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# A generalization of a Tauberian theorem by Pleijel

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## 1. Introduction

In [4], [5] Pleijel proved the following Tauberian theorem. (The factor $|\lambda|^{-h}$ in [4] is here included in the measure $d\sigma$.)

*If* $s<1$, $\sigma(\lambda)\in T^s$, $\sigma(-\lambda)\in T^s$ *and*

$$\int_{-\infty}^{+\infty}(\lambda+t)^{-1}\,d\sigma(\lambda)=t^{-1}p(t^{-1})+o(t^{s-1}), \tag{1}$$

*when* $t\to\infty$ *along all non-real halfrays from the origin, then* $\sigma(\lambda)\in I^s$ *and* $\sigma(-\lambda)\in I^s$. *If* $s$ *is an integer,* $I^{-s}(\sigma(\lambda)-(-1)^s\sigma(-\lambda))\in\omega^0$.

In (1) $p$ is a real polynomial, and the relation $\varphi\in\omega^s$ means that $\varphi(\lambda)=$ constant $+\,o(\lambda^s)$ when $\lambda\to+\infty$. $I^k$ is defined by $dI^k\varphi=\lambda^k d\varphi$. If $I^k\varphi\in\omega^{k+s}$ for one value of $k$, then the same relation is valid for all $k\neq -s$ and we write $\varphi\in I^s$. When $s\neq 0$, $I^s=\omega^s$, but $\omega^0$ is a proper subset of $I^0$. The integral in (1) is convergent if and only if $I^{-1}\sigma\in\omega^0$. Finally, $\varphi\in T^s_+(T^s_-)$ means that there is a constant $C$ such that $d\varphi+C\lambda^{s-1}d\lambda$ is $\geqslant 0\ (\leqslant 0)$ for sufficiently large positive $\lambda$, and $T^s=T^s_+\cup T^s_-$.

In the proof in [4] of the Tauberian theorem, the asymptotic relation (1) is used only along the imaginary halfrays except in two exceptional cases, which occur when $s$ is an integer. In these cases (1) is used for the four halfrays $\arg t=\frac{1}{4}\pi+n\cdot\frac{1}{2}\pi$, $n=0,1,2,3$. Thus the conditions in the theorem are unnecessarily strong.

A closer study of the question has shown that it is sufficient to have the asymptotic relation along an arbitrary pair of non-real halfrays which are separated by the imaginary axis. This includes the case when one or both halfrays are purely imaginary. Moreover, the use of a larger Tauberian class than $T^s$ led to a more natural proof of the Tauberian theorem.

We can suppose that the halfrays are the imaginary halfrays. For if (1) is valid along another pair of rays, the relation holds also along the complex conjugate rays as the polynomial $p$ is real. The integral and the polynomial in (1) increase so slowly when $t\to\infty$ that the Phragmén–Lindelöf principle can be used in the angles formed in the upper and in the lower halfplane by the four halfrays. These angles contain the imaginary halfrays.

## 2. Tauberian classes

We use Tauberian classes which are closely connected with the slowly decreasing functions.

**Definition.** *A function $\varphi$ belongs to the class $U^s_+$ if*

$$\lim_{\mu\to 1+0}\ \varliminf_{x\to\infty}\ \inf_{x\leqslant y\leqslant \mu x}\ x^{-s}(\varphi(y)-\varphi(x))\geqslant 0.$$

*We put $U^s_-=-U^s_+$, $U^s=U^s_+\cup U^s_-$ and $U^s_0=U^s_+\cap U^s_-$.*

The class $U^0_+$ consists of the slowly decreasing functions and the class $U^0_0$ of the slowly oscillating functions (on the interval $(0,\infty)$).

The classes $U^s_+$ and $U^s_-$ are convex cones, $U^s_0$ is linear and if $\varphi_1$ and $\varphi_2$ belong to $U^s$, then either $\varphi_1+\varphi_2$ or $\varphi_1-\varphi_2$ belongs to $U^s$. If $\varphi\in U^s$ and $\psi\in U^s_0$, then $\varphi+\psi$ belongs to $U^s$.

The classes $T^s_+(T^s_-)$ are included in the classes $U^s_+(U^s_-)$, for take, for instance, $\varphi\in T^s_+$. Then $d\varphi+C\lambda^{s-1}d\lambda\geqslant 0$ for $\lambda\geqslant\lambda_0$, and we may evidently assume that $C>0$. Then

$$x^{-s}(\varphi(\lambda)-\varphi(x))\geqslant -Cx^{-s}\int_x^y\lambda^{s-1}d\lambda\geqslant -C\int_1^\mu u^{s-1}du,$$

when $\lambda_0\leqslant x\leqslant y\leqslant\mu x$. The last integral tends to 0 when $\mu\to 1+0$.

We shall need the following properties of the classes $U$.

