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Morrey spaces in harmonic analysis

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Abstract. Through a geometric capacitary analysis based on space dualities, this paper addresses several fundamental aspects of functional analysis and potential theory for the Morrey spaces in harmonic analysis over the Euclidean spaces.

1. Introduction

Let us start with the motivation and the structure of this paper.

1.1. Motivation

A real-valued function f is said to belong to the Morrey space $L^{p,\lambda}$ on the N-dimensional Euclidean space $\mathbb{R}^N$ provided the following norm is finite:

$$\|f\|_{L^{p,\lambda}}=\left(\sup_{(x,r)\in\mathbb{R}^N\times\mathbb{R}_+} r^{\lambda-N}\int_{B(x,r)}|f(y)|^p\,dy\right)^{1/p}.$$

Here $1\le p<\infty$, $0<\lambda\le N$, $\mathbb{R}_+=(0,\infty)$, and $B(x,r)$ is a ball in $\mathbb{R}$ centered at x of radius r. This class of functions was first used in a 1938 paper by C. B. Morrey [26] to show that certain systems of partial differential equations (PDEs) had Hölder continuous solutions. Though Morrey worked mainly in two dimensions for his results, the concept of the integral average over a ball having a certain growth has found many applications over the years; see e.g. [7], [10], [36], and [38]. In fact, the main fame that rests with Morrey spaces is the following celebrated lemma.

Morrey's lemma. *Let the function u satisfy $|\nabla u|\in L^{p,\lambda}$ (even locally) with $\lambda<p$. Then u is Hölder continuous of exponent $\alpha=1-\lambda/p$.*

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Then, in the early 1960s, to handle appropriately the cases $\lambda \geq p$, S. Campanato [13] defined a scale of spaces $\mathcal{L}^{p,\lambda}$ that included the distinguished quartet:

$$L^p - L^{p,\lambda} - \mathrm{BMO} - C^\alpha.$$

His definition uses the modified mean oscillation: $f \in \mathcal{L}^{p,\lambda}$ if and only if

$$\left(\sup_{(x,r)\in\mathbb{R}^N\times\mathbb{R}_+} r^{\lambda-N} \int_{B(x,r)} |f(y)-f_r(x)|^p \, dy \right)^{1/p} < \infty,$$

where $f_r(x)$ denotes the integral mean of f over the ball $B(x,r)$, and $1 \leq p < \infty$, $-p < \lambda \leq N$. As a matter of fact, we have the following:

(i) when $\lambda = N$, $\mathcal{L}^{p,\lambda}$ coincides with L^p;

(ii) when $0 < \lambda < N$, $\mathcal{L}^{p,\lambda}$ equals $L^{p,\lambda}$;

(iii) when $\lambda = 0$, $\mathcal{L}^{p,\lambda}$ is precisely BMO (the John–Nirenberg space of functions with bounded mean oscillation [19]);

(iv) when $-p < \lambda < 0$, $\mathcal{L}^{p,\lambda}$ becomes C^α—the class of all Hölder continuous functions with exponent $\alpha = -\lambda/p$.

These results are nicely summarized in J. Peetre's 1969 survey paper [28]. Furthermore, these spaces gained some popularity in the late 1960s and early 1970s, especially because of the inclusion of the space BMO. And, this was greatly enhanced by the discovery, in the early 1970s, of the predual to BMO by C. Fefferman and E. M. Stein [20], namely, the real Hardy space H^1; see the excellent treatises [34] and [37] that summarize this theory.

But in recent years, the excitement over the Morrey spaces has diminished—save for many attempts to generalize the growth of the integral mean (or other averages in other norms). The reason for this seems to be that either the predual to $L^{p,\lambda}$ was unknown or there was no version that really fit in well with the theory of function spaces of harmonic analysis, as defined by [37] and [34]. Indeed, early versions of the predual were given as early as 1986; see [3], [40], [23], [11], and [6]. Here we announce a new formulation of the predual that seems to be more in the main stream of harmonic analysis (as well as PDEs). It uses the original construction of [23], which comes from PDEs, though the weight functions here are different. Our space $H^{p,\lambda}$ is a modification of that given in [6]. Here we make use of A_p-weights of harmonic analysis:

$$H^{p,\lambda} = \left\{ f \in L^p_{\mathrm{loc}} : \|f\|_{H^{p,\lambda}} = \inf_w \left(\int_{\mathbb{R}^N} |f(y)|^p w(y)^{1-p} \, dy \right)^{1/p} < \infty \right\},$$

where L^p_{loc} means the class of all p-locally integrable functions on $\mathbb{R}^N$ and the infimum is over all non-negative weights w that belong to the class A_1 and satisfy

$$\int_{\mathbb{R}^N} w \, d\Lambda^{(\infty)}_{N-\lambda} = \int_0^\infty \Lambda^{(\infty)}_{N-\lambda}(\{x \in \mathbb{R}^N : w(x) > t\}) \, dt \leq 1.$$

Here and henceforth, for $0<\lambda<N$ the symbol $\Lambda^{(\infty)}_{N-\lambda}$ stands for the Hausdorff capacity with order $N-\lambda$, as a set function on $\mathbb{R}^N$. By the theory of A_1-weights (see also [34] or [37]):

$$w\in A_1 \quad\Longrightarrow\quad w^{1-p}\in A_p, \quad \text{where } 1<p<\infty.$$

With the previous definition we have the following duality formula:

$$(H^{p',\lambda})^*=L^{p,\lambda}, \quad \text{where } p'=\frac{p}{p-1}.$$

Further history of the Morrey spaces has included the proof of boundedness of the classical operators of harmonic analysis on these spaces: Calderón–Zygmund singular integral operators, maximal operators—especially the Hardy–Littlewood maximal operator, and the potential operators; see [28], [3], [15], and others including e.g. [17] and [25]. However, with the predual given above, these results are mere corollaries to the A_p-weight theory of harmonic analysis. Here it should be also mentioned that we can now develop the interpolation theory of operators of G. Stampacchia (cf. [32]) to include the case where Zorko's subspaces (arising from [40]) of the Morrey spaces can lie in the domains of the operator. Previously, interpolation only worked when the operator had some L^p-space as domain and a Morrey space as range, but there were negative results when an $\mathcal{L}^{p,\lambda}$ acts as a domain space—see Stein–Zygmund [35] (for $-p<\lambda<0$), Ruiz–Vega [29] (for $0<\lambda\le 1/p$, $N>1$), and Blasco–Ruiz–Vega [11] (for $0<\lambda\le 1/p$, $N=1$).

1.2. The rest of the paper

The follow-up of this introductory section comprises five sections where the analytic and geometric essentials of an associated capacity play an important role. Section 2 reviews some fundamental properties of the Hausdorff capacity, its induced Choquet integrals and the A_p-weights, but also deals with the Sobolev-type imbeddings via the (α,p)-Riesz kernels/potentials. As the central issue of this article, Section 3 investigates the dual theory for the spaces $H^{p,\lambda}$, $L^{p,\lambda}$ and the Zorko spaces $L^{p,\lambda}_0$ (just like D. Sarason's space of functions with vanishing mean oscillation (VMO) [31])—giving such a duality triplet:

$$L^{p,\lambda}_0-H^{p',\lambda}-L^{p,\lambda} \quad \text{in analogy to} \quad \mathrm{VMO}-H^1-\mathrm{BMO}\,.$$

Two natural and interesting applications of this new duality relation are included respectively in: Section 4 handling the continuity of the fractional order maximal operators and Riesz potential operators on the dual pairs $(L^{p,\lambda},H^{p',\lambda})$; and Section 5 treating the interpolation of operators with the Zorko spaces as the interpolation domains. Finally, Section 6 is concerned with the Morrey–Sobolev capacities as a continuation of both Section 4 and the results presented earlier in [6], but also includes size estimates for the Riesz potential operators in terms of the Wolff-type potentials.

Notation. Throughout this article, in most cases we use $U \lesssim V$, $U \gtrsim V$, and $U \sim V$ to denote that there is a constant $c>0$ such that $U \le cV$, $U \ge cV$, and $c^{-1}V \le U \le cV$, respectively.

2. Background material

The main new development in the theory of Morrey spaces is an effective use of the Hausdorff capacity and its induced Choquet integrals, the A_p-weights, and the Sobolev imbeddings via Riesz potentials.

2.1. Hausdorff capacity and its Choquet integrals

Given $0<\alpha<N$, the αth order *Hausdorff capacity* at the level $\varepsilon \in (0,\infty]$ of a subset E of $\mathbb{R}^N$ is determined by:

$$\Lambda_\alpha^{(\varepsilon)}(E) = \inf\left\{\sum_{j=1}^\infty r_j^\alpha : E \subseteq \bigcup_{j=1}^\infty B(x_j, r_j) \text{ and } r_j \le \varepsilon, j=1,2,...\right\}.$$

Recall that $\Lambda_\alpha^{(\varepsilon)}(\,\cdot\,)$ has the same null sets as its more well-known cousin—the α-order Hausdorff measure (see e.g. [18] and [14]):

$$\Lambda_\alpha^{(0)}(E) = \lim_{\varepsilon \to 0} \Lambda_\alpha^{(\varepsilon)}(E).$$

However, $\Lambda_\alpha^{(\infty)}$ is finite on all bounded sets, and yet $\Lambda_\alpha^{(0)}$ is only finite on special α-dimensional sets although it is a metric outer measure there.

