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Generalized zeta-functions

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1. Introduction

In this paper, we will show that some of the features common to certain zeta-functions which occur in analysis are properties of much more general objects. As it turns out, a considerable unity of treatment can be achieved with surprisingly mild assumptions, and when possible, we will point out how familiar cases fall within the more general framework. Our starting point will be Bochner's paper [1], and I would like here to express my gratitude to Professor Bochner for his generous advice during the preparation of this paper, and to thank Professors Edward Nelson and Robert Langlands for several informative conversations.

2. Some examples

1. The Riemann zeta-function can be defined in a right half-plane by a Dirichlet series

$$2\zeta(2s) = \Sigma' n^{-2s},$$

where the prime indicates that the summation is over all non-zero integral lattice-points in E^1. In particular, the series is of the general type

$$\Sigma'[P(n)]^{-s},$$

where P is a positive form in E^1.

2. In [3], P. Epstein discussed, among other things, series of the type

$$\Sigma'[P(n)]^{-s},$$

where P is a positive-definite quadratic form in E^k, and where the summation is over all non-zero integral lattice-points in E^k. He was able to show that the function corresponding to the series (the latter clearly converges uniformly in some right half-plane) is meromorphic in s, and satisfies a functional equation analogous to that for the Riemann zeta-function.

3. Bochner [1] considered series of the type

$$\Sigma'[P(n)]^{-s},$$

where P is simply required to be a positive homogeneous form in E^k, not necessarily quadratic. He showed that the function corresponding to the series is meromorphic in the s-plane, with a single simple pole at $s=k/N$, where N is the degree of the form. In addition, he showed that the function

$$F(x,s)=\lim_{\varepsilon\to 0}\ \Sigma' e^{2\pi i\langle x,n\rangle-|\varepsilon n|^2}[P(n)]^{-s}$$

exists for arbitrary complex s and x not an integral lattice-point, and is, for fixed s, real-analytic in x with certain growth properties in the neighborhood of a lattice-point. (Here "$|\cdot|$" denotes the Euclidean norm). If s is a positive integer, the function $F(x,s)$ corresponds to a fundamental solution on the k-torus for the operator

$$\left[P\left(\frac{1}{2\pi i}\frac{\partial}{\partial x_j}\right)\right]^s.$$

3. A Generalization

Let B be a (not necessarily commutative) Banach algebra with unit. Let $\beta(x)$ be a B-valued function of $x=(x_1,\dots,x_k)$ in $E^k-\{0\}$ satisfying the following conditions:

(a). (Homogeneity) There exists a positive number N, such that $\beta(rx)=r^N\beta(x)$ for $r>0$ and $x\neq 0$.

(b). $\beta(x)$ is C^∞ in $E^k-\{0\}$. (By C^∞ we will always mean that all mixed derivatives exist in the strong topology.)

(c). For all $x\neq 0$, the spectrum of $\beta(x)$ is contained in C^1-R_-, where $R_-=$ the non-positive reals. (It will develop that there are possible variants of this condition.)

Examples

1. $B=C^1$, $\beta(x)=x^2$. (This will correspond to the Riemann zeta-function.)
2. $B=C^1$, $\beta(x)=P(x)$, where P is a positive homogeneous form on E^k. (Epstein zeta-functions; the zeta-functions introduced in [1].)
3. $B=$ The algebra of complex $n\times n$ matrices, $\beta(x)=$ a matrix of homogeneous Nth order polynomials satisfying (c).

Now the principal branch of λ^{-s} is defined in C^1-R_-, and we can adopt the usual convention and define the $-s$th power of $\beta(x)$ (s arbitrary complex) by a contour integral

$$\frac{1}{2\pi i}\int_C \frac{\lambda^{-s}}{\lambda e-\beta(x)}\,d\lambda, \tag{1}$$

where e is the unit of B, and C is a smooth curve in C^1-R_- enclosing the spectrum of $\beta(x)$ (which is, of course, compact). This is well-defined in the sense that it is independent of C.

Lemma 3.1. *Let $x \neq 0$. Suppose s fixed and $r > 0$. Then*

$$[\beta(rx)]^{-s} = r^{-Ns}[\beta(x)]^{-s}.$$

Proof. This is an easy consequence of (1) and condition (a).

Lemma 3.2. *For fixed $x \neq 0$, $[\beta(x)]^{-s}$ is an entire, B-valued function of s.*

Proof. This follows immediately from (1).

