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On the intersection of classes of infinitely differentiable functions

By JAN BOMAN

1. Introduction

It is well known that the intersection of all non-quasianalytic classes of functions is equal to the class of all real analytic functions (see e.g. Bang [1]). In the present paper we shall describe the intersection of more restricted families of non-quasianalytic classes of functions.

If L: $k \to L(k)$, $k=0, 1, 2, \ldots$ is a sequence of positive numbers, and Ω is an open subset of R^n, we define $C^L = C^L(\Omega)$ as the set of infinitely differentiable functions u such that to every compact set $F \subset \Omega$ there exists a constant C such that

$$|D^k u| \leqslant C^{k+1} L(k)^k, \quad \text{if} \quad x \in F \quad (k=0, 1, 2, \ldots).$$

Here D^k denotes an arbitrary partial derivative of order k. If $L(k)=k$ when $k \geqslant 1$, then C^L is equal to the class of all real analytic functions on Ω.

Put C_0^L = the set of all functions in C^L whose supports are compact subsets of Ω.

Definition 1. *The class C^L is said to be quasianalytic, if C_0^L contains no function except the zero-function.*

A complete characterisation of the sequences L such that the class C^L is quasianalytic was given in 1926 by the following theorem.

Denjoy–Carleman Theorem. (*Carleman* [*3*].) *The class C^L is quasianalytic if and only if $\sum_{k=0}^{\infty} \underline{L}(k)^{-1}$ is divergent, where $\underline{L}$ denotes the largest increasing minorant sequence of L.*

Theorem 1. *Let M and N be two positive sequences such that $\sum M(k)^{-1} = \infty$, $\sum N(k)^{-1} < \infty$ and N/M is increasing. Denote by $\mathcal{K}(M, N)$ the following set of sequences L:*
$\mathcal{K}(M, N) = \{L;\ L/M$ is increasing, L/N is decreasing, $\sum L(k)^{-1} < \infty\}$.

Then
$$\bigcap_{L \in \mathcal{K}(M,N)} C^L = C^{\sup(\hat{M}, \check{N})}, \tag{1}$$

where
$$\hat{M}(k) = M(k) \sum_{0}^{k} M(j)^{-1} \quad (k \geqslant 0),$$

and
$$\check{N}(k) = N(k) \sum_{k}^{\infty} N(j)^{-1} \quad (k \geqslant 0),$$

Note that if M is increasing the Denjoy-Carleman theorem shows that the condition $\Sigma L(k)^{-1} < \infty$ in the definition of $\mathcal{K}(M, N)$ is equivalent to the condition that C^L is non-quasianalytic.

From Theorem 1 we formally obtain Theorem 2 and Theorem 3 by deleting the condition that L/N is decreasing and that L/M is increasing respectively.

Theorem 2. *Let M be a positive sequence such that $\Sigma M(k)^{-1} = \infty$. Put $\mathcal{K}^+(M) = \{L;\ L/M$ is increasing, $\Sigma L(k)^{-1} < \infty\}$.*

Then
$$\bigcap_{L \in \mathcal{K}^+(M)} C^L = C^{\hat{M}}.$$

Theorem 3. *Let N be a positive sequence such that $\Sigma N(k)^{-1} < \infty$. Put $\mathcal{K}^-(N) = \{L;\ L/N$ is decreasing, $\Sigma L(k)^{-1} < \infty\}$.*

Then
$$\bigcap_{L \in \mathcal{K}^-(N)} C^L = C^{\check{N}}.$$

Taking $M(k) = 1$ for every k gives $\hat{M}(k) = k+1$, and the class in the right-hand side of (1) becomes $C^{\sup(\{k+1\}, \check{N})}$. In some applications it is useful to know conditions on N in order that this class be equal to the analytic class. It is obvious that this is the case if $\check{N}(k) < C(k+1)$ for some C. However, this condition turns out to be also necessary, as is expressed by the following theorem (see the remark after Theorem 1).