Also, given $1 \le p < \infty$, we define

$$\int_{\mathbb{R}^N} |f|^p \, d\Lambda_\alpha^{(\infty)} = \int_0^\infty \Lambda_\alpha^{(\infty)}(\{x \in \mathbb{R}^N : |f(x)| > t\}) \, dt^p$$

as the Choquet p-integral with respect to $\Lambda_\alpha^{(\infty)}$ of f in C_0 (the class of continuous functions with compact support in $\mathbb{R}^N$). The class $L^p(\Lambda_\alpha^{(\infty)})$ is the closure of C_0 in the quasi-norm

$$\|\cdot\|_{L^p(\Lambda_\alpha^{(\infty)})} = \left(\int_{\mathbb{R}^N} |\cdot|^p \, d\Lambda_\alpha^{(\infty)}\right)^{1/p}.$$

Thus $f\in L^p(\Lambda_\alpha^{(\infty)})$ is automatically $\Lambda_\alpha^{(\infty)}$-quasi-continuous, i.e., given $\varepsilon>0$ there is a set E with $\Lambda_\alpha^{(\infty)}(E)<\varepsilon$ and the restriction of f to $\mathbb{R}^N\setminus E$ is continuous there.

$\Lambda_\alpha^{(\infty)}(\,\cdot\,)$ is only a capacity in the sense of N.G. Meyers, i.e., it satisfies

(i) $\Lambda_\alpha^{(\infty)}(\varnothing)=0$—zero property;

(ii) $E_1\subseteq E_2\Rightarrow\Lambda_\alpha^{(\infty)}(E_1)\le\Lambda_\alpha^{(\infty)}(E_2)$—monotonicity;

(iii) $\Lambda_\alpha^{(\infty)}\big(\bigcup_{j=1}^\infty E_j\big)\le\sum_{j=1}^\infty \Lambda_\alpha^{(\infty)}(E_j)$—countable subadditivity.

In particular, $\Lambda_\alpha^{(\infty)}$ is not an additive measure. Therefore, we must be careful when working with the Choquet integral against this capacity. Nevertheless, we have Choquet's main result regarding this extension of the standard integral.

Theorem 1. (Choquet) *If $C(\,\cdot\,)$ is a capacity in the sense of Meyers, then the Choquet integral $\int_{\mathbb{R}^N} f\,dC$ of $f\ge 0$ with respect to $C(\,\cdot\,)$ is sublinear if and only if $C(\,\cdot\,)$ is strongly subadditive.*

For a proof of this assertion see [16] and [4] as well as [8]. Here, the capacity $C(\,\cdot\,)$ is *strongly subadditive* if for any two sets $E_1,E_2\subseteq\mathbb{R}^N$ it follows that

$$C(E_1\cup E_2)+C(E_1\cap E_2)\le C(E_1)+C(E_2).$$

Moreover, the Choquet integral $\int_{\mathbb{R}^N} f\,dC$ of $f\ge 0$ is sublinear when and only when

$$\int_{\mathbb{R}^N}(f_1+f_2)\,dC\le\int_{\mathbb{R}^N} f_1\,dC+\int_{\mathbb{R}^N} f_2\,dC\quad\text{for } f_1,f_2\ge 0.$$

Now it is well known that $\Lambda_\alpha^{(\infty)}$ is not, in general, strongly subadditive, but an equivalent version is the *dyadic Hausdorff capacity* $\tilde\Lambda_\alpha^{(\infty)}$ which is defined in the following manner. If $\{Q_j\}_{j=1}^\infty$ denotes a family of dyadic cubes in $\mathbb{R}^N$, i.e., those cubes congruent to

$$[0,1)^N=\underbrace{[0,1)\times\ldots\times[0,1)}_{N\text{ copies}}$$

and whose vertices lie on the lattice $\mathbb{Z}^n$ dilated by a factor 2^{-k} where $k\in\mathbb{Z}$, then

$$\tilde\Lambda_\alpha^{(\infty)}(E)=\inf\sum_{j=1}^\infty \ell(Q_j)^\alpha,$$

where $\ell(Q_j)$ is the side length of each Q_j and $E\subseteq\operatorname{Int}\big(\bigcup_{j=1}^\infty Q_j\big)$, where the infimum is taken over all such families of dyadic cubes. From [21] and most recently [39] we read off the following result.

Theorem 2. (Yang–Yuan) *If* $0<\alpha<N$*, then* $\tilde{\Lambda}_\alpha^{(\infty)}(\cdot)$ *is strongly subadditive and there exists a constant* $c>0$ *depending only on* α *and* N *such that*

$$\frac{1}{c}\tilde{\Lambda}_\alpha^{(\infty)} \le \Lambda_\alpha^{(\infty)} \le c\tilde{\Lambda}_\alpha^{(\infty)}.$$

Hence by this result we have that if $(\alpha,p)\in(0,N)\times(1,\infty)$, then the quasi-sublinearity and the quasi-Hölder inequality for the Choquet integrals of two real-valued functions f_1 and f_2 on $\mathbb{R}^N$:

$$\int_{\mathbb{R}^N} |f_1+f_2|\,d\Lambda_\alpha^{(\infty)} \lesssim \int_{\mathbb{R}^N} |f_1|\,d\Lambda_\alpha^{(\infty)} + \int_{\mathbb{R}^N} |f_2|\,d\Lambda_\alpha^{(\infty)}$$

and

$$\int_{\mathbb{R}^N} |f_1 f_2|\,d\Lambda_\alpha^{(\infty)} \lesssim \left(\int_{\mathbb{R}^N} |f_1|^p\,d\Lambda_\alpha^{(\infty)}\right)^{1/p} \left(\int_{\mathbb{R}^N} |f_2|^{p'}\,d\Lambda_\alpha^{(\infty)}\right)^{1/p'},$$

respectively, hold—they are the main estimates required in the subsequent sections.

2.2. Weight functions

Also, we need a review of the so-called A_p-weights, $1\le p<\infty$. A non-negative function w is an A_1-*weight* if for all coordinate cubes $Q\subseteq\mathbb{R}^N$ one has

$$\ell(Q)^{-N}\int_Q w\,dy \le c_1 \inf_Q w$$

for some constant $c_1>0$. An equivalent version is

$$M_0 w \le c_2 w \quad \text{a.e. on } \mathbb{R}^N$$

for some constant $c_2>0$. Here $M_0 w$ is the Hardy–Littlewood maximal function of w, i.e.,

$$M_0 w(x) = \sup_{r\in\mathbb{R}_+} r^{-N}\int_{B(x,r)} |w(y)|\,dy.$$

Next, for $p\in(1,\infty)$, we say that a non-negative function w is an A_p-weight provided there is a constant $c_{p,N}>0$ depending on p and N such that for all coordinate cubes Q,

$$\left(\ell(Q)^{-N}\int_Q w(y)\,dy\right)\left(\ell(Q)^{-N}\int_Q w(y)^{1/(1-p)}\,dy\right)^{p-1} \le c_{p,N}.$$

A remarkable fact is the following implication (cf. [34] or [37]):

$$w\in A_1 \quad\Longrightarrow\quad w\in A_p.$$

Now, an important result in our characterization of the predual space $H^{p',\lambda}$ is the following assertion (due to Adams [3] when $p=1$ and Orobitg–Verdera [27] for $p<1$; and the cases $p>1$ being well known—see for instance [34] or [37]).

Theorem 3. (Adams–Orobitg–Verdera) *Let $0<\alpha<N$ and $(N-\alpha)/N<p<\infty$. Then there is a constant $c_{\alpha,p,N}>0$ depending only on α, p and N such that*

$$\int_{\mathbb{R}^N} (M_0 f)^p \, d\Lambda^{(\infty)}_{N-\alpha} \le c_{\alpha,p,N} \int_{\mathbb{R}^N} |f|^p \, d\Lambda^{(\infty)}_{N-\alpha}$$

holds for all real-valued functions f with the right-hand-integral being finite.

2.3. Sobolev-type imbeddings

For $\alpha\in(0,N)$, the local integrability of $|x|^{-\alpha}$ generates a Riesz operator or a negative power of the Laplace operator, denoted by $\mathcal{I}_\alpha$ or $(-\Delta)^{-\alpha/2}$, via the Fourier transform (cf. [5] or [33]):

$$\widehat{\mathcal{I}_\alpha f} = \widehat{(-\Delta)^{-\alpha/2} f} = |x|^{-\alpha} \hat{f}$$

for any $f\in\mathcal{S}$ (the Schwartz class of rapidly decreasing C^∞ functions on $\mathbb{R}^N$). If

$$I_\alpha(x) = \frac{\Gamma\big(\frac{1}{2}(N-\alpha)\big)}{\pi^{N/2} 2^\alpha \Gamma\big(\frac{1}{2}\alpha\big)} |x|^{\alpha-N},$$

denotes the αth order Riesz kernel, then any $f\in\mathcal{S}$ can be represented as a Riesz potential:

$$f(x) = \mathcal{I}_\alpha (-\Delta)^{\alpha/2} f(x) = \int_{\mathbb{R}^N} (-\Delta)^{\alpha/2} f(y) I_\alpha(x-y)\, dy.$$

In other words, if $g=(-\Delta)^{\alpha/2} f$, then

$$u(x) = \int_{\mathbb{R}^N} I_\alpha(x-y) g(y)\, dy$$

solves the $\frac{1}{2}\alpha$th order Laplace equation

$$(-\Delta)^{\alpha/2} u = g,$$

and hence $I_\alpha(x,y)=I_\alpha(|x-y|)$ can be treated as the Green function for this generalized Laplace equation on $\mathbb{R}^N$. Here and henceforth, the symbol $(-\Delta)^{\alpha/2}$ stands for the $\frac{1}{2}\alpha$th order Laplacian defined by

$$\widehat{(-\Delta)^{\alpha/2} f} = |x|^\alpha \hat{f}, \quad f\in\mathcal{S}.$$

For its applications to a singular obstacle problem [12], this operator can be evaluated via

$$\lim_{r\downarrow 0}\int_{\mathbb{R}^N\setminus B(x,r)}\frac{f(x)-f(y)}{|x-y|^{N+\alpha}}\,dy.$$

Next, for $\alpha\in(0,N)$ and $1<p<\infty$ denote by

$$\dot{L}^{\alpha,p}=\{f:f=\mathcal{I}_\alpha g \text{ and } g\in L^p\}$$

the Sobolev space of all (α,p)-potentials on $\mathbb{R}^N$, where L^p stands for the Lebesgue space of all p-integrable functions on $\mathbb{R}^N$. Then $\dot{L}^{\alpha,p}$ induces the following (α,p)-Riesz capacity

$$C_\alpha(E;L^p)=\inf\left\{\int_{\mathbb{R}^N}g(y)^p\,dy:1_E\le\mathcal{I}_\alpha g\in\dot{L}^{\alpha,p} \text{ and } g\ge 0\right\}$$

of a set $E\subseteq\mathbb{R}^N$, where 1_E is the characteristic function of E.

