely, every morphism  $\alpha \in \text{Mor}_{\mathcal{F}}(P, Q)$  is the composition of an isomorphism  $\tilde{\alpha} \in \text{Mor}_{\mathcal{F}}(P, \alpha(P))$  with an inclusion  $\alpha(P) \leq Q$ . Then  $\tilde{\alpha} = \phi_k \circ \phi_{k-1} \circ \dots \circ \phi_1$  for some  $\phi_i \in \text{Aut}_{\mathcal{F}}(P_i)$  with  $P_i$  being an  $\mathcal{F}$ -centric subgroup of  $S$  for  $1 \leq i \leq k$ . Since  $V$  is central in  $S$ , we must have  $V \leq P_i$ , and  $\phi_i(V) \leq V$  because  $V$  is weakly closed in  $(S, \mathcal{F}, \mathcal{L})$ . Thus  $\tilde{\alpha}$  can be extended to a morphism from  $VP$  to  $V\alpha(P)$  and therefore  $\alpha$  can be extended to a morphism from  $VP$  to  $VQ$ .  $\square$

A straightforward consequence is the following  $p$ -local version of Grün's theorem [Go, Theorem 7.5.2].

**Proposition 5.7.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and let  $A \leq Z(S)$  be weakly closed in  $\mathcal{F}$ . Then  $\text{Foc}(\mathcal{F}) = \text{Foc}(N_{\mathcal{F}}(A))$ .*

*Proof.* By Lemma 5.6,  $A$  is normal in  $\mathcal{F}$ , so the fusion systems  $\mathcal{F}$  and  $N_{\mathcal{F}}(A)$  actually coincide.  $\square$

We are now ready to give our “global fusion criteria”. At the level of groups, the first criterion is due to Huppert and is proven in [H, Satz IV.4.9] by means of

the abelian transfer. We propose also a variation. The third one is called Frobenius  $p$ -nilpotency criterion in the literature, see for example [R, Theorem 10.3.2], and the fourth criterion is a stronger form.

**Theorem 5.8.** *A  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if one of the following four equivalent conditions is satisfied:*

- (1) *two elements  $a, b \in S$  are  $\mathcal{F}$ -conjugate if and only if they are  $S$ -conjugate;*
- (2) *two  $n$ -tuples of commuting elements of  $S$  are  $\mathcal{F}$ -conjugate if and only if they are  $S$ -conjugate,  $n > 1$ ;*
- (3) *for every subgroup  $P \leq S$ ,  $\text{Aut}_{\mathcal{F}}(P)$  is a  $p$ -group;*
- (4) *for every  $\mathcal{F}$ -centric subgroup  $P \leq S$ ,  $\text{Aut}_{\mathcal{F}}(P)$  is a  $p$ -group.*

*Proof.* It is clear that all four conditions hold for a nilpotent  $p$ -local finite group. Let us thus assume that condition (2) holds. In particular for any two elements  $a$  and  $b$  in  $S$  which are  $\mathcal{F}$ -conjugate, the  $n$ -tuples  $(a, \dots, a)$  and  $(b, \dots, b)$  are also  $\mathcal{F}$ -conjugate. Therefore they must be  $S$ -conjugate, which proves that  $a$  and  $b$  are  $S$ -conjugate. This shows that (2) implies (1).

Assume now that condition (1) holds, so that the image of any element under an  $\mathcal{F}$ -automorphism in  $S$  is a conjugate of that element by some element in  $S$ , that is, both the focal and hyperfocal subgroups are contained in the commutator subgroup  $[S, S]$ . Therefore, according to Proposition 5.3, the identity of  $S/[S, S]$  factors through the map  $H_1(Bi; \mathbb{Z}_p^\wedge): S/[S, S] \rightarrow H_1(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge) \cong S/\text{Foc}(\mathcal{F})$ . We have seen in Remark 3.4 that this map is always an epimorphism. It is hence an isomorphism and we have proven that the  $p$ -adic characterization (1) in Corollary 4.3 holds.