Theorem 4. *Let N be a positive increasing sequence such that $\Sigma N(k)^{-1} < \infty$. Then the intersection of all non-quasianalytic classes C^L, where L is increasing and L/N is decreasing, is equal to the analytic class if and only if $\check{N}(k) < C(k+1)$ for some constant C, or, which is equivalent*
$$\sum_k^{\infty} N(j)^{-1} < Ck/N(k) \quad (k = 1, 2, \ldots). \tag{2}$$

Theorem 5. *Under the conditions of Theorem 1 the classes $C^{\hat{M}}$, $C^{\check{N}}$ and $C^{\sup(\hat{M}, \check{N})}$ are quasianalytic.*

Note that the quasianalyticity of two classes C^A and C^B does not imply the quasianalyticity of the class $C^{\sup(A, B)}$. In Theorem 5 the quasianalyticity of the class $C^{\sup(\hat{M}, \check{N})}$ follows from the quasianalyticity of the classes $C^{\hat{M}}$ and $C^{\check{N}}$ and the fact that the sequences $\hat{M}$ and $\check{N}$ are related by the condition that N/M is increasing.

In the next section we give proofs of the theorems. In section 3 we discuss a number of special cases and applications.

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2. Proofs of the theorems

We first deduce some formulas which connect the sequences M and N with their respective transforms $\hat{M}$ and $\check{N}$. From the definition of $\hat{M}$ we obtain

$$1 - \hat{M}(k)^{-1} = 1 - \left(M(k)^{-1} \Big/ \sum_0^k M(j)^{-1} \right) = \sum_0^{k-1} M(j)^{-1} \Big/ \sum_0^k M(j)^{-1}, \quad \text{if} \quad k \geqslant 1.$$

Hence $$\prod_1^k (1-\hat{M}(j)^{-1}) = (M(0)\sum_0^k M(j)^{-1})^{-1}. \tag{3}$$

Similarly we obtain

$$\prod_0^{k-1} (1-\check{N}(j)^{-1}) = \sum_k^\infty N(j)^{-1} \Big/ \sum_0^\infty N(j)^{-1}, \quad k \geqslant 1. \tag{4}$$

(Note that $\hat{M}(k) > 1$ when $k \geqslant 1$ and $\check{N}(k) > 1$ for every k.) From these formulas it follows immediately that $\Sigma \hat{M}(k)^{-1}$ and $\Sigma \check{N}(k)^{-1}$ are divergent. In fact, since $\Sigma M(k)^{-1}$ is divergent by assumption, (3) proves that $\Pi(1-\hat{M}(k)^{-1})$ is divergent to zero and hence that $\Sigma \hat{M}(k)^{-1}$ is divergent. Similarly (4) shows that $\Sigma \check{N}(k)^{-1}$ is divergent, since $\Sigma N(k)^{-1}$ is convergent.

Using (3) and the definition of $\hat{M}$ we can express $M(k)$ when $k \geqslant 1$ in terms of $M(0)$ and $\hat{M}$:

$$M(k) = M(0)\,\hat{M}(k) \prod_1^k (1-\hat{M}(j)^{-1}). \tag{5}$$

Similarly we obtain from (4)

$$N(k) = \left(\sum_0^\infty N(j)^{-1}\right)^{-1} \check{N}(k) \Big/ \prod_0^{k-1} (1-\check{N}(j)^{-1}). \tag{6}$$

Formulas (5) and (6) show that to any given sequence A_k such that $A_k > 1$ and $\Sigma A_k^{-1} = \infty$ there exist positive sequences M and N (not uniquely determined), such that $\hat{M}(k) = \check{N}(k) = A_k$ when $k \geqslant 1$, $\Sigma M(k)^{-1} = \infty$ and $\Sigma N(k)^{-1} < \infty$.