The following Sobolev-type imbedding is a new variant of [5, Theorem 7.2.2].

Theorem 4. *Given $1<p<q<\infty$ and $0<\alpha p<N$, let μ be a non-negative Borel measure on $\mathbb{R}^N$ and $L^q(\mu)$ be the Lebesgue space of all q-integrable functions on $\mathbb{R}^N$ with respect to μ. Then the following properties are mutually equivalent*:

(i) *$\mathcal{I}_\alpha$ is a continuous operator from L^p into $L^q(\mu)$*;

(ii) *The global Riesz's kernel decay inequality*

$$\mu(\{y\in\mathbb{R}^N:I_\alpha(x,y)>t\})\lesssim t^{-q(N-\alpha p)/p(N-\alpha)}$$

holds for all $t\in\mathbb{R}_+$;

(iii) *The isocapacitary-type inequality*

$$\mu(K)\lesssim C_\alpha(K;L^p)^{q/p}$$

holds for all compact sets $K\subseteq\mathbb{R}^N$;

(iv) *The Faber–Krahn type inequality*

$$\mu(\Omega)^{p/q-1}\lesssim\lambda_{\alpha,p,\mu}(\Omega)$$

holds for all bounded open sets $\Omega\subseteq\mathbb{R}^N$, where

$$\lambda_{\alpha,p,\mu}(\Omega)=\inf\left\{\frac{\int_\Omega|f(y)|^p\,dy}{\int_\Omega|\mathcal{I}_\alpha f|^p\,d\mu}:f\in C_0^\infty(\Omega) \text{ and } f\not\equiv 0 \textit{ on } \Omega\right\}$$

and $C_0^\infty(\Omega)$ stands for the class of all C^∞ functions with compact support in Ω.

Proof. Note that (i) ⇔ (ii) ⇔ (iii) is a consequence of Theorems 7.2.1–7.2.2 and Proposition 5.1.2 of [5]. So, it remains to verify (i) ⇔ (iv).

Suppose (i) is valid. Using this assumption and Hölder's inequality we obtain that for $f\in C_0^\infty(\Omega)$ with $f\not\equiv0$ on any bounded open set Ω,

$$\begin{aligned}\int_\Omega |\mathcal{I}_\alpha f|^p\,d\mu &\le \biggl(\int_\Omega |\mathcal{I}_\alpha f|^q\,d\mu\biggr)^{p/q}\mu(\Omega)^{1-p/q}\\ &\lesssim \biggl(\int_{\mathbb{R}^N} |f(y)|^p\,dy\biggr)\mu(\Omega)^{1-p/q}\\ &\lesssim \biggl(\int_\Omega |f(y)|^p\,dy\biggr)\mu(\Omega)^{1-p/q},\end{aligned}$$

whence getting

$$\lambda_{\alpha,p,\mu}(\Omega)^{-1}\lesssim\mu(\Omega)^{1-p/q},$$

i.e., (iv) is true. Conversely, suppose (iv) is valid. Then, for any $f\in C_0^\infty(\Omega)$ with $f\not\equiv0$ and Ω being a bounded open set, we have

$$\lambda_{\alpha,p,\mu}(\Omega)\le\frac{\int_\Omega |f(y)|^p\,dy}{\int_\Omega |\mathcal{I}_\alpha f(y)|^p\,d\mu(y)}\le\frac{\int_{\mathbb{R}^N} |f(y)|^p\,dy}{\mu(\Omega)}$$

provided that $\mathcal{I}_\alpha f\ge1$ on Ω. This, in turn, shows that

$$\lambda_{\alpha,p,\mu}(\Omega)\le\frac{C_\alpha(\Omega;L^p)}{\mu(\Omega)}.$$

Now, an application of the assumption (iv) gives

$$\mu(\Omega)\lesssim C_\alpha(\Omega;L^p)^{q/p}.$$

Consequently, (i) is true, due to (iii) ⇔ (i). □

Remark 5. Here it is worth making two comments on Theorem 4.

(i) If $d\mu(y)=|f(y)|^r\,dy$ and $\gamma=N-q(N-\alpha p)/p>0$ then condition (ii) of Theorem 4 says that f belongs to $L^{r,\gamma}$. In other words, this Morrey space describes the corresponding Sobolev imbedding.

(ii) $\lambda_{\alpha,p,\mu}(\Omega)^{1/p}$ is a variant of the first eigenvalue of $(-\Delta)^{\alpha/2}$—in particular, when $p=2$, $q=2N/(N-2)$, $\alpha=1$ and $d\mu(y)=dy$ in Theorem 4, one has

$$\lambda_{\alpha,p,\mu}(\Omega)^{1/p}\gtrsim\mu(\Omega)^{-1/N}.$$

Corresponding to this, the first eigenvalue $\beta_1(\Omega)$ of the $\frac12$th order Laplacian $(-\Delta)^{1/2}$ on Ω (a special case of the Klein–Gordon operator) enjoys (cf. [22])

$$\beta_1(\Omega)\gtrsim\mu(\Omega)^{-1/N}.$$

3. Preduals and Zorko spaces

This section treats the matter of the predual space of a Morrey space, and introduces the Zorko space whose dual is identical to the predual.

3.1. Predual spaces

To prove that $H^{p',\lambda}$ introduced in Section 1 is the predual of $L^{p,\lambda}$, let us first set

$$A_1^{(N-\lambda)} = \left\{ w \in A_1 : \int_{\mathbb{R}^N} w \, d\Lambda_{N-\lambda}^{(\infty)} \le 1 \right\}.$$

Note that $A_1^{(N-\lambda)} \subseteq L^1(\Lambda_{N-\lambda}^{(\infty)})$, the quasi-continuity being immediate by Theorem 3. Also, we need the following preliminary result from [6, Theorem 2.3].

Theorem 6. (Adams–Xiao) *For*

$$1 < p < \infty, \quad p' = \frac{p}{p-1}, \quad \text{and} \quad 0 < \lambda < N,$$

the space

$$\widetilde{H}^{p',\lambda} = \left\{ g \in L^{p'}_{\mathrm{loc}} : \|g\|_{\widetilde{H}^{p',\lambda}} = \inf_w \left(\int_{\mathbb{R}^N} |g(y)|^{p'} w(y)^{1-p'} \, dy \right)^{1/p'} < \infty \right\}$$

is the predual of $L^{p,\lambda}$, where w satisfies:

$$w \in L^1(\Lambda_{N-\alpha}^{(\infty)}) \quad \text{such that} \quad \int_{\mathbb{R}^N} w \, d\Lambda_{N-\alpha}^{(\infty)} \le 1.$$

In particular,

$$\|f\|_{L^{p,\lambda}} = \sup_g \left| \int_{\mathbb{R}^N} f(y) g(y) \, dy \right| \quad \text{for } f \in L^{p,\lambda},$$

where the supremum is over all $g \in \widetilde{H}^{p',\lambda}$ such that $\|g\|_{\widetilde{H}^{p',\lambda}} \le 1$.

The above consideration leads to the following equivalence.

Theorem 7. *Let $1<p<\infty$, $p'=p/(p-1)$, and $0<\lambda<N$. Then the space $\widetilde{H}^{p',\lambda}$ is equivalent to the space $H^{p',\lambda}$.*

Proof. Clearly, $\|f\|_{H^{p',\lambda}} \ge \|f\|_{\widetilde{H}^{p',\lambda}}$ since we are taking the infimum over a larger set on the right-hand side of this inequality. For the reverse inequality, given a weight w used to define $\widetilde{H}^{p',\lambda}$, we construct a weight in A_1:

$$w_\theta = (M_0 w^{1/\theta})^\theta, \quad 0 < \theta < 1.$$

On the one hand, there is a constant $c_0>0$ such that

$$\int_{\mathbb{R}^N} w_\theta \, d\Lambda_{N-\lambda}^{(\infty)} \le c_0 \int_{\mathbb{R}^N} w \, d\Lambda_{N-\lambda}^{(\infty)} \le c_0 \quad \text{for } \frac{\lambda}{N} < \theta,$$

by Theorem 3. On the other hand, for any $\theta\in(0,1)$ we have $w_\theta \in A_1$ by the classical construction of Coifman–Rochberg; see [37] or [34]. So, w_θ/c_0 belongs to $A_1^{(N-\lambda)}$. Since $w_\theta \gtrsim w$ holds almost everywhere, we conclude that

$$\|f\|_{H^{p',\lambda}}^{p'} \le \int_{\mathbb{R}^N} |f(y)|^{p'} w_\theta(y)^{1-p'} \, dy \lesssim \int_{\mathbb{R}^N} |f(y)|^{p'} w^{1-p'}(y) \, dy$$

thereby reaching $\|f\|_{H^{p',\lambda}} \lesssim \|f\|_{\widetilde{H}^{p',\lambda}}$. Therefore we have in the notation of functional analysis

$$(H^{p,\lambda})^* = L^{p,\lambda}. \quad \square$$

3.2. Zorko spaces

Now as pointed out in [40], C_0, is not dense in $L^{p,\lambda}$. So, Zorko defined a subspace of $L^{p,\lambda}$ using

$$\lim_{y\to 0} \|f(y+\cdot\,)-f(\,\cdot\,)\|_{L^{p,\lambda}} = 0 \quad \text{for } f\in L^{p,\lambda}.$$

This method appeared also in [31] to characterize VMO. For the purpose of this paper, we write $L_0^{p,\lambda}$ for the Zorko space that is now defined as the closure of C_0 in the $L^{p,\lambda}$-norm. It then follows from

$$\left| \int_{\mathbb{R}^N} f(y) g(y) \, dy \right| \le \|f\|_{L^{p,\lambda}} \|g\|_{H^{p',\lambda}}$$

that $L_0^{p,\lambda}$ is the predual to $H^{p',\lambda}$. Hence in particular the three spaces

$$L_0^{p,\lambda} - H^{p',\lambda} - L^{p,\lambda}$$

have a relationship akin to

$$\text{VMO} - H^1 - \text{BMO}$$

in harmonic analysis; see [34] or [37].