It is obvious that (3) implies (4). Finally, if condition (4) holds, and  $P \leq S$  is an  $\mathcal{F}$ -centric subgroup of  $S$ , then  $P \leq N_S(P)$  and so  $\text{Inn}(P) \leq \text{Aut}_S(P)$ . In particular,  $\text{Out}_S(P)$  is a non-trivial subgroup of  $\text{Out}_{\mathcal{F}}(P)$  and so  $\text{Out}_{\mathcal{F}}(P)$  is a non-trivial  $p$ -group, hence  $P$  is not  $\mathcal{F}$ -radical. Therefore the only  $\mathcal{F}$ -radical  $\mathcal{F}$ -centric subgroup is  $S$ . By Alperin’s fusion theorem, Theorem 2.6, every morphism in  $\mathcal{F}$  is the restriction of an  $\mathcal{F}$ -automorphism of  $S$ . On the other hand, since  $S$  is fully normalized in  $\mathcal{F}$ ,  $\text{Inn}(S) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(S))$ , which implies  $\text{Aut}_{\mathcal{F}}(S) = \text{Inn}(S)$ . Hence every morphism in  $\mathcal{F}$  is induced by  $S$ -conjugation, that is,  $\mathcal{F} = \mathcal{F}_S(S)$ .  $\square$

The following corollary is a very simple criterion to determine nilpotency of  $p$ -local finite groups with an abelian Sylow  $p$ -subgroup.

**Corollary 5.9.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group with  $S$  abelian. Then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if one of the following two equivalent conditions is satisfied:*

- (1) two elements  $a, b \in S$  are  $\mathcal{F}$ -conjugate if and only if they are equal;
- (2)  $\text{Aut}_{\mathcal{F}}(S)$  is the trivial group.

Theorem 5.8 allows us to obtain for example a  $p$ -local version of Huppert's result [H, Satz III.12.1].

**Proposition 5.10.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and  $n$  be a positive integer such that  $p^n > 2$ . Suppose that every element of order  $p^n$  in  $S$  is central in  $\mathcal{F}$ . Then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent.*

*Proof.* We use the Frobenius-type characterization (4) in Theorem 5.8 of nilpotent  $p$ -local finite groups. Let  $K$  be the subgroup generated by all the elements of order  $p^n$  in  $S$ . Then  $K$  is maximal with respect to being an abelian subgroup and having exponent  $p^n$ . Let  $P \leq S$  be an  $\mathcal{F}$ -centric subgroup, and let  $\alpha \in \text{Aut}_{\mathcal{F}}(P)$ . As every element of order  $p^n > 2$  is central in  $\mathcal{F}$ ,  $\alpha$  acts trivially on every element of order  $p^n$  in  $P$ . As  $K \leq P$ , it acts trivially on  $K$  and, by [Bl],  $\alpha$  is an element of order  $p$ , i.e.,  $\text{Aut}_{\mathcal{F}}(P)$  is a  $p$ -group.  $\square$

## 6. Quillen's first criterion

The criterion we offer in this section is an extension of Quillen's [Q, Theorem 1.5]. The following lemma was originally [Q, Proposition 4.1], in which Quillen only considers elementary abelian subgroups of the Sylow  $p$ -subgroup. Here the exponent is arbitrary.

**Lemma 6.1.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and let  $A$  be a subgroup of  $S$  maximal subject to being normal abelian and of exponent  $p^n > 2$ . Then  $A$  is also maximal subject to being abelian and of exponent  $p^n$  in  $(S, \mathcal{F}, \mathcal{L})$ , that is, any  $\mathcal{F}$ -conjugate of  $A$  is maximal subject to being abelian and of exponent  $p^n$  in  $S$ .*

*Proof.* Assume that there exist abelian subgroups  $A' \leq W' \leq S$  of exponent  $p^n > 2$  such that  $A$  and  $A'$  are  $\mathcal{F}$ -conjugate by the morphism  $\varphi: A' \rightarrow A$ . Since  $W' \leq C_S(A') \leq N_{\varphi}$ , the subgroup  $W'$  is  $\mathcal{F}$ -conjugate to another abelian subgroup  $W$  that contains  $A$ . But according to [A] (see also [H, Satz III.12.1]),  $A$  is maximal in  $S$  subject to being abelian and of exponent  $p^n$ . Hence  $W = A$  and  $W' = A'$ , and the result follows.  $\square$

The following result provides a way to show that certain subgroups are central in a  $p$ -local finite group.

**Proposition 6.2.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and let  $V \leq Z(S)$  be an  $\mathcal{F}$ -weakly closed subgroup such that  $\text{Aut}_{\mathcal{F}}(V)$  is a  $p$ -group. Then  $V$  is central in  $(S, \mathcal{F}, \mathcal{L})$ , that is,  $(S, \mathcal{F}, \mathcal{L}) = (C_S(V), C_{\mathcal{F}}(V), C_{\mathcal{L}}(V))$ .*

*Proof.* By Lemma 5.6 we know that  $(S, \mathcal{F}, \mathcal{L}) = (S, N_{\mathcal{F}}(V), N_{\mathcal{L}}(V))$ . Note that  $V$  is  $\mathcal{F}$ -weakly closed, and so it is fully normalized and fully centralized. Therefore  $\text{Aut}_S(V) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(V))$ . Since  $\text{Aut}_{\mathcal{F}}(V)$  is a  $p$ -group, it must be  $\text{Aut}_S(V) = N_S(V)/C_S(V) = S/S = \{1\}$ . We will show that the centralizer and normalizer  $p$ -local finite groups  $(S, C_{\mathcal{F}}(V), C_{\mathcal{L}}(V))$  and  $(S, N_{\mathcal{F}}(V), N_{\mathcal{L}}(V))$  are equal.

