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On lacunary power series

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§ 1. Introduction

In this paper we shall study the behavior of certain lacunary power series $S(x)=\sum_{k=1}^{\infty} c_k e^{in_k x}$, where the frequencies n_k form a sequence of positive real numbers satisfying $n_{k+1}/n_k > q > 1$, $k=1, 2, 3, \ldots$. It has been shown by R. Salem and A. Zygmund (see [2]) that if q is large enough and certain conditions are imposed on the coefficients c_k, then there exists an interval on which the values $S(x)$ fill an open set; that is, $S(x)$ gives us an example of a Peano curve. If other conditions are imposed, to wit, $\sum_{k=1}^{\infty} |c_k| = \infty$ and $\lim_{k\to\infty} c_k = 0$, then a theorem of R. E. A. C. Paley says that every complex number ζ can be obtained as a sum $S(x)$ (see [3]).

It will be our aim to extend these two results in several directions. In the first case we shall show that the result of Salem and Zygmund is valid for all $q>1$. In the second case we shall show that by considering the sets of limit points of the partial sums of $S(x)$, as x varies throughout a large enough interval, we obtain exactly the family of all closed connected subsets of the extended plane (or, as we shall show to be equivalent, all sets that arise as the collection of limit points of the partial sums of a series $\sum_1^{\infty} \alpha_k$ with $\lim_{k\to\infty} \alpha_k = 0$). Furthermore, we shall show that, in both cases, this behaviour takes place in certain sets of measure zero that are constructed by a process similar to that which gives us the Cantor set.

Let us first consider the behaviour of $S(x)$ on the above mentioned intervals. Before announcing the precise statements of the theorems we have just described, however, let us point out that their proofs will be based on the following result, which is of interest in itself:

Theorem I. *Suppose $Q(x)=\sum_1^N c_k e^{in_k x}$, where $\dfrac{n_{k+1}}{n_k} > q > 1$, then there exist two constants $A=A_q$ and $A'=A'_q$, depending only on q, such that, whenever an interval I has length $|I| \geqslant A'/n_1$ then*

This research was conducted while the last-named author was a post-doctoral fellow of the National Science Foundation.

$$\sum_{1}^{N} |c_k| \leqslant A \sup_{x \in I} \mathcal{R}\{Q(x)\}.^1 \tag{1.1}$$

The result dealing with Peano curves is the following:

Theorem II. *Suppose* $S(x) = \sum_1^\infty c_k e^{in_k x}$, $n_{k+1}/n_k > q > 1$, *is an absolutely convergent lacunary power series. Then there exist constants* γ, ξ *and* ν, *depending only on* q, *such that if* $|c_k| \leqslant \gamma \sum_{j=k+1}^{\infty} |c_j|$, $k = 1, 2, \ldots$, I *is an interval of length at least* ξ/n_1 *and* w *a complex number satisfying* $|w| \leqslant \nu \sum_1^\infty |c_k|$ *then there exists* $x \in E$ *such that* $S(x) = w$.

For example, it follows from this theorem that the series

$$S(x) = \sum_{k=1}^{\infty} \frac{e^{in_k x}}{k^p}$$

defines such a Peano curve for all $q > 1$ and $p > 1$. Another simple example of a series satisfying the hypothesis of theorem II is the Weierstrass function

$$S(x) = \sum_{k=1}^{\infty} e^{-\lambda k} e^{in_k x}$$

whenever $\lambda \leqslant \log(1+\gamma)$.

Perhaps the simplest formulation of the above-mentioned generalization of the theorem of Paley is the following:

Theorem III. *Suppose* $S(x) = \sum_1^\infty c_k e^{in_k x}$, $n_{k+1}/n_k > q > 1$, *is a lacunary power series satisfying* $\sum_1^\infty |c_k| = \infty$ *and* $\lim_{k\to\infty} c_k = 0$. *Then there exists a constant* $A = A_q$ *such that, if* $\mathfrak{A} = \sum_1^\infty \alpha_k$ *is any numerical series with (complex) terms tending to* 0 *and* I *any interval of length* A/n_1, *we can then find* $x \in I$ *so that the set of limit points of the partial sums of* $S(x)$ *coincides with the set of limit points of the partial sums of* $\mathfrak{A}$.

We shall show that for, not only the last two, but for all three of these theorems the behaviour exhibited by the lacunary polynomial or series in question actually takes place on certain Cantor-type sets. The simplest of these type of sets are constructed in the following way: Let us fix an interval I (which shall be called the *support* of our set) and a constant K smaller than $\frac{1}{3}$ (which shall be called the *removal ratio* of our set).[2] Let us now remove from I a sub-

[1] If $z = u + iv$ is a complex number then $\mathcal{R}\{z\} = u$ will denote the real part of z.

[2] An example showing that this restriction is necessary, at least for the generalization of theorem I, can be found in § VI of [1].

interval situated in its middle and such that the ratio of its length to that of I is smaller than K. This leaves us with two similar intervals with each of which we can repeat this procedure (the ratio of the length of the removed to that of the previously remaining interval need not be the same as before, as long as it is smaller than K). Keeping up this process we obtain a certain set $E \subset I$. We shall refer to the removed intervals as *black* intervals, while the intervals from which a black interval is removed shall be called *white* intervals. More generally, our methods for proving these theorems will not require that the black intervals be taken away from the middle of white intervals; however, we shall be forced to assume that at each stage the length of the smaller remaining white interval divided by the length of the larger one majorizes some fixed positive number that depends on the removal ratio. We shall call these sets supertriadic (for a previous study connecting these type of sets with lacunary series see [1].)

The organization of this paper follows that of this outline. We shall devote the next section to the proofs of theorems I, II and III. Theorem I is not new (see [3]) but the proof we shall give is different from the original one. This proof is a refinement of an argument given in [4] (see pages 247–249) which is there used to obtain a somewhat weaker result. In the third, and last, section we extend these three theorems to the case of the Cantor-type sets just described. The proofs of these more general theorems are considerably more complicated than those of the theorems announced above which concern the behavior of our lacunary series on intervals. It should be pointed out that the method of Salem and Zygmund, as mentioned in their paper, can be used to show that, for q large enough, we can substitute certain sets of measure zero for the interval I in the statement of theorem II. These sets, however, are supertriadic sets of the type used in our generalization of theorem II.

§ 2. The interval case

(i) Let us first observe that theorem I is an immediate consequence of the following fact:

Given real numbers $n_1 < n_2 < \dots < n_N$ satisfying $\dfrac{n_{k+1}}{n_k} > q > 1$ and frequencies $\varphi_1, \varphi_2, \dots, \varphi_N$ then there exists a constant A'_q such that for each interval I of length $|I| \geqslant A'_q/n_1$ we can find a finite non-negative measure μ, with support included in I, whose Fourier transform

$$M(u) = \int_I e^{iux}\, d\mu(x) = \int_{-\infty}^{\infty} e^{iux}\, d\mu(x)$$

satisfies

$$\mathcal{R}\{M(n_k)\, e^{i\varphi_k}\} \geqslant 1/C_q, \tag{2.1}$$

where C_q is some positive constant depending only on q.

For, letting φ_k be defined by $c_k = |c_k|\, e^{i\varphi_k}$ and $T(x) = \mathcal{R}\{Q(x)\}$, we have

$$\mu(I)\left[\sup_{x\in I} T(x)\right] \geqslant \int_I T(x)\,d\mu(x) = \mathcal{R}\left\{\sum_1^N |c_k|\, e^{i\varphi_k} M(n_k)\right\} \geqslant \frac{1}{C_q}\sum_1^N |c_k|$$

which gives us (1.1) with $A = A_q \geqslant C_q\,\mu(I)$.

In the sequel we shall be dealing with power polynomials obtained by taking finite sums of the form $\sum e^{i\nu_k x} = P(x)$. We shall call the set of frequencies $\{\nu_k\}$ the *spectrum* of this polynomial and shall denote it by $Sp(P)$. The number $\delta = \min_{k\neq j}\{|\nu_k - \nu_j|\}$ will then be called the *step* of this spectrum.

Having made these definitions, let us begin our proof of theorem I. By the observation we have just made, this proof reduces to constructing the measure μ. This construction will depend on two lemmas concerning the F. Riesz products:

$$P_N(x) = \prod_1^N (1 + \cos[n_k x + \varphi_k]) = \prod_1^N (1 + \tfrac{1}{2}[e^{in_k x} e^{i\varphi_k} + e^{-in_k x} e^{-i\varphi_k}])$$

$$= 1 + \sum_1^N \tfrac{1}{2} e^{-i\varphi_k} e^{-in_k x} + \dots .$$

The first lemma, which can be easily proved by induction on N, collects several well known facts (see [4], pages 247–249):

Lemma (2.1). *If $q > 3$ then $P_N(x)$ has coefficients of modulus not greater than 1; in particular, the coefficients associated with the frequencies 0 and $-n_k$ are 1 and $\frac{1}{2}e^{-i\varphi_k}$ respectively. Furthermore, $Sp(P_N)$ has step no smaller than $n_1\dfrac{q-3}{q-1}$ and is contained, with the exception of 0, within the segments*

$$\pm\left[n_k\left(1 - \frac{1}{q} - \frac{1}{q^2} - \dots - \frac{1}{q^{k-1}}\right),\ n_k\left(1 + \frac{1}{q} + \frac{1}{q^2} + \dots + \frac{1}{q^{k-1}}\right)\right]^1. \tag{2.2}$$

The second lemma (also essentially contained in the same passage of [4]) is an easy consequence of the first:

Lemma (2.2). *Suppose $q > 1$ and let p be a positive integer satisfying $q^p > q^{p-1} + 2$. Define*

$$P_j(x) = \prod_{k=0} (1 + \cos[n_{j+kp}\,x + \varphi_{j+kp}]) \quad j = 1, 2, \dots, p.$$

Then the sets $Sp(P_j) - \{0\}$ are disjoint, their union has step $\geqslant \delta = a_q n_1$, where a_q is a positive constant that depends only on q, and $Q(x) = \sum_{j=1}^p P_j(x)$ has the form

$$Q(x) = p + \tfrac{1}{2}\sum_1^N e^{-i\varphi_k} e^{-in_k x} + \sum \alpha_j e^{-i\lambda_j x},$$

where $|\alpha_j| \leqslant 1$. A fortiori, $Sp(Q) - \{0\}$ has step $\geqslant \delta$.