Now by condition (b), $\beta(x)$ is norm continuous on $|x| = 1$, and by a combination of condition (c), the continuity of the spectrum ([4], p. 118), and the fact that the spectrum of $\beta(x)$ is compact for each x, we see that the totality of the spectral values of $\beta(x)$ on $|x| = 1$ constitutes a compact subset of $C^1 - R_-$. We enclose this subset in a smooth curve C in $C^1 - R_-$. Then by the homogeneity of the spectrum, for x in a neighborhood of $|x| = 1$,

$$[\beta(x)]^{-s} = \frac{1}{2\pi i}\int_C \frac{\lambda^{-s}}{\lambda e - \beta(x)}\, d\lambda. \tag{1'}$$

In particular, it is clear from this that $[\beta(x)]^{-s}$ is C^∞ in x in some neighborhood of $|x| = 1$, for fixed s, and so by Lemma 3.1, $[\beta(x)]^{-s}$ is, in fact, C^∞ in x for x in $E^k - \{0\}$.

Lemma 3.3. *Let D be a disc in C^1. Then $[\beta(x)]^{-s}$, or more generally, any fixed partial derivative with respect to x of $[\beta(x)]^{-s}$, is bounded in norm for s in D and $|x| = 1$.*

Proof. This follows immediately from (1′).

Lemma 3.4. *Let $a_1, \ldots, a_k$ be fixed non-negative integers, and let D be a disc in C^1. Then there exists $A > 0$ such that*

$$\left\| \frac{\partial^{a_1 + \cdots + a_k}}{\partial x_1^{a_1} \ldots \partial x_k^{a_k}} [\beta(x)]^{-s} \right\| < A\, |x|^{-(N \operatorname{Re} s + a_1 + \cdots + a_k)}$$

for $x \neq 0$ and s in D, where "$\|\cdot\|$" is the algebra norm. In particular, if $a_1, \ldots, a_k = 0$,

$$\|[\beta(x)]^{-s}\| < A\, |x|^{-N \operatorname{Re} s}.$$

Proof. In view of Lemmas 3.1 and 3.3, it will suffice to show that if $F(x)$ is a C^∞ B-valued function of x in $E^k - \{0\}$ of weight M, i.e., such that $F(rx) = r^M F(x)$ for some fixed complex M and any $r > 0$, then the operation of taking a partial derivative with respect to x_i yields a C^∞ B-valued function of x in $E^k - \{0\}$ of weight $M - 1$ $(i = 1, \ldots, k)$. Let $x_0 = (x_1^0, \ldots, x_k^0) \neq 0$. Without loss of generality, we can suppose $i = 1$. Now

$$\left.\frac{\partial F(x)}{\partial x_1}\right|_{x_0} = \lim_{h \to 0} \frac{F(x_1^0 + h, \ldots, x_k^0) - F(x_1^0, \ldots, x_k^0)}{h},$$

and
$$\left.\frac{\partial F(x)}{\partial x_1}\right|_{rx_0} = \lim_{h\to 0} \frac{F(rx_1^0+h, \ldots, rx_k^0) - F(rx_1^0, \ldots, rx_k^0)}{h}$$

$$= \lim_{h\to 0} \frac{F\left(rx_1^0+\dfrac{rh}{r}, \ldots, rx_k^0\right) - F(rx_1^0, \ldots, rx_k^0)}{h}$$

$$= r^{M-1} \lim_{h\to 0} \frac{F\left(x_1^0+\dfrac{h}{r}, \ldots, x_k^0\right) - F(x_1^0, \ldots, x_k^0)}{\dfrac{h}{r}}$$

$$= r^{M-1} \left.\frac{\partial F(x)}{\partial x_1}\right|_{x_0}. \qquad \text{Q.E.D.}$$

Consider now the series
$$Z(s) = \Sigma' [\beta(n)]^{-s}.$$

By Lemma 3.4, it locally converges absolutely and uniformly in some right half-plane Re $s > \sigma_0$, and by Lemma 3.2, represents a holomorphic, B-valued function there.

