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On the boundary values of harmonic functions in R^3

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1. Introduction

The purpose of this paper is to exhibit three theorems about the boundary values of harmonic functions, defined in R^3 regions which are bounded by Liapunov surfaces. Theorem 1 shows the existence almost everywhere of non-tangential boundary values of positive harmonic functions. A full proof of this theorem is given. Theorem 2 assures the existence almost everywhere of non-tangential boundary values for functions bounded in cones with vertex on the boundary and lying in the region. Theorem 3, finally, gives a necessary and sufficient condition for the existence of non-tangential boundary values, originally derived by Marcinkiewicz and Zygmund and later generalized by Stein. As the proofs of the two latter theorems differ from proofs published elsewhere only in the technical aspect, these are not included here.

2. Definitions

We consider an open region Ω_1, bounded by a Liapunov surface S_1. By Liapunov surface we mean a closed, bounded surface with the following properties:

1°. At every point of S_1 there exists a uniquely defined tangent plane, and thus also a normal.

2°. There exist two constants $C'>0$ and λ, $0<\lambda\leqslant 1$, such that if θ is the angle between two normals, and r is the distance between their foot points, the following inequality holds $\theta < C'\cdot r^{\lambda}$.

3°. There exists a constant $d>0$ such that if Σ is a sphere with radius d and center Q_0 on the surface, a line parallel to the normal at Q_0 meets S_1 at most once inside Σ. It is easily realized that d may be chosen arbitrarily small.

For the properties of Liapunov surfaces see Gunther [5]. In the sequel we shall consider only inner normals, which will simply be referred to as normals. We denote by $V(Q, \alpha, h)$ a right circular cone having vertex at $Q\in S_1$, axis along the normal at Q, altitude h, generating angle $=\alpha$ and being contained in Ω_1. Non-tangential approach to the boundary means approach inside some $V(Q, \alpha, h)$. $r(P, Q)$ will be the distance between $P\in\Omega_1$ and the tangent plane at $Q\in S_1$. The volume element in R^3 will be denoted by dv and the surface element by dS.

3. Lemmata

If $Q \in S_1$ we introduce "the local coordinate system" with the origin at Q, with the (xy)-plane in the tangent plane at Q, and with the positive z-axis along the normal at Q. Inside the Liapunov sphere, S_1 may be represented on the form $z = f(x, y)$.

Lemma 1. *If $C'd^\lambda < 1$ we have*

$$|f(x, y)| \leqslant 2C'(x^2 + y^2)^{\frac{\lambda+1}{2}}$$

and

$$|f(x, y)| \leqslant 2C'|P - Q|^{1+\lambda}$$

where $P = (x, y, f(x, y))$.

Proof: Cf. Smirnov [7], p. 490.

Lemma 2. *Let ω be $< \pi/2$. Then there is a constant d_1 such that if $d < d_1$, a line making an angle $\leqslant \omega$ with the normal at an arbitrary point $Q \in S_1$, will meet S_1 at most once inside the sphere with center Q and radius d.*

Proof: See Gunther [5], p. 6.

Denote by $G(P, Q)$ the Green's function of Ω_1.

Lemma 3. *There is a constant c_1, depending on Ω_1 only, such that for any points P, Q we have*

$$\left|\frac{\partial G}{\partial x_\theta}(P, Q)\right| \leqslant \frac{c_1}{|P - Q|^2},$$

where $(\partial G/\partial x_\theta)$ (P, Q) denotes differentiation in an arbitrary direction at P. In particular, for $P \in S_1$ we have

$$\left|\frac{\partial G}{\partial n_P}(P, Q)\right| \leqslant \frac{c_1}{|P - Q|^2},$$

$\partial G/\partial n_P$ denoting the normal derivative at P.

Proof: See Eidus [3].