[1] If $a < b$, $+[a, b]$ shall denote the interval $[a, b]$ while $-[a, b]$ denotes the interval $[-b, -a]$.

For our hypothesis on p implies $q^p > 3$. Thus, we can apply lemma (2.1) to each of the polynomials P_j. Hence, $Sp(P_j) - \{0\}$ is contained within the intervals (2.2) (with q replaced by q^p). Lemma (2.2) will then certainly be true if the intervals associated with P_j and $P_{j'}$, $j \neq j'$, are disjoint and at a distance at least δ from each other (δ shall be determined presently). But this is easily seen to be the case, since, assuming, say, that $\alpha = j + kp > j' + \varrho p = \beta$,

$$a_\alpha \left(1 - \frac{1}{q^p} - \frac{1}{q^{p2}} - \dots - \frac{1}{q^{p(k-2)}}\right) - n_\beta \left(1 + \frac{1}{q^p} + \frac{1}{q^{p2}} + \dots + \frac{1}{q^{p(\varrho-2)}}\right)$$

$$> n_\beta \left[\frac{n_\alpha}{n_\beta}\left(1 - \frac{1}{q^p - 1}\right) - \frac{q^p}{q^p - 1}\right] > n_\beta \left[q^{\alpha-\beta}\left(\frac{q^p - 2}{q^p - 1}\right) - \frac{q^p}{q^p - 1}\right]$$

$$\geqslant n_1 \left[\frac{q(q^p - 2) - q^p}{q^p - 1}\right] = n_1 \left[\frac{q(q^p - q^{p-1} - 2)}{q^p - 1}\right] = \delta > 0.$$

We note that, according to our proof, $a_q = \min \left\{\dfrac{q^{p+1} - q^p - 2q}{q^p - 1}, \dfrac{q^p - 3}{q^p - 1}\right\}$.

The construction of μ is now straight forward. We first remark that it is sufficient to consider an interval I centered about the origin, since the general case would then follow by a translation of the variable x. Thus, we shall assume that $I = [-2\varrho, 2\varrho]$, where a lower bound on ϱ shall be given presently (thus giving us a value for A_q'). Let $a(x)$ be the triangular function

$$a(x) = \begin{cases} \dfrac{x}{4\varrho^2} + \dfrac{1}{2\varrho}, & -2\varrho \leqslant x \leqslant 0 \\[2ex] \dfrac{1}{2\varrho} - \dfrac{x}{4\varrho^2}, & 0 \leqslant x \leqslant 2\varrho \\[2ex] 0 & , |x| > 2\varrho. \end{cases}$$

Thus, $A(u) = \int_{-\infty}^{\infty} e^{iux} a(x)\,dx = \dfrac{\sin^2 \varrho x}{\varrho^2 x^2}$ and, in particular, $\int_{-\infty}^{\infty} a(x)\,dx = A(0) = 1$

We define μ by letting $\dfrac{d\mu}{dx} = Q(x)\,a(x)$. Thus,

$$M(u) = \int_{-\infty}^{\infty} e^{iux}\,d\mu(x) = p\,A(x) + \tfrac{1}{2}\sum_{1}^{N} e^{-i\varphi_k} A(x - n_k) + \sum \alpha_j A(x - \lambda_j).$$

But, in general, if $\nu_0, \nu_1, \nu_2, \dots$ is a sequence with step $\geqslant \delta$ and $\{b_k\}$ is any sequence of real numbers, we have

$$\left|\sum_{h \neq m} b_h A(\nu_m - \nu_h)\right| \leqslant \sup |b_h| \sum_{h \neq m} A(\nu_m - \nu_h) \leqslant \{\sup |b_h|\} \frac{2}{\varrho^2} \sum_{n=1}^{\infty} \frac{1}{n^2 \delta^2}.$$

Thus, letting the ν_k's be the frequencies and the b_k's the coefficients of $Q(x)$ we obtain

$$|M(n_k)-\tfrac{1}{2}e^{-i\varphi_k}|=\left|p\,A(n_k)+\tfrac{1}{2}\sum_{j\neq k}e^{-i\varphi_j}A(n_k-n_j)+\sum\alpha_j A(n_k-\lambda_j)\right|\leqslant$$

$$\leqslant p\left(\frac{2}{\delta^2\varrho^2}\right)\sum_1^\infty\frac{1}{n^2}=\frac{p\pi}{3\,\delta^2\varrho^2} \tag{2.3}$$

where $\delta=a_q n_1$. Hence, if we choose ϱ such that $\frac{p\pi}{3\,\delta^2\varrho^2}<\frac{1}{4}$, inequality (2.3) can be written in the form

$$|M(n_k)\,e^{i\varphi_k}-\tfrac{1}{2}|<\tfrac{1}{4},$$

which clearly implies $\mathcal{R}\{M(n_k)\,e^{i\varphi_k}\}\geqslant\frac{1}{4}$. Hence, inequality (2.1) is satisfied with $C_q=4$. The lower bound on A_q' obtained by this method is easily seen to be $\frac{4}{a_q}\sqrt{\frac{p\pi}{3}}$. Theorem I is thus proved.

In the sequel we shall apply theorem I in slightly different, but equivalent, forms. The following three corollaries are the restatements of this result we shall use.

Corollary (2.1). *Suppose* $Q(x)=\sum_1^N c_k e^{in_kx}$, *where* $\frac{n_{k+1}}{n_k}>q>1$, *then there exist two constants,* $\alpha=\alpha_q$, $0<\alpha<\pi/2$, *and* $A'=A_q'$, *depending only on* q, *such that whenever a complex number* Z *has modulus* $|Z|=\frac{\Delta}{\cos\alpha}$, *where* $\Delta=\sum_1^N|c_k|$, *and an interval* I *has length* $|I|\geqslant A'/n_1$ *then there exists* $x\in I$ *such that*

$$|Q(x)-Z|\leqslant\Delta\ \mathrm{tg}\ \alpha. \tag{2.4}$$

By multiplying both $Q(x)$ and Z by $e^{-i\arg Z}$ we can reduce the problem to the case when Z is real. To see that theorem I implies corollary (2.1) (the direction we need) is now immediate. For, letting A' be the constant of theorem I, we know by that theorem (see inequality (1.1)) that there exists an $x\in I$ such that $Q(x)$ lies to the right of the vertical line ϱ passing through the point $(\Delta/A,\,0)$. On the other hand, clearly $|Q(x)|\leqslant\Delta$, which implies that $Q(x)$ must lie in the intersection of the half-plane to the right of ϱ and the disc about 0 of radius Δ (the shaded region in fig. 1). Defining α by the equation $\cos\alpha=1/A$, inequality (2.4) follows from the observation that the distance between a point of this region and Z cannot exceed $\omega=\Delta\tan\alpha$ (see fig. 1).

Corollary (2.2). *Suppose* $Q(x)=\sum_1^N c_k e^{in_kx}$, *where* $\frac{n_{k+1}}{n_k}>q>1$, *then there exist two constants* $A''=A_q''$ *and* $B=B_q\geqslant 1$, *depending only on* q, *such that each interval* I *of length* $|I|\geqslant A''/n_1$ *contains a subinterval* J *of length* $\frac{2}{B\,n_N}$ *such that*

$$\sum_1^N|c_k|\leqslant B\,\mathcal{R}\{Q(y)\} \tag{2.5}$$

for each $y\in J$.

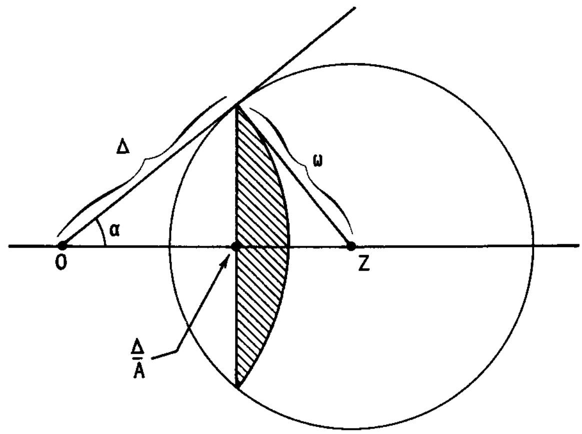

Fig. 1.

For, if I' is an interval of length at least A'/n_1, where A' is the constant of theorem I, then, by that theorem we know that there exists $x \in I'$ such that $\sum_1^N |c_k| \leqslant A\,\mathcal{R}\{Q(x)\}$. But, if y is any other point, using the mean value theorem, we obtain

$$|Q(x)-Q(y)| \leqslant |x-y| \sum_1^N n_k |c_k| \leqslant n_N |x-y| \sum_1^N |c_k|.$$

Thus,

$$\mathcal{R}\{Q(y)\} = \mathcal{R}\{Q(x)\} - \mathcal{R}\{Q(x)-Q(y)\} \geqslant \frac{1}{A}\sum_1^N |c_k| - n_N|x-y|\sum_1^N |c_k|.$$

Hence, letting $B = 2A$ and $y \in J = \left[x - \dfrac{1}{Bn_N},\ x + \dfrac{1}{Bn_N}\right]$ we obtain

$$\frac{1}{B}\sum_1^N |c_k| = \frac{1}{2A}\sum_1^N |c_k| = \left(\frac{1}{A} - \frac{n_N}{2A\,n_N}\right)\sum_1^N |c_k| \leqslant \left(\frac{1}{A} - n_N|x-y|\right)\sum_1^N |c_k| \leqslant \mathcal{R}\{Q(y)\},$$

which is inequality (2.5). If I' is chosen as the central subinterval of an interval I of length at least A''/n_1, where $A'' = A' + 2/B$, then, clearly, $J \subset I$ and the corollary is proved.

