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Universality and fine zero spacing on general sets

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Abstract. A recent approach of D. S. Lubinsky yields universality in random matrix theory and fine zero spacing of orthogonal polynomials under very mild hypothesis on the weight function, provided the support of the generating measure μ is $[-1,1]$. This paper provides a method with which analogous results can be proven on general compact subsets of $\mathbf{R}$. Both universality and fine zero spacing involves the equilibrium measure of the support of μ. The method is based on taking polynomial inverse images, by which results can be transferred from $[-1,1]$ to a system of intervals, and then to general sets.

1. Introduction

In [12] D. Lubinsky found a stunningly simple approach to universality limits. His technique has completely reshaped the subject, and its importance is hard to overestimate. Among the highlights of this new approach is universality under the sole continuity of the weight [12] (previously analyticity was required!), fine spacing of zeros of orthogonal polynomials by Levin and Lubinsky in [10], and universality for exponential weights in [11]. The present work was motivated by these results, and it would not have been possible without them.

Let μ be a positive finite Borel measure with compact support E on the real line. We assume that E consists of infinitely many points, and then we can form the orthonormal polynomials $p_n(\mu;x)=\gamma_n(\mu)x^n+\dots$ with respect to μ.

We shall denote by $\operatorname{cap}(E)$ the logarithmic capacity of E. For the leading coefficients $\gamma_n(\mu)$ of $p_n(\mu;x)$ it is known ([18, Corollary 1.1.7]) that

$$\liminf_{n\to\infty}\gamma_n(\mu)^{1/n}\geq\frac{1}{\operatorname{cap}(E)},$$

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and the measure μ is called *regular* (from the point of view of orthogonal polynomials) if

$$\lim_{n\to\infty}\gamma_n(\mu)^{1/n}=\frac{1}{\operatorname{cap}(E)} \tag{1.1}$$

and the right-hand side is finite. This is a rather mild assumption, and it holds under fairly general conditions on μ (see [18]). For various properties of orthogonal polynomials with respect to regular measures see [18]. In particular, if ν and μ have the same support, $\nu\ge\mu$ and μ is regular, then so is ν (since then $\gamma_n(\nu)\le\gamma_n(\mu)$).

Let $x_{n,1}<x_{n,2}<...<x_{n,n}$ be the zeros of $p_n(\mu;x)$, and

$$K_n(\mu;x,y)=\sum_{j=0}^{n-1}p_j(\mu;x)p_j(\mu;y) \tag{1.2}$$

be the associated reproducing kernel. It has been known that some universality questions in random matrix theory can be expressed in terms of orthogonal polynomials, in particular in the off-diagonal behavior of the reproducing kernel (see [3], [4], [9] and [13]). When $E=[-1,1]$ and $d\mu(x)=w(x)\,dx$, a form of universality in random matrix theory can be stated as

$$\lim_{n\to\infty}\frac{K_n\left(x+\dfrac{a}{w(x)K_n(x,x)},x+\dfrac{b}{w(x)K_n(x,x)}\right)}{K_n(x,x)}=\frac{\sin\pi(a-b)}{\pi(a-b)} \tag{1.3}$$

(with $K_n(x,y)=K_n(\mu;x,y)$). This had first been proven under strong conditions on w by various authors (the first rigorous proofs seem to have been given in [4], [3] and [9]; see [12] for a discussion), and more recently by Lubinsky [12] under continuity of w. More precisely, Lubinsky proved that (1.3) holds uniformly in $x\in S$ and locally uniformly in $a,b\in\mathbf{R}$ provided μ is a regular measure with support $[-1,1]$, $S\subset(-1,1)$ is a compact set, μ is absolutely continuous in a neighborhood of S and its density (i.e. Radon–Nikodym derivative) w is positive and continuous on S. E. Levin and D. Lubinsky [10] used this to prove the following strong asymptotics for the spacing of zeros: under the previous conditions if $\operatorname{dist}(x_{n,k},S)=O(n^{-1})$, then

$$\lim_{n\to\infty}(x_{n,k+1}-x_{n,k})\frac{n}{\pi\sqrt{1-x_{n,k}^2}}=1. \tag{1.4}$$

See [10] for predecessor results as well as for similar zero spacing at ±1 and for exponential weights on $\mathbf{R}$.