Clearly  $\text{Hom}_{C_{\mathcal{F}}(V)}(P, Q) \subseteq \text{Hom}_{N_{\mathcal{F}}(V)}(P, Q)$  for all  $P, Q \leq S$ . Let  $\phi: P \rightarrow Q$  be a morphism in  $N_{\mathcal{F}}(V)$ , then there exists  $\psi: PV \rightarrow QV$  in  $\mathcal{F}$  such that  $\psi|_P = \phi$  and  $\psi(V) \leq V$ . Since  $\psi|_V$  is a morphism in  $\mathcal{F}$ , it must be the identity, and hence  $\phi$  is a morphism in  $C_{\mathcal{F}}(V)$ . Now, if  $P$  and  $Q$  are  $\mathcal{F}$ -centric, then the morphisms for  $C_{\mathcal{L}}(V)$  and  $N_{\mathcal{L}}(V)$  as defined in [BLO2, Definitions 2.4 and 6.1] are given respectively by

$$\begin{aligned} \text{Mor}_{C_{\mathcal{L}}(V)}(P, Q) &= \pi^{-1}(\text{Hom}_{C_{\mathcal{F}}(V)}(P, Q)), \\ \text{Mor}_{N_{\mathcal{L}}(V)}(P, Q) &= \pi^{-1}(\text{Hom}_{N_{\mathcal{F}}(V)}(P, Q)), \end{aligned}$$

where  $\pi: \mathcal{L} \rightarrow \mathcal{F}^c$  is the projection functor from Definition 2.10. This shows that  $C_{\mathcal{L}}(V) = N_{\mathcal{L}}(V)$ .  $\square$

Before stating Quillen’s characterization, we start with a special case.

**Lemma 6.3.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group at an odd prime  $p$ . Assume that there exists a maximal elementary abelian normal subgroup  $V \triangleleft S$  which is  $\mathcal{F}$ -normal and such that  $\text{Aut}_{\mathcal{F}}(V)$  is a  $p$ -group. Then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent.*

*Proof.* We show that for every  $P \leq S$ ,  $\text{Aut}_{\mathcal{F}}(P)$  is a  $p$ -group, and thereby verifying condition (3) in Theorem 5.8. Let  $\alpha \in \text{Aut}_{\mathcal{F}}(P)$  be an element of order  $q$  prime to  $p$ . Then it is the restriction of some element  $\tilde{\alpha} \in \text{Aut}_{\mathcal{F}}(PV)$ . Let the order of  $\tilde{\alpha}$  be  $q^n p^k t$  where  $n > 0$  and  $t$  is prime to  $q$  and  $p$ . In particular,  $r = p^k t$  is prime to  $q$  and so there is a positive integer  $l$  prime to  $q$  such that  $lr$  is congruent to 1 mod  $q$ . The order of  $\tilde{\alpha}^{lr}$  divides  $a$ , in particular it is prime to  $p$ , and the restriction to  $P$  is  $\alpha$  again. Therefore, we may assume that  $\tilde{\alpha}$  has order prime to  $p$ .

Now,  $\tilde{\alpha}|_V$  is an element in the  $p$ -group  $\text{Aut}_{\mathcal{F}}(V)$ , which must be the identity. As  $p > 2$ , Lemma 6.1 applies and so  $V$  is a maximal elementary abelian subgroup of  $S$  and hence of  $PV$ . We deduce from [Bl] that the order of  $\tilde{\alpha}$  is a power of  $p$ . Hence  $\tilde{\alpha} = 1$  and therefore  $\alpha = 1$ .  $\square$

In the proof of the following theorem we will use quotients of fusion systems and centric linking systems. We refer the reader to Definition 2.8 and Theorem 2.9 in Section 2 and [BLO2, Lemma 5.6] for more details.

**Theorem 6.4.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group at an odd prime  $p$ . Then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if every elementary abelian normal subgroup  $V \triangleleft S$  is  $\mathcal{F}$ -weakly closed and  $\text{Aut}_{\mathcal{F}}(V)$  is a  $p$ -group.*

*Proof.* This condition is necessary in view of Theorem 5.8. Let thus  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group such that every elementary abelian normal subgroup  $V \triangleleft S$  is  $\mathcal{F}$ -weakly closed and  $\text{Aut}_{\mathcal{F}}(V)$  is a  $p$ -group. We show that it must then be nilpotent.