Corollary (2.3). *Suppose* $Q(x) = \sum_1^N c_k e^{in_k x}$, *where* $\dfrac{n_{k+1}}{n_k} > q > 1$, *then there exist constants* $A'' = A''_q$, $\alpha = \alpha_q$ $(0 < \alpha < \pi/2)$ *and* $B' = B'_q$, *depending only on* q, *such that whenever an interval* I *and a complex number* ζ *satisfy* $|I| \geqslant \dfrac{A''}{n_1}$ *and* $|\zeta| > \dfrac{\Delta}{\cos\alpha} = \dfrac{\sum_1^N |c_k|}{\cos\alpha}$, *then there exists a subinterval* $J \subset I$ *of length* $|J| = \dfrac{3B'}{n_N}$ *such that*

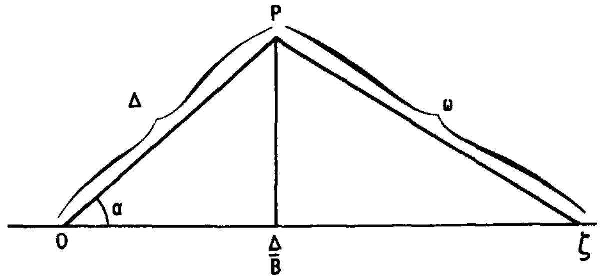

Fig. 2.

$$|\zeta - Q(x)| \leqslant |\zeta| - \frac{\cos\alpha}{2}\Delta \tag{2.6}$$

for all $x \in J$.

This corollary can be derived from the preceding one by a geometrical argument that is very similar to the proof of corollary (2.1). Again, we may assume that ζ is real. Let us define α by letting $\cos\alpha = 1/B$ (where B is the constant in (2.5)) and let us construct the triangle $OP\zeta$ of sides of length Δ and $|\zeta|$ forming an angle α between them (see figure 2). The fact that $|\zeta| > \dfrac{\Delta}{\cos\alpha}$ implies that the angle $OP\zeta$ is greater than $\pi/2$ and we certainly have $\omega/|\zeta| < 1$. Using these two inequalities and the identity $|\zeta|^2 - \omega^2 = 2\Delta|\zeta|\cos\alpha - \Delta^2$ (i.e. the law of cosines) we thus obtain

$$|\zeta| - \omega = \frac{\Delta(2|\zeta|\cos\alpha - \Delta)}{|\zeta| + \omega} = \Delta\left(\frac{2\cos\alpha - \dfrac{\Delta}{|\zeta|}}{1 + \dfrac{\omega}{|\zeta|}}\right) > \frac{\Delta}{2}(2\cos\alpha - \cos\alpha) = \frac{\cos\alpha}{2}\Delta.$$

But, by exactly the same type of argument used at the end of the proof of corollary (2.1), we obtain the fact that whenever $\sum_1^N |c_k| \leqslant B\mathcal{R}\{Q(x)\}$ (see inequality (2.5)) then the distance between $Q(x)$ and ζ cannot exceed ω. Thus, we have shown $|Q(x) - \zeta| \leqslant \omega \leqslant |\zeta| - \dfrac{\cos\alpha}{2}\Delta$ for all such values of x. But, by collary (2.2), given any interval I of length at least A''/n_1, such x's will fill a subinterval J of length at least $\dfrac{2}{Bn_N}$. This proves our corollary with $B' = \dfrac{2}{3B}$.

(ii) Let us now turn to the proof of theorem II. We shall decompose the series $S(x)$ in question into successive lacunary blocks of its terms

$$S(x) = Q_1(x) + Q_2(x) + \ldots + Q_j(x) + \ldots .$$

Theorem I, in the form of corollary (2.1), will be applied inductively to these blocks in order to "aim" toward our point w. We skall thus obtain a convergent

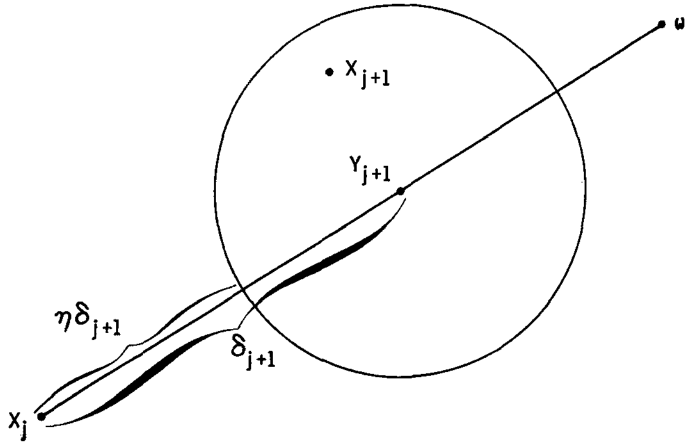

Fig. 3.

sequence of real numbers, $\{x_j\}$, such that the sequence of partial sums $S_j(x_j) = Q_1(x_j) + Q_2(x_j) + \ldots + Q_j(x_j) = X_j$ converges to w. From this it will follow easily that $S(x) = w$, where $x = \lim_{j=\infty} x_j$.

The following elementary lemma contains the purely geometric aspects of the argument involved in the construction of this sequence $\{X_j\}$:

Lemma (2.3). *Let $\delta_j \geqslant 0$ and $p_k = \sum_{j=k+1}^{\infty} \delta_j < \infty$ $(k=0, 1, \ldots)$. If w is a complex number satisfying $|w| \leqslant \eta \varrho_0$ and $\delta_k \leqslant (\eta/2)\varrho_k$, $k = 1, 2, \ldots$, where $0 < \eta < 1$, then for all sequences of complex numbers $\{X_j\}$ and $\{Y_j\}$ constructed in such a fashion that $X_0 = 0$,*

$$Y_{j+1} = X_j + \delta_{j+1}\left\{\frac{w - X_j}{|w - X_j|}\right\}$$

$$|X_{j+1} - Y_{j+1}| \leqslant (1-\eta)\,\delta_{j+1}, \quad j = 0, 1, 2, \ldots$$

we have
$$|w - X_j| \leqslant \eta\,\varrho_j, \quad j = 0, 1, 2, \ldots.$$

In particular, $\lim_{j\to\infty} X_j = w$.

The proof follows immediately by an induction argument. By assumption, $|w - X_0| = |w| \leqslant \eta\,\varrho_0$. We shall show that the inequality $|w - X_j| \leqslant \eta\,\varrho_0$ implies $|w - X_{j+1}| \leqslant \eta\,\varrho_{j+1}$. Suppose, first, that $|w - X_j| \geqslant \delta_{j+1}$. Then, $|w - X_{j+1}| \leqslant |w - X_j| - \eta\,\delta_{j+1}$ (see fig. 3). Thus, $|w - X_{j+1}| \leqslant \eta\,\varrho_j - \eta\,\delta_{j+1} = \eta\,\varrho_{j+1}$. If, on the other hand, $|w - X_j| < \delta_{j+1}$ then

$$|w - X_{j+1}| \leqslant |w - Y_{j+1}| + |Y_{j+1} - X_{j+1}| \leqslant \delta_{j+1} + (1-\eta)\,\delta_{j+1} < 2\,\delta_{j+1} \leqslant \eta\,\varrho_{j+1}$$

which completes the proof.

Before beginning our construction of the sequence $\{X_j\}$ let us first show that

$$|c_k| \leqslant \gamma \sum_{k+1}^{\infty} |c_j| \tag{2.7}$$

implies

$$|c_1| n_1 + |c_2| n_2 + \dots + |c_k| n_k \leqslant \beta n_k \sum_{j=k+1}^{\infty} |c_j| \tag{2.8}$$

provided

$$\beta \geqslant \frac{\gamma q}{q-1-\gamma} > 0. \tag{2.9}$$

When $k=1$ (2.7) clearly implies (2.8) as long as $\beta \geqslant \gamma$, which certainly holds if $\beta \geqslant \dfrac{\gamma q}{q-1-\gamma}$. Now let us suppose (2.8) true for k; we shall show that it is true for $k+1$. By our hypotheses, we obtain

$$|c_1| n_1 + \dots + |c_k| n_k + |c_{k+1}| n_{k+1} \leqslant n_k \beta \sum_{j=k+1}^{\infty} |c_j| + |c_{k+1}| n_{k+1} =$$

$$= n_k \beta \sum_{j=k+2}^{\infty} |c_j| + n_k \beta |c_{k+1}| + n_{k+1} |c_{k+1}| \leqslant n_{k+1} \left(\frac{\beta}{q} + \frac{\gamma \beta}{q} + \gamma \right) \sum_{j=k+2}^{\infty} |c_j|$$

$$\leqslant n_{k+1} \beta \sum_{j=k+2}^{\infty} |c_j|,$$

which is inequality (2.8) for $k+1$.

The following notation will be useful. We shall denote by n_j' and n_j'' the smallest and the largest of the frequencies of the block Q_j, by c_j' and c_j'' the coefficients of $e^{in_j'x}$ and $e^{in_j''x}$, and by Δ_j the sum of the absolute values of the coefficients of Q_j. Thus, $Q_j(x) = c_j' e^{in_j'x} + \dots + c_j'' e^{in_j''x}$ and $\Delta_j = |c_j'| + \dots + |c_j''|$.