Lemma 4. *The derivatives of $G(P, Q)$ are continuous in $\Omega_1 \cup S_1$.*

Proof: According to Schauder [6], a harmonic function in Ω_1, having tangential derivatives along the boundary which satisfy a Hölder condition, has derivatives of the first order that are continuous in $\Omega_1 \cup S_1$. Thus it suffices to prove that $1/|P-Q|$, $Q \in \Omega_1$, $P \in S_1$, has tangential derivatives of the above-mentioned kind. But that is clear, because the tangential derivatives may be expressed as a linear combination of the partial derivatives of $1/|P-Q|$ with functions which themselves satisfy Hölder conditions, as coefficients.

Lemma 5. *Let ω be $< \pi/2$. Then there exist two constants $\delta > 0$ and $c_{13} > 0$ such that if $Q \in S_1$, $B \in \Omega_1$, $|B-Q| < \delta$ and if the angle between $\overrightarrow{QB}$ and the normal at Q is $\leqslant \omega$ we have*

$$\frac{\partial G}{\partial n_Q}(Q, B) \geqslant \frac{c_{13}}{|Q-B|^2}.$$

Proof: It suffices to prove the lemma in the case $\omega = 0$. A simple application of Harnack's inequality then gives the general case. We may write (cf. Gunther [5], p. 202)

$$G(P, P') = \frac{1}{|P-P'|} - \frac{1}{4\pi} \int_{S_1} \frac{1}{|P-T|} \cdot \frac{\partial G}{\partial n_T}(T, P')\, dS(T).$$

As we may regard the integral as the potential of a simple layer, we get (cf. Gunther [5], p. 62):

$$\frac{\partial G}{\partial n_Q}(Q, B) = \frac{1}{|Q-B|^2} - \frac{1}{4\pi} \int_{S_1} \frac{\cos\varphi}{|Q-T|^2} \cdot \frac{\partial G}{\partial n_T}(T, B)\, dS(T) + \frac{1}{2} \frac{\partial G}{\partial n_Q}(Q, B)$$

i.e.
$$\frac{\partial G}{\partial n_Q}(Q, B) = \frac{2}{|Q-B|^2} - \frac{1}{2\pi} \int_{S_1} \frac{\cos\varphi}{|Q-T|^2} \frac{\partial G}{\partial n_T}(T, B)\, dS(T),$$

where φ is the angle between $\overrightarrow{QT}$ and the normal at Q. Choose $\sigma > 0$ so that $32\, c_1 \cdot C' \cdot \sigma^\lambda < 1$ and $\sigma < d$. If $T = (\xi, \eta, \zeta)$ we have by Lemma 1, when $\xi^2 + \eta^2 \leqslant \sigma^2$,

$$\zeta < 2C' |Q-T|^{1+\lambda}.$$

Moreover, we have $\cos\varphi = \dfrac{\zeta}{|Q-T|}$, which gives us

$$|\cos\varphi| \leqslant 2C' |Q-T|^\lambda.$$

For δ small enough we evidently have

$$|B-T| \geqslant \frac{1}{\sqrt{2}} |B-Q|.$$

Hence

$$\left| \int_{|Q-T| \leqslant \sigma} \frac{\cos\varphi}{|Q-T|^2} \frac{\partial G}{\partial n_T}(T, B)\, dS(T) \right| \leqslant \frac{4c_1 \cdot C'}{|B-Q|^2} \int_{|Q-T| \leqslant \sigma} |Q-T|^{\lambda-2}\, dS(T)$$

$$\leqslant \frac{16 c_1 \cdot C'}{|B-Q|^2} \sigma^\lambda \cdot 2\pi < \frac{\pi}{|B-Q|^2}.$$

Having fixed σ we find

$$\left| \int_{|Q-T| \geqslant \sigma} \frac{\cos\varphi}{|Q-T|^2} \frac{\partial G}{\partial n_T}(T, B)\, dS(T) \right| \leqslant \frac{1}{\sigma^2} \cdot 4\pi.$$