Now $[\beta(x)]^{-s}$ will generally not be defined for $x=0$, and to avoid complications arising from this fact, we take once and for all some fixed C^∞ smoothing function $\chi(x)$, which is identically 1 in the complement of a small ball around the origin, and zero in a neighborhood of the origin, and multiply $[\beta(x)]^{-s}$ by $\chi(x)$. (We assume that the ball does not contain any integral lattice-points other than the origin.) To avoid additional notation, we continue to use the symbol $[\beta(x)]^{-s}$ to denote what should actually be written as $\chi(x)[\beta(x)]^{-s}$. Note that with this convention, $[\beta(x)]^{-s}$ is defined for all x and s, is C^∞ in x for fixed s, is entire in s for fixed x, and we can easily show that Lemma 3.4 holds for all x, though with perhaps a different A. In particular, the terms of the series $\Sigma'[\beta(n)]^{-s}$ are unaltered.

Lemma 3.5. *If f is a C^∞ B-valued function of x in E^k, which, together with its derivatives, tends to zero sufficiently rapidly as $|x| \to \infty$, then the Poisson summation formula holds for f, where the Fourier transform is defined by*

$$\hat{f}(y) = \int_{E^k} f(x)\, e^{-2\pi i \langle x, y\rangle}\, dV_x.$$

Proof. The proof proceeds exactly as in the ordinary case.

Theorem 3.6. *$Z(s)$ is a B-valued meromorphic function of s. It has a simple pole at $s = k/N$ with the B-valued residue*

$$N^{-1} \int_{|x|=1} [\beta(\alpha)]^{-k/N}\, dS_\alpha$$

and no other poles. (dS_α = the natural $(k-1)$-dimensional volume element on S^{k-1}).

Proof. (Since the proof for $B=C^1$ is contained in [1], and since only obvious modifications are required for the general case, we only indicate the highlights.) It is known from Lemma 3.4 that sufficiently high x-derivatives of $[\beta(x)]^{-s}$ are uniformly L^1 for s in any disc we choose. Supposing $[\beta(x)]^{-s}$ satisfies the conditions for the Poisson summation formula in some half-plane, this suggests applying the formula there and then integrating the expression for the Fourier transform of $[\beta(x)]^{-s}$ repeatedly by parts, in order to extend the representation to the left. The term $\hat{}[\beta(0)]^{-s}$ must, of course, be treated separately, but this turns out to be no problem, on account of the homogeneity of $[\beta(x)]^{-s}$ away from the origin. On the other hand, this procedure, although conceptually straightforward, involves certain details which can be avoided through the following device:

Consider the B-valued functions

$$K_\varepsilon(x,s)=e^{-|\varepsilon x|^2}[\beta(x)]^{-s}.$$

By Lemma 3.4, $K_\varepsilon(x,s)$ is uniformly L^1 for s in any fixed disc and $\varepsilon\neq 0$ fixed. Moreover, $K_\varepsilon(x,s)$ satisfies the conditions for the Poisson summation formula in x, and $\hat{K}_\varepsilon(x,s)$ is entire in s, for fixed x and ε.

Now for Re $s>\sigma_0$, $\qquad \Sigma'[\beta(n)]^{-s}=\Sigma[\beta(n)]^{-s}$

locally converges absolutely and uniformly, and since

$$\begin{aligned}\Sigma[\beta(n)]^{-s}&=\lim_{\varepsilon\to 0}\Sigma K_\varepsilon(n,s)\\&=\lim_{\varepsilon\to 0}\Sigma \hat{K}_\varepsilon(n,s),\end{aligned}$$

the problem becomes one of showing that $\lim_{\varepsilon\to 0}\hat{K}_\varepsilon(n,s)$ is meromorphic, with a single simple pole at $s=k/N$ having residue

$$N^{-1}\int_{|x|=1}[\beta(\alpha)]^{-k/N}\,dS_\alpha.$$

Now $\qquad \Sigma\hat{K}_\varepsilon(n,s)=\hat{K}_\varepsilon(0,s)+\Sigma'\hat{K}_\varepsilon(n,s)$

and $\qquad \hat{K}_\varepsilon(0,s)=\int_{E^k}e^{-|\varepsilon x|^2}[\beta(x)]^{-s}\,dV_x.$

For Re s large, $\qquad \int_{E^k}\|[\beta(x)]^{-s}\|\,dV_x<\infty,$

so in a suitable half-plane,

$$\lim_{\varepsilon\to 0}\hat{K}_\varepsilon(0,s)=\int_{E^k}[\beta(x)]^{-s}\,dV_x.$$