Letting $\delta_j = \dfrac{\Delta_j}{\cos \alpha}$. where α is the constant of corollary (2.1), and putting $\varrho_k = \sum_{j=k+1}^{\infty} \delta_j = \dfrac{1}{\cos \alpha} \sum_{k+1}^{\infty} \Delta_j = \dfrac{R_k}{\cos \alpha}$ we now construct inductively sequences $\{X_j\}$ and $\{Y_j\}$ satisfying the hypotheses of lemma (2.3). Since $X_0 = 0$, we have $Y_1 = X_0 + \delta_1 \dfrac{w - X_0}{|w - X_0|} = \delta_1 \dfrac{w}{|w|}$. We select an arbitrary real number x_0 and apply corollary (2.1) to $Q = Q_1$, $I = I_1 = \left[x_0 - \dfrac{A'}{2 n_1'}, x_0 + \dfrac{A'}{2 n_1'} \right]$ and $Z = Z_1 = \delta_1 \dfrac{w}{|w|} = \delta_1 \dfrac{w - X_0}{|w - X_0|}$. We obtain $x_1 \in I$ such that

$$|Q_1(x_1) - Z_1| \leqslant \Delta_1 \operatorname{tg} \alpha = \delta_1 \sin \alpha.$$

We then define $X_1 = S_1(x_1) = Q_1(x_1)$ and, clearly,

$$|X_1 - Y_1| = |Q_1(x_1) - Z_1| \leqslant \delta_1 \sin \alpha.$$

Y_2 is then determined by $Y_2 = X_1 + \delta_2 \dfrac{w - X_1}{|w - X_1|}$. Now we apply corollary (2.1) again; this time to $Q = Q_2$, $I = I_2 = \left[x_1 - \dfrac{A'}{2n_2'}, \ x_1 + \dfrac{A'}{2n_2'}\right]$ and $Z = Z_2 = \delta_2 \dfrac{w - X_1}{|w - X_1|}$. We obtain an $x_2 \in I$ such that

$$|Q_2(x_2) - Z_2| \leqslant \Delta_2 \operatorname{tg} \alpha = \delta_2 \sin \alpha. \tag{2.10}$$

We then define $X_2 = S_2(x_2) = S_1(x_2) + Q_2(x_2)$. Hence,

$$|X_2 - Y_2| \leqslant |S_1(x_2) - S_1(x_1)| + |Q_2(x_2) - Z_2|.$$

But, by inequality (2.8),

$$|S_1'(x)| = |in_1 c_1 e^{in_1 x} + \ldots + in_1'' c_1'' e^{in_1'' x}| \leqslant n_1'' \beta \sum_{n_2'}^{\infty} |c_k| \leqslant n_1'' \beta \varrho_1$$

and, moreover, x_2 being a point of I_2, $|x_2 - x_1| \leqslant \dfrac{A_1}{2n_2'}$. Thus, by the mean value theorem

$$|S_1(x_2) - S_1(x_1)| \leqslant \frac{A' n_1'' \beta \varrho_1}{2n_2'} < \frac{A'}{2} \beta \varrho_1.$$

Using this estimate and (2.10), therefore, we obtain

$$|X_2 - Y_2| \leqslant \frac{A' \beta \varrho_1}{2} + \delta_2 \sin \alpha.$$

Continuing in this fashion, we obtain a sequence $\{x_j\}$ with

$$x_{j+1} \in I_{j+1} = \left[x_j - \frac{A'}{2n_{j+1}'}, \ x_j + \frac{A'}{2n_{j+1}'}\right]$$

(from which we easily see that

$$|x_k - x_{\varrho-1}| < \frac{A'}{n_\varrho'} \frac{q}{q-1}, \quad 1 \leqslant \varrho < k, \tag{2.11}$$

and, consequently, $x = \lim x_j$ exists and lies in an interval about x_0 of length $\left.\dfrac{2q}{q-1} \dfrac{A'}{n_1}\right)$ and such that if $X_j = S_j(x_j)$, $Y_j = Y_{j-1} + \delta_j \dfrac{w - X_{j-1}}{|w - X_{j-1}|}$, $j \geqslant 1$, then

$$|X_j - Y_j| \leqslant \frac{A' \beta}{2} \varrho_{j-1} + \delta_j \sin \alpha.$$

We could then apply lemma (2.3), and obtain the conclusion of theorem II, with $x = \lim_{j\to\infty} x_j$,[1] if

$$\delta_j \leqslant \frac{\eta}{2}\varrho_j, \tag{2.12}$$

and
$$\frac{A'\beta}{2}\varrho_{j-1} + \delta_j \sin\alpha \leqslant (1-\eta)\,\delta_j, \quad j = 1, 2, \dots . \tag{2.13}$$

In other words, we must show that we can choose $\gamma = \gamma(q)$, $\eta = \eta(q)$ and the sequence $\{n_j''\}$ in such a way that (2.12) and (2.13) hold. We first note that for (2.13) to hold we must choose η to be a positive number less than $1 - \sin\alpha$. Having chosen such an η, n_j'' shall be chosen inductively to be the largest number of the part of the sequence $\{n_k\}$ following n_{j-1}'' such that (2.12) holds (which can obviously be done since $\sum |c_k| < \infty$). Thus, $\Delta_j + |c_{j+1}'| > (\eta/2)(R_j - |c_{j+1}'|)$. But, by (2.7), $|c_{j+1}'| \leqslant \gamma(R_j - |c_{j+1}'|)$. Hence, we should impose the condition $\gamma < \eta/2$. We also note that these two inequalities imply

$$\Delta_j > \left(\frac{\eta}{2} - \gamma\right) R_j. \tag{2.14}$$

Rewriting (2.13) in the form

$$[1 - \eta - \sin\alpha]\,\delta_j \geqslant \frac{A'\beta}{2}[\varrho_j + \delta_j]$$

or, after multiplying both sides by $\cos\alpha$, in the form

$$\left[1 - \eta - \sin\alpha - \frac{A'\beta}{2}\right]\Delta_j \geqslant \frac{A'\beta}{2} R_j,$$

we see that (2.14) implies (2.13) provided

$$\frac{A'\beta}{2\left(1 - \eta - \sin\alpha - \dfrac{A'\beta}{2}\right)} \leqslant \frac{\eta}{2} - \gamma.$$

But, since $\lim_{\gamma\to 0} \dfrac{\gamma q}{q - 1 - \gamma} = 0$, we can certainly find γ so small, and a β satisfying (2.9), such that this last inequality and the condition $\gamma < \eta/2$ be satisfied. This proves theorem II with this value of γ, $\nu = \dfrac{\eta}{\cos\alpha}$ and $\xi = 2A'\dfrac{q}{q-1}$.

[1] For $|S(x) - S_j(x_j)| \leqslant |S_j(x) - S_j(x_j)| + |S(x) - S_j(x)| \leqslant |S_j(x) - S_j(x_j)| + R_j \leqslant \dfrac{A' n_j'' \beta q}{n_{j+1}'(q-1)}\varrho_j + {}$ ${}+ R_j \to 0$ as $j \to \infty$, the last estimate making use of (2.8), the mean value theorem and the limiting case of (2.11), when $k \to \infty$.

(iii). The proof of theorem III has many points in common with the preceding one but contains ideas that are new. Instead of lemma (2.3), for example, we shall use the following analogous result:

Lemma (2.4). *Let $\{\delta_j\}$, $j=1, 2, \ldots, m$, be a positive sequence, and suppose that w and X_0 are two complex numbers satisfying $|w-X_0| \leqslant \eta \sum_1^m \delta_j$, $0<\eta<1$; then, whenever two sequences $\{X_j\}$ and $\{Y_j\}$ satisfy*

$$Y_{j+1} = X_j + \delta_{j+1} \frac{w-X_j}{|w-X_j|} \text{ and } |X_{j+1} - Y_{j+1}| \leqslant (1-\eta)\,\delta_{j+1},$$

$j=0, 1, 2, \ldots, m-1$, we have $|w-X_m| \leqslant 2 \sup_j \delta_j$.

For, if $|w-X_{m-1}|<\delta_m$ then, as in the proof of lemma (2.3), $|w-X_m|<2\,\delta_m$ and the lemma is true. On the other hand, if $|w-X_{m-1}| \geqslant \delta_m$ then (see fig. 3) $|w-X_m| \leqslant |w-X_{m-1}| - \eta\,\delta_m$. In this last case, applying this argument again we obtain the result that, if $|w-X_{m-2}|<\delta_{m-1}$ then $|w-X_{m-1}|<2\,\delta_{m-1}$ which implies $|w-X_m|<2\,\delta_{m-1}-\eta\,\delta_m<2\,\delta_{m-1}$ and the lemma is proved; or, if $|w-X_{m-2}| \geqslant \geqslant \delta_{m-1}$ then

$$|w-X_m| \leqslant |w-X_{m-1}| - \eta\,\delta_m \leqslant |w-X_{m-2}| - \eta\,\delta_{m-1} - \eta\,\delta_m.$$

Thus, we see that by preceding in this way we obtain the desired result if, for some j, $|w-X_j|<\delta_{j+1}$. On the other hand, if this inequality is satisfied by no value of j, $1 \leqslant j \leqslant m-1$, we obtain the contradiction

$$|w-X_m| \leqslant |w-X_0| - \eta \sum_1^m \delta_j \leqslant \eta \sum_1^m \delta_j - \eta \sum_1^m \delta_j = 0.$$

Our proof of theorem III shall be based on this lemma and corollary (2.1). Let A' and α, then, be the constants involved in this last corollary and let $\beta>0$ and $\eta>0$ be two numbers satisfying

$$(\cos\alpha)\frac{A'\beta}{2} + \sin\alpha = 1-\eta. \tag{2.15}$$

Because of our assumptions on the sequences $\{c_k\}$ and $\{n_k\}$ we have

$$|c_1|\,n_1 + |c_2|\,n_2 + \ldots + |c_\varrho|\,n_\varrho = o(n_\varrho), \tag{2.16}$$

as $\varrho \to \infty$. Now let us define the sequence of integers $\{\varrho_j\}$ by the conditions $\varrho_1 = 1$ and ϱ_{j+1}, $j=1, 2, \ldots$, the least integer for which

$$\Delta_j = |c_{\varrho_j+1}| + |c_{\varrho_j+2}| + \ldots + |c_{\varrho_{j+1}}|$$

satisfies

$$|c_1|\, n_1 + |c_2|\, n_2 + \ldots + |c_{\varrho_j}|\, n_{\varrho_j} \leqslant \beta\, n_{\varrho_j}\, \Delta_j. \tag{2.17}$$