**Proposition.**

(**a**) *If $\varphi\in U^q_+$, then $I^p\varphi\in U^{p+n}_+$ for $p\geqslant 0$.*

(**b**) *If $\varphi(\lambda)\in U^q_+$, then $\varphi(\lambda^\alpha)\in U^{q\alpha}_+$ for $\alpha>0$.*

(**c**) *$U^p_+\subset U^q_+$ if $p\leqslant q$.*

(**d**) *$I^q\subset U^q_0$.*

(**a**), (**b**) *and* (**c**) *are valid also for the classes $U_-$.*

*Proof.*

(**a**) Suppose $\varphi\in U^q_+$. A simple calculation shows that

$$x^{-p-q}(I^p\varphi(y)-I^p\varphi(x))=x^{-q}(\varphi(y)-\varphi(x))+x^{-p-q}\int_x^y(\varphi(y)-\varphi(\lambda))\,d(\lambda^p).$$

Thus if $x^{-q}(\varphi(y)-\varphi(x))\geqslant -c$ for $\lambda_0\leqslant x\leqslant y\leqslant\mu x$, where $c\geqslant 0$, then

$$x^{-p-q}(I^p\varphi(y)-I^p\varphi(x))\geqslant -c+x^{-p-q}\int_x^y -c\lambda^q d(\lambda^p)$$
$$=-c\left(1+p\int_1^{y/x}u^{p+q-1}du\right)\geqslant -c\left(1+p\int_1^\mu u^{p+q-1}du\right),$$

when $\lambda_0\leqslant x\leqslant y\leqslant\mu x$. This proves (**a**).

(b) Suppose $\varphi \in U^q_+$. Evidently

$$x^{-\alpha q}(\varphi(y^\alpha) - \varphi(x^\alpha)) = X^{-q}(\varphi(Y) - \varphi(X)) \geqslant -c,$$

when $\lambda_0 \leqslant X \leqslant Y \leqslant \mu X$, where $X = x^\alpha$, $Y = y^\alpha$, that is, when $\lambda_0^{1/\alpha} \leqslant x \leqslant y \leqslant \mu^{1/\alpha} x$, and (b) follows because $\mu \to 1+0$ when $\mu^{1/\alpha} \to 1+0$.

(c) Obvious.

(d) If $q \neq 0$, then $I^q = \omega^q$, and if $\varphi \in I^q$, then

$$|x^{-q}(\varphi(y) - \varphi(x))| = |(y/x)^q \varphi(y)/y^q - \varphi(x)/x^q| \leqslant ((y/x)^q + 1)\, o(1) \to 0$$

when $x \to +\infty$ and $x \leqslant y \leqslant \mu x$. Suppose secondly that $\varphi \in I^0$. Then $x^{-1}\int_0^x \lambda\, d\varphi(\lambda) = o(1)$ when $x \to +\infty$. Evidently

$$x^{-1}\int_0^x \lambda\, d\varphi(\lambda) = x^{-1}\left(x\varphi(x) - \int_0^x \varphi(\lambda)\, d\lambda\right)$$

and thus $$\varphi(x) = x^{-1}\int_0^x \varphi(\lambda)\, d\lambda + o(1), \text{ when } x \to +\infty.$$

We get $$\varphi(y) - \varphi(x) = o(1) + y^{-1}\int_0^y \varphi(\lambda)\, d\lambda - x^{-1}\int_0^x \varphi(\lambda)\, d\lambda$$

$$= o(1) + y^{-1}\int_x^y \varphi(\lambda)\, d\lambda + (x/y - 1)\, x^{-1}\int_0^x \varphi(\lambda)\, d\lambda$$

$$= o(1) + y^{-1}\int_x^y \varphi(\lambda)\, d\lambda + (x/y - 1)\, \varphi(x)$$

$$= o(1) + y^{-1}\int_x^y (\varphi(\lambda) - \varphi(x))\, d\lambda,$$

when $x \to +\infty$, $x \leqslant y \leqslant \mu x$. This gives

$$\inf_{x \leqslant y \leqslant \mu x} (\varphi(y) - \varphi(x)) \geqslant o(1) + (1 - \mu^{-1}) \inf_{x \leqslant \lambda \leqslant y \leqslant \mu x} (\varphi(\lambda) - \varphi(x)).$$

Thus $$\inf_{x \leqslant y \leqslant \mu x} (\varphi(y) - \varphi(x)) \geqslant \mu o(1).$$

The same reasoning gives an upper bound. The result is

$$\lim_{x \to +\infty} \sup_{x \leqslant y \leqslant \mu x} |\varphi(y) - \varphi(x)| = 0,$$

which means that $\varphi$ is slowly oscillating.

### 3. The unilateral Tauberian theorem

We shall use the following unilateral Tauberian theorem.

**Theorem.** *If $s \leqslant 1$ and $\sigma \in U^s$, then*

$$\int_0^\infty (\lambda+t)^{-1}\, d\sigma(\lambda) = t^{-1} p(t^{-1}) + o(t^{s-1}),\ t \to +\infty,$$

*implies that $\sigma \in I^s$. If $s$ is an integer, the result can be improved to $I^{-s}\sigma \in \omega^0$.*

For $s=0$ and for $0<s<1$, the theorem is announced by Hardy and Littlewood [1] and by Karamata [2], but no explicit proof is given. We sketch the proof. Let $0 \leqslant s < 1$. A partial integration (justified by the fact that $I^{-1}\sigma \in \omega^0$) gives

$$t^{-1}\int_0^\infty (\lambda/t)^s (1+\lambda/t)^{-2} (\sigma(\lambda)/\lambda^s)\, d\lambda = o(1) + K, \tag{2}$$

where the constant $K=0$ if $s>0$.