Proof of Theorem 1. First we prove that if $L \in \mathcal{K}(M,N)$, then $C^L \supset C^{\sup(\hat{M},\check{N})}$. Since $\Sigma L(k)^{-1} = C < \infty$ and L/M is increasing, we have

$$C > \sum_0^k L(j)^{-1} = \sum_0^k (M(j)/L(j))\, M(j)^{-1} \geqslant (M(k)/L(k)) \sum_0^k M(j)^{-1} = L(k)^{-1}\hat{M}(k). \tag{7}$$

Similarly, since L/N is decreasing

$$C > \sum_k^\infty L(j)^{-1} = \sum_k^\infty (N(j)/L(j))\, N(j)^{-1} \geqslant (N(k)/L(k)) \sum_k^\infty N(j)^{-1} = L(k)^{-1}\check{N}(k). \tag{8}$$

Thus $CL(k) > \max\,(\hat{M}(k), \check{N}(k))$, which proves that $C^L \supset C^{\sup(\hat{M},\check{N})}$ and hence that $\cap\, C^L \supset C^{\sup(\hat{M},\check{N})}$.

To prove that $\cap\, C^L \subset C^{\sup(\hat{M},\check{N})}$ we shall prove that to an arbitrary function $g \notin C^{\sup(\hat{M},\check{N})}$ there exists a sequence L such that $L \in \mathcal{K}(M,N)$ and $g \notin C^L$. If $g \notin C^{\sup(\hat{M},\check{N})}$, there exists a compact set $F \subset \Omega$, such that $G(k) = (\sup_{x \in F} |D^k g(x)|)^{1/k}$ satisfies

$$\overline{\lim_{k\to\infty}}\, (G(k)/\max\,(\hat{M}(k), \check{N}(k))) = \infty. \tag{9}$$

We have to find a sequence $L \in \mathcal{K}(M, N)$ such that

$$\overline{\lim_{k\to\infty}}\,(G(k)/L(k)) = \infty. \tag{10}$$

We may assume that G/N is bounded, since otherwise (10) is satisfied with $L=N$, and clearly $N \in \mathcal{K}(M,N)$. Let a_j and b_j be sequences of positive numbers, such that $a_j \to \infty$ and $b_j \to 0$ when $j\to\infty$, and $\Sigma(a_j b_j)^{-1} < \infty$. In view of (9) we can find an increasing sequence of indices k_j such that

$$G(k_j)/\max(\hat{M}(k_j),\ \check{N}(k_j)) \geqslant a_j \quad (j=1,2,\ldots). \tag{11}$$

Then $b_j G(k_j)/M(k_j) \to \infty$ by virtue of (11) and the fact that $\hat{M}/M$ is increasing. Also, $b_j G(k_j)/N(k_j) \to 0$, since G/N is bounded. Thus by taking a subsequence if necessary we can always obtain that

$$\bar{b}_j G(k_j)/M(k_j) \quad \text{is increasing,} \tag{12}$$

$$\bar{b}_j G(k_j)/N(k_j) \quad \text{is decreasing, and} \tag{13}$$

$$G(k_j)/\max(\hat{M}(k_j),\ \check{N}(k_j)) \geqslant \bar{a}_j, \tag{14}$$

where $\bar{a}_j$ and $\bar{b}_j$ are subsequences of the sequences a_j and b_j respectively and hence satisfy $\bar{b}_j \to 0$ and $\Sigma(\bar{a}_j\bar{b}_j)^{-1} < \infty$.

Assume that $k_1 = 0$ and put

$$L'(k) = \bar{b}_j G(k_j) N(k)/N(k_j),\ k_j \leqslant k < k_{j+1} \quad (j=1,2,\ldots),$$

$$L''(k) = \bar{b}_{j+1} G(k_{j+1}) M(k)/M(k_{j+1}),\ k_j < k \leqslant k_{j+1} \quad (j=1,2,\ldots),$$

$$L''(0) = \bar{b}_1 G(0), \text{ and}$$

$$L(k) = \min\,(L'(k), L''(k)) \quad (k=0,1,2,\ldots). \tag{15}$$

Then it is obvious that (10) is fulfilled, since $L(k_j) = \bar{b}_j G(k_j)$ for every j and $\bar{b}_j \to 0$.