An immediate consequence of (2.16) is that $\Delta_j = o(1)$. We now define

$$Q_j(x) = \sum_{\varrho_j+1}^{\varrho_j+1} c_k e^{i n_k x}$$

$$q_j = \text{degree of } Q_j = n_{\varrho_j+1}$$

$$S_m^* = Q_1 + Q_2 + \ldots + Q_m.$$

The theorem will then be proved if we can show that there exists a sequence of real numbers, $\{x_\gamma\}$, and one of integers, $\{m_\gamma\}$, such that

$$\begin{aligned} &\text{a)}\quad S_{m_\gamma}^*(x_\gamma) = w_\gamma + o(1) \\ &\text{b)}\quad x_{\gamma+1} - x_\gamma = 0\left(\frac{1}{q_{m_\gamma}}\right) \\ &\text{c)}\quad \Delta_{m_\gamma+1} + \Delta_{m_\gamma+2} + \ldots + \Delta_{m_{\gamma+1}} = o(1), \end{aligned} \tag{2.18}$$

where $w_\gamma = \sum_1^\gamma \alpha_k$ is the γ^{th} partial sum of $\mathcal{A}$. For, by b), $\{x_\gamma\}$ will converge and its limit x will satisfy $x - x_\gamma = 0(q_{m_\gamma}^{-1})$.[1] But, since by (2.16), the derivative of $S_{m_\gamma}^*(x)$ satisfies $\|S_{m_\gamma}^{*\prime}\|_\infty = o(q_{m_\gamma})$, we must have (using the mean value theorem)

$$S_{m_\gamma}^*(x) - S_{m_\gamma}^*(x_\gamma) = o(1);$$

thus, by a)

$$S_{m_\gamma}^*(x) = w_\gamma + o(1) \tag{2.19}$$

Now, let $\nu_\gamma = \varrho_{m_\gamma+1}$, so that the partial sum S_{λ_γ} coincides with $S_{m_\gamma}^*$. Condition c), then, implies that

$$\sup_{\nu_\gamma \leqslant \nu \leqslant \nu_{\gamma+1}} |S_\nu(x) - S_{\nu_\gamma}(x)| = o(1), \tag{2.20}$$

which, by (2.19), implies

$$\sup_{\nu_\gamma \leqslant \nu \leqslant \nu_{\gamma+1}} |S_\nu(x) - w_\gamma| = o(1),$$

and this certainly implies the conclusion of theorem III.

After selecting x_1 and m_1 arbitrarily let us now show how we can choose $x_{\gamma+1}$ and $m_{\gamma+1}$ once x_γ and m_γ are given. Let us put

[1] As we shall see, x_1, the first number of the sequence $\{x_\gamma\}$, can be selected arbitrarily. Thus I can be taken as an interval about x_1 and its length should be large enough to guarantee that $x \in I$. Thus, $A = A_Q$ can be determined by the relation $|x - x_1| \leqslant A/q_{m_1} 2$.

$$\delta_j = \frac{1}{\cos\alpha}\Delta_{m_\gamma+j},\ \theta_0 = x_\gamma$$

$$X_0 = S^*_{m_\gamma}(\theta_0),\quad Z_1 = \delta_1 \frac{w_{\gamma+1} - X_0}{|w_{\gamma+1} - X_0|}.$$

By corollary (2.1) there exists a point θ_1 such that $|\theta_1 - \theta_0| \leqslant \dfrac{A'}{2q_{m_\gamma}}$ and

$$|Q_{m_\gamma+1}(\theta_1) - Z_1| \leqslant \delta_1 \sin\alpha.$$

Let us define $X_1 = S^*_{m_\gamma+1}(\theta_1)$, $Z_2 = \delta_2 \dfrac{w_{\gamma+1} - X_1}{|w_{\gamma+1} - X_1|}$ and repeat this argument: In general, having obtained

$$X_j = S^*_{m_\gamma+j}(\theta_j)$$

and defined
$$Z_{j+1} = \delta_{j+1} \frac{w_{\gamma+1} - X_j}{|w_{\gamma+1} - X_j|},$$

an application of corollary (2.1) gives us a point θ_{j+1} such that

$$\begin{cases} |Q_{m_\gamma+j+1}(\theta_{j+1}) - Z_{j+1}| < \delta_{j+1} \sin\alpha \\ |\theta_{j+1} - \theta_j| \leqslant \dfrac{A'}{2\,q_{m_\gamma+j}}. \end{cases} \tag{2.21}$$

Putting $Y_{j+1} = X_j + Z_{j+1}$ and $X_{j+1} = S^*_{m_\gamma+j+1}(\theta_{j+1})$ we have

$$|X_{j+1} - Y_{j+1}| \leqslant |S^*_{m_\gamma+j}(\theta_{j+1}) - S^*_{m_\gamma+j}(\theta_j)| + |Q_{m_\gamma+j+1}(\theta_{j+1}) - Z_{j+1}|.$$

The first term of the right hand side of this inequality is majorized by $|\theta_{j+1} - \theta_j|$ $\|S^{*\prime}_{m_\gamma+j}\|_\infty$ and, thus, by (2.17) and (2.21), by

$$\frac{A'}{2q_{m_\gamma+j}}\beta\, q_{m_\gamma+j}\,\Delta_{m_\gamma+j+1} = \frac{A'\beta}{2}\delta_{j+1}\cos\alpha,$$

since $q_{m_\alpha+j} = n_{\varrho_{m_\alpha+j+1}}$. Thus, keeping this in mind and using (2.21) to estimate the second term, we obtain

$$|X_{j+1} - Y_{j+1}| \leqslant \left(\frac{A'\beta}{2}\cos\alpha + \sin\alpha\right)\delta_{j+1} = (1-\eta)\,\delta_{j+1}. \tag{2.22}$$

Now let us choose the least integer m such that

$$|w_{\gamma+1} - X_0| \leqslant \eta \sum_1^m \delta_j. \tag{2.23}$$

We can thus apply lemma (2.4) and obtain

$$|w_{\gamma+1} - X_m| \leqslant 2 \sup_j \delta_j \leqslant \frac{2}{\cos \alpha} \sup_{k > m_\gamma} \Delta_k. \tag{2.24}$$

On the other hand, our choice of m is such that

$$\sum_1^m \delta_j < \delta_m + \frac{1}{\eta} |w_{\gamma+1} - X_0|. \tag{2.25}$$

We now choose $x_{\gamma+1} = \theta_m$ and $m_{\gamma+1} = m_\gamma + m$ and shall show that, with these choices, (2.18) is true, thus proving the theorem: Part a) is an immediate consequence of (2.24) and the fact that $\Delta_j = o(1)$; part b) follows from the second part of (2.21) (since $x_{\gamma+1} - x_\gamma = \theta_m - \theta_0$); finally, part c) now follows from (2.25), the already established part a) and the assumption that the terms $\alpha_{\gamma+1} = w_{\gamma+1} - w_\gamma$ of the series $\mathcal{A}$ tend to 0.

As was mentioned in the introduction, theorem III is equivalent to the following result:

If $\mathcal{D}$ is a closed connected subset of the extended plane then there exists a point of our interval I such that the set of limit points of the partial sums of $S(x)$ coincides with $\mathcal{D}$.

We see this immediately by establishing the following

Lemma (2.5). *A subset $\mathcal{D}$ of the extended plane is closed and connected if and only if it is the set of limit points of the partial sums of a series $\mathcal{A} = \sum_1^\infty \alpha_k$ whose (complex) terms tend to zero.*

The fact that the set of limit points of the partial sums of such a series is closed and connected in the extended plane is a very elementary result and, thus, we shall only sketch the proof of the converse.

Let us first observe that if $\mathcal{D}$ is bounded, x_0 a point of $\mathcal{D}$ and ϱ a positive number then there exists a finite sequence $S = \{x_0, x_1, \dots, x_n = x_0\} \subset \mathcal{D}$ such that $|x_{k+1} - x_k| < \varrho$ and such that each $x \in \mathcal{D}$ is at a distance less than ϱ from S. For it is sufficient to choose a finite number of $y_\nu \in \mathcal{D}$ such that each $x \in \mathcal{D}$ is at a distance less than ϱ from some y_ν and then construct such a ϱ-chain (that is, a sequence of points of $\mathcal{D}$ such that consecutive members are at a distance less than ϱ from each other) joining x_0 and y_ν; the reordered union of these ϱ-chains can be taken as S. Now choose a sequence $\varrho_n \to 0$ and corresponding ϱ_n-chains S_n, which we reorder into a single sequence $\{w_k\}$ by taking S_{k+1} after S_k. By putting $\alpha_k = w_k - w_{k-1}$ we obtain the desired series $\mathcal{A} = \sum_1^\infty \alpha_k$. In case $\mathcal{D}$ is unbounded we select a point $x_0 \in \mathcal{D}$ and for each $n > |x_0|$ we consider the closed connected region $\mathcal{D}_n$ formed by intersecting $\mathcal{D}$ with the disc $|Z| \leqslant n$ and then completed with the circle $|Z| = n$. Selecting $\varrho_n \to 0$ and defining S_n with respect to $\mathcal{D}_n$, ϱ_n and x_0 we then continue as before to obtain the desired series $\mathcal{A}$.

§ 3. The general case

(i) We now pass to the extension of these theorems discussed in the introduction. First we shall prove the following generalization of theorem I:

Theorem I′. *Suppose* $Q(x)=\sum_1^N c_k e^{in_k x}$, *where* $\dfrac{n_{k+1}}{n_k}>q>1$, *then there exist constants* $A=A_q$, $A'=A'_q$ *and* $K=K_q$, *depending only on* q, *such that whenever* E *is a supertriadic set whose support is an interval of length at least* A'/n_1 *and whose removal ratio is* K *then*

$$\sum_1^N |c_k| \leqslant A \sup_{x\in E} \mathcal{R}\{Q(x)\}.$$

In order to derive this theorem from theorem I we shall need a more intricate decomposition of a general lacunary power series into successive blocks than the one used in the proof of theorem II. This decomposition will be obtained by means of the following lemma:

Lemma (3.1). *Suppose we are given an* $\varepsilon>0$, *an integer* $s>0$ *and that*

$$S(x)=\sum_{k=1}^{\infty} c_k e^{in_k x},\quad \frac{n_{k+1}}{n_k}>q>1.$$

is a lacunary power series, then there exists a constant R, *depending on* q, ε *and* s, *such that* $S(x)$ *can be written as a sum of successive lacunary blocks (corresponding to the same* q*)*

$$S(x)=Q_1(x)+Q_1^*(x)+\dots+Q_j(x)+Q_j^*(x)+\dots,\tag{3.1}$$

with the following properties:

$$\text{(i)}\ \ n_j''/n_j'<R;\quad \text{(ii)}\ \ r<\frac{n'_{j+1}}{n_j''}<r^2;\quad \text{(iii)}\ \ \Delta_j^*\leqslant\varepsilon(\Delta_j+\Delta_{j+1}),$$

where n_j' *and* n_j'' *denote the lowest and the highest frequency of* $Q_j(x)$, Δ_j *and* Δ_j^* *denote the sums of the absolute values of the coefficients of* Q_j *and* Q_j^*, *respectively, and* $r=q^{s+1}$.