M. Findley [6] proved a local version of (1.3) and (1.4) under the condition that $\log w \in L^1$ in a neighborhood of x and x is a Lebesgue point for both w and its local outer function (see below for a precise formulation).

The main objective of this paper is to extend all these results to the case when the support E of μ is an arbitrary compact subset of the real line (of positive capacity). As we shall see, in this case the role of $1/\pi\sqrt{1-x^2}$ is assumed by the equilibrium density of E (of course, $1/\pi\sqrt{1-x^2}$ is the equilibrium density for $[-1,1]$).

2. Results

Let E be a compact subset of the real line of positive capacity, and let μ_E be its equilibrium measure (see e.g. [18]). In what follows $\mathrm{Int}(E)$ denotes the interior of E in $\mathbf{R}$.

Let, as before, μ be a finite Borel measure with compact support $E \subset \mathbf{R}$. We shall always assume that μ is regular in the sense of (1.1), hence E is of positive capacity. If μ is absolutely continuous with respect to Lebesgue measure on an interval $I \subset \mathrm{Int}(E)$, then we call its Radon–Nikodym derivative $d\mu(x)/dx$ with respect to Lebesgue measure its *density*, and we denote it by $w(x)$. We shall denote the density of the equilibrium measure μ of E by ω_E. It exists (with the choice (2.3)) everywhere on $\mathrm{Int}(E)$ (and it is continuous – actually C^∞ – there).

Theorem 2.1. *Let μ be a regular measure of compact support $E \subset \mathbf{R}$, $S \subset \mathrm{Int}(E)$ be a compact subset of the interior of E, and assume that μ is absolutely continuous in a neighborhood of S and its density $w(x)$ is continuous and positive on S. Then for any L have*

$$\lim_{n\to\infty} n(x_{n,k+1}-x_{n,k})\omega_E(x) = 1 \tag{2.1}$$

uniformly in $x \in S$ and $|x_{n,k}-x| \le L/n$.

Thus, the spacing of zeros is

$$x_{n,k+1}-x_{n,k} = \frac{1+o(1)}{n\omega_E(x)}$$

in any L/n-neighborhood of $x \in S$, and this spacing is uniform in x. It will also follow from the proof that for all large n there are zeros $x_{n,k}$ with $|x-x_{n,k}| \le L/n$ (as long as $L > 1/2\omega_E(x)$).

Theorem 2.2. *Under the conditions of Theorem* 2.1 *we have*

$$\lim_{n\to\infty}\frac{1}{n}K_n\left(\mu;x+\frac{a}{n},x+\frac{b}{n}\right)=\frac{\sin\pi\omega_E(x)(a-b)}{\pi(a-b)w(x)} \tag{2.2}$$

uniformly in $x\in S$ *and locally uniformly in* $a,b\in\mathbf{R}$.

The last clause means that if L is any number, then the convergence is uniform in $x\in S$ and $a,b\in[-L,L]$.

Since $K_n(\mu;x,x)/n\to\omega_E(x)/w(x)$ as $n\to\infty$ (see Theorem 3.1), this is equivalent to the universality limit (1.3) (with $K_n(x,y)=K_n(\mu;x,y)$).

Since for $E=[-1,1]$ the equilibrium density is $\omega_{[-1,1]}(x)=1/\pi\sqrt{1-x^2}$, in this case these theorems give back the original theorems of Lubinsky [12] and Levin and Lubinsky [10] on $[-1,1]$.

The following local result uses even less conditions on μ. To formulate it we introduce the following terminology. It is known that

$$w(x_0):=\lim_{t\to 0}\frac{\mu([x_0-t,x_0+t])}{2t} \tag{2.3}$$

exists almost everywhere on $\mathbf{R}$, and it coincides a.e. with the Radon–Nikodym derivative (with respect to Lebesgue measure) of the absolutely continuous part of μ ([14, Theorem 7.14]). For simplicity we *call this w the Radon–Nikodym derivative, or density of μ*. We shall also assume that in a neighborhood I of a point this w satisfies the local Szegő property, i.e. $\log w\in L^1(I)$. Then

$$H_w(z)=\exp\left(\frac{i}{2\pi}\int_I\frac{1}{z-t}\log w(t)\,dt\right),\quad \operatorname{Im} z>0,$$