If now δ is chosen small enough, we get

$$\frac{1}{\sigma^2}\cdot 4\pi < \frac{2\pi}{2\delta^2}.$$

Thus

$$\frac{\partial G}{\partial n_Q}(Q, B) = \frac{2}{|Q-B|^2} - \frac{1}{2\pi}\int_{S_1}\frac{\cos\varphi}{|Q-T|^2}\frac{\partial G}{\partial n_T}(T, B)\,dS(T) \geqslant \frac{1}{|Q-B|^2}.$$

Lemma 6. *Let* P_0 *be a fixed point of* Ω_1. *There exists a constant* $c_2 > 0$ *such that*

$$\frac{\partial G}{\partial n_Q}(Q, P_0) \geqslant c_2$$

for all $Q \in S_1$.

Proof: Let δ be as in Lemma 5. We may assume that $|P_0 - Q| > \delta$ for all $Q \in S_1$. The set of points P, belonging to Ω_1, and for which $|P-Q| \geqslant \delta/2$ when $Q \in S_1$, is compact and contains P_0. Harnack's inequality gives

$$\frac{1}{c_3}u(P_0) \leqslant u(P) \leqslant c_3 \cdot u(P_0)$$

for every positive harmonic function and every P belonging to the set. Let Q be an arbitrary point $\in S_1$. $(\partial G/\partial n_Q)(Q, P)$ is harmonic and positive. Moreover, there is a point P on the normal at Q, for which $\delta/2 \leqslant |P-Q| \leqslant \delta$, and by Lemma 5 we get

$$\frac{\partial G}{\partial n_Q}(Q, P_0) \geqslant \frac{1}{c_3}\cdot\frac{\partial G}{\partial n_Q}(Q, P) \geqslant \frac{c_{13}\cdot 4}{c_3 \cdot \delta^2} = c_2 > 0.$$

Lemma 7. *If* Q *and* $Q' \in S_1$, $|Q-Q'| \geqslant \varrho > 0$,

$$\lim_{P \to Q'}\frac{\partial G}{\partial n_Q}(Q, P) = 0$$

uniformly in Q *and* Q'.

Proof: As in the proof of Lemma 5 we have

$$\frac{\partial G}{\partial n_Q}(Q, P) = \frac{\partial}{\partial n_Q}\frac{2}{|Q-P|} - \frac{1}{2\pi}\int_{S_1}\frac{\cos\varphi}{|Q-T|^2}\frac{\partial G}{\partial n_T}(T, P)\,dS(T).$$

Choose an arbitrary $\varepsilon > 0$. It is easily realized, that $(\partial/\partial n_Q)(1/|Q-P|)$ is uniformly continuous in $\{P \mid |Q-P| \geqslant \varrho/2\}$. Choose σ_1, $0 < \sigma_1 < \varrho/3$, so small that

$$\left|\frac{\partial}{\partial n_Q}\frac{1}{|Q-P|} - \frac{\partial}{\partial n_Q}\frac{1}{|Q-P'|}\right| < \frac{\varepsilon}{6}$$

if $|P-P'|<2\sigma_1$. Finally choose σ_2, $0<\sigma_2<\varrho/3$ so that

$$\frac{144\cdot c_1\cdot C'\cdot\pi}{\varrho^2}\sigma_2^\lambda<\frac{\varepsilon}{3}.$$

For $|P-Q'|<\sigma_1/2$ we now have

$$\int_{|T-Q'|\geqslant\sigma_1}\frac{\partial G}{\partial n_T}(T,P)\,dS(T)\leqslant A\cdot\frac{|P-Q'|^\lambda}{\sigma_1^2},$$

where A is a constant not depending on Q' and σ_1. In fact, this inequality is a lemma of Liapunov's. It is proved in Gunther |5|, p. 200 for the case when P is on the normal at Q', but the proof is easily extended to the more general case. Hence we get

$$\int_{\substack{|T-Q'|\geqslant\sigma_1\\|T-Q|\geqslant\sigma_2}}\frac{\cos\varphi}{|Q-T|^2}\cdot\frac{\partial G}{\partial n_T}(T,P)\,dS(T)\leqslant A\cdot|P-Q'|^\lambda\cdot\frac{1}{\sigma_1^2}\cdot\frac{1}{\sigma_2^2}.$$