In order to prove this lemma let us first add to $S(x)$ terms with $c_k=0$, if necessary, so that

$$q<\frac{n_{k+1}}{n_k}<q^2,\quad k=1,2,\dots.\tag{3.2}$$

Let us now choose a positive integer m satisfying $1/m<\varepsilon$ and an arbitrary positive integer s. We then define r to be q^{s+1}. Let us split $S(x)$ into successive blocks, each containing $(m+1)s$ terms, and from the second one choose the power polynomial of s successive terms having a minimum Δ (= sum of absolute values of its coefficients). We do the same for the fourth block of $(m+1)s$ terms, the sixth block, etc. … . We shall denote these polynomials by $Q_1^*, Q_2^*, Q_3^*, \dots$ and the blocks preceding them by $Q_1, Q_2, Q_3, \dots$. Each Q_j^*, then, has s terms and, by (3.2), the ratio of the highest to the lowest frequencies of this polynomial is between $q^{(s-1)}$ and $q^{2(s-1)}$. On the other hand, each Q_j has less than $3(m+1)s$ terms, as can be easily seen from our construction. Let $R=q^{6(m+1)s}$. The lemma now follows easily: letting $n_j'=\lambda_1^{(j)}<\lambda_2^{(j)}<\ldots<\lambda_{k_j-1}^{(j)}<\lambda_{k_j}^{(j)}=n_j''$ by the frequencies of Q_j, we just observed that $k_j<3(m+1)s$; consequently, using (3.2), we have

$$\frac{n_j''}{n_j'}=\frac{\lambda_{k_j}^{(j)}}{\lambda_{k_j-1}^{(j)}}\cdot\frac{\lambda_{k_j-1}^{(j)}}{\lambda_{k_j-2}^{(j)}}\cdots\frac{\lambda_2^{(j)}}{\lambda_1^{(j)}}\leqslant(q^2)^{k_j-1}<(q^2)^{3(m+1)s}=R,$$

which is inequality (i).

Similarly, in order to prove (ii), suppose $\nu_1^{(j)}<\nu_2^{(j)}<\ldots<\nu_s^{(j)}$ are the frequencies of Q_j^*. Then, again using (3.2),

$$r=q^{s+1}<\frac{n_{j+1}'}{\nu_s^{(j)}}\frac{\nu_s^{(j)}}{\nu_{s-1}^{(j)}}\cdots\frac{\nu_2^{(j)}}{\nu_1^{(j)}}\frac{\nu_1^{(j)}}{n_j''}=\frac{n_{j+1}'}{n_j''}\leqslant q^{2(s+1)}=r^2.$$

Lastly, in order to derive inequality (iii) let us observe that, since Q_j^* was chosen from the $2j^{th}$ block of $(m+1)s$ terms of $S(x)$ in such a way that Δ_j^* was minimal, Δ_j^* does nos exceed any of the sums, $\Delta_1^{(j)}, \Delta_2^{(j)}, \ldots, \Delta_{m+1}^{(j)}$, of the absolute values of the coefficients of the $m+1$ consecutive polynomials of s terms obtainable from this $2j^{th}$ block. Thus, if $\Delta_i^{(j)}=\min_{1\leqslant k\leqslant m+1}\{\Delta_k^{(j)}\}$,

$$\Delta_j^*\leqslant\Delta_i^{(j)}\leqslant\frac{1}{m}\sum_{i\neq k}\Delta_k^{(j)}\leqslant\varepsilon(\Delta_j+\Delta_{j+1}),$$

and the lemma is proved.

We now pass to the proof of the theorem. We shall apply lemma (3.1) to $S(x)=Q(x)$ with $\varepsilon=1/4B$ and $r=q^{s+1}$ where $s+1$ is the smallest integer such that $r=q^{s+1}\geqslant\frac{3}{2}A''B$, where A'' and B are the constants of corollary (2.2). We thus obtain a decomposition of $Q(x)$ into a finite number of blocks

$$Q(x)=Q_1(x)+Q_1^*(x)+\ldots+Q_n(x)+Q_n^*(x).$$

We choose an arbitrary interval I of length at least A''/n_1 aus our support of our supertriadic set. The theorem will then be proved if we can show[1] that there exists a sequence $I_1\supset I_2\supset\ldots\supset I_n$ of subintervals of white intervals of no more than three time their lengths such that

$$\Delta_j\leqslant B\,\mathcal{R}\{Q_j(x)\}$$

if $x\in I_j$, $j=1, 2, \ldots, n$. For I_n will then certainly have points in common with E and, if $x\in I_n$,

[1] For simplicity we shall assume throughout that the black intervals are removed from the middle of white intervals. The reader will find it easy to see that the methods used extend to include the more general sets described at the beginning.

$$\sum_1^N |c_k| = \sum_1^n (\Delta_j + \Delta_j^*) \leqslant (1+2\varepsilon) \sum_1^n \Delta_j \leqslant (1+2\varepsilon) B \sum_{j=1}^n \mathcal{R}\{Q_j(x)\}$$

$$= (1+2\varepsilon) B \left[\mathcal{R}\{Q(x)\} - \sum_1^n \mathcal{R}\{Q_j^*(x)\}\right] \leqslant (1+2\varepsilon) B \mathcal{R}\{Q(x)\}$$

$$+ (1+2\varepsilon) B \sum_1^n \Delta_j^* \leqslant (1+2\varepsilon) B \mathcal{R}\{Q(x)\} + (1+2\varepsilon) 2\varepsilon B \sum_1^N |c_k|$$

$$\leqslant \tfrac{3}{2} B \mathcal{R}\{Q(x)\} + \tfrac{3}{4} \sum_1^N |c_k|,$$

where in the last estimate we used the fact that $B \geqslant 1$ and, hence, $\varepsilon \leqslant \frac{1}{4}$. Thus,

$$\sum_1^N |c_k| - \tfrac{3}{4} \sum_1^N |c_k| \leqslant \tfrac{3}{2} B \mathcal{R}\{Q(x)\}$$

and the inquality of theorem I′ holds for $A = 6B$.

We shall construct inductively this sequence of intervals, together with an accompanying sequence of integers $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n$, in such a way that $|I_j| = \dfrac{2}{3 n_j'' B}$ and such that, after performing the first α_j stages of removing black intervals, I_j is contained in one of the white intervals of length $3|I_j|$. Let $K = 1/3 R r^2$ and apply corollary (2.2) to $Q(x) = Q_1(x)$ and our chosen interval I. We thus obtain a subinterval J_1 of length $2/Bn_1''$ in which $\mathcal{R}\{Q_1(x)\} \geqslant \Delta_1/B$. Observe that, by lemma (3.1) and our choice of r,

$$\frac{|J_1|}{|I|} \geqslant \frac{2}{A''B} \frac{n_1'}{n_1''} \geqslant \frac{3}{r} \frac{1}{R} \geqslant \frac{3}{r^2 R} = 9K.$$

We now begin our dissection of I until one of the black intervals intersects J_1. Since the removal ratio in no larger than K this black interval cannot remove more than $1/9^{\text{th}}$ of J_1. Thus, we are able to pick a subinterval $I_1 \subset J_1$ of length $|I_1| = \dfrac{|J_1|}{3} = \dfrac{2}{3Bn_1''}$ and such that it has an end point, but no other point, in common with the removed black interval. Now we remove black intervals up to and including the α_1^{th} stage, where the $(\alpha_1+1)^{\text{st}}$ is the first stage which has a black interval intersecting I_1. Clearly, since the removal ratio is less than $1/3$, I_1 is contained in a white interval of the α_1^{th} stage of length $\leqslant 3|I_1|$.

Now suppose appropriate sequences $\{I_1, I_2, \ldots, I_j\}$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_j\}$ have been defined. Let us construct I_{j+1} and α_{j+1}. Since, by our choice of r and lemma (3.1),

$$|I_j| = \frac{2}{3n_j''B} \geqslant \frac{A''}{rn_j''} > \frac{A''}{n_{j+1}'}$$

we can apply corollary (2.2) to $Q(x) = Q_{j+1}(x)$ and $I = I_j$. We obtain a subinterval

$J_{j+1} \subset I_j$ of length $|J_{j+1}| = \dfrac{2}{Bn''_{j+1}}$ on which $\mathcal{R}\{Q_{j+1}(x)\} \geq \Delta_{j+1}/B$. Let us now continue our dissection until a black interval intersects J_{j+1}. If b is the length of this black interval, then $b \leq \dfrac{|J_{j+1}|}{9}$ as can be seen by the following argument: Letting w denote the length of the white interval of the α_j^{th} stage containing I_j, and keeping in mind that $w \leq 3|I_j|$, we have

$$\frac{b}{|J_{j+1}|} \leq \frac{b}{w} \cdot \frac{w}{|J_{j+1}|} \leq K \frac{3|I_j|}{|J_{j+1}|} \leq K \frac{\left(\dfrac{2}{3n''_j B}\right)}{\left(\dfrac{2}{Bn''_{j+1}}\right)} = K \frac{n''_{j+1}}{3n''_j} = \frac{K}{3} \frac{n''_{j+1}}{n'_{j+1}} \frac{n'_{j+1}}{n''_j} < \frac{K}{3} Rr^2 = \tfrac{1}{9}.$$

Thus, we are able to choose a subinterval $I_{j+1} \subset J_{j+1}$ of length $\dfrac{|J_{j+1}|}{3} = \dfrac{2}{3Bn''_{j+1}}$ having an end point, but no other point, in common with the black interval in question. We now remove black intervals until and including the α_{j+1}^{th} stage where the $(\alpha_{j+1}+1)^{st}$ is the first stage for which a black interval intersects I_{j+1}. Clearly, I_{j+1} is in a white interval of the α_{j+1}^{th} stage of length no more than $3|I_{j+1}|$. This finishes our induction and the theorem is proved.