Next we prove that L/M is increasing if L is defined by (15). In view of (12) L''/M is increasing. L'/M is increasing in every interval $k_j \leqslant k < k_{j+1}$, since L'/N is constant in that interval and N/M is increasing according to the assumption. Noting that $L'(k_j) = L''(k_j)$ for every j we conclude that L/M is increasing in the whole interval $k \geqslant 0$. Using (13) we can prove in an exactly analogous way that L/N is decreasing.

Finally we prove that $\Sigma L(k)^{-1}$ is convergent. From the definition of L we obtain $L(k)^{-1} \leqslant L'(k)^{-1} + L''(k)^{-1}$ and since $L'(k_j) = L''(k_j)$

$$\sum_{k_j \leqslant k < k_{j+1}} L(k)^{-1} \leqslant \sum_{k_j \leqslant k < k_{j+1}} L'(k)^{-1} + \sum_{k_j < k < k_{j+1}} L''(k)^{-1}$$

$$= \bar{b}_j^{-1} G(k_j)^{-1} N(k_j) \sum_{k_j \leqslant k < k_{j+1}} N(k)^{-1} + \bar{b}_{j+1}^{-1} G(k_{j+1})^{-1} M(k_{j+1}) \sum_{k_j < k < k_{j+1}} M(k)^{-1}$$

$$< \bar{b}_j^{-1} G(k_j)^{-1} \check{N}(k_j) + \bar{b}_{j+1}^{-1} G(k_{j+1})^{-1} \hat{M}(k_{j+1}).$$

This together with (14) gives

$$\sum_{k_j \leqslant k < k_{j+1}} L(k)^{-1} \leqslant (\bar{a}_j \bar{b}_j)^{-1} + (\bar{a}_{j+1}\bar{b}_{j+1})^{-1},$$

which proves that $\Sigma L(k)^{-1}$ is convergent, since $\Sigma(\tilde{a}_j\bar{b}_j)^{-1}$ is convergent. This completes the proof of Theorem 1.

Proof of Theorem 2. Formula (7) proves that

$$\bigcap_{\mathcal{K}^+(M)} C^L \supset C^{\hat{M}}.$$

On the other hand, if we can find a sequence N such that N/M is increasing, $\check{N} \leqslant \hat{M}$ and $\Sigma N(k)^{-1} < \infty$, we obtain from Theorem 1

$$\bigcap_{\mathcal{K}^+(M)} C^L \subset \bigcap_{\mathcal{K}(M,N)} C^L = C^{\sup(\hat{M},\check{N})} = C^{\hat{M}}.$$

Put $A_k = \min(\hat{M}(k), (M(k)+M(k+1))/M(k+1))$. Since $\Sigma A_k^{-1} \geqslant \Sigma \hat{M}(k)^{-1} = \infty$, the remark following formula (6) shows that there exists a sequence N such that

$$\check{N}(k) = A_k \quad (k = 1, 2, \ldots) \tag{16}$$

and $\Sigma N(k)^{-1} < \infty$. It is obvious that $\check{N}(k) \leqslant \hat{M}(k)$ for every k, so it only remains to prove that N/M is increasing. In fact, from (6) we obtain

$$\frac{N(k+1)}{M(k+1)} : \frac{N(k)}{M(k)} = \frac{M(k)}{M(k+1)} \cdot \frac{\check{N}(k+1)}{\check{N}(k)} \cdot \frac{1}{(1-\check{N}(k)^{-1})} > \frac{M(k)}{M(k+1)(\check{N}(k)-1)} \geqslant 1.$$

The last inequality follows from (16). The proof is complete.

Proof of Theorem 3. Formula (8) shows that

$$\bigcap_{\mathcal{K}^-(N)} C^L \supset C^{\check{N}}.$$

The opposite inclusion will follow in exactly the same way as in the proof of Theorem 2 if we can find, for a given sequence N, a sequence M such that $\hat{M} \leqslant \check{N}$, $\Sigma M(k)^{-1} = \infty$ and N/M is increasing. By the remark following formula (6) we can find a sequence M such that $\Sigma M(k)^{-1} = \infty$ and

$$\hat{M}(k) = \min((N(k-1)+N(k))/N(k-1), \check{N}(k)). \tag{17}$$

Then we obtain from (5):

$$\frac{N(k+1)}{M(k+1)} : \frac{N(k)}{M(k)} = \frac{N(k+1)}{N(k)} \cdot \frac{\hat{M}(k)}{\hat{M}(k+1)} \cdot \frac{1}{1-\hat{M}(k+1)^{-1}} > \frac{N(k+1)}{N(k)(\hat{M}(k+1)-1)} \geqslant 1,$$

where the last inequality follows from (17). This proves Theorem 3.