Consider first the elementary abelian normal subgroup  $V_0 = \Omega_1(Z(S))$ . It is weakly closed in  $(S, \mathcal{F}, \mathcal{L})$  by hypothesis, and thus normal in  $\mathcal{F}$  by Lemma 5.6. Given that  $\text{Aut}_{\mathcal{F}}(V_0)$  is a  $p$ -group, Proposition 6.2 applies. We have the equality  $(S, \mathcal{F}, \mathcal{L}) = (S, C_{\mathcal{F}}(V_0), C_{\mathcal{L}}(V_0))$ . Hence  $V_0$  is central in  $(S, \mathcal{F}, \mathcal{L})$ .

Let  $V \leq S$  be a maximal normal elementary abelian subgroup of  $S$  containing  $V_0$ . If  $V = V_0$ , then it is  $\mathcal{F}$ -normal and, by Lemma 6.3,  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent. Let us thus assume that  $V_0$  is strictly contained in  $V$ .

Consider the quotient  $p$ -local finite group  $(S_1, \mathcal{F}_1, \mathcal{L}_1) = (S/V_0, \mathcal{F}/V_0, \mathcal{L}/V_0)$ , as in [BLO2, Lemma 5.6]. From the fibration

$$BV_0 \longrightarrow |\mathcal{L}|_p^\wedge \longrightarrow |\mathcal{L}_1|_p^\wedge,$$

we see that  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if  $(S_1, \mathcal{F}_1, \mathcal{L}_1)$  is so. Let  $V_1$  be the only subgroup satisfying  $V_0 < V_1 \leq V$  and  $V_1/V_0 = V/V_0 \cap Z(S_1)$ .

By construction,  $V_1$  is elementary abelian and normal in  $S$ . It is also strictly larger than  $V_0$  because  $V/V_0$  is a non-trivial normal subgroup of  $S/V_0$  and therefore it has non-trivial intersection with its center [Go, Theorem 2.6.4]. By the hypothesis,  $V_1$  is weakly closed in  $\mathcal{F}$ , so  $V_1/V_0$  is weakly closed in  $\mathcal{F}_1$ . By Lemma 5.6,  $V_1/V_0$  is normal in  $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ .

Let  $f: P/V_0 \rightarrow Q/V_0$  be a map in  $\mathcal{F}_1$ . Since  $V_1/V_0$  is normal in  $\mathcal{F}_1$ ,  $f$  is the restriction of a map  $g: PV_1/V_0 \rightarrow QV_1/V_0$  in  $\mathcal{F}_1 = \mathcal{F}/V_0$ . This map must be induced by a morphism  $h: PV_1 \rightarrow QV_1$  in  $\mathcal{F}$ , which must satisfy  $h(P) \leq Q$  and  $h(V_1) = V_1$ , since  $V_1$  is  $\mathcal{F}$ -weakly closed and both  $P$  and  $Q$  contain  $V_0$ . This shows that  $\mathcal{F}_1 = N_{\mathcal{F}}(V_1)/V_0$ , and so there is a fibration

$$BV_0 \longrightarrow |N_{\mathcal{L}}(V_1)|_p^\wedge \longrightarrow |\mathcal{L}_1|_p^\wedge,$$

from which we deduce that  $(S_1, \mathcal{F}_1, \mathcal{L}_1)$  is nilpotent if and only if  $(S, \mathcal{F}^1, \mathcal{L}^1) = (S, N_{\mathcal{F}}(V_1), N_{\mathcal{L}}(V_1))$  is nilpotent. The fusion system  $\mathcal{F}^1$  is a subcategory of  $\mathcal{F}$  with



the same objects, so it also satisfies that every elementary abelian normal subgroup  $W \triangleleft S$  is  $\mathcal{F}^1$ -weakly closed and  $\text{Aut}_{\mathcal{F}^1}(W)$  is a  $p$ -group. Moreover,  $V_1$  is normal in  $(S, \mathcal{F}^1, \mathcal{L}^1)$ .

If  $V_1 \neq V$  we can iterate the process by defining  $V_2$  to be the only subgroup satisfying  $V_1 < V_2 \leq V$  and  $V_2/V_1 = V/V_1 \cap Z(S/V_1)$ . The  $p$ -local finite group  $(S, \mathcal{F}^1, \mathcal{L}^1)$  is nilpotent if and only if a new  $p$ -local finite group  $(S, \mathcal{F}^2, \mathcal{L}^2)$ , which normalizes  $V_2$  and satisfies that every elementary abelian normal subgroup  $W \triangleleft S$  is  $\mathcal{F}^2$ -weakly closed and  $\text{Aut}_{\mathcal{F}^2}(W)$  is a  $p$ -group, is nilpotent.