The following tells us more about the foregoing new triplet.

Theorem 8. *Let* $(\lambda,p)\in(0,N)\times(1,\infty)$ *and* $\lambda p/(p-1)<N$. *Then*

$$\mathcal{I}_{N-\lambda} : H^{p,\lambda} \to L_0^{p,\lambda}$$

is continuous.

Proof. If $f\geq 0$, $(x,r)\in\mathbb{R}^N\times\mathbb{R}_+$, and $w\geq 0$, we estimate

$$\begin{aligned}\int_{B(x,r)}(\mathcal{I}_{N-\lambda}f(y))^p\,dy &\lesssim \int_{B(x,r)}\left(\int_{\mathbb{R}^N}|y-z|^{-\lambda}f(z)\,dz\right)^p dy\\ &\lesssim \int_{B(x,r)}\left(\int_{\mathbb{R}^N}|y-z|^{-\lambda p/(p-1)}w(z)\,dz\right)^{p-1}\left(\int_{\mathbb{R}^N}|f(z)|^p w(z)^{1-p}\,dz\right)dy\\ &\lesssim \left(\int_{\mathbb{R}^N}|f(z)|^p w(z)^{1-p}\,dz\right)\int_{B(x,r)}(\mathcal{I}_{N-\lambda p/(p-1)}w(y))^{p-1}\,dy.\end{aligned}$$

But, from [27] and [4] it follows that

$$\int_{\mathbb{R}^N} w(y)^{N/(N-\lambda)}\,dy\lesssim\left(\int_{\mathbb{R}^N} w\,d\Lambda^{(\infty)}_{N-\lambda}\right)^{N/(N-\lambda)}.$$

Hence by Theorem 4 with

$$d\mu(y)=dy,\quad p=\frac{N}{N-\lambda},\quad q=\frac{pN}{N-\alpha p},\quad \text{and}\quad \alpha=N-\frac{\lambda p}{p-1},$$

we see that

$$w\in L^{N/(N-\lambda)}\quad\Longrightarrow\quad \mathcal{I}_{N-\lambda p/(p-1)}w\in L^{(p-1)N/\lambda}.$$

Consequently, if

$$\int_{\mathbb{R}^N} w\,d\Lambda^{(\infty)}_{N-\lambda}\leq 1$$

then

$$\begin{aligned}\int_{B(x,r)}(\mathcal{I}_{N-\lambda p/(p-1)}w)^{p-1}\,dy &\lesssim r^{N-\lambda}\left(\int_{B(x,r)}(\mathcal{I}_{N-\lambda p/(p-1)}w)^{N(p-1)/\lambda}\,dy\right)^{\lambda/N}\\ &\lesssim r^{N-\lambda}\left(\int_{\mathbb{R}^N} w(y)^{N/(N-\lambda)}\,dy\right)^{(N-\lambda)/N}\\ &\lesssim r^{N-\lambda},\end{aligned}$$

which, along with Theorem 6 or 7, completes the estimate

$$\|\mathcal{I}_{N-\lambda}f\|_{L^{p,\lambda}}\lesssim\|f\|_{H^{p,\lambda}}.$$

The fact that $\mathcal{I}_{N-\lambda}$ maps $H^{p,\lambda}$ into $L_0^{p,\lambda}$ is a consequence of this last a priori estimate since C_0 is dense in $H^{p,\lambda}$. □

Remark 9. Let $1<p<\infty$ and $\lambda/p<\alpha\le N/p$. Then in view of Morrey's lemma recalled in Section 1, $\mathcal{I}_\alpha:L^{p,\lambda}\to C^{\alpha-\lambda/p}$ is continuous. This result can be established via [24, Lemma 1.34]. Therefore, we may view the Morrey space $L^{p,\lambda}$ as a "Morrey-bridge" from $H^{p,\lambda}$ to the Hölder continuous functions via

$$H^{p,\lambda}\xrightarrow{\mathcal{I}_{N-\lambda}}L^{p,\lambda}\xrightarrow{\mathcal{I}_\alpha}C^{\alpha-\lambda/p},$$

where $\lambda/p<\alpha<\lambda<N(p-1)/p$.

4. Maximal and potential operators

In this section, we consider the general order maximal and potential operators acting on the Morrey spaces and their preduals through Section 3.

4.1. Hardy–Littlewood maximal operators

First of all, we note that the Hardy–Littlewood maximal operator M_0 is bounded on the $L^{p,\lambda}$ spaces, a fact already recorded in [15], but now we use the duality theory established in Section 3 to obtain it. The following appeared first in [6, Theorem 2.2].

Theorem 10. (Adams–Xiao) *For $1<p<\infty$ and $0<\lambda<N$ let $f\in L^{p,\lambda}$. Then*

$$\sup_w\int_{\mathbb{R}^N}|f(y)|^p w(y)\,dy=\|f\|^p_{L^{p,\lambda}},$$

where the supremum is taken over all non-negative $w\in B_1^{(N-\lambda)}$, i.e.,

$$0\le w\in L^1(\Lambda^{(\infty)}_{N-\lambda})\quad \textit{such that}\quad \int_{\mathbb{R}^N}w\,d\Lambda^{(\infty)}_{N-\lambda}\le 1.$$

This result is used to prove the following lemma.

Lemma 11. *For $1<p<\infty$ and $0<\lambda<N$ let $f\in L^{p,\lambda}$. Then*

$$\|f\|^p_{L^{p,\lambda}}=\sup_w\int_{\mathbb{R}^N}|f(y)|^p w(y)\,dy,$$

where the supremum is taken over all $w \in A_1^{(N-\lambda)}$, *i.e.*,

$$w \in A_1 \quad \textit{such that} \quad \int_{\mathbb{R}^N} w \, d\Lambda_{N-\lambda}^{(\infty)} \le 1.$$

Proof. Clearly, the class $A_1^{(N-\lambda)}$ is smaller than $B_1^{(N-\lambda)}$. Hence

$$\sup_{w \in A_1^{N-\lambda}} \int_{\mathbb{R}^N} |f(y)|^p w(y) \, dy \le \|f\|_{L^{p,\lambda}}^p.$$

Once again, constructing the weight

$$w_\theta = (M_0 w^{1/\theta})^\theta, \qquad \frac{N-\lambda}{N} < \theta < 1,$$

we have by Theorem 3 that

$$\int_{\mathbb{R}^N} w_\theta \, d\Lambda_{N-\lambda}^{(\infty)} \le c_0 \int_{\mathbb{R}^N} w \, d\Lambda_{N-\lambda}^{(\infty)} \le c_0$$

holds for some constant $c_0 > 0$, and so that $w_\theta / c_0 \in A_1^{(N-\lambda)}$. Accordingly,

$$\frac{1}{c_0} \int_{\mathbb{R}^N} |f(y)|^p w(y) \, dy \le \sup_{\omega \in A_1^{(N-\lambda)}} \int_{\mathbb{R}^N} |f(y)|^p \omega(y) \, dy.$$

This shows that

$$\sup_{w \in A_1^{N-\lambda}} \int_{\mathbb{R}^N} |f(y)|^p w(y) \, dy \ge \|f\|_{L^{p,\lambda}}^p. \qquad \square$$

The last lemma yields the following boundedness of M_0 acting on the dual pair $(H^{p',\lambda}, L^{p,\lambda})$.

Theorem 12. *Let* $p \in (1,\infty)$, $p' = p/(p-1)$, *and* $\lambda \in (0,N)$. *Then*
(i) M_0 *is bounded on* $L^{p,\lambda}$;
(ii) M_0 *is bounded on* $H^{p',\lambda}$.

Proof. (i) Because $w \in A_1^{(N-\lambda)}$ yields $w^{1-p} \in A_p$, we find that

$$\int_{\mathbb{R}^N} (M_0 f(y))^p w(y) \, dy \lesssim \int_{\mathbb{R}^N} |f(y)|^p w(y) \, dy$$

and use Lemma 11 to derive the boundedness of M_0 on $L^{p,\lambda}$.

(ii) Since $w \in A_1$ implies $w^{1-p'} \in A_{p'}$, we conclude that

$$\int_{\mathbb{R}^N} (M_0 f(y))^{p'} w(y)^{1-p'} \, dy \lesssim \int_{\mathbb{R}^N} |f(y)|^{p'} w(y)^{1-p'} \, dy,$$

whence getting (ii) via the definition of $H^{p',\lambda}$ and Muckenhoupt's original argument for boundedness of M_0 on a weighted Lebesgue space. $\square$

The previous idea leads to the following more general result.