As in Lemma 5 we also get

$$\int_{|T-Q|\leqslant\sigma_2}\frac{\cos\varphi}{|Q-T|^2}\frac{\partial G}{\partial n_T}(T,P)\,dS(T)\leqslant\frac{9\cdot c_1}{\varrho^2}\cdot 16\,C'\pi\cdot\sigma_2^\lambda<\frac{\varepsilon}{3}.$$

Finally

$$\left|2\frac{\partial}{\partial n_Q}\frac{1}{|Q-P|}-\frac{1}{2\pi}\int_{|T-Q'|\leqslant\sigma_1}\frac{\partial}{\partial n_Q}\frac{1}{|Q-T|}\frac{\partial G}{\partial n_T}(T,P)\,dS(T)\right|$$

$$\leqslant\frac{1}{2\pi}\int_{|T-Q'|\leqslant\sigma_1}\left|\frac{\partial}{\partial n_Q}\frac{1}{|Q-P|}-\frac{\partial}{\partial n_Q}\frac{1}{|Q-T|}\right|\frac{\partial G}{\partial n_T}(T,P)\,dS(T)$$

$$+\frac{\partial}{\partial n_Q}\frac{1}{|Q-P|}\cdot\frac{1}{2\pi}\int_{|T-Q'|\geqslant\sigma_1}\frac{\partial G}{\partial n_T}(T,P)\,dS(T)\leqslant\frac{\varepsilon}{3}+\frac{4}{\varrho^2}\cdot A\cdot\frac{|P-Q'|^\lambda}{\sigma_1^2}$$

provided that $|P-Q'|<\sigma_1/2$.

If P is chosen so close to Q' that $A\cdot|P-Q'|^\lambda\frac{1}{\sigma_1^2}\cdot\frac{1}{\sigma_2^2}<\frac{\varepsilon}{6}$, the lemma is proved by inserting the resulting estimates in the integral representation of $(\partial G/\partial n_Q)(Q,P)$.

4. Theorem 1

If u_1 is a positive harmonic function in Ω_1, then u_1 has non-tangential boundary values almost everywhere on S_1. Moreover, the boundary values $\in L^1(S_1)$.

Remark. Fatou proved in 1906 that a bounded harmonic function in the unit disk has non-tangential boundary values a.e. [4]. Using conformal mapping, Fatou's theorem may be stated for quite general plane regions. Calderon [1] and Carleson [2] have proved similar theorems for regions in R^n bounded by planes, but have weaker assumptions on the function. Tsuji [10] and Solomencev [8] have studied regions bounded by curved surfaces, but are assuming stronger regularity conditions than those in this paper. The method of proof used here is strongly related to the paper of prof. Carleson, to whom the author also owes his interest in this problem.

Proof: Let Q' be an arbitrary point $\in S_1$. It suffices to prove the existence of boundary values in a neighbourhood of Q'. Assume d to be so small that

1°. $d < d_1 =$ the number appearing in Lemma 2, if $\omega = \operatorname{arctg} 1/2$.

2°. $8C'd^\lambda < 1$, i.e. $2C'd^{1+\lambda} < d/4$.