Iterating this process a finite number of times, we end up with a  $p$ -local finite group  $(S, \tilde{\mathcal{F}}, \tilde{\mathcal{L}})$  for which  $V$  is  $\tilde{\mathcal{F}}$ -normal and such that  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if  $(S, \tilde{\mathcal{F}}, \tilde{\mathcal{L}})$  is so. We conclude by Lemma 6.3.  $\square$

Note that  $p > 2$  is a necessary condition, since the 2-local finite group induced by the semidirect product  $Q_8 \rtimes 3$  provides a counterexample at the prime 2.

### 7. Cohomological characterizations in high degrees

In contrast to the results in Section 4, we now look at cohomological characterizations of nilpotent  $p$ -local finite groups in high degrees. The proofs follow the lines of Quillen’s arguments in the case of finite groups [Q]. Quillen attributes the first criterion to Atiyah. Let  $\mathcal{F}$  be a saturated fusion system. We will use the ring of stable elements

$$H^*(\mathcal{F}; \mathbb{F}_p) = \varinjlim_{\mathcal{O}(\mathcal{F})} H^*(-; \mathbb{F}_p),$$

as introduced in [BLO2, Section 5]. It only depends on the fusion system, but the main theorem of the cited section identifies it with the mod- $p$  cohomology of  $|\mathcal{L}|$  when a centric linking system  $\mathcal{L}$  associated with  $\mathcal{F}$  exists.

**Theorem 7.1.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and  $Bi: BS \rightarrow |\mathcal{L}|$  be the standard inclusion. Then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if one of the following equivalent three conditions is satisfied:*

- (1)  $Bi^*: H^n(|\mathcal{L}|; \mathbb{F}_p) \rightarrow H^n(BS; \mathbb{F}_p)$  is an isomorphism for all sufficiently large integers  $n$ ;
- (2)  $Bi^*: H^n(|\mathcal{L}|; \mathbb{Z}_p^\wedge) \rightarrow H^n(BS; \mathbb{Z}_p^\wedge)$  is an isomorphism for all sufficiently large integers  $n$ ;
- (3) if  $p > 2$ , then for each  $x \in H^{\text{even}}(BS; \mathbb{F}_p)$  there exists a power  $q$  of  $p$  such that  $x^q \in \text{Im}(Bi^*)$ , that is,  $Bi^*$  is an  $F$ -epimorphism.

*Proof.* Conditions (1) and (2) are equivalent by a universal coefficient theorem argument. We work therefore with  $p$ -adic coefficients and assume now that the  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  satisfies condition (2). By Castellana and Morales' theorem [CM], the  $p$ -adic  $K$ -theory of  $|\mathcal{L}|$  can be computed with stable elements and so  $K^0(|\mathcal{L}|; \mathbb{Z}_p^\wedge)$  is a free  $\mathbb{Z}_p^\wedge$ -module and its rank is the number of  $\mathcal{F}$ -conjugacy classes of elements in  $S$ . However, the map  $Bi: BS \rightarrow |\mathcal{L}|$  induces a morphism of Atiyah–Hirzebruch spectral sequences [K, Theorem 4.2.7], converging to the map of  $p$ -completed  $K$ -theory rings  $K^*(|\mathcal{L}|; \mathbb{Z}_p^\wedge) \rightarrow K^*(BS; \mathbb{Z}_p^\wedge)$  induced by  $Bi$ . This map has finite kernel and cokernel in each dimension by assumption, so  $K^0(|\mathcal{L}|; \mathbb{Z}_p^\wedge)$  and  $K^0(BS; \mathbb{Z}_p^\wedge)$  must have the same rank. Two elements in  $S$  are hence  $\mathcal{F}$ -conjugate if and only if they are  $S$ -conjugate and this is the characterization of nilpotency given by part (1) of Theorem 5.8.

By Proposition 3.3, a nilpotent  $p$ -local finite group satisfies condition (3). To prove that condition (3) characterizes nilpotency for  $p > 2$ , we show that it implies the condition appearing in Theorem 6.4. Let  $V \triangleleft S$  be an elementary abelian normal subgroup of  $S$  and consider the ideal

$$\mathfrak{p}_V = \{u \in H^{\text{even}}(BS; \mathbb{F}_p) \mid u|_{H^{\text{even}}(BV; \mathbb{F}_p)} \text{ is nilpotent}\}$$

in  $H^{\text{even}}(BS; \mathbb{F}_p)$ . It is a prime ideal since restriction to  $V$  induces an injection  $H^{\text{even}}(BS; \mathbb{F}_p)/\mathfrak{p}_V \rightarrow H^{\text{even}}(BV; \mathbb{F}_p)/\mathfrak{n} = S(V^\#)$ , where  $S(V^\#)$  is the symmetric algebra on  $V^\# = \text{Rep}(V, \mathbb{Z}/p)$  with the elements of  $V^\#$  in degree 2 [Q] and  $\mathfrak{n}$  is the nilradical. For simplicity of notation, let us denote by  $Bi^{-1}(X)$  the set  $(Bi^*)^{-1}(X)$  for any  $X \subseteq H^{\text{even}}(BS; \mathbb{F}_p)$ .