Moreover,
$$\int_{E^k} = \int_{|x|<1} + \int_{|x|\geqslant 1},$$
and by Fubini's theorem,
$$\int_{|x|\geqslant 1} [\beta(x)]^{-s} dV_x = \int_{|x|=1} dS_\alpha \int_{r=1}^{\infty} [\beta(r\alpha)]^{-s} r^{k-1} dr.$$

By Lemma 3.1, this is equal to
$$\int_{|x|=1} dS_\alpha \int_{r=1}^{\infty} [\beta(\alpha)]^{-s} r^{k-Ns-1} dr$$
$$= (Ns-k)^{-1} \int_{|x|=1} [\beta(\alpha)]^{-s} dS_\alpha \quad (\text{Re } s \text{ large}).$$

Now both
$$\int_{|x|<1} [\beta(x)]^{-s} dV_x$$
and
$$\int_{|x|=1} [\beta(\alpha)]^{-s} dS_\alpha$$
are entire in s, so
$$\lim_{\varepsilon \to 0} \hat{K}_\varepsilon(0, s)$$
is a B-valued meromorphic function of s having a single simple pole at $s = k/N$ with residue
$$N^{-1} \int_{|x|=1} [\beta(\alpha)]^{-k/N} dS_\alpha.$$

To complete the proof of the theorem, we have to show that
$$\lim_{\varepsilon \to 0} \Sigma' \hat{K}_\varepsilon(n, s)$$
is entire, and to do this, it suffices to show that given any disc D,
$$\lim_{\varepsilon \to 0} \Sigma' \hat{K}_\varepsilon(n, s)$$
exists and is holomorphic in D.

Now for fixed $\varepsilon \neq 0$, $\Sigma' \hat{K}_\varepsilon(n, s)$ converges uniformly in any disc, and is thus an entire function. Moreover,
$$\Sigma' \hat{K}_\varepsilon(n, s) = \Sigma' \int_{E^k} e^{-|\varepsilon x|^2} [\beta(x)]^{-s} e^{-2\pi i \langle x, n \rangle} dV_x,$$

and appropriate integrations by parts show that

$$\Sigma' \hat{K}_\varepsilon(n,s) = \frac{(-1)^h}{(2\pi)^{2h}} \Sigma' |n|^{-2h} \int_{E^k} \Delta_x^h \{e^{-|\varepsilon x|^2}[\beta(x)]^{-s}\} e^{-2\pi i <x,n>} dV_x,$$

where Δ_x^h is the h-times iterated Laplacian with respect to x.

From this it is clear that to complete the proof, we only need to verify that for h large,

$$\lim_{\varepsilon\to 0} \int_{E^k} \|\Delta_x^h \{e^{-|\varepsilon x|^2}[\beta(x)]^{-s}\} - \Delta_x^h[\beta(x)]^{-s}\| dV_x = 0$$

uniformly in D, since then the family (in ε) of entire functions $\{\Sigma' \hat{K}_\varepsilon(n,s)\}$ will be uniformly convergent in D. For the proof of this last fact, which is straightforward, the reader is referred to [1], pp. 34–35.

Remark. Using the same methods as above, it is possible to prove the following extension of Theorem 3.6:

Let $\beta(x)$ be as before, and let $\tau(x)$ be a homogeneous C^∞ B-valued function of x in $E^k - \{0\}$ of weight M, i.e., such that $\tau(rx) = r^M \tau(x)$ for $r > 0$ $(x \neq 0)$. Define

$$Z(s) = \Sigma' \tau(n) [\beta(n)]^{-s}$$

in a right half-plane. Then $Z(s)$ is a B-valued meromorphic function of s. It has a single simple pole at $s = (M+k)/N$ with residue

$$N^{-1} \int_{|x|=1} \tau(\alpha) [\beta(\alpha)]^{-(M+k)/N} dS_\alpha$$

and no other poles.

4. An example

It is natural to ask whether or not the case $N=2$ exhibits special properties, such as the presence of a functional equation, and in this section we will supply a partial answer to this question. Throughout what follows, B is assumed to be both commutative and semi-simple.