We introduce the local coordinate system with the origin at Q'. Let $O = (0, 0, 3d/4)$. The condition 2° shows that a line through O and a point on S_1 inside the sphere Σ with center Q' and radius d will make an angle with the z-axis which is $< \operatorname{arctg} 1/2 = \omega$. Condition 1° and Lemma 2 show that such a line meets S_1 at most once inside Σ. We now introduce polar coordinates with origin $= O$, φ being the angle between radius vector and the plane through O parallel with the (x, y)-plane. What we just proved shows that the part of S_1 which is inside Σ may be represented by $\varrho = \varrho_1(\theta, \varphi)$. Let S be the part of S_1 which is inside the cylinder, which in the local system has the equation $x^2 + y^2 = d^2/4$. The surface of this cylinder may be represented in polar coordinates: $\varrho = \varrho_2(\theta, \varphi)$. The values of the vector (θ, φ) for which $(\theta, \varphi, \varrho_1(\theta, \varphi)) \in S$ will be called "(θ, φ)-values belonging to S". Evidently, $(\theta, -(\pi/4))$ do not belong to S. Now let $\Psi(\theta, \varphi)$ be an infinitely differentiable function, such that $0 \leqslant \Psi \leqslant 1$, $\Psi(\theta, \varphi) = 0$ for $\varphi \geqslant -(\pi/4)$, and $\Psi(\theta, \varphi) = 1$ for (θ, φ) belonging to S. The existence of such a function is easily proved. The surface generated by

$$\varrho = \varrho(\theta, \varphi) = \varrho_1(\theta, \varphi)\, \Psi(\theta, \varphi) + \varrho_2(\theta, \varphi)[1 - \Psi(\theta, \varphi)]$$

satisfies the conditions of a Liapunov surface, apart from not being bounded. If we insert the halfsphere with radius $d/2$ and center $(0, 0, d/2)$ however, we get a new region Ω, bounded by a Liapunov surface. We denote by Γ the part of $\partial\Omega$ which is not S and assume that the normals of $S \cup \Gamma$ satisfy $\theta \leqslant C \cdot r^\lambda$.

Consider the restriction u (as a function of the local coordinates) of u_1 to Ω. Of course, u is harmonic and positive. Put $u_\varepsilon(x, y, z) = u(x, y, z + \varepsilon)$. For ε small enough this function is defined in $\Omega \cup S \cup \Gamma$, and thus it may be represented as an integral (cf. Gunther [5], p. 202)

$$u_\varepsilon(P) = \frac{1}{4\pi} \int_{\Gamma \cup S} u_\varepsilon(Q) \frac{\partial G}{\partial n_Q}(Q, P)\, dS(Q).$$

The measures $d\mu_\varepsilon(Q) = u_\varepsilon(Q)\, dS(Q)$ have compact support and uniformly bounded total mass. For by Lemma 6 and the fact that u is positive, we get

$$\int_{S \cup \Gamma} |d\mu_\varepsilon| = \int u_\varepsilon(\theta)\, dS(Q) \leqslant \frac{1}{c_2} \int \frac{\partial G}{\partial n_Q}(Q, O)\, u_\varepsilon(Q)\, dS(Q) = \frac{4\pi}{c_2} u_\varepsilon(O).$$

But $u_\varepsilon(O) \to u(O)$ and thus the total mass is uniformly bounded. Hence we may choose a weakly convergent subsequence, $d\mu_\varepsilon \to d\mu$. A well-known theorem of Lebesgue allows us to decompose $d\mu$:

$$d\mu(Q) = f(Q)\, dS(Q) + d\sigma(Q),$$

where $f(Q) \in L^1(S \cup \Gamma)$, and $d\sigma$ is singular. This gives us

$$u(P) = \frac{1}{4\pi} \int_{\Gamma \cup S} \frac{\partial G}{\partial n_Q}(Q, P)\,[f(Q)\, dS(Q) + d\sigma(Q)].$$

We assert that $u(P) \to f(Q)$ when $P \to Q$ non-tangentially, a.e. on S. By another well-known theorem of Lebesgue we have for almost all $T \in S$:

$$\int_{|Q-T|<\varepsilon} \{|f(Q) - f(T)|\, dS(Q) + d\sigma(Q)\} = o(\varepsilon^2).$$

Assume Q_0 to be such a point and choose an arbitrary $\alpha < \pi/2$. Let h be so small that $V(Q_0, \alpha/2 + \pi/4, h)$ is contained in Ω. Suppose now that $\delta > 0$ is chosen so that