Let  $A$  be an elementary abelian subgroup of  $S$  which is  $\mathcal{F}$ -isomorphic to  $V$ . Then we have  $Bi^{-1}(\mathfrak{p}_A) = Bi^{-1}(\mathfrak{p}_V)$ , since  $H^*(|\mathcal{L}|; \mathbb{F}_p)$  is computed by stable elements. Given  $u \in \mathfrak{p}_V$ , condition (3) implies that  $u^q \in \text{Im}(Bi^*)$  for some  $q$  which is a power of  $p$ . Say  $Bi^*(v) = u^q$ . That means that  $v \in Bi^{-1}(\mathfrak{p}_V) = Bi^{-1}(\mathfrak{p}_A)$  and so  $u^q \in \mathfrak{p}_A$ . But  $\mathfrak{p}_A$  is a prime ideal of  $H^{\text{even}}(BS; \mathbb{F}_p)$ , so  $u \in \mathfrak{p}_A$ . We conclude that  $\mathfrak{p}_A = \mathfrak{p}_V$ . Applying [Q, Theorem 2.7] we obtain that  $A$  and  $V$  must be conjugate subgroups in  $S$ . Since  $V \triangleleft S$ ,  $A = V$  and  $V$  is  $\mathcal{F}$ -weakly closed.

It remains to prove that  $\text{Aut}_{\mathcal{F}}(V)$  is a  $p$ -group. We will actually show that  $\text{Aut}_{\mathcal{F}}(V) = \text{Aut}_S(V)$ . We proceed as Quillen in [Q, Theorem 2.10]. Consider the maps  $H^{\text{even}}(|\mathcal{L}|; \mathbb{F}_p)/Bi^{-1}(\mathfrak{p}_V) \rightarrow H^{\text{even}}(BS; \mathbb{F}_p)/\mathfrak{p}_V \rightarrow S(V^\#)$  and the associated extensions of quotient fields

$$k(Bi^{-1}(\mathfrak{p}_V)) \subseteq k(\mathfrak{p}_V) \subseteq k(V).$$

Note that the groups of automorphisms of the extensions  $k(\mathfrak{p}_V) \subseteq k(V)$  and  $k(Bi^{-1}(\mathfrak{p}_V)) \subseteq k(V)$  are  $\text{Aut}_S(V)$  and  $\text{Aut}_{\mathcal{F}}(V)$ , respectively [Q, Theorem 2.10]. The proof of [BLO2, Proposition 5.2] shows that the ring  $H^{\text{even}}(BS; \mathbb{F}_p)$  is integral

over  $H^{\text{even}}(|\mathcal{L}|; \mathbb{F}_p)$  and thus  $k(Bi^{-1}(\mathfrak{p}_V)) \subseteq k(\mathfrak{p}_V)$  is an algebraic extension. Since any element in  $H^{\text{even}}(BS; \mathbb{F}_p)$  is in  $H^{\text{even}}(\mathcal{L}; \mathbb{F}_p)$  after being raised to a certain power of  $p$ , the extension  $k(Bi^{-1}(\mathfrak{p}_V)) \subseteq k(\mathfrak{p}_V)$  is purely inseparable [I, Theorem 19.10]. Therefore,  $\text{Aut}_{\mathcal{F}}(V) = \text{Aut}_S(V)$ .  $\square$

We also offer a characterization of nilpotency in terms of the Morava  $K$ -theory  $K(n)$ , for any  $n \geq 1$ . For groups, this criterion was discovered by Brunetti in [Br]. His proof is based on the beautiful result of Hopkins, Kuhn, and Ravenel [HKR] on generalized characters of finite groups.

**Theorem 7.2.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and let  $Bi: BS \rightarrow |\mathcal{L}|$  be the standard inclusion. Then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if  $Bi^*: K(n)^*(|\mathcal{L}|) \rightarrow K(n)^*(BS)$  is an isomorphism.*

*Proof.* By [W], if we have a map of spaces inducing an isomorphism on the  $n$ th Morava  $K$ -theory, then it also induces an isomorphism on the first Morava  $K$ -theory. By [CM], an isomorphism in  $K(1)$  implies that two elements are conjugate in  $S$  if and only if they are conjugate in  $\mathcal{F}$ , that is, condition (1) holds in Theorem 4.2.