differs from the local outer function (see [8, p. 133]) associated with the restriction $w^{1/2}|_I$ of $w^{1/2}$ to I on the upper half-plane only in a constant multiple of modulus 1, therefore the nontangential limit $H_w(x)$ of $H_w(z)$ exists at almost every $x\in\mathbf{R}$ as $z\to x\in\mathbf{R}$. Furthermore $|H_w(x)|^2=w(x)$ for almost every $x\in I$, and almost all points $x_0\in I$ are Lebesgue points for H_w:

$$\lim_{h\to 0}\frac{1}{h}\int_{-h}^{h}|H_w(x_0+t)-H_w(x_0)|\,dt=0. \tag{2.4}$$

The function H_w is also closely related to the local Szegő function associated with $w|_I$ (see [6]); in particular, they have the same Lebesgue points in I (see [6, Lemma 12]).

Theorem 2.3. *Let μ be a regular measure of compact support $E\subset\mathbf{R}$, and let $x_0\in\mathrm{Int}(E)$ be a Lebesgue point for the Radon–Nikodym derivative $w(x)=d\mu(x)/dx$ of μ. Assume further that $w(x_0)>0$, $\log w$ is integrable in a neighborhood I of x_0 and* (2.4) *is satisfied at x_0. Then* (2.2) *holds at $x=x_0$ locally uniformly in $a,b\in\mathbf{R}$. Furthermore, for any $L>2\omega_E(x_0)$ and sufficiently large n, there are zeros $x_{n,k}$ of $p_n(\mu;z)$ in $[x_0-L/n, x_0+L/n]$ and* (2.1) *holds at $x=x_0$ uniformly in $|x_{n,k}-x_0|\leq L/n$.*

Implicit among the conditions of this theorem is that the limit (2.3) exists at x_0.

While preparing this work, we learned about a simultaneous parallel paper by B. Simon [17]. Simon proves Theorem 2.2 for his regular sets (in the terminology of [17] this means that the essential support has absolutely continuous equilibrium measure), and as a consequence obtains Theorem 2.1 in this case. Originally he had also planned a local extension in the spirit of Theorem 2.3, but having learned about Findley's manuscript [6] (which is basically Theorem 2.3 for $E=[-1,1]$), he generously abandoned that direction. The author is grateful to him for pointing out a subtle point regarding the local version Theorem 2.3. The method of the present paper and that of Simon [17] are so vastly different (modulo the background work of Lubinsky), that it would not have made much sense to unite them. But we believe that both methods are worth publishing and will produce further results in the future.

3. Outline of the proof

A crucial role will be played by the Christoffel functions

$$\lambda_n(\mu;x)=\frac{1}{K_n(x,x)}=\left(\sum_{j=0}^{n-1}p_j(x)^2\right)^{-1}. \tag{3.1}$$

It is well known that $\lambda_n(x)$ is the minimum value in the following extremal problem:

$$\lambda_n(\mu;x)=\inf_{P_{n-1}(x)=1}\int_{\mathbf{R}}P_{n-1}^2\,d\mu,$$

where the infimum is taken over all polynomials P_{n-1} of degree at most $n-1$. Thus, $\lambda_n(\mu;x)$ is a monotone function both in μ and in n, and this will be our standard tool below.

Lubinsky's proof was based on the following result (see [12, (3.5)]).

Lubinsky's inequality. *If $\sigma \le \sigma^*$, then*

$$\lambda_m(\sigma;u)|K_m(\sigma;u,v)-K_m(\sigma^*;u,v)| \le \left(\frac{\lambda_m(\sigma;u)}{\lambda_m(\sigma;v)}\right)^{1/2}\left[1-\frac{\lambda_m(\sigma;u)}{\lambda_m(\sigma^*;u)}\right]^{1/2}. \tag{3.2}$$

The point in this inequality is that the Christoffel functions $\lambda_n(\mu;x)$ behave much more nicely than the orthogonal polynomials or the reproducing kernels because they are monotonic in the measure μ. In particular, they have strong localization, which, in turn of Lubinsky's inequality, yields some kind of localization (in x) on $K_n(\mu;x+a/n,x+b/n)/n$ ([12, Theorem 3.1]). This easily allows one to deduce universality for general measures if one knows it for a single measure, and Lubinsky used the Legendre weight for this comparison. This approach works on $[-1,1]$.