Theorem 13. *Let $\lambda\in(0,N)$ and let $p\in(1,\infty)$. If T is a classical Calderón–Zygmund operator, then T is bounded on $L^{p,\lambda}$ and $H^{p',\lambda}$.*

Proof. Suppose T^* is the adjoint of T. Then for $f\in L^{p,\lambda}$ and $g\in H^{p',\lambda}$ one has

$$\begin{aligned}\left|\int_{\mathbb{R}^N}(Tf)(y)g(y)\,dy\right| &= \left|\int_{\mathbb{R}^N} f(y)(T^*g)(y)\,dy\right| \\ &\le \|f\|_{L^{p,\lambda}}\|T^*g\|_{H^{p',\lambda}} \\ &\lesssim \|f\|_{L^{p,\lambda}}\|g\|_{H^{p',\lambda}}.\end{aligned}$$

The last inequality follows from (ii) above and the well-known A_p-weight estimates arising from

$$w\in A_1 \quad\Longrightarrow\quad w\in A_{p'} \quad\Longrightarrow\quad w^{1-p}\in A_p.$$

Thus, T is bounded on $L^{p,\lambda}$, and similarly on $H^{p',\lambda}$. $\square$

4.2. αth maximal and potential operators

For $\alpha\in(0,N)$, let μ be a non-negative Borel measure on $\mathbb{R}^N$, and denote by

$$\mathcal{I}_\alpha\mu(x)=\frac{\Gamma\left(\frac{1}{2}(N-\alpha)\right)}{\pi^{N/2}2^\alpha\Gamma\left(\frac{1}{2}\alpha\right)}\int_{\mathbb{R}^N}|x-y|^{\alpha-N}\,d\mu(y),\quad x\in\mathbb{R}^N$$

and

$$M_\alpha\mu(x)=\sup_{r\in\mathbb{R}_+} r^{\alpha-N}\mu(B(x,r))$$

its αth order Riesz potential and αth order fractional maximal function, respectively.

Theorem 14. *With $\lambda\in(0,N)$, $p\in(1,\infty)$, $M_\alpha\mu$, and $\mathcal{I}_\alpha\mu$ as above, we have*
(i) $\|\mathcal{I}_\alpha\mu\|_{L^{p,\lambda}}\lesssim\|M_\alpha\mu\|_{L^{p,\lambda}}$;
(ii) $\|\mathcal{I}_\alpha\mu\|_{H^{p,\lambda}}\lesssim\|M_\alpha\mu\|_{H^{p,\lambda}}$.

Proof. We begin the proof of (i) by slightly modifying the Stein duality inequality (see [34, pp. 146–148]): for $f\in L^{p_1}+L^{p_2}$, where $1<p_1<p_2<\infty$, and $g\in L^p$, where $p>1$, one has

$$\left|\int_{\mathbb{R}^N} f(y)g(y)\,dy\right| \lesssim \int_{\mathbb{R}^N} f^{\#}(y)M_0 g(y)\,dy.$$

Here and henceforth, $f^{\#}$ is the standard Fefferman–Stein sharp function

$$f^{\#}(x)=\sup_{r>0} r^{-N}\int_{B(x,r)} |f(y)-f_r(x)|\,dy,$$

and $f\in L^{p_1}+L^{p_2}$ means that f can be written as a sum $f=f_1+f_2$, where $f_j\in L^{p_j}$, $j=1,2$. For us in this proof, we take $f=\mathcal{I}_\alpha\mu$, μ with compact support and $1<p_1<N/(N-\alpha)<p_2<\infty$, and g will be a non-negative function with compact support. It then follows from the last duality inequality involving # and M_0 that

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{I}_\alpha\mu(y)g(y)\,dy &\lesssim \int_{\mathbb{R}^N} (\mathcal{I}_\alpha\mu)^{\#}(y)M_0 g(y)\,dy\\ &\lesssim \int_{\mathbb{R}^N} M_\alpha\mu(y)M_0 g(y)\,dy\\ &\lesssim \|M_\alpha\mu\|_{L^{p,\lambda}}\|M_0 g\|_{H^{p',\lambda}}\\ &\lesssim \|M_\alpha\mu\|_{L^{p,\lambda}}\|g\|_{H^{p',\lambda}}.\end{aligned}$$

Here we have used the equivalent estimate in [1]:

$$(\mathcal{I}_\alpha\mu)^{\#}\sim M_\alpha\mu,$$

but also Theorem 12(ii). The final result of (i) follows from the monotone convergence theorem and the $H^{p',\lambda}-L^{p,\lambda}$ duality.

For (ii), we proceed as above, but now we choose a suitable $f\in L_0^{p,\lambda}$ and we invoke the $L_0^{p,\lambda}-H^{p',\lambda}$ duality. □

Remark 15. Here, it is appropriate to point out that Theorem 14(i) appeared first in [6] but its proof contained an incomplete inequality (4.8) whose accurate form is

$$\int_Q |\mathcal{I}_\alpha\mu(y)-(\mathcal{I}_\alpha\mu)_Q|^p\,dy \lesssim \int_Q ((\mathcal{I}_\alpha\mu)^{\#}(y))^p\,dy,$$

which implies Theorem 14(i) right away, where

$$(\mathcal{I}_\alpha\mu)_Q=\frac{1}{|Q|}\int_Q \mathcal{I}_\alpha\mu(y)\,dy$$

and $|E|$ is the Lebesgue measure of a set $E\subseteq\mathbb{R}^N$. To see the last inequality, we need a simple revision of [6, Lemma 4.1(ii)] and its proof. In fact, given a cube Q we may assume $(\mathcal{I}_\alpha\mu)_Q=0$, otherwise $\mathcal{I}_\alpha\mu$ is replaced by $\mathcal{I}_\alpha\mu-(\mathcal{I}_\alpha\mu)_Q$. According to the Calderón–Zygmund decomposition for Q, $t>0$, and $\mathcal{I}_\alpha\mu$, described in the argument for [6, Lemma 4.1(ii)], we have

$$Q=P^t\cup Q^t \quad \text{and} \quad Q^t=\bigcup_{k=1}^{\infty} Q_k^t.$$

Over there, redefining $M(t)$ as $\sum_{k=1}^{\infty}|Q_k^t|$ and setting $\widetilde{Q}_k^t=2Q_k^t$ (the cube with twice the side length and the same center as Q_k^t), we find that $x\notin\bigcup_{k=1}^{\infty}\widetilde{Q}_k^t$ implies $\mathcal{I}_\alpha\mu(x)\le 5^N t$ and hence

$$|\{x\in Q:|\mathcal{I}_\alpha\mu(x)|>5^N t\}|\le\left|\bigcup_{k=1}^{\infty}\widetilde{Q}_k^t\right|=2^N M(t);$$

see also [30, p. 217]. Note that [6, (4.5)] is still valid for the above redefined $M(t)$. So, the correct estimate in the statement of [6, Lemma 4.1(ii)] is

$$M(t)\le\left|\left\{x\in Q:(\mathcal{I}_\alpha\mu)^{\#}(x)>\tfrac{1}{2}\varepsilon t\right\}\right|+\varepsilon M(2^{-N-1}t).$$

Choosing $\varepsilon=2^{-1-p(N+1)}$ in the above yields

$$\int_0^\infty M(t)\,dt^p\le 2^{1+p(2+p(N+1))}\int_Q((\mathcal{I}_\alpha\mu)^{\#}(x))^p\,dx.$$

Consequently, the final group of inequalities on [6, p. 1641] is rewritten as

$$\begin{aligned}\int_Q|\mathcal{I}_\alpha\mu(y)|^p\,dy&=5^{pN}\int_0^\infty|\{x\in Q:|\mathcal{I}_\alpha\mu(x)|>5^N t\}|\,dt^p\\&\le 2^N 5^{pN}\int_0^\infty M(t)\,dt^p\\&\le 5^{pN}2^{N+1+p(2+p(N+1))}\int_Q(\mathcal{I}_\alpha\mu)^{\#}(x)^p\,dx,\end{aligned}$$

reaching the desired inequality.

Now, we turn to the action of the Riesz potential operator $\mathcal{I}_\alpha$ on both $L^{p,\lambda}$ and $H^{p,\lambda}$. The first part of the following result is known (see for instance [1]), but we give a proof following our ideas above.

Theorem 16. *Let* $\alpha\in(0,N)$ *and* $1<p<\min\{\lambda/\alpha,(p-1)\lambda/\alpha\}$. *Then*

(i)

$$\mathcal{I}_\alpha : L^{p,\lambda} \longrightarrow L^{\lambda p/(\lambda-\alpha p),\lambda}\cap L^{p,\lambda-\alpha p}$$

is continuous;

(ii)

$$\mathcal{I}_\alpha : H^{p,\lambda} \longrightarrow H^{\lambda p/(\lambda-\alpha p),\lambda}\cap H^{p,\lambda-\alpha p/(p-1)}$$

is continuous.

Proof. For (i), we use the following fundamental inequality for any $f\geq 0$,

$$\mathcal{I}_\alpha f(x)\lesssim (M_{\lambda/p}f(x))^{\alpha p/\lambda}(M_0 f(x))^{1-\alpha p/\lambda},$$

which was proved in [1] and [2].

To get the boundedness of

$$\mathcal{I}_\alpha :\ L^{p,\lambda} \longrightarrow L^{\lambda p/(\lambda-\alpha p),\lambda}$$

we use the last estimate to obtain

$$\int_{B(x,r)} (\mathcal{I}_\alpha f(y))^{\lambda p/(\lambda-\alpha p)}\,dy \lesssim \|f\|_{L^{p,\lambda}}^{\alpha p/\lambda}\int_{B(x,r)}(M_0 f(y))^p\,dy$$

whence getting the result via Theorem 12(i).