Conversely, if a  $p$ -local finite group is nilpotent, we have seen in Proposition 3.3 that  $Bi$  induces an isomorphism in mod- $p$  cohomology. It must then be a  $K(n)$ -equivalence as well, see for example [M, Corollary 1.5].  $\square$

### 8. Quillen categories

For a finite  $p$ -group  $S$ , let  $\varepsilon_S$  denote the category whose objects are the elementary abelian  $p$ -subgroups of  $S$  and whose morphisms are given by conjugation. Similarly, for a  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$ , let  $\varepsilon_{\mathcal{F}}$  be the category with the same objects considered as a full subcategory of  $\mathcal{F}$ . They are the Quillen categories of  $S$  and  $\mathcal{F}$ , respectively. Recall that a ring homomorphism  $\gamma: B \rightarrow A$  is an  $F$ -isomorphism if each element in  $\text{Ker}(\gamma)$  is nilpotent and if for all  $a \in A$  there is some  $k > 0$  such that  $a^k \in \text{Im}(\gamma)$ . Following [BLO2], we define

$$H_{\varepsilon}^*(S; \mathbb{F}_p) = \varprojlim_{\varepsilon_S} H^*(-; \mathbb{F}_p) \quad \text{and} \quad H_{\varepsilon}^*(\mathcal{F}; \mathbb{F}_p) = \varprojlim_{\varepsilon_{\mathcal{F}}} H^*(-; \mathbb{F}_p).$$

*Definition 8.1.* Let  $\mathcal{C}$  and  $\mathcal{D}$  be two categories equipped with functors  $\mathcal{C} \xrightarrow{\gamma} \text{Gr}$  and  $\mathcal{D} \xrightarrow{\delta} \text{Gr}$  to the category of groups. A functor  $\Psi: \mathcal{C} \rightarrow \mathcal{D}$  is *isotypical* if  $\gamma$  is naturally isomorphic to  $\delta \circ \Psi$ .

**Theorem 8.2.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group,  $p > 2$ , and  $Bi: BS \rightarrow |\mathcal{L}|$  be the standard inclusion. Then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if one of the following equivalent conditions is satisfied:*

- (1) *the Quillen categories of  $S$  and  $\mathcal{F}$  are isotypically equivalent;*
- (2)  *$Bi^*: H^*(|\mathcal{L}|; \mathbb{F}_p) \rightarrow H^*(BS; \mathbb{F}_p)$  is an  $F$ -isomorphism.*

*Proof.* By Broto, Levi, and Oliver [BLO2, Proposition 5.1], there are  $F$ -isomorphisms  $\lambda_S: H^*(BS) \rightarrow H^*_\varepsilon(S)$  and  $\lambda_{\mathcal{F}}: H^*(\mathcal{F}) \rightarrow H^*_\varepsilon(\mathcal{F})$ . The stable element formula [BLO2, Theorem 5.8] shows that the natural map  $R_{\mathcal{L}}: H^*(|\mathcal{L}|) \rightarrow H^*(\mathcal{F})$  is an isomorphism. So in fact, we have the commutative diagram

$$\begin{array}{ccc}
 H^*(|\mathcal{L}|) & \xrightarrow{\lambda_{\mathcal{F}} R_{\mathcal{L}}} & H^*_\varepsilon(\mathcal{F}) \\
 Bi^* \downarrow & & \downarrow j \\
 H^*(BS) & \xrightarrow{\lambda_S} & H^*_\varepsilon(S).
 \end{array}$$

If the Quillen categories of  $S$  and  $\mathcal{F}$  are isotypically equivalent, then the map  $j$  is an isomorphism and therefore the map  $Bi^*$  is an  $F$ -isomorphism. Assume now that  $Bi^*$  is an  $F$ -isomorphism. Then condition (3) in Theorem 7.1 holds and so  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent. It is obvious that a nilpotent  $p$ -local group satisfies condition (1).  $\square$

Let  $\mathcal{C}$  be a class of finite subgroups of  $S$ . We say that a subgroup  $H$  of  $S$  controls fusion of  $\mathcal{C}$ -groups in  $\mathcal{F}$  if the following conditions hold:

- any  $\mathcal{C}$ -subgroup of  $S$  is  $\mathcal{F}$ -conjugate to a subgroup of  $H$ ;
- for any  $\mathcal{C}$ -subgroup  $P$  of  $H$  and any  $f: P \rightarrow S$  in  $\mathcal{F}$  such that  $f(P) \leq H$ , there exists  $h \in H$  such that  $f(x) = h x h^{-1}$  for all  $x \in P$ .