When the measure is supported on more general sets, then localization still holds, but there does not seem to be any easy measure around which could be used for comparison. But, as we shall see – and this is the essence of our method – there is no need to have one, for in this case we essentially transform Lubinsky's theorem from $[-1,1]$ to general sets. This transformation is done by applying polynomial mappings, properties of which have been established in the paper [7] by Geronimo and Van Assche. The polynomial inverse image method has been applied to transfer polynomial inequalities in [20], but it seems to be quite a bit of luck that it works in our present case, as well.

The method is the following: start from the result on $[-1,1]$, and apply it to a polynomial mapping $y=T_N(x)$ with a polynomial T_N (of degree N) with some special properties (see the next section). This results in a statement on the inverse image $F:=T_N^{-1}[-1,1]$. In this step the interval splits up into more than one part. With such polynomial inverse images one can approximate arbitrary sets of finite intervals (see [19]), and this sometimes allows one to prove the statement for general subsets of the real line.

In the present case let ν_0 be a measure on $[-1,1]$ and let ν be its pull-back on $F=T_N^{-1}[-1,1]$ under the mapping $y=T_N(x)$. This pull-back transformation preserves the equilibrium measure (this was one of the crucial observations of Geronimo and Van Assche in [7]). Some of the orthogonal polynomials also transform nicely: $p_{nN}(\nu;x)=p_n(\nu_0;T_N(x))$. For other indices, i.e. for $p_{nN+j}(\nu;z)$ with $j=1,\dots,N-1$, there is also a relation, but it is rather implicit and it is not possible to use. So we rely on this relation only for the indices nN. Note that here ν is not our given measure μ (which may not be a pull-back of anything), but we can make μ and ν coincide on some small interval by appropriately prescribing ν_0. Unfortunately, the reproducing kernels do not seem to transform in a manageable way. Note however, that if one argument of the mth reproducing kernel is a zero of the mth orthogonal polynomial, then the formula for the reproducing kernel is greatly simplified, so we fix one of the arguments to be a zero. This needs to be done both for the original measure μ, as well as for the pull-back measure ν, so the points that are fixed are different in these two cases, which necessitates an additional shift. With this transform technique and with Lubinsky's localization we obtain something close to what we want but only for the sequence $\{nN\}_{n=1}^{\infty}$ of indices. To cover the full sequence, i.e. indices of the form $nN+j$ with some j, we factor out j zeros of the $(nN+j)$-th orthogonal polynomial, and note that then the rest is the nNth orthogonal polynomial but with respect to a measure that varies with n and that includes the factored zeroes. Then we shall do the previous transform but for these varying measures, and we get the result for the full sequence.

Levin and Lubinsky derived zero spacing from universality limits, but we have found it more convenient to deal first directly with Theorem 2.1 on zero spacing and then proving Theorem 2.2.

On the other hand, when proving Theorem 2.3 we already have lots of measures for comparison at our disposition (this is the content of Theorem 2.2), so we can directly apply Lubinsky's localization technique in this case.

Just as in [12], the proofs are based on asymptotics for Christoffel functions, more precisely on the following results.

Theorem 3.1. *With the assumptions of Theorem* 2.1 *we have*

$$\lim_{n\to\infty} n\lambda_n\Big(\mu; x+\frac{a}{n}\Big) = \frac{w(x)}{\omega_E(x)} \tag{3.3}$$

uniformly in $x\in S$ *and locally uniformly in* $a\in\mathbf{R}$.

Theorem 3.2. *With the assumptions of Theorem* 2.3 *we have* (3.3) *at* $x=x_0$ *locally uniformly in* $a\in\mathbf{R}$.

4. Preliminaries

It is known that the orthonormal polynomials satisfy a three-term recurrence relation

$$xp_n(\mu;x) = a_n(\mu)p_{n+1}(\mu;x)+b_n(\mu)p_n(\mu;x)+a_{n-1}(\mu)p_{n-1}(\mu;x),$$

and with these numbers $a_n(\mu)>0$ we have the Christoffel–Darboux formula

$$K_n(\mu;x,y) = a_n(\mu)\frac{p_n(\mu;x)p_{n-1}(\mu;y)-p_{n-1}(\mu;x)p_n(\mu;y)}{x-y} \tag{4.1}$$

for the nth reproducing kernel.