Proof of Theorem 4. As we have already mentioned it follows from Theorem 1 and the Denjoy–Carleman theorem that the intersection studied in Theorem 4 is equal to $C^{\sup(\{k+1\},\check{N})}$. Clearly this class always contains the analytic class. Hence what we have to prove is that $C^{\sup(\{k+1\},\check{N})}$ is contained in the analytic class if and only if (2) holds. The sufficiency of (2) is trivial. In proving the necessity we shall use the following lemma.

Lemma 1. *Assume that $B(k) \geqslant k$ and that B is almost increasing in the sense that*

$$B(k+1) \geqslant B(k) - a \tag{18}$$

with some constant a independent of k. Then C^B is contained in the analytic class if and only if $B(k) \leqslant Ck$ for some C and all $k \geqslant 1$.

To simplify some formulas we shall consider the sequence $\{k\}$, although it does not take a positive value when $k=0$; thus in a number of formulas k should take the values $k \geqslant 1$ instead of $k \geqslant 0$.

To deduce Theorem 4 from Lemma 1 it is sufficient to show that $\check{N}$ is almost increasing in the sense of (18), and hence that the same is true of the sequence $B = \sup(\{k+1\}, \check{N})$. And this follows from (6) and the fact that N is increasing:

$$\frac{N(k+1)}{N(k)} = \frac{\check{N}(k+1)}{\check{N}(k)\,(1-\check{N}(k)^{-1})} = \frac{\check{N}(k+1)}{\check{N}(k)-1} \geqslant 1.$$

For the proof of Lemma 1 we need this well-known result (see e.g. Bang [1]).

Lemma 2. *If $E(k)^k$ and $F(k)^k$ are logarithmically convex, then C^E is contained in C^F if and only if $E(k) \leqslant CF(k)$ for some C.*

If $B(k)^k$ had been assumed to be logarithmically convex, then Lemma 1 would have followed immediately from Lemma 2, since k^k is logarithmically convex.

We have to make a simple computation, the result of which can be expressed as follows.

Lemma 3. *Let m and n be positive integers, such that $n/m > e$, and let $G(k)$ be defined for $m \leqslant k \leqslant n$, in such a way that $G(m) \geqslant m$, $G(n) \geqslant n$, and $k \log G(k)$ is linear. Then*

$$\max_k\,(G(k)/k) > (2e)^{-1}\frac{n/m}{\log\,(n/m)}. \tag{19}$$

We now prove Lemma 1 using Lemma 2 and Lemma 3. We need of course only prove the necessity of the condition $B(k) \leqslant Ck$. Given the sequence B, define B^0 as the largest sequence such that $B^0(k) \leqslant B(k)$ and $k \log B^0(k)$ is convex. In other words $B^0(k)^k$ is the largest logarithmically convex minorant of $B(k)^k$. Let k_j, $j=1, 2, \ldots$, be the increasing sequence of all $k \geqslant 1$ such that $B(k) = B^0(k)$. Assume that C^B is contained in the analytic class A, that $B(k) \geqslant k$ and that B satisfies (18). Then obviously $C^{B^0} \subset A$ and by Lemma 2 there is a constant C such that $B^0(k) \leqslant Ck$. Further, since $B(k_j) = B^0(k_j)$ for every j, we have

$$\frac{B(k)}{k} \leqslant \frac{B(k_{j+1}) + (k_{j+1} - k_j)\max\,(a, 0)}{k_j}$$

$$\leqslant (C + \max\,(a, 0))\,(k_{j+1}/k_j), \quad \text{if} \quad k_j \leqslant k \leqslant k_{j+1}.$$

To prove Lemma 1 it thus only remains to show that k_{j+1}/k_j must be bounded if $B^0(k)/k$ is bounded. But this is immediately seen from Lemma 3, if we take $m = k_j$, $n = k_{j+1}$ and $B^0(k) = G(k)$. (Note that $B^0(k) \geqslant k$ for every k, since $B(k) \geqslant k$ and k^k is logarithmically convex.)