When  $H = S$ , the first condition is a tautology, so  $S$  itself controls fusion if any  $\mathcal{F}$ -morphism  $P \rightarrow S$  from a group  $P \in \mathcal{C}$  is conjugation by an element in  $S$ .

*Definition 8.3.* Let  $G$  be a group. We say  $x \in G$  is a  $\mathcal{C}_p$ -element when  $x^p = 1$  if  $p$  is odd or  $x^4 = 1$  if  $p = 2$ . A subgroup of  $G$  generated by a  $\mathcal{C}_p$ -element is called a  $\mathcal{C}_p$ -subgroup and we denote by  $\mathcal{C}_p$  the class of  $\mathcal{C}_p$ -subgroups.

Given  $K \leq S$  and  $f: K \rightarrow S$ , let us denote by  $[K, f]$  the subgroup of  $S$  generated by the elements of the form  $[a, f] = a^{-1} f(a)$  with  $a \in K$ . We also denote by  $[K, f, f]$  the subgroup  $[[K, f], f]$ .

**Theorem 8.4.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group. Then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent if and only if  $S$  controls fusion of  $\mathcal{C}_p$ -subgroups.*

*Proof.* We follow the strategy in [GS, Theorem 2]. We will show that the control of the fusion condition implies condition (3) in Theorem 5.8. Let  $P \leq S$ , and let  $f \in \text{Aut}_{\mathcal{F}}(P)$  be an automorphism of order prime to  $p$ . There exists some  $l \geq 1$  such that  $P$  is contained in a subgroup  $Z_l(S)$  of the upper central series, defined inductively by  $Z_0(S) = 1$  and  $Z_l(S) = \{x \in S \mid [x, y] \in Z_{l-1}(S) \text{ for all } y \in S\}$ . We will prove by induction on  $l$  that  $f$  is the identity map.

Suppose first that  $P \leq Z(S)$  and consider a  $C_p$ -element  $a \in P$ . Since  $S$  controls fusion of  $C_p$ -subgroups, there is an  $s \in S$  such that  $f(a) = sas^{-1}$ . But  $a \in Z(S)$ , so  $f(a) = a$ . By [Go, Theorem 5.2.4],  $f = 1_P$ .

Now consider  $P \leq Z_l(S)$  and suppose that the result is known for subgroups of  $Z_{l-1}(S)$ . Let  $K$  be the subgroup of  $P$  generated by  $C_p$ -elements. Both  $K$  and  $[K, f]$  are stabilized by  $f$ .

Consider a  $C_p$ -element  $a \in P$ . Since  $S$  controls fusion of  $C_p$ -subgroups, there is  $s \in S$  such that  $f(a) = sas^{-1}$  and so  $[a, f] = [a, s] \in Z_{l-1}(S)$ . Note that  $[K, f]$  is generated by elements of the form  $b[a, f]b^{-1}$ , where  $a, b \in K$  and  $a$  is a  $C_p$ -element. Therefore  $[K, f] \leq Z_{l-1}(S)$ , so that the induction hypothesis applies:  $f$  restricted to  $[K, f]$  must be the identity. Since  $f$  stabilizes  $[K, f]$ , we have that  $[K, f, f] = 1$ . But by [Go, Theorem 5.3.6],  $[K, f, f] = [K, f]$  under these circumstances, so  $f$  restricted to  $K$  is the identity as well. In particular,  $f$  fixes the  $C_p$ -elements of  $P$ . Now [H, Satz IV.5.12] implies that  $f$  is the identity.  $\square$

*Definition 8.5.* ([BCGLO1]) For any saturated fusion system  $\mathcal{F}$  over a finite  $p$ -group  $S$ , the *center of  $\mathcal{F}$*  is the subgroup

$$Z_{\mathcal{F}}(S) = \{x \in Z(S) \mid f(x) = x \text{ for all } f \in \text{Mor}(\mathcal{F}^c)\} = \varprojlim_{\mathcal{F}^c} Z(-).$$

The characteristic subgroup  $\Omega_i(S)$  is the subgroup generated by all elements  $x$  such that  $x^{p^i} = 1$ . The following result is now a straightforward consequence of Theorem 8.4.

**Corollary 8.6.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group. If the  $C_p$ -elements of  $S$  are in the center of  $\mathcal{F}$ , then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent. In particular, if  $p$  is odd and  $\Omega_1(S) \subseteq Z_{\mathcal{F}}(S)$  (respectively  $p=2$  and  $\Omega_2(S) \subseteq Z_{\mathcal{F}}(S)$ ), then  $(S, \mathcal{F}, \mathcal{L})$  is nilpotent.*

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