(ii) Next, we shall prove the following extension of theorem II:

Theorem II′. *Suppose* $S(x) = \sum_1^\infty c_k e^{in_k x}$, $\dfrac{n_{k+1}}{n_k} > q > 1$, *is an absolutely convergent lacunary power series. Then there exist constants* γ, ξ, ν *and* K, *depending only on* q, *such that if* $|c_k| \leq \gamma \sum_{k+1}^\infty |c_j|$, $k = 1, 2, \ldots$, w *is a complex number satisfying* $|w| \leq \nu \sum_1^\infty |c_j|$ *and* E *is a supertriadic set of support at least* ξ/n_1 *and removal ratio* K *then there exists* $x \in E$ *such that* $S(x) = w$.

In order to prove this theorem we shall make use of corollary (2.3). Let us first observe that if this corollary is valid for some value of B' then it certainly holds for all smaller values of B'. Thus, letting α and A'' be the constants in this corollary, we choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{8} \cdot \dfrac{\cos\alpha}{\cos\alpha + 2}$ and B' so small that $B' \cdot \dfrac{q}{q-1}(1+2\varepsilon) < \dfrac{\cos\alpha}{8}$. Moreover, let s be a positive integer large enough so that $r = q^{s+1} > A''/B'$. We now apply lemma (3.1) and thus obtain a number R and a decomposition

$$S(x) = Q_1(x) + Q_1^*(x) + \ldots + Q_j(x) + Q_j^*(x) + \ldots$$

satisfying (i), (ii) and (iii). From the order of our choices we see that all our constants, including R, depend only q.

Now, letting $K = \min\left\{\dfrac{1}{3Rr^2}, \dfrac{B'}{3A''R}\right\}$ and E a supertriadic set constructed on an interval I of length A''/n_1, we shall show that when w satisfies

$$|w| < \frac{\cos\alpha}{8} \sum_1^\infty |c_k| \tag{3.3}$$

then there exists $x \in E$ such that $S(x) = w$. Thus, values of ξ and ν that will be admissible are A'' and $\frac{\cos\alpha}{8}$, respectively.

We shall use the notation of lemma (3.1). In addition we shall let $c_j'' =$ coefficient of $e^{in_j''x} =$ last coefficient of Q_j, $n_j^* =$ smallest frequency of Q_j^* and $c_j^* =$ coefficient of $e^{in_j^*x} =$ first coefficient of Q_j^*.

It is clear that it suffices to show that there exists a sequence of subintervals of I, $I_1 \supset I_2 \supset \ldots \supset I_N \supset \ldots$, each contained in a white interval, such that

(α) there exists N for which

$$d_N = \max_{x \in I_N} |w - [Q_1(x) + Q_j^* + \ldots + Q_N(x)]| \leqslant \frac{\Delta_{N+1}}{\cos\alpha},$$

and

(β) If there exists N_1 for which (α) holds then there exists $N_2 > N_1$ for which (α) holds.

It will be convenient to write $F_N = Q_1 + Q_1^* + \ldots + Q_N$ and denote by $x_N \in I_N$ a point at which the maximum in (α) is attained. Thus, $d_N = |w - F_N(x_N)|$. Furthermore, we define

$$T_N = \frac{|c_1| n_1 + |c_2| n_2 + \ldots + |c_N''| n_N''}{n_N''}.$$

As in the last proof, we shall construct the intervals $I_1 \supset I_2 \supset \ldots \supset I_N \supset \ldots$ inductively together with a sequence of integers $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_N \leqslant \ldots$ such that, if we remove the first α_N stages of black intervals, then I_N is contained in one of the white intervals of length $\leqslant 3|I_N|$. We shall also construct them in such a way that $|I_N| = \frac{B'}{n_N''}$.

Let us first suppose that $|w| > \frac{\Delta_1}{\cos\alpha}$. In this case we can apply corollary (2.3) to I, $Q_1(x)$ and w. Thus, we obtain a subinterval $J_1 \subset I$ of length $|J_1| = \frac{3B'}{n_1''}$ such that

$$\max_{x \in J_1} |w - Q_1(x)| \leqslant |w| - \frac{\Delta_1}{2} \cos\alpha_1. \tag{3.4}$$

Clearly,
$$\frac{|J_1|}{|I|} = \frac{3B_1}{n_1''} \frac{n_1}{A''} \geqslant \frac{3B'}{A''R} \geqslant 9K.$$

We now begin our dissection of I until one of the black intervals intersects J_1. Since the removal ratio is K, this black interval cannot remove more than $(\frac{1}{9})^{th}$

of J_1. We pick I_1 to be a subinterval of J_1 of length $|I_1| = \frac{|J_1|}{3} = \frac{B'}{n_1''}$ which has an end point in common with the removed black interval but no other point. Now we remove black intervals up to and including the α_1^{th} stage, where the $(\alpha_1+1)^{st}$ stage is the first stage which has a black interval intersecting I_1. Clearly, I_1 lies within a white interval of the α_1^{th} stage of length $\leqslant 3|I_1|$. Furthermore, I_1 being a subinterval of J_1, inequality (3.4) holds for I_1:

$$d_1 = \max_{x \in I_1} |w - Q_1(x)| = |w - Q_1(x_1)| \leqslant |w| - \frac{\Delta_1}{2} \cos \alpha. \tag{3.5}$$

Now, if $d_1 \leqslant \frac{\Delta_2}{\cos \alpha}$ then condition (α) is satisfied with $N = 1$ and we stop this type of construction. On the other hand, if $d_1 > \frac{\Delta_2}{\cos \alpha}$ we can apply corollary (2.3) to I_1 $\left(\text{since } |I_1| = \frac{B'}{n_1''} \geqslant \frac{A''}{n_2'}, \text{ by our choice of } r \text{ and by lemma (3,1)}\right)$, $Q_2(x)$ and $w - Q_1(x_1)$. We obtain a subinterval $J_2 \subset I_1$ of length $|J_2| = \frac{3B'}{n_2''}$ such that

$$|w - Q_1(x_1) - Q_2(x)| \leqslant |w - Q_1(x_1)| - \frac{\Delta_2}{2} \cos \alpha = d_1 - \frac{\Delta_2}{2} \cos \alpha \tag{3.6}$$

for all $x \in J_2$.

By exactly the same process we have just described (and is explicitly carried out in the last proof) we obtain a subinterval $I_2 \subset J_2$ length $\frac{B'}{n_2''}$ and an integer α_2. Since inequality (3.6) is valid when $x \in I_2 \subset J_2$

$$\begin{aligned} d_2 &= \max_{x \in I_2} |w - Q_1(x) - Q_1^*(x) - Q_2(x)| \\ &\leqslant \max_{x \in I_2} |w - Q_1(x_1) - Q_2(x)| + \max_{x \in I_2} |Q_1^*(x)| + \max_{x \in I_2} |Q_1(x_1) - Q_1(x)| \\ &\leqslant d_1 - \frac{\Delta_2}{2} \cos \alpha + \Delta_1^* + |I_1| \max_{x \in I_1} |Q_1'(x)| \leqslant d_1 - \frac{\Delta_2}{2} \cos \alpha + \Delta_1^* + B'T_1 \\ &\leqslant |w| - \frac{\cos \alpha}{2} (\Delta_1 + \Delta_2) + \Delta_1^* + B'T_1/, \end{aligned}$$

where, in obtaining the last inequality we made use of (3.5).

Proceeding in this way, if $d_j > \frac{\Delta_{j+1}}{\cos \alpha}$, $j = 2, 3, \ldots, N-1$, we obtain a sequence of intervals $I_2 \supset I_3 \supset \ldots \supset I_N$, of length $|I_j| = \frac{B'}{n_j''}$, and a sequence of integers $\alpha_2 \leqslant \alpha_3 \leqslant \ldots \leqslant \alpha_N$ with the properties described above such that

$$d_N \leqslant |w| - \frac{\cos\alpha}{2}(\Delta_1+\Delta_2+\ldots+\Delta_N)+(\Delta_1^*+\ldots+\Delta_{N-1}^*)+B'(T_1+\ldots+T_{N-1}). \tag{3.7}$$

Let us, for the moment, assume that

$$\sum_{j=1}^{N-1} T_j \leqslant \frac{q}{q-1}(\Delta_1+\Delta_1^*+\Delta_2+\Delta_2^*+\ldots+\Delta_{N-1}). \tag{3.8}$$

Then, combining (3.7) and (3.8) and remembering that

$$\frac{q}{q-1}B' < \frac{\cos\alpha}{8} \quad \text{and} \quad \varepsilon < \frac{1}{8}\frac{\cos\alpha}{(\cos\alpha)+2},$$

we obtain

$$\begin{aligned} d_N &\leqslant |w| - \frac{\cos\alpha}{2}(\Delta_1+\Delta_1^*+\Delta_2+\Delta_2^*+\ldots+\Delta_N)+\frac{\cos\alpha+2}{2}\sum_{j=1}^{N-1}\Delta_j^* \\ &\quad + B'\frac{q}{q-1}(\Delta_1+\Delta_1^*+\Delta_2+\Delta_2^*+\ldots+\Delta_N) \\ &\leqslant |w| - \frac{\cos\alpha}{2}(\Delta_1+\Delta_1^*+\Delta_2+\Delta_1^*+\ldots+\Delta_N)+\varepsilon(\cos\alpha+2)\sum_{j=1}^{N}\Delta_j \\ &\quad + \frac{\cos\alpha}{8}(\Delta_1+\Delta_1^*+\Delta_2+\Delta_2^*+\ldots+\Delta_N) \\ &\leqslant |w| - \frac{\cos\alpha}{4}(\Delta_1+\Delta_1^*+\Delta_2+\Delta_2^*+\ldots+\Delta_N). \end{aligned}$$

But, since $|w| < \frac{\cos\alpha}{8}\sum_1^\infty |c_k|$, by assumption, the last expression is negative if N is large enough. Since $d_N \geqslant 0$ this is impossible. Hence, we must have $d_N \leqslant \frac{\Delta_{N+1}}{\cos\alpha}$ for some N; that is (α) must hold for some N.