One of the best ways to think about orthogonal polynomials is the following: if $p_n(\mu;x)=\gamma_n(\mu)x^n+\ldots$, then the monic orthogonal polynomials $p_n(\mu;x)/\gamma_n(\mu)$ uniquely solve the following extremal problem with, $1/\gamma_n(\mu)^2$ being the extremum value,

$$\inf_{P_n(x)=x^n+\ldots}\int_{\mathbf{R}}|P_n|^2\,d\mu=\int_{\mathbf{R}}\left(\frac{p_n(\mu;x)}{\gamma_n(\mu)}\right)^2d\mu(x)=\frac{1}{\gamma_n(\mu)^2}. \tag{4.2}$$

We shall use the following lemmas several times. To the content of the lemmas note that if H is a compact subset of the real line, then its equilibrium measure μ_H is absolutely continuous in $\mathrm{Int}(H)$, and its density $\omega_H(x)$ is continuous there.

Lemma 4.1. *If $H\subset H'$ are compact subsets of $\mathbf{R}$ of positive capacity, then $\mu_{H'}|_H\le\mu_H$, and, as a consequence, if $I\subset\mathrm{Int}(H)$ is an interval, then $\omega_{H'}(x)\le\omega_H(x)$ for all $x\in I$.*

This is so because μ_H is the balayage of $\mu_{H'}$ out of $\mathbf{C}\setminus H$, see e.g. [16, Theorem IV.1.6(e)].

Lemma 4.2. *Let H and H_k, $k=1,2,\ldots$, be compact subsets of the real line lying in some fixed interval. Let furthermore O be an open set that is contained in all H_k and H. Assume that $\mathrm{cap}(H_k)\to\mathrm{cap}(H)$, and either $H_k\subseteq H$ for all k, or $H\subseteq H_k$ for all k. Then, as $k\to\infty$, we have $\omega_{H_k}(x)\to\omega_H(x)$ locally uniformly in O.*

Proof. When $H\subset H_k$ for all k this is [19, (30)] (local uniformity follows from the proof and was actually stated on the top of p. 296). When $H_k\subset H$ the proof is just the same. □

We shall use Markov's inequality

$$\sum_{j=1}^{k-1}\lambda_m(\mu;x_{m,j})\le\int_{-\infty}^{x_{m,j}}d\mu\le\sum_{j=1}^{k}\lambda_m(\mu;x_{m,j}). \tag{4.3}$$

If we apply this with the index k and the index i, then subtraction gives

$$\sum_{j=i+1}^{k-1}\lambda_m(\mu;x_{m,j})\le\int_{x_{m,i}}^{x_{m,k}}d\mu\le\sum_{j=i}^{k}\lambda_m(\mu;x_{m,j}). \tag{4.4}$$

In particular,

$$\lambda_m(\mu;x_{m,k})\le\int_{x_{m,k-1}}^{x_{m,k+1}}d\mu, \tag{4.5}$$

and

$$\int_{x_{m,k-1}}^{x_{m,k}}d\mu\le\lambda_m(\mu;x_{m,k-1})+\lambda_m(\mu;x_{m,k}). \tag{4.6}$$

Lemma 4.3. *Suppose that μ is a regular measure with compact support E, and that I and I' are two closed subintervals of* $\mathrm{Int}(E)$ *such that I lies in the interior of I'. Let us assume that μ is absolutely continuous on I' and for its density $w(x)$ we have $M_1 \le w(x) \le M_2$ on I' with some positive constants M_1 and M_2. Let further B_1 and B_2 be positive lower and upper bounds for $\omega_E(x)$ on I'. Then there is an m_0 such that for $m \ge m_0$,*

(i) *no subinterval of I of length $<M_1/2M_2B_2m$ contains more than two zeros of $p_m(\mu;z)$;*

(ii) *any subinterval of I of length $>4M_2/M_1B_1m$ contains at least one zero of $p_m(\mu;z)$.*