It remains to prove Lemma 3. First note that the inequality

$$k \log G(k) \geq k \log n - m \log (n/m), \tag{20}$$

holds for each k, since it obviously holds for $k=m$ and $k=n$, and both sides are linear in k. Taking $k_0=[m\log(n/m)]+1$, where $[x]$ denotes the integral part of x, we have $m<k_0<n$, and we obtain from (20)

$$\max_k \log (G(k)/k) > \log n - 1 - \log k_0 > \log (n/m) - 1 - \log 2 - \log\log (n/m),$$

which is the same as (19).

Proof of Theorem 5. In view of the Denjoy–Carleman theorem it is enough to prove that the series $\Sigma \hat{M}(k)^{-1}$, $\Sigma \check{N}(k)^{-1}$ and $\Sigma(\max(\hat{M}(k), \check{N}(k)))^{-1}$ are divergent. We have already proved that $\Sigma \hat{M}(k)^{-1}$ and $\Sigma \check{N}(k)^{-1}$ are divergent, so it only remains to prove that $\Sigma(\max(\hat{M}(k), \check{N}(k)))^{-1}$ is divergent.

Set $\hat{M}(k)^{-1}=a_k$, $\check{N}(k)^{-1}=b_k$ and $d_k=\min(a_k, b_k)$. The condition that N/M is increasing can be formulated in terms of a_k and b_k by means of the formulas (5) and (6):

$$\frac{N(k+1)}{M(k+1)} : \frac{N(k)}{M(k)} = \frac{a_{k+1}}{b_{k+1}} \cdot \frac{b_k}{a_k} \cdot \frac{1}{(1-a_{k+1})(1-b_k)} \geq 1,\ k \geq 1. \tag{21}$$

We first prove that if (21) is valid, then the following inequality holds

$$\frac{d_{k+1}}{b_{k+1}} \cdot \frac{b_k}{d_k} \geq (1-d_{k+1})(1-b_k), \tag{22}$$

i.e. (21) is valid with $d_k=\min(a_k, b_k)$ instead of a_k. To see this, put $c_j=d_j/b_j$, and note that (22) must hold if

$$c_{k+1}/c_k \geq 1. \tag{23}$$

Since $c_j \leq 1$ for each j, it is clear that (23) holds if $a_{k+1} \geq b_{k+1}$, because then $c_{k+1}=1$. On the other hand, if $a_{k+1}<b_{k+1}$, then $d_{k+1}=a_{k+1}$, and by applying (21) and the fact that $d_k \leq a_k$ we obtain (22), which proves the assertion.

Now, if a_k were $>b_k$ only for a finite number of k, it would be trivial that Σd_k is divergent, since Σa_k is divergent. Hence we can assume that $d_k=b_k$ for infinitely many k. It is obvious that we can also assume that $d_k=b_k<\frac{1}{2}$ for infinitely many k. Let m denote any of those indices k. Let n be the smallest integer such that $\Sigma_m^n b_k \geq \frac{1}{2}$. Then $n>m$ and we have $\Sigma_m^{n-1} d_k \leq \Sigma_m^{n-1} b_k < \frac{1}{2}$. Since $\Pi(1-c_k)>1-\Sigma c_k$ for arbitrary c_k such that $0<c_k<1$, we obtain

$$\prod_m^j (1-b_k) \prod_m^j (1-d_k) > \tfrac{1}{4}, \quad \text{if} \quad m \leq j < n. \tag{24}$$