1°. $4\delta < d$ and $4\delta < \inf |Q_0 - Q|$, $Q \in \Gamma$.
2°. Lemma 5 is valid for $\omega = \pi/4$.
3°. $\sqrt{1 + \frac{9}{4} - 3\cos 2\beta} < \frac{5}{8}$ and $\sqrt{1 + \frac{16}{9} - \frac{8}{3}\cos 2\beta} < \frac{5}{12}$ where $\beta = \operatorname{arctg}\,[2C\,\delta^\lambda]$.
4°. $C\,\delta^\lambda < \dfrac{\sqrt{15}}{14} < \dfrac{\pi}{8}$.

We assume A to be an arbitrary point $\in V(Q_0, \alpha, h)$, such that $|A - Q_0| = a < \delta/2$. Define

$$L_0 = \{P \mid P \in S,\ |Q_0 - P| < a\}.$$

$$L_\nu = \{P \mid P \in S,\ 2^{\nu-1} a \leqslant |Q_0 - P| < 2^\nu a\}$$

for $\nu = 1, 2, \ldots, N$ where $2^N a \leqslant \delta/2 < 2^{N+1} a$ and

$$L = \{\Gamma \cup S\} - \{\bigcup_{\nu=0}^{N} L_\nu\}.$$

As we can write

$$f(Q_0) = \frac{1}{4\pi} \int_{S \cup \Gamma} f(Q_0) \frac{\partial G}{\partial n_Q}(Q, A)\, dS(Q) \quad \text{we get}$$

$$|u(A) - f(Q_0)| \leqslant \frac{1}{4\pi} \int_{S \cup \Gamma} |f(Q) - f(Q_0)| \frac{\partial G}{\partial n_Q}(Q, A)\, dS(Q) + \frac{1}{4\pi} \int_{S \cup \Gamma} \frac{\partial G}{\partial n_Q}(Q, A)\, d\sigma(Q)$$

$$\leqslant \frac{1}{4\pi} \sum_{\nu=0}^{N} \left\{ \sup_{Q \in L_\nu} \frac{\partial G}{\partial n_Q}(Q, A) \right\} \int_{\bigcup_{i=0}^{\nu} L_i} \{|f(Q) - f(Q_0)|\, dS(Q) + d\sigma(Q)\}$$

$$+O(1)\sup_{Q\in L}\frac{\partial G}{\partial n_Q}(Q,A)=\varepsilon\left(\frac{\delta}{2}\right)\sum_{\nu=0}^{N}\left\{\sup_{Q\in L\nu}\frac{\partial G}{\partial n_Q}(Q,A)\right\}2^{2\nu}\cdot a^2$$

$$+O(1)\sup_{Q\in L}\frac{\partial n_Q}{G\partial}(Q,A). \tag{1}$$

To simplify the notations we put $(\partial G/\partial n_Q)\,(Q,P)=K(Q,P)$. In the sequel c_i denote constants not depending on a,ν or δ. We need estimates of $K(Q,A)$ for $Q\in L_\nu$. For the two cases $\nu=0$ and $\nu=1$ Lemma 5 gives

$$K(Q,A)\leqslant\frac{c_1}{|Q-A|^2}\leqslant\frac{c_1}{a^2\sin^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}=\frac{c_4}{a^2}. \tag{2}$$

Choose ν, $2<\nu\leqslant N$, let Q_ν be an arbitrary point $\in L_\nu$, and let B be the point on the normal at Q_ν, for which we have $|Q_\nu-B|=2^\nu a$. As $2^\nu a\leqslant 2^N a<\delta<d/4$, B is in Ω. It follows from Lemma 3 that