In order to derive the boundedness of

$$\mathcal{I}_\alpha :\ L^{p,\lambda} \longrightarrow L^{p,\lambda-\alpha p}$$

we once again use the fundamental estimate to obtain

$$\int_{B(x,r)} (\mathcal{I}_\alpha f(y))^{p}\,dy \lesssim \|f\|_{L^{p,\lambda}}^{\alpha p^2/\lambda}\int_{B(x,r)}(M_0 f(y))^{p(\lambda-\alpha p)/\lambda}\,dy,$$

whence deriving the result via Hölder's inequality and Theorem 12(i).

For (ii), we just use the duality $(H^{p',\lambda})^*=L^{p,\lambda}$, (i), and the following estimate

$$\left|\int_{\mathbb{R}^N}\mathcal{I}_\alpha f(y)g(y)\,dy\right| = \left|\int_{\mathbb{R}^N} f(y)\mathcal{I}_\alpha g(y)\,dy\right| \lesssim \|f\|_{H^{p,\lambda}}\|\mathcal{I}_\alpha g\|_{L^{p',\lambda}}.$$

There are two cases handled as follows.

Case 1. Choosing $\tilde{p}=p\lambda/(p\alpha+(p-1)\lambda)$, we use the first imbedding of (i) to obtain

$$\|\mathcal{I}_\alpha g\|_{L^{p',\lambda}}\lesssim \|g\|_{L^{\tilde{p},\lambda}},\quad g\in L_0^{\tilde{p},\lambda},$$

whence getting via duality of the Morrey spaces,

$$\|\mathcal{I}_\alpha f\|_{H^{\lambda/p(\lambda-\alpha p)},\lambda}\lesssim \|f\|_{H^{p,\lambda}},\quad f\in H^{p,\lambda}.$$

Case 2. Choosing $\tilde{\lambda}=\lambda-p\alpha/(p-1)$, we use the second imbedding of (i) to derive

$$\|\mathcal{I}_\alpha g\|_{L^{p',\lambda}} \lesssim \|g\|_{L^{p',\tilde{\lambda}}}, \quad g\in L_0^{p',\tilde{\lambda}},$$

whence finding via duality of the Morrey spaces,

$$\|\mathcal{I}_\alpha f\|_{H^{p,\lambda-\alpha/p(p-1)}} \lesssim \|f\|_{H^{p,\lambda}}, \quad f\in H^{p,\lambda}. \quad \square$$

5. Interpolation of operators

In this section, we work on the interpolation theory of operators on the Morrey spaces, now with the $L_0^{p,\lambda}$ spaces in the domain rather than the range.

5.1. Atomic decomposition for $L^1(\Lambda_\alpha^{(\infty)})$

We need the following atomic decomposition of this Choquet space (provided in [6, Remark 3.4]), which differs from Zorko's $(p', N-\lambda)$-atomic decomposition of the predual of a Morrey space.

Lemma 17. (Adams–Xiao) *Let* $\alpha\in(0,N)$. *Then* $w\in L^1(\Lambda_\alpha^{(\infty)})$ *if and only if* $w=\sum_{k=1}^\infty b_k a_k$, *where* $\{b_k\}_{k=1}^\infty\in l^1$ *and the* a_k *are* (∞,α)*-atoms, i.e., for each natural number* k *there is a cube* Q_k *such that*

$$\operatorname{supp} a_k \subseteq Q_k \quad \textit{with} \quad \|a_k\|_{L^\infty(Q_k)} \le \ell(Q_k)^{-\alpha}.$$

Also, the infimum $\inf\sum_{k=1}^\infty |b_k|$ *over all possible such* $\{b_k\}_{k=1}^\infty$ *is comparable to the norm of* $L^1(\Lambda_\alpha^{(\infty)})$, *i.e.,*

$$\int_{\mathbb{R}^N} |w|\, d\Lambda_\alpha^{(\infty)} \sim \inf \sum_{k=1}^\infty |b_k|.$$

5.2. Interpolation for $(L_0^{p,\lambda}, L^q)$

As noticed in Section 1, there are λ and T such that T maps $L^{p_i,\lambda}$ into L^{q_i} but T does not send $L^{p_\theta,\lambda}$ into L^{q_θ}, where p_θ and q_θ are the standard intermediate values for p_i and q_i. Therefore, the following result appears very natural.

Theorem 18. *Let* T *be a linear operator with boundedness of*

$$T\colon L_0^{p_i,\lambda_i} \longrightarrow L^{q_i}, \quad \textit{where } 1<p_i,q_i<\infty,\ i=0,1,$$

i.e., there are two positive constants c_1 and c_2 such that

$$\left(\int_{\mathbb{R}^N} |Tf(y)|^{q_i}\,dy\right)^{1/q_i} \le c_i \sup_{(x,r)\in\mathbb{R}^N\times\mathbb{R}_+} \left(r^{\lambda_i - N}\int_{B(x,r)} |f(y)|^{p_i}\,dy\right)^{1/p_i}$$

for all functions $f\in C_0$, then $T\colon L^{p_\theta,\lambda_\theta}\to L^{q_\theta}$ is bounded, where

$$\frac{\lambda_\theta}{p_\theta} = (1-\theta)\frac{\lambda_0}{p_0} + \theta\frac{\lambda_1}{p_1}, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Proof. Our hypothesis implies that the duality operator T^* of T enjoys

$$T^*\colon L^{q_i'} \longrightarrow H^{p_i',\lambda_i},$$

where

$$1 < p_i' = \frac{p_i}{p_i - 1}, \text{ and } q_i' = \frac{q_i}{q_i - 1} < \infty, \quad i = 0, 1,$$

i.e., duality gives

$$T\colon\ L^q \longrightarrow L^{p,\lambda} \qquad \Longleftrightarrow \qquad T^*\colon\ H^{p',\lambda} \longrightarrow L^{q'}$$

but

$$T^*\colon\ L^{q'} \longrightarrow H^{p',\lambda} \quad \Longrightarrow \quad T\colon\ L^{p,\lambda} \longrightarrow L^q,$$

and

$$T\colon\ L_0^{p,\lambda} \longrightarrow L^q \qquad \Longrightarrow \qquad T^*\colon\ L^{q'} \longrightarrow H^{p',\lambda}.$$

G. Stampacchia [32] obtained the case

$$T\colon L^{q_i} \longrightarrow L^{p_i,\lambda_i}, \quad i = 0, 1,$$

and then concluded that

$$T\colon L^{q_\theta} \longrightarrow L^{p_\theta,\lambda_\theta}.$$

There, the proof was easy:

$$\sup_{(x,r)\in\mathbb{R}^N\times\mathbb{R}_+} \left(r^{\lambda_i - N}\int_{B(x,r)} |f(y)|^{p_i}\,dy\right)^{1/p_i} \le \tau_i \left(\int_{\mathbb{R}^N} |f(y)|^{q_i}\,dy\right)^{1/q_i}, \quad i = 0, 1,$$

implies

$$\int_{B(x,r)} |f(y)|^{p_i}\,dy \le \tau_i^{p_i} r^{N-\lambda_i} \int_{\mathbb{R}^N} |f(y)|^{q_i}\,dy,$$

where τ_1 and τ_2 are positive constants. Consequently, by the standard interpolation theory, the norm M_θ for $L^{p_\theta}(B(x,r))$ (the Lebesgue p_θ-space over $B(x,r)$) satisfies

$$\mathsf{M}_\theta \le \mathsf{M}_0^{1-\theta}\mathsf{M}_1^{\theta},$$

where

$$\mathsf{M}_i = \tau_i r^{(N-\lambda_i)/p_i}, \quad i=0,1.$$

So, the result follows.

This approach does not work if a Morrey space is in the domain of T. Nevertheless, for this case, we use duality to get

$$T^*\colon L^{q_i'} \longrightarrow H^{p_i',\lambda_i}, \quad i=0,1.$$

Then for each i there is a $w_i \in A_1^{(N-\lambda_i)}$ such that

$$\left(\int_{\mathbb{R}^N} |T^* f(y)|^{p_i'} w_i(y)^{1-p_i'}\,dy\right)^{1/p_i'} \le \tau_i \left(\int_{\mathbb{R}^N} |f(y)|^{q_i'}\,dy\right)^{1/q_i'}.$$

So, if we apply Stein's interpolation theorem (with change of measure, see [9]), we get

$$\left(\int_{\mathbb{R}^N} |T^* f(y)|^{p_\theta'} (u_0(y)^{1-\theta} u_1(y)^{\theta})^{p_\theta'}\,dy\right)^{1/p_\theta'} \le \tau_\theta \left(\int_{\mathbb{R}^N} |f(y)|^{q_\theta'}\,dy\right)^{1/q_\theta'},$$

where $\tau_\theta > 0$ is a constant,

$$u_0 = w_0^{-1/p_0}, \quad \text{and} \quad u_1 = w_1^{-1/p_1}.$$

If

$$\Theta = \frac{p_\theta}{p_1}\theta,$$

then

$$\frac{p_\theta}{p_0}(1-\theta) = 1-\Theta \quad \text{and} \quad \lambda_\theta = (1-\Theta)\lambda_0 + \Theta\lambda_1,$$

and hence

$$(u_0^{1-\theta}u_1^{\theta})^{-p_\theta} = w_0^{(1-\theta)p_\theta/p_0} w_1^{\theta p_\theta/p_1} = w_0^{1-\Theta} w_1^{\Theta}.$$

So the desired result

$$\|T^* f\|_{H^{p_\theta',\lambda_\theta}} \le \tau_\theta \|f\|_{L^{q_\theta'}}, \quad 0 \le \theta \le 1,$$

or

$$T\colon L^{p_\theta,\lambda_\theta} \longrightarrow L^{q_\theta}.$$

follows from the inequality

$$\int_{\mathbb{R}^N} w_0^{1-\Theta} w_1^{\Theta}\, d\Lambda^{(\infty)}_{(N-\lambda_0)(1-\Theta)+(N-\lambda_1)\Theta} \lesssim \left(\int_{\mathbb{R}^N} w_0\, d\Lambda^{(\infty)}_{N-\lambda_0}\right)^{1-\Theta} \left(\int_{\mathbb{R}^N} w_1\, d\Lambda^{(\infty)}_{N-\lambda_1}\right)^{\Theta}.$$