Theorem 4.1. *Let $\beta(x)$ be a B-valued function of $x = (x_1, \ldots, x_k)$ satisfying conditions (a), (b), and (c) of (III), with $N=2$. Suppose that for each homomorphism T in the maximal ideal space of B, $T[\beta(x)]$ is a positive definite quadratic form $< F_T x, x >$, where F_T is a positive definite symmetric real matrix whose dependence on T is indicated by the subscript. Then there exists a B-valued function $\beta^*(x)$ satisfying conditions (a), (b), and (c) of (III), with $N=2$, and a fixed element of B, which we shall call $\Delta^{-\frac{1}{2}}$, such that if $Z_\beta(s)$ and $Z_\beta^*(s)$ are respectively the zeta-functions (in the sense of (III)) associated with $\beta(x)$ and $\beta^*(x)$, then for each s,*

$$\Delta^{-\frac{1}{2}} \frac{\Gamma(k/2-s)}{\pi^{k/2-s}} Z_\beta^*(k/2-s) = \frac{\Gamma(s)}{\pi^s} Z_\beta(s).$$

(*Cf.* [3], p. 625.)

Proof. Assuming that both $\Delta^{-\frac{1}{2}}$ and $\beta^*(x)$ exist, we know from [3] how a homomorphism T should act on them. Specifically, writing Δ^{-1} for $(\Delta^{-\frac{1}{2}})^2$, we should have $T[\Delta^{-1}] = (\det F_T)^{-1}$, and $T[\beta^*(x)] = \langle F_T^{-1}x, x \rangle$.

To show that there exist objects in B which behave this way under homomorphism, it will suffice to show that, as functions of T, both $(\det F_T)^{-1}$, and, for fixed x, $\langle F_T^{-1}x, x \rangle$, are in the transform algebra $\hat{B}$ of B, and then take their inverses, which will be unique, since B is semi-simple. Now for every fixed x, $\langle F_T x, x \rangle$ is, by hypothesis, in $\hat{B}$. On the other hand, it is trivial, (for example, by induction on k), to see that in general, if $\langle Ax, x \rangle$ is a quadratic form and α is any entry in the matrix A, then there exist a finite number of fixed non-zero vectors $x_1^0, \ldots, x_M^0$, depending on the location of α but not on its value, such that α is a fixed linear combination of the numbers $\langle Ax_i^0, x_i^0 \rangle$ $(i = 1, \ldots, M)$. In particular, therefore, both $(\det F_T)^{-1}$, and, for each x, $\langle F_T^{-1}x, x \rangle$, are expressible as rational functions (with positive denominators) of objects already in $\hat{B}$, and since $\hat{B}$ is closed under the application of holomorphic functions ([5], p. 78), they have unique preimages, Δ^{-1} and $\beta^*(x)$, in B. Now it is clear from the behavior of $\beta^*(x)$ under homomorphism, and from its form as the preimage in B of $\langle F_T^{-1}x, x \rangle$, that $\beta^*(x)$ satisfies conditions (*a*), (*b*), and (*c*) of (III), with $N = 2$, and by a second application of the fact that $\hat{B}$ is closed under composition with holomorphic functions, we find that there exists an element $\Delta^{-\frac{1}{2}}$ of B which corresponds in $\hat{B}$ to the positive square root of $(\det F_T)^{-1}$. Setting up as before the zeta-functions $Z_\beta(s)$ and $Z_\beta^*(s)$ corresponding to $\beta(x)$ and $\beta^*(x)$, we have from [3] that for each homomorphism T,

$$T\left[\Delta^{-\frac{1}{2}} \frac{\Gamma(k/2 - s)}{\pi^{k/2-s}} Z_\beta^*(k/2 - s)\right] = T\left[\frac{\Gamma(s)}{\pi^s} Z_\beta(s)\right],$$

and since B is semi-simple, we conclude that in fact

$$\Delta^{-\frac{1}{2}} \frac{\Gamma(k/2 - s)}{\pi^{k/2-s}} Z_\beta^*(k/2 - s) = \frac{\Gamma(s)}{\pi^s} Z_\beta(s). \quad \text{Q.E.D.}$$