By multiplying the inequalities (22) for $k=m, m+1, \ldots, j-1$ and using (24) and the fact that $b_m=d_m$ we obtain

$$\frac{d_j}{b_j} \geq \prod_m^{j-1} (1-b_k) \prod_{m+1}^{j} (1-d_k) > \tfrac{1}{4}, \quad \text{if} \quad m<j<n \tag{25}$$

and
$$\frac{d_n}{b_n} \geqslant \prod_{m}^{n-1} (1-b_k) \prod_{m+1}^{n-1} (1-d_k)(1-d_n) > \tfrac{1}{4}(1-d_n),$$

which gives
$$d_n > \tfrac{1}{8} \min(1, b_n). \tag{26}$$

Now recall that n was chosen so that $\Sigma_m^n b_k \geqslant \frac{1}{2}$. Together with (25) and (26) this gives
$$\sum_m^n d_k > \tfrac{1}{4} \sum_m^{n-1} b_k + \tfrac{1}{8} \min(1, b_n) > \tfrac{1}{16}. \tag{27}$$

Since (27) can be proved for an infinite number of indices m, it follows that Σd_k is divergent. This completes the proof of Theorem 5.

3. Applications

We will now study a number of special cases of our theorems.

I. Taking $M(k) = 1$ for every k gives $\hat{M}(k) = k+1$. Taking into account the Denjoy-Carleman theorem (see the remark after Theorem 1) we can express the corresponding special case of Theorem 2 as follows.

Theorem 6 (*see e.g. Bang [1]*). *The intersection of all non-quasianalytic classes C^L, where L is increasing, is equal to the class of all real analytic functions.*

II. Taking $M(0) = 1$ and $M(k) = k$ when $k \geqslant 1$ gives $\hat{M}(k) = k(1 + \Sigma_1^k(1/j))$ if $k \geqslant 1$. Since there are constants C_1 and C_2 such that $C_1 \log k \leqslant \Sigma_1^k(1/j) \leqslant C_2 \log k$, Theorem 2 gives

Theorem 7. *The intersection of all non-quasianalytic classes C^L, where $L(k)/k$ is increasing, is equal to the class $C^{\{k \log k\}}$.*

The class C^L is said to be inverse closed, if $u \in C^L$ and $u \neq 0$ implies that $1/u \in C^L$. Rudin [5] proved that if $L(k)^k$ is logarithmically convex and C^L is non-quasianalytic, then C^L is inverse closed if and only if $L(k)/k$ is almost increasing in the following sense: there exists a constant C such that $L(j)/j \leqslant CL(k)/k$ when $j \leqslant k$. Using this result Rudin proved the following theorem, which is closely related to Theorem 7.

Theorem 8. *The intersection of all inverse closed non-quasianalytic classes C^L, where $L(k)^k$ is logarithmically convex, is equal to the class $C^{\{k \log k\}}$.*

III. We indicate two applications of Theorem 4. First, take $N(k) = k^a$, $(k \geqslant 1)$, where $a > 1$. This gives $\check{N}(k) < 2k/(a-1)$, so that (2) is satisfied. This special case of Theorem 4 can be used in the study of the propagation of analyticity of solutions of linear partial differential equations of general type (see Boman [2]).

Ehrenpreis uses another special case of Theorem 4 in studying the range of convolution operators [4]. He considers the intersection of all non-quasianalytic classes C^L, where L is increasing and satisfies $L(k+1) < CL(k)$ for some constant C. This case one can obtain from Theorem 4 by taking $N(k) = C^k$ where $C > 1$, which gives $\check{N}(k) = C/(C-1)$ for every k.

REFERENCES

1. Bang, Th., Om quasi-analytiske funktioner. Copenhagen, 1946.
2. Boman, J., On the propagation of analyticity of solutions of differential equations with constant coefficients. Ark. Mat. *5*, 271–279 (1964).
3. Carleman, T., Fonctions quasi analytiques. Paris, 1926.
4. Ehrenpreis, L., Solution of some problems of division. IV. Invertible and elliptic operators. Amer. J. Math. *82*, 522–588 (1960).
5. Rudin, W., Division in algebras of C^{∞}-functions. MRC Techn. Rep., Nov. 1961.

Tryckt den 10 april 1964

Uppsala 1964. Almqvist & Wiksells Boktryckeri AB