To see the last inequality, we use Lemma 17 with

$$w_i = \sum_{k=1}^{\infty} b_k^{(i)} a_k^{(i)} \in L^1(\Lambda^{(\infty)}_{N-\lambda_i})$$

to get two positive constants c_1 and c_2 such that

$$\frac{1}{c_i} \inf \sum_{k=1}^{\infty} |b_k^{(i)}| \le \int_{\mathbb{R}^N} w_i\, d\Lambda^{(\infty)}_{N-\lambda_i} \le c_i \inf \sum_{k=1}^{\infty} |b_k^{(i)}|, \quad i=0,1.$$

Then, setting

$$b_k = |b_k^{(0)}|^{1-\Theta} |b_k^{(1)}|^{\Theta},$$

we have $\{b_k\}_{k=1}^{\infty} \in l^1$ and

$$\begin{aligned}\sum_{k=1}^{\infty} b_k &\le \left(\sum_{k=1}^{\infty} |b_k^{(0)}|\right)^{1-\Theta} \left(\sum_{k=1}^{\infty} |b_k^{(1)}|\right)^{\Theta} \\ &\le c_0^{1-\Theta} c_1^{\Theta} \left(\int_{\mathbb{R}^N} w_0\, d\Lambda^{(\infty)}_{N-\lambda_0}\right)^{1-\Theta} \left(\int_{\mathbb{R}^N} w_1\, d\Lambda^{(\infty)}_{N-\lambda_1}\right)^{\Theta} \\ &\le c_0^{1-\Theta} c_1^{\Theta},\end{aligned}$$

since each w_i is chosen for each H^{p_i,λ_i}. And if

$$a_k = |a_k^{(0)}|^{1-\Theta} |a_k^{(1)}|^{\Theta},$$

then we easily see that a_k is an $(\infty, N-\lambda_\theta)$-atom, and

$$\int_{\mathbb{R}^N} a_k\, d\Lambda^{(\infty)}_{N-\lambda_\theta} \le 1.$$

Hence by the quasi-Hölder inequality and the quasi-sublinearity for the integral $\int_{\mathbb{R}^N} w\, d\Lambda^{(\infty)}_{N-\lambda}$, we get

$$\begin{aligned}\int_{\mathbb{R}^N} w_0^{1-\Theta} w_1^{\Theta}\, d\Lambda^{(\infty)}_{N-\lambda_\theta} &\lesssim \int_{\mathbb{R}^N} \sum_{k=1}^{\infty} (b_k^{(0)} a_k^{(0)})^{1-\Theta} (b_k^{(1)} a_k^{(1)})^{\Theta}\, d\Lambda^{(\infty)}_{N-\lambda_\theta} \\ &\lesssim \left(\int_{\mathbb{R}^N} w_0\, d\Lambda^{(\infty)}_{N-\lambda_0}\right)^{1-\Theta} \left(\int_{\mathbb{R}^N} w_1\, d\Lambda^{(\infty)}_{N-\lambda_1}\right)^{\Theta},\end{aligned}$$

as desired. □

Remark 19. In a manner similar to the above, we get that

$$T\colon H^{p_i',\lambda_i} \longrightarrow H^{q_i',\mu_i}, \quad i=0,1,$$

implies

$$T\colon H^{p_\theta',\lambda_\theta} \longrightarrow H^{q_\theta',\mu_\theta}, \quad 0<\theta<1,$$

where in addition to the formulas on p_θ and λ_θ we assume

$$\frac{\mu_\theta}{q_\theta} = (1-\theta)\frac{\mu_0}{q_0} + \theta\frac{\mu_1}{q_1}.$$

In fact, since any w with

$$0 \le w \in L^1(\Lambda_{d_\theta}^{(\infty)}), \quad \text{where } d_\theta = (1-\theta)d_0 + \theta d_1,$$

can be written as

$$w = \sum_{k=1}^{\infty} b_k a_k \le \left(\sum_{k=1}^{\infty} b_k a_{k,0}\right)^{1-\theta} \left(\sum_{k=1}^{\infty} b_k a_{k,1}\right)^{\theta},$$

where a_k is a (∞, d_θ)-atom and $a_{k,i}$ is a (∞, d_i)-atom. Hence

$$w \le w_0^{1-\theta} w_1^{\theta} \quad \text{with } 0 \le w_i \in L^1(\Lambda_{d_i}^{(\infty)}),\ i=0,1.$$

As a result, we have

$$\int_{\mathbb{R}^N} |f(y)|^{p_\theta'} (w_0^{1-\theta}(y) w_1^{\theta}(y))^{1-p_\theta'}\, dy \le \int_{\mathbb{R}^N} |f(y)|^{p_\theta'} w(y)^{1-p_\theta'}\, dy.$$

Here we have once again used Stein's interpolation with change of measure (cf. [10, p. 213, Theorem 3.6]).

6. Morrey–Sobolev capacities and Wolff-type potentials

In [6], we introduced two versions of a (Morrey–Sobolev) capacity associated with potentials of functions in $H^{p,\lambda}$—type I capacities—and with potentials of functions in $L^{p,\lambda}$—type II capacities. Based on the previous discussions, we develop some further results here complimenting those of [6].

6.1. Local estimates for capacities

Like the (α,p)-Riesz capacity reviewed in Section 2, if X is either $L_0^{p,\lambda}$ or $H^{p,\lambda}$, we set

$$C_\alpha(E;X)=\inf\{\|f\|_X^p:\mathcal{I}_\alpha f\ge 1 \text{ on } E\},$$

where $E\subseteq\mathbb{R}^N$. The min-max theorem gives an equivalent version

$$\operatorname{Cap}_\alpha(E;X)=\sup\{\|\mu\|_1:\operatorname{supp}\mu\subseteq E \text{ and } \|\mathcal{I}_\alpha\mu\|_{X^*}\le 1\}.$$

Here X^* is the dual of X. It is not hard to see that

$$C_\alpha(E;L^{p,\lambda})\sim C_\alpha(E;L_0^{p,\lambda})$$

holds because if $f\in L^{p,\lambda}$, then $\phi_\varepsilon*f$ is continuous for ϕ_ε—an approximation of the identity—and

$$\mathcal{I}_\alpha(\phi_\varepsilon*f)=\mathcal{I}_\alpha\phi_\varepsilon*f\to\mathcal{I}_\alpha f \quad \text{a.e.}$$

which follows from

$$|\phi_\varepsilon*f|\lesssim M_0 f,\quad \varepsilon>0.$$

In fact, the min-max theorem gives

$$C_\alpha(E;X)=\operatorname{Cap}_\alpha(E;X^*)^p.$$

Theorem 20. *Let $(\alpha,p)\in(0,N)\times(1,\infty)$, $\alpha p<\lambda<N$, $(x,r)\in\mathbb{R}^N\times\mathbb{R}_+$, and let $E\subseteq\mathbb{R}^N$ be compact. Then*

(i)

$$\operatorname{Cap}_\alpha(E\cap B(x,r);L^{p',\lambda})\lesssim r^{N-\lambda}\operatorname{Cap}_\alpha(E\cap B(x,r);H^{p',\lambda}),$$

that is, type II *capacities are stronger than type* I *capacities.*

(ii)

$$\frac{C_\alpha(E\cap B(x,r);H^{p,\lambda})}{r^{(N-\lambda-\alpha)p+\lambda}}\lesssim\frac{C_\alpha(E\cap B(x,r);L^p)}{r^{N-\alpha p}}\lesssim\frac{C_\alpha(E\cap B(x,r);L^{p,\lambda})}{r^{\lambda-\alpha p}}.$$

Proof. (i) If

$$\operatorname{Cap}_\alpha(E\cap B(x,r);L^{p',\lambda})>0,$$

then there is a non-negative Borel measure μ supported on $E\cap B(x,r)$ such that

$$\int_{B(x,r)}(\mathcal{I}_\alpha\mu(y))^{p'}\,dy\le r^{N-\lambda}.$$

Thus, setting

$$\nu = r^{\lambda-N}\mu \quad \text{and} \quad w = r^{\lambda-N} 1_{B(x,r)},$$

we get

$$\int_{\mathbb{R}^N} (\mathcal{I}_\alpha \nu(y))^{p'} w(y)^{-1/(p-1)}\, dy \le 1.$$

This in turn yields

$$r^{\lambda-N}\operatorname{Cap}_\alpha(E\cap B(x,r); L^{p',\lambda}) \lesssim \operatorname{Cap}_\alpha(E\cap B(x,r); H^{p',\lambda}).$$

(ii) A similar argument gives

$$C_\alpha(E\cap B(x,r); L^p) \lesssim r^{N-\lambda} C_\alpha(E\cap B(x,r); L^{p,\lambda})$$

and

$$C_\alpha(E\cap B(x,r); H^{p,\lambda}) \lesssim r^{(N-\lambda)(p-1)} C_\alpha(E\cap B(x,r); L^p).$$

Consequently, the desired estimate follows. □

6.2. Wolff-type potentials

Note that the estimates in Theorem 20 are precisely the type of estimate that one needs to compare a reasonable notion of "thinness" associated with each of these capacities—for the concept of thinness for the capacity $C_\alpha(\,\cdot\,; L^p)$ see [5]. We will postpone a complete discussion of thinness to the capacities $C_\alpha(\,\cdot\,; L^{p,\lambda})$ to another time. But we think it is appropriate here to record the Wolff potentials associated with each of these Morrey–Sobolev capacities. Actually, the Wolff potential for the type I case was already given in [6]. Both of these depend on the results of Section 4 regarding

$$\|\mathcal{I}_\alpha\mu\|_{L^{p',\lambda}} \quad \text{and} \quad \|\mathcal{I}_\alpha\mu\|_{H^{p',\lambda}}.$$