As an illustration, suppose H is a Hilbert space, and $L = \int_a^b \lambda dE_\lambda$ is a bounded self-adjoint operator on H. Let B be the algebra of continuous functions of L, i.e., operators of the form

$$\int_a^b g(\lambda)\, dE_\lambda,$$

where g is a complex-valued continuous function on $[a, b]$. Define

$$\beta(x) = \int_a^b \langle F_\lambda x, x \rangle dE_\lambda.$$

where for each λ, F_λ is a real positive definite symmetric $k \times k$ matrix, whose entries are continuous functions of $\lambda (a \leqslant \lambda \leqslant b)$. Then in the terminology of this section, we find that for this particular case,

$$\beta^*(x) = \int_a^b < F_\lambda^{-1} x, x > dE_\lambda,$$

and

$$\Delta^{-\frac{1}{2}} = \int_a^b (\det F_\lambda)^{-\frac{1}{2}} dE_\lambda.$$

5

In this concluding section, we show that the results in sections § 3 and § 4 of [1] generalize to our context. We begin by recalling Theorem 8 of [1], which, in a specialized version, goes as follows:

If $f(x)$ is a complex-valued real analytic function in $E^k - \{0\}$, and if for some complex s and $N > 0$, $f(rx) = r^{-Ns} f(x)$ $(r > 0)$, then for x not an integral lattice-point,

$$\lim_{\varepsilon \to 0} \Sigma' e^{-|\varepsilon n|^2} f(n) e^{2\pi i < x, n >}$$

exists and is real analytic.

The proof of this is far from simple, and rather than prove the corresponding theorem for B-valued functions directly, we will assume the above theorem as a lemma. We require one additional lemma.

Lemma 5.1. *Let B be a complex Banach space, and let $F(x)$ be a function from an open set U in E^k to B. Suppose $F(x)$ is weakly real analytic in U, i.e., for every T in B^*, $T[F(x)]$ is real analytic in U. Then $F(x)$ is strongly real analytic in U, i.e., it has a power series representation at every point in U, where the coefficients of the power series are elements of B.*

Proof. Let $x_0 \varepsilon U$. We will show that $F(x)$ is strongly real analytic in a neighborhood of x_0. Suppose U is identified with a subset of C^k in the usual way. Let $P(r, x_0)$ denote the open polycylinder in C^k of radius r about x_0. For positive integers m and n, define $E_{m,n}$ to be the set of elements T of B^* such that $T[F(x)]$ has a holomorphic extension, which we will denote by $T[F(z)]$, to $P(1/n, x_0)$, and such that $|T[F(z)]| \leqslant m$ in $P(1/n, x_0)$. Now each $E_{m,n}$ is closed in B^*, for suppose $T_n \to T$ and $T_n \in E_{m_0, n_0}$. Then the family $\{T_n[F(z)]\}$ is bounded and hence normal in $P(1/n_0, x_0)$, so there exists a subsequence $\{T_{n_k}[F(z)]\}$ which converges uniformly on compact subsets of $P(1/n_0, x_0)$. On the other hand, $T[F(x)]$ has a holomorphic extension $T[F(z)]$ to some complex neighborhood of x_0 and so by Vitali's theorem, $T_{n_k}[F(z)]$ converges either to $T[F(z)]$ or to a continuation of $T[F(z)]$ in $P(1/n_0, x_0)$, and the limit function is clearly less than or equal to m_0 in modulus, which shows that E_{m_0, n_0} is closed. By applying the Baire category theorem to the sets $E_{m,n}$, which clearly exhaust B^*, we see that one of them contains a ball in B^*. Transferring, if necessary, this ball to the origin, we find that there exists a complex polycylinder R about x_0, and $M > 0$, such that for any $T \in B^*$, $T[F(x)]$ has a holomorphic extension $T[F(z)]$ to R, and $|T[F(z)]| \leqslant M \|T\|$ for $z \in R$. Thus to each $z \in R$ corresponds an element $F(z)$ of B^{**}, and $F(z)$, regarded as a function from R to B^{**}, is weakly holomorphic in the restricted sense that if $T \in B^*$, then $T[F(z)]$ is holomorphic in R. Now B^* is a so-called determining manifold for B^{***}, and by a known theorem ([2], p. 354), this implies that $F(z)$ is in fact strongly holomorphic as a function from R to B^{**}. (The proof in the reference is only for $k=1$, but it generalizes, via Hartog's theorem, to several variables.) To complete the proof, we need to show that for $z \in R$, $F(z)$ is, in fact, in B. By the Hahn–Banach theorem, it suffices to show that any $T \in B^{***}$ which annihilates B, annihilates $F(z)$. Now $F(z) \in B$ for z in a real neighborhood N of x_0, so if $T \in B^{***}$ annihilates B, it annihilates $F(z)$, for $z \in N$. But this clearly implies that T annihilates $F(z)$, for $z \in R$. Q.E.D.