Before considering the remaining cases let us prove a result that we shall need which includes, as a special case, inequality (3.8). Define

$$G_k = \frac{|c_1|n_1+|c_2|n_2+\ldots+|c_k|n_k}{n_k}$$

and let us estimate the sums $\sum_p^{p+\varrho} G_k$. If $j \geqslant p$ then $|c_j|$ appears in all the G_k's of this sum and its coefficient is

$$n_j\left(\frac{1}{n_p}+\ldots+\frac{1}{n_{p+\varrho}}\right) < \frac{n_j}{n_p}\left(1+\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^\varrho}\right) < \frac{n_j}{n_p}\frac{q}{q-1}.$$

On the other hand, if $j > p$ then $|c_j|$ occurs only in $G_j, G_{j+1}, \ldots, G_{p+\varrho}$ and its coefficients is $\frac{n_j}{n_j} + \frac{n_j}{n_{j+1}} + \ldots + \frac{n_j}{n_{p+1}} < 1 + \frac{1}{q} + \ldots + \frac{1}{q^{p+\varrho-j}} < \frac{q}{q-1}$. Thus

$$\sum_p^{p+\varrho} G_k < \frac{q}{q-1} \frac{1}{n_p} \sum_{j=1}^{p} n_j |c_j| + \frac{q}{q-1} \sum_{j=p+1}^{p+\varrho} |c_j|.$$

Hence,

$$\begin{aligned} \sum_{j=p}^{p+k} T_j \leqslant \sum_{n_j = n_p''}^{n_{p+k}''} G_j &< \frac{q}{q-1} (T_p + |c_p^*| + \ldots + |c''_{p+k}|) \\ &= \frac{q}{q-1} (T_p + \Delta_p^* + \Delta_{p+1} + \Delta_{p+1}^* + \ldots + \Delta_{p+k}). \end{aligned} \tag{3.9}$$

Since $T_1 < \Delta_1$ (3.9) certainly implies (3.8).

Now suppose that either (a) $|w| \leqslant \frac{\Delta_1}{\cos \alpha}$ or that (b) $d_p \leqslant \frac{\Delta_{p+1}}{\cos \alpha}$ for some $p \geqslant 1$. Putting $d_0 = |w|$ and, for later use, $I_0 = I$, we see that inequality (b), for $p \geqslant 0$, includes both cases. Let us define $J_{p+1} \subset I_p$ (assuming $I_1, \ldots, I_p$ were obtained by the previous process, when $p \geqslant 1$) to be any subinterval of length $|J_{p+1}| = \frac{3B'}{n''_{p+1}}$ and we then construct $I_{p+1} \subset J_{p+1}$, of length $\frac{B'}{n''_{p+1}}$, as before (together with the integer α_{p+1}). If $d_{p+1} \leqslant \frac{\Delta_{p+2}}{\cos \alpha}$ then, when $p = 0$, we have condition (α) with $N = 1$ and, when $p > 0$, condition (β) is satisfied with $N_1 = p$ and $N_2 = p + 1$. It only remains to examine the case $d_{p+1} > \frac{\Delta_{p+2}}{\cos \alpha}$. But in this case, as before, we can apply corollary (2.3) to I_{p+1}, $Q_{p+2}(x)$ and the point $w - F_{p+1}(x_{p+1})$. By our previous methods we thus obtain subintervals $I_{p+2} \subset J_{p+2} \subset J_{p+1}$ (and an integer α_{p+2}) of lengths $\frac{B'}{n''_{p+2}}$ and $\frac{3B'}{n'_{p+2}}$, respectively, for which

$$\max_{x \in I_{p+2}} |w - F_{p+1}(x_{p+1}) - Q_{p+2}(x)| \leqslant d_{p+1} - \frac{\Delta_{p+2}}{2} \cos \alpha.$$

Hence, as before,

$$d_{p+2} = |w - F_{p+2}(x_{p+2})| = \max_{x \in I_{p+2}} |w - F_{p+2}(x)| \leqslant d_{p+1} - \frac{\Delta_{p+2}}{2} \cos \alpha + \Delta_{p+1}^* + B' T_{p+1}.$$

If $d_{p+2} \leqslant \frac{\Delta_{p+3}}{\cos \alpha}$ we're through. Otherwise, repeating this argument, we obtain

$$d_{p+3} \leqslant d_{p+1} - \frac{\cos \alpha}{2} (\Delta_{p+2} + \Delta_{p+3}) + (\Delta_{p+1}^* + \Delta_{p+2}^*) + B'(T_{p+1} + T_{p+2}).$$

Continuing in this way $\left(\text{assuming we keep obtaining } d_{p+j} > \dfrac{\Delta_{p+j+1}}{\cos\alpha}\right)$, and using the inequality

$$d_{p+1} = \max_{x\in I_{p+1}} |w - \{Q_1(x) + \dots + Q_p(x) + Q_p^*(x) + Q_{p+1}(x)\}|$$
$$\leqslant \max_{x\in I_{p+1}} |w - F_p(x)| + \Delta_p^* + \Delta_{p+1} \leqslant d_p + \Delta_p^* + \Delta_{p+1} \leqslant \frac{\Delta_{p+1}}{\cos\alpha} + \Delta_p^* + \Delta_{p+1},$$

we obtain (by (3.9), lemma (3.1) and the inequalities defining ε and B')

$$\begin{aligned} d_{p+k} &\leqslant \left(\frac{1}{\cos\alpha}+1\right)\Delta_{p+1} + \Delta_p^* - \frac{\cos\alpha}{2}(\Delta_{p+1} + \dots + \Delta_{p+k}) + (\Delta_{p+1}^* + \dots + \Delta_{p+k-1}^*) \\ &\qquad + B'(T_{p+1} + \dots + T_{p+k-1}) \\ &< \left(\frac{1+\cos\alpha}{\cos\alpha}\right)\Delta_{p+1} + \Delta_p^* - \frac{\cos\alpha}{2}(\Delta_{p+2} + \dots + \Delta_{p+k}) + (\Delta_{p+1}^* + \dots + \Delta_{p+k-1}) \\ &\qquad + B'\frac{q}{q-1}(T_{p+1} + \Delta_{p+1}^* + \Delta_{p+2} + \dots + \Delta_{p+k-1}) \\ &< \left(\frac{1+\cos\alpha}{\cos\alpha}\right)\Delta_{p+1} + 2\varepsilon\sum_{j=0}^{k}\Delta_{p+j} + B'\frac{q}{q-1}T_{p+1} + B'\frac{q}{q-1}(1+2\varepsilon)\sum_{j=1}^{k}\Delta_{p+j} \\ &\qquad - \frac{\cos\alpha}{2}\sum_{j=2}^{k}\Delta_{p+j} \\ &< \left(\frac{1+\cos\alpha}{\cos\alpha}\right)\Delta_{p+1} + 2\varepsilon\Delta_{p+1} + \frac{\cos\alpha}{8}\Delta_{p+1} + 2\varepsilon\Delta_p + \frac{qB'}{q-1}T_{p+1} \\ &\qquad + \frac{\cos\alpha}{8}\sum_{j=2}^{k}\Delta_{p+j} - \frac{\cos\alpha}{2}\sum_{j=2}^{k}\Delta_{p+j}. \end{aligned}$$

But it follows immediately from lemma (3.1) that $\Delta_{p+1} < R\,T_{p+1}$ and $\Delta_p < r^2 R\,T_{p+1}$. Thus, we have shown

$$d_{p+k} \leqslant C\,T_{p+1} - \tfrac{3}{8}\cos\alpha\sum_{j=2}^{k}\Delta_{p+j}, \tag{3.10}$$

where $C = \dfrac{qB'}{q-1} + \dfrac{1+\cos\alpha}{\cos\alpha}R + 2\varepsilon R + \dfrac{\cos\alpha}{8}R + 2\varepsilon r^2 R.$

We shall now show that, for an appropriate value of the constant γ in Theorem II′, the right hand side will be negative if large enough. This will finish our proof since it implies that we must have the desired inequality $d_{p+j} \leqslant \dfrac{\Delta_{p+j+1}}{\cos\alpha}$ for some $j \geqslant 1$.

Since $\sum_{j=2}^{k-1}(\Delta_{p+j} + \Delta_{p+j}^*) \leqslant (1+2\varepsilon)\sum_{j=2}^{k}\Delta_{p+j}$ the right hand side of (3.10) is majorized by

$$C T_{p+1} - \frac{3 \cos \alpha}{8(1+2\varepsilon)} \sum_{j=2}^{k-1} (\Delta_{p+j} + \Delta^*_{p+j}).$$

But this expression will certainly be negative for k large if the coefficients of $S(x)$ satisfy a relation of the form

$$\frac{|c_1| n_1 + |c_2| n_2 + \dots + |c_k| n_k}{n_k} \leqslant \beta \sum_{k+1}^{\infty} |c_j|,$$

$k = 1, 2, \dots,$ where $\beta < \dfrac{3 \cos \alpha}{8 C(1+2\varepsilon)}$. But we have already shown that the inequality $|c_k| \leqslant \gamma \sum_{k+1}^{\infty} |c_j|$, $k = 1, 2, \dots,$ does imply such a relation (see (2.7) and (2.8)) when γ is so small that

$$\beta \geqslant \frac{\gamma_q}{q-1-\gamma} > 0.$$

(iii) By an almost exact repetition of these arguments we now can obtain the following generalization of theorem III:

Theorem III′. *Suppose* $\mathfrak{A} = \sum_1^{\infty} \alpha_k$ *is a series of complex numbers tending to* 0 *and that* $S(x) = \sum_1^{\infty} c_k e^{in_k x}$, $n_{k+1}/n_k > q > 1$, *is lacunary power series satisfying* $\sum_1^{\infty} |c_k| = \infty$ *and* $\lim_{k \to \infty} c_k = 0$. *Then in any supertriadic set E of the type described in the previous theorem we can find a point x such that the set of limit points of the partial sums of* $S(x)$ *coincides with the set of limit points of the partial sums of* $\mathfrak{A}$.

If we use the previous argument in order to successively approximate the partial sums of $\mathfrak{A}$ (taking care, at each stage, to use as little of $S(x)$ as possible) we obtain the proof of this theorem. We leave the details to the reader.

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