Theorem 21. *Let $(\alpha,\lambda,p)\in(0,N)\times(0,N)\times(1,\infty)$, $\alpha p<N$, $p'=p/(p-1)$, and let μ be a non-negative Borel measure on $\mathbb{R}^N$. Then*

(i)

$$\|\mathcal{I}_\alpha\mu\|_{L^{p',\lambda}}^{p'} \sim \int_{\mathbb{R}^N} W^{\mu}_{\alpha,p,\lambda}(y)\, d\mu(y);$$

(ii)

$$\|\mathcal{I}_\alpha\mu\|_{H^{p',\lambda}}^{p'} \sim \inf_{w\in A_1^{(N-\lambda)}} \int_{\mathbb{R}^N} W^{\mu,w}_{\alpha,p,\lambda}(y)\, d\mu(y).$$

Here

$$W^{\mu}_{\alpha,p,\lambda}(y)=\int_0^\infty\left(\frac{\mu(B(y,r))}{r^{\lambda+p(N-\lambda-\alpha)}}\right)^{p'-1}\frac{dr}{r}$$

and

$$W^{\mu,w}_{\alpha,p,\lambda}(y)=\int_0^\infty\left(\frac{r^{\alpha p}\mu(B(y,r))}{\int_{B(y,r)}w(z)\,dz}\right)^{p'-1}\frac{dr}{r}\quad \textit{for } w\in A_1^{(N-\lambda)}.$$

Proof. We first find upper bounds. Taking a close look at Theorem 14, where $\|\mathcal{I}_\alpha\mu\|_{X^*}$ is measured in terms of $\|M_\alpha\mu\|_{X^*}$, we can focus on $\|M_\alpha\mu\|^{p'}_{L^{p',\lambda}}$. Note that

$$M_\alpha\mu(x)\le\left(\int_0^\infty(t^{\alpha-N}\mu(B(x,t)))^{p'}\,\frac{dt}{t}\right)^{1/p'}.$$

So, if we set

$$I(x,r,t)=\int_{B(x,r)}\mu(B(y,t))^{p'}\,dy,$$

then we have

$$\|M_\alpha\mu\|^{p'}_{L^{p',\lambda}}\le\sup_{(x,r)\in\mathbb{R}^N\times\mathbb{R}_+}r^{\lambda-N}\int_0^\infty t^{p'(\alpha-N)}I(x,r,t)\,\frac{dt}{t}$$

and

$$\begin{aligned}I(x,r,t)&=\int_{B(x,r)}(\mu(B(y,t)))^{p'-1}\left(\int_{B(x,t)}d\mu(y)\right)dy\\&\le\int_{\mathbb{R}^N}(\mu(B(y,t)))^{p'-1}|B(x,r)\cap B(y,t)|\,d\mu(y).\end{aligned}$$

Rewriting

$$\|M_\alpha\mu\|^{p'}_{L^{p',\lambda}}\le\sup_{(x,r)\in\mathbb{R}^N\times\mathbb{R}_+}r^{\lambda-N}\left(\int_0^r+\int_r^\infty\right)t^{p'(\alpha-N)}I(x,r,t)\,\frac{dt}{t},$$

we control the two integrals on the right-hand side separately.

For the first integral, we have

$$\begin{aligned}\sup_{(x,r)\in\mathbb{R}^N\times\mathbb{R}_+}r^{\lambda-N}\int_0^r(\dots)&\lesssim\int_0^\infty\left(t^{N+p'(\alpha-N)}t^{\lambda-N}\int_{\mathbb{R}^N}\mu(B(y,2t))^{p'-1}\,d\mu(y)\right)\frac{dt}{t}\\&\lesssim\int_{\mathbb{R}^N}W^{\mu}_{\alpha,p,\lambda}\,d\mu.\end{aligned}$$

For the second integral, we have

$$\sup_{(x,r)\in\mathbb{R}^N\times\mathbb{R}_+} r^{\lambda-N}\int_r^\infty (\dots) \lesssim \int_0^\infty \left(t^{p'(\alpha-N)+\lambda}\int_{\mathbb{R}^N}\mu(B(y,2t))^{p'-1}\,d\mu(y)\right)\frac{dt}{t}$$

$$\lesssim \int_{\mathbb{R}^N} W^\mu_{\alpha,p,\lambda}\,d\mu.$$

A combined use of the last three estimates yields the required upper bound of $\|M_\alpha\mu\|^{p'}_{L^{p',\lambda}}$.

Meanwhile, we look for an upper bound for

$$\|M_\alpha\mu\|^{p'}_{H^{p',\lambda}} \le \inf_{w\in A_1^{(N-\lambda)}}\int_{\mathbb{R}^N}\left(\int_0^\infty (t^{\alpha-N}\mu(B(x,t)))^{p'}\frac{dt}{t}\right)w(x)^{1-p'}\,dx.$$

Now, setting

$$J(t)=\int_{\mathbb{R}^N}\mu(B(x,t))^{p'}w(x)^{1-p'}\,dx,$$

we estimate

$$\begin{aligned} J(t) &\le \int_{\mathbb{R}^N}\left(\mu(B(x,t))^{p'-1}w(x)^{1-p'}\int_{B(x,r)}d\mu\right)dx\\ &= \int_{\mathbb{R}^N}\left(\int_{B(y,t)}\mu(B(x,t))^{p'-1}w(x)^{1-p'}\,dx\right)d\mu(y)\\ &\lesssim \mu(B(y,2t))^{p'-1}t^N\int_{B(y,t)}w(x)^{-1/(p-1)}\,dx. \end{aligned}$$

Since $w\in A_1$ and hence $w\in A_p$, we get

$$J(t)\lesssim \mu(B(y,2t))^{p'-1}t^N\left(t^{-N}\int_{B(y,t)}w(x)\,dx\right)^{1-p'},$$

whence obtaining the result.

On the other hand, we need to look for the corresponding lower bounds. This is easy. In fact,

$$\begin{aligned} \|\mathcal{I}_\alpha\mu\|^{p'}_{L^{p',\lambda}} &\gtrsim \sup_{(x,r)\in\mathbb{R}^N\times\mathbb{R}_+} r^{\lambda-N}\int_{B(x,r)}\left(\int_0^\infty\left(t^{\alpha-N}\mu(B(z,t))\,\frac{dt}{t}\right)^{p'}\right)dz\\ &\gtrsim \sup_{x\in\mathbb{R}^N}\int_0^\infty r^{\alpha+\lambda-2N}\left(\int_{B(x,r)}\left(\frac{\mu(B(z,2r))}{r^{N-\alpha}}\right)^{p'-1}\mu(B(z,r))\,dz\right)\frac{dr}{r} \end{aligned}$$

$$\gtrsim \int_0^\infty r^{\lambda-N} r^N r^{(\alpha-N)p'} \left(\int_{\mathbb{R}^N} \mu(B(y,r))^{p'-1}\, d\mu(y)\right) \frac{dr}{r}$$
$$\gtrsim \int_{\mathbb{R}^N} W^{\mu}_{\alpha,p,\lambda}\, d\mu.$$

And for the other, we have

$$\|\mathcal{I}_\alpha \mu\|^{p'}_{H^{p',\lambda}} \gtrsim \inf_{w\in A_1^{(N-\lambda)}} \int_{\mathbb{R}^N} \left(\int_0^\infty t^{\alpha-N} \mu(B(x,t))\, \frac{dt}{t}\right)^{p'} w(x)^{1-p'}\, dx$$
$$\gtrsim \inf_{w\in A_1^{(N-\lambda)}} \int_0^\infty r^{(\alpha-N)(1+p')} \int_{\mathbb{R}^N} \frac{\int_{B(z,r)} w(x)^{1-p'}\, dx}{\left(\int_{B(z,3r)} d\mu(y)\right)^{1-p'}}\, d\mu(z)\, \frac{dr}{r}$$
$$\gtrsim \inf_{w\in A_1^{(N-\lambda)}} \int_{\mathbb{R}^N} \int_0^\infty \frac{r^{(\alpha-N)p'}}{\mu(B(z,r))^{1-p'}} \left(\int_{B(z,r)} w(x)^{1-p'}\, dx\right) \frac{dr}{r}\, d\mu(z)$$
$$\gtrsim \inf_{w\in A_1^{(N-\lambda)}} \int_{\mathbb{R}^N} \int_0^\infty \frac{\left(\frac{r^{(\alpha p-N)/(p-1)}}{\mu(B(z,r))^{1/(1-p)}}\right)}{\left(r^{-N}\int_{B(z,r)} w(y)\, dy\right)^{1/(p-1)}}\, \frac{dr}{r}\, d\mu(z)$$
$$\gtrsim \inf_{w\in A_1^{(N-\lambda)}} \int_{\mathbb{R}^N} W^{\mu,w}_{\alpha,p,\lambda}\, d\mu.$$

In the last two estimates we again invoke the A_p condition for w. □

Remark 22. Recall from [5, Theorem 4.5.4] that if

$$\dot{W}^{\mu}_{\alpha,p}(y) = \int_0^\infty \left(\frac{\mu\big(B(y,r)\big)}{r^{N-\alpha p}}\right)^{1/(p-1)} \frac{dr}{r}$$

is the so-called Wolff potential, then

$$\|\mathcal{I}_\alpha \mu\|^{p'}_{L^{p'}} \sim \int_{\mathbb{R}^N} \dot{W}^{\mu}_{\alpha,p}(y)\, d\mu(y).$$

This type of estimate may be regarded as a special case $\lambda = N$ of Theorem 21.

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