The kernel $g(x) = x^s(1+x)^{-2}$ is a Wiener kernel since

$$\int_0^\infty g(t)\cdot t^{ui}\, dt = \frac{\pi(s+ui)}{\sin \pi(s+ui)} \neq 0 \quad \text{for } u \text{ real.}$$

If $s=0$, the theorem follows from a theorem by Karamata [3]. If $s>0$, $\sigma \in U^s_+$ gives that $\sigma(\lambda)/\lambda^s$ is bounded below (Karamata [2] p. 36). From this combined with (2) we get that $\int_0^t \sigma(\lambda)\, d\lambda = O(t^{s+1})$ as $t \to +\infty$. Hence

$$\begin{aligned}(\mu-1)\cdot t\sigma(t) &= \left((\mu t - t)\,\sigma(t) - \int_t^{\mu t} \sigma(\lambda)\, d\lambda\right) + O(t^{s+1}) \\ &= -\int_t^{\mu t} (\sigma(\lambda)-\sigma(t))\, d\lambda + O(t^{s+1}) \leqslant C\cdot \int_t^{\mu t} t^s d\lambda + O(t^{s+1}) = O(t^{s+1})\end{aligned}$$

as $\sigma \in U^s_+$. We conclude that $\sigma(t) = O(t^s)$. If $\sigma \in U^s$ and $\sigma(\lambda)/\lambda^s$ is bounded, it is easy to see that $\sigma(\lambda)/\lambda^s \in U^0$. The theorem by Karamata then gives the wanted result.

The case $s<0$ is reduced to the case $0 \leqslant s < 1$ in the same way as in Pleijel [4] p. 565. This reduction uses the properties (a) and (c) of the Tauberian classes.

### 4. The bilateral Tauberian theorem

Suppose that (1) holds along the imaginary halfrays. In the same way as in [4] it follows that

$$\int_0^\infty (\lambda+t)^{-1}\, dA(\sqrt{\lambda}) = t^{-1} p_1(t^{-1}) + o(t^{s/2-1}), \tag{3}$$

$$\int_0^\infty (\lambda+t)^{-1}\sqrt{\lambda}\,dS(\sqrt{\lambda}) = t^{-1}p_2(t^{-1}) + o(t^{(s+1)/2-1}), \tag{4}$$

when $t \to \infty$ along the positive real axis. In these formulas $p_1$ and $p_2$ are real polynomials and

$$S(\lambda) = \sigma(\lambda) + \sigma(-\lambda), \tag{5}$$

$$A(\lambda) = \sigma(\lambda) - \sigma(-\lambda). \tag{6}$$

We replace the Tauberian condition in the theorem in section 1 by the condition $\sigma(\lambda) \in U^s$, $\sigma(-\lambda) \in U^s$. From this follows that either $S(\lambda)$ or $A(\lambda) \in U^s$. Suppose that $S$, say, belongs to $U^s$. Then $IS(\lambda) \in U^{s+1}$ by the proposition (**a**), and $IS(\sqrt{\lambda}) \in U^{\frac{1}{2}(s+1)}$ by (**b**). The unilateral theorem now can be applied to (4). This gives $IS(\sqrt{\lambda}) \in I^{\frac{1}{2}(s+1)}$ which is the same as $S(\lambda) \in I^s$, and by (**d**) it follows that $S(\lambda) \in U_0^s$.

From (5) and (6) follows that $A(\lambda) = 2\sigma(\lambda) - S(\lambda)$, where $2\sigma(\lambda) \in U^s$ and $-S(\lambda) \in U^s$. This implies that $A(\lambda) \in U^s$ and $A(\sqrt{\lambda}) \in U^{s/2}$. From (3) we obtain, using the unilateral theorem, that $A(\lambda) \in I^s$, and when $s$ is an even integer, $I^{-s}A \in \omega^0$.

The case in which $A(\lambda) \in U^s$ follows from the original Tauberian conditions is treated in a completely analogous way, starting from (2) instead of from (3).

We have the following bilateral Tauberian and Abelian theorem.

**Theorem.** *Let $s < 1$ and $\sigma(\lambda)$, $\sigma(-\lambda) \in U^s$. Then the following conditions are equivalent*

1. *The relation* (1) *is valid along one pair of non-real halfrays from the origin separated by the imaginary axis.*
2. *The relation* (1) *is valid along all non-real halfrays from the origin.*
3. *$\sigma(\lambda)$ and $\sigma(-\lambda) \in I^s$, and $I^{-s}(\sigma(\lambda) - (-1)^s\sigma(\lambda)) \in \omega^0$ if $s$ is an integer.*

We have proved the Tauberian part, $2. \Rightarrow 1. \Rightarrow 3.$ The Abelian part, $3. \Rightarrow 2.$, is the Abelian theorem in Pleijel [4] p. 568.

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