$$K(Q_\nu,B)\leqslant\frac{c_1}{2^{2\nu}\cdot a^2}. \tag{3}$$

We denote by the $(\xi\,\eta\,\zeta)$-system the local coordinate system with the origin at Q_0. Let M_ν be the point of L_ν, whose projection along the ζ-axis on the (ξ,η)-plane is on the same line through Q_0 as Q_ν and which lies at a distance $=3\cdot 2^{\nu-2}a$ from Q_0. Letting Γ_1 be the sphere with center M_ν and radius $3\cdot 2^{\nu-3}a$, we find that B does not lie inside Γ_1. Now there exists a constant c_5 such that $K(Q_\nu,P)\leqslant c_5\cdot K(Q_\nu,B)$ for $P\notin\Gamma_1$. For by Lemma 5 we have

$$K(Q_\nu,B)\geqslant\frac{c_{13}}{|Q_\nu-B|^2}=c_{13}\cdot 2^{-2\nu}\cdot a^{-2}. \tag{4}$$

From assumption 3° about δ it follows that $|P-Q_\nu|\geqslant 2^{\nu-4}a$ for $P\notin\Gamma_1$, and from this fact and Lemma 3 it follows that

$$K(Q_\nu,P)\leqslant\frac{c_1}{|Q_\nu-P|^2}\leqslant c_1\cdot 2^{-2\nu+8}\cdot a^{-2}. \tag{5}$$

(4) and (5) give

$$\frac{K(Q_\nu,P)}{K(Q_\nu,B)}\leqslant\frac{c_1\,a^{-2\nu+8}\cdot a^{-2}}{c_{13}\cdot 2^{-2\nu}\cdot a^{-2}}=\frac{c_1\cdot 2^8}{c_{13}}=c_5. \tag{6}$$

Denoting by $h_\nu(P)$ the harmonic measure of $L_{\nu-1}\cup L_\nu\cup L_{\nu+1}$ we find for $P\in\partial\Gamma_1\cap\Omega$

$$h_\nu(P)\geqslant c_6>0. \tag{7}$$

For introduce the (X,Y,Z)-system $=$ the local coordinate system with the origin at M_ν. The part of S, for which $X^2+Y^2\leqslant(7\cdot 2^{\nu-4}a)^2$ is contained in $L_{\nu-1}\cup L_\nu\cup L_{\nu+1}$.

This follows from assumption 4° about δ; if Q is on said part of S we have $|Q-M_\nu|^2 \leqslant (7\cdot 2^{\nu-4}\cdot a)^2+(2C[7\cdot 2^{\nu-4}a]^{1+\lambda})^2<2^{\nu-1}a$, while for $Q\in S$ but $Q\notin L_{\nu-1} \cup L_\nu \cup L_{\nu+1}$ we evidently have $|Q-M_\nu|\geqslant 2^{\nu-1}a$. If P is an arbitrary point of $\partial\Gamma_1\cap\Omega$, we find for $P\in S$ $h_\nu(P)=1$, and hence we may assume that $P\notin S$. Put $P'=$ the projection of P on the (X, Y)-plane and $\overrightarrow{P'P}=l$, $|P'-P|=|l|$. We proved above that

$$U=\left\{Q\,|\,Q\in S,\; Q=(X,Y,Z)\Rightarrow |(X,Y,0)-P'|\leqslant\frac{|l|}{12}\right\}\subset L_{\nu-1}\cup L_\nu\cup L_{\nu+1}.$$

If $P''\in U$ the angle between $\overrightarrow{P''P}$ and l is $<\pi/8$, and from assumption 4° about δ it follows that the angle between l and the normal at P'' is $<\pi/8$. Hence the angle between $\overrightarrow{P''P}$ and the normal at P'' is $<\pi/4$. Lemma 5 and assumption 2° about δ then give

$$\frac{\partial G}{\partial n_{P''}}(P'',P)\geqslant\frac{c_{13}}{16\,|l|^2}.$$

Thus

$$h_\nu(P)=\frac{1}{4\pi}\int_{L_{\nu-1}\cup L_\nu\cup L_{\nu+1}}\frac{\partial G}{\partial n_Q}(Q,P)\,dS(Q)\geqslant\frac{c_{13}}{4\pi\cdot 16\,|l|^2}\cdot\frac{\pi\cdot|l|^2}{144}=c_6>0.$$