Theorem 5.2. *Let B be a Banach algebra with unit. Suppose $\beta(x)$ is a strongly real analytic B-valued function of x in $E^k - \{0\}$ which satisfies properties (a) and (c) of (III). Then for x not an integral lattice-point and fixed complex s,*

$$\lim_{\varepsilon \to 0} \Sigma' e^{-|\varepsilon n|^2} \frac{e^{2\pi i <x, n>}}{[\beta(n)]^s} = F(x, s) \tag{2}$$

exists and is strongly real analytic.

Example. If B is the algebra of complex $n \times n$ matrices, s is a positive integer, and $\beta(x)$ is a suitable matrix of homogeneous polynomials, then $F(x, s)$ is a fundamental matrix on the k-torus for the system

$$\left[\beta\left(\frac{1}{2\pi i}\frac{\partial}{\partial x_j}\right)\right]^s \vec{f} = 0.$$

Proof of Theorem 5.2. Exactly as in section III, we can show by the Poisson summation formula that for fixed s, the limit in (2) exists uniformly in ε for x in the complement of any fixed "periodic" neighborhood of the integral lattice-points. I.e., there exists $F(x, s) \in B$ which satisfies (2). By applying bounded linear functionals to (2), noting that $[\beta(x)]^{-s}$ is real analytic in x for $x \neq 0$, and invoking Lemma 5.1 and Theorem 8 of [1], we obtain the theorem. Q.E.D.

We next give a generalization of Theorem 13 of [1]. Theorem 13, in a specialized form, states that:

If $f(x)$ is a C^∞ complex-valued function in $E^k - \{0\}$, and if for some complex s and $N > 0$, $f(rx) = r^{-Ns} f(x)\,(r > 0)$, then for x not an integral lattice-point,

$$\lim_{\varepsilon \to 0} \Sigma' e^{-|\varepsilon n|^2} f(n)\, e^{2\pi i <x, n>} = F(x)$$

exists, and in a neighborhood of the origin, we have the estimates:

$$F(x) = 0(|x|^{N \operatorname{Re} s - k}), \quad \text{if } k - N \operatorname{Re} s > 0;$$

$$F(x) = 0(\log |x|^{-1}), \quad \text{if } k - N \operatorname{Re} s = 0.$$

We generalize this as follows:

Theorem 5.3. *Let B be a Banach algebra with unit, and let $\beta(x)$ be a B-valued function of x in $E^k - \{0\}$, which has properties (a), (b), and (c) of section III. Then for x not an integral lattice-point, and any complex s,*

$$\lim_{\varepsilon \to 0} \sum{}' e^{-|\varepsilon n|^2} \frac{e^{2\pi i \langle x, n \rangle}}{[\beta(n)]^s} = F(x, s) \tag{3}$$

exists, and in a neighborhood of the origin, we have the estimates:

$$\|F(x,s)\| = 0(|x|^{N \operatorname{Re} s - k}), \quad \text{if } k - N \operatorname{Re} s > 0;$$

$$\|F(x,s)\| = 0(\log |x|^{-1}), \quad \text{if } k - N \operatorname{Re} s = 0.$$

Proof. As in section III, we can show by the Poisson summation formula that for fixed s, the limit in (3) exists uniformly in ε for x in the complement of any fixed "periodic" neighborhood of the integral lattice-points, and it is easy to show that it is, in fact, C^∞. That is, there exists $F(x, s) \in B$ which satisfies (3). Now it turns out in the proof of Theorem 18 of [1] that the estimates depend only on bounds for the derivatives of $f(x)$. If now D is a constant-coefficient linear differential operator, and $T \in B^*$ is such that $\|T\| \leqslant 1$, then $|DT[[\beta(x)]^{-s}]| = |T[D[\beta(x)]^{-s}]| \leqslant \|D[\beta(x)]^{-s}\|$. In particular, by the Hahn–Banach theorem, for fixed s and x_0 not an integral lattice-point, there exists $T \in B^*$ such that $\|T\| = 1$, and $|T[F(x_0, s)]| = \|F(x_0, s)\|$. Since for $\|T\| \leqslant 1$, the derivatives of $T[[\beta(x)]^{-s}]$ are bounded by the norms of the corresponding derivatives of $[\beta(x)]^{-s}$, Theorem 5.3 follows. Q.E.D.

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