For $P\in\partial\Gamma\cap\Omega$ it follows from (6) and (7)

$$\frac{K(Q_\nu,P)}{K(Q_\nu,B)}\leqslant c_5=\frac{c_5}{c_6}\cdot c_6\leqslant c_7\cdot h_\nu(P).$$

If Q is any other boundary point of $\Omega\backslash\Gamma_1$, we have $\underline{\lim}_{P\to Q}\, h_\nu(P)\geqslant 0$, while by Lemma 7 we have

$$\lim_{P\to Q}\frac{K(Q_\nu,P)}{K(Q_\nu,B)}=0.$$

Together with (3), the maximum principle then gives for $P\in\Omega\backslash\Gamma_1$

$$K(Q_\nu,P)\leqslant c_7\cdot h_\nu(P)\cdot K(Q_\nu,B)\leqslant c_7\cdot h_\nu(P)\cdot c_1\cdot 2^{-2\nu}\cdot a^{-2}.$$

Particularly we get for $P=A$

$$K(Q_\nu,\,A)\leqslant c_7\cdot h_\nu(A)\cdot c_1\cdot 2^{-2\nu}\cdot a^{-2}. \tag{8}$$

Inserting (2) and (8) in (1) we find

$$|u(A)-f(Q_0)|\leqslant c_8\cdot\varepsilon\left(\frac{\delta}{2}\right)+\varepsilon\left(\frac{\delta}{2}\right)\cdot c_9\cdot\sum_{\nu=2}^{N}h_\nu(A)+O(1)\cdot\sup_{Q\in L}\frac{\partial G}{\partial n_Q}(Q,A).$$

Now $\sum_{\nu=2}^N h_\nu(P)\leqslant 3$, as we may regard $\sum_{\nu=2}^N h_\nu(P)$ as the harmonic measure of a certain subset of S, each point being counted at most three times. From Lemma 7 it follows

$$\overline{\lim_{A\to Q_0}}\left\{\sup_{Q\in L}\frac{\partial G}{\partial n_Q}(Q,A)\right\}=0.$$

Hence $$\overline{\lim_{A\to Q_0}}\,|u(A)-f(Q_0)|\leqslant c_{10}\cdot\varepsilon\left(\frac{\delta}{2}\right).$$

But $\varepsilon(\delta/2)\to 0$ when $\delta\to 0$, and thus the first part of the theorem is proved.

We remarked above that the boundary values $\in L^1(S\cup\Gamma)$ and thus also $\in L^1(S)$. The latter part of the theorem now follows from the simple fact that we can cover S_1 with a finite number of such neighbourhoods S.

5. Theorem 2

If u is harmonic in Ω_1, and if for almost all Q there is a cone $V(Q,\alpha,h)$ in which u is bounded, then u has non-tangential boundary values almost everywhere on S_1.

Remark. The analogue of this theorem in the case S_1 is a plane was proved by Calderon [1]. Having proved Theorem 1, Theorem 2 follows as in Calderon's paper, his proof needing only minor modifications to be applicable in this case. We therefore omit the proof.

6. Theorem 3

The harmonic function u in Ω_1 has non-tangential boundary values if and only if

$$\int_{V(Q,\alpha,h)}\frac{|\operatorname{grad} u(P)|^2}{r(P,Q)}\,dv(P)<\infty$$

almost everywhere in Q.

Remark. The values of α and h may depend on Q. The formulation of this theorem is chosen because of its resemblance to a similar theorem by Stein [9], valid when S_1 is a plane. Note that in this case the theorem of Carleson [2] contains that of Stein. The proof of Theorem 3 in principle follows Stein's proof, the difference being of a technical kind. However, the technical difficulties are overcome by the use of the methods developed in the lemmas and in Theorem 1. Hence we here are content with the mere statement of the theorem.

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