Proof. Let $I'' \subset \mathrm{Int}(I')$ be another closed interval that contains I in its interior. Let μ_1 and μ_2 be the measures that agree with μ outside I', but on I' they have densities M_1 and M_2, respectively. These measures are regular (see the localization theorem of [18, Theorem 5.3.3]), so we can apply Theorem 3.1 (the proof of which does not use the present lemma) to deduce that $m\lambda_m(\mu_1;x) \to M_1/\omega_E(x)$ and $m\lambda_m(\mu_2;x) \to M_2\omega_E(x)$, as $m \to \infty$, and the convergence is uniform on I''. Since $\lambda_m(\mu;x)$ lies in between $\lambda_m(\mu_1;x)$ and $\lambda_m(\mu_2;x)$, it follows that there is an m_1 such that for $m \ge m_1$ we have

$$\frac{M_1}{2B_2m} \le \lambda_m(\mu;x) \le \frac{2M_2}{B_1m}, \quad x \in I''.$$

Now suppose that $J \subset I$ contains at least three zeros, say $x_{m,k-1}$, $x_{m,k}$, $x_{m,k+1}$. Apply (4.5) to them, and notice that the right-hand side is at most $|J|M_2$, while, as we have just seen, the left-hand side is at least $M_1/2B_2m$. These are compatible only for $|J| \ge M_1/2M_2B_2m$, and this proves (i).

In proving (ii) we note first of all that all parts of the support of μ attract zeros of $p_m(\mu;z)$ for large m (see e.g. [15]), which implies that for some m_2 and $m \ge m_2$ there is at least one-one zero in the two subintervals of $I'' \setminus I$. Thus, if for $m \ge \max\{m_1, m_2\}$ there is no zero in a $J \subset I$, and $x_{m,k-1}$ is the largest zero of $p_m(\mu;z)$ lying to the left of J, then $x_{m,k-1}, x_{m,k} \in I''$. So (4.6) gives that

$$2\frac{2M_2}{B_1n} \ge \lambda_m(\mu;x_{m,k-1}) + \lambda_m(\mu;x_{m,k}) \ge \int_{x_{m,k-1}}^{x_{m,k}} d\mu \ge \int_J d\mu \ge |J|M_1,$$

which is impossible for $|J| > 4M_2/M_1B_1n$, and this is (ii). $\square$

Lemma 4.4. *Let $L>0$ be a fixed number. With the assumptions of Theorem* 2.3 *there is an n_0 such that for $n \ge n_0$,*

(i) *no subinterval of $[x_0 - L/n, x_0 + L/n]$ of length $<1/3n\omega_E(x_0)$ contains more than two zeros of $p_n(\mu;z)$;*

(ii) $[x_0 - 9/n\omega_E(x_0), x_0 + 9/n\omega_E(x_0)]$ *contains at least one zero of $p_n(\mu;z)$.*

The content of (ii) in the lemma is to guarantee at least one zero. Then it follows from Theorem 2.3 that any subinterval of $[x_0-L/n, x+L/n]$ of length Δ/n with $\Delta>1/\omega_E(x_0)$ contains at least one zero of $p_n(\mu;z)$.

Proof. The proof is along the same lines as the preceding proof. By Theorem 3.2 we have

$$\frac{w(x_0)}{2m\omega_E(x_0)} \le \lambda_m\Big(x+\frac{a}{m}\Big) \le \frac{2w(x_0)}{m\omega_E(x_0)} \tag{4.7}$$

for $|a|\le L/m$ and $m\ge m_0$. Assume now that for some large n there is no zero of $p_n(\mu;z)$ in $[x_0-K/n, x_0+K/n]$. Let $x_{n,k-1}<x_0-K/n$ be the largest zero to the left of this interval. Then $x_{n,k}>x_0+K/n$, and $x_{n,k}-x_{n,k-1}>2K/n$. Let m be selected so that

$$\frac{1}{m} \le x_{n,k}-x_{n,k-1} < \frac{2}{m}.$$

Then $m<n/K$, but for large n we have $m\ge m_0$ (recall that in any interval which intersects the support of μ in an infinite number of points there is at least one zero of $p_n(\mu;z)$ for large n). Then

$$x_{n,k-1}, x_{n,k} \in \Big[x_0-\frac{2}{m}, x_0+\frac{2}{m}\Big],$$

and so from (4.6), (4.7) and the monotonicity of Christoffel functions we can infer

$$\int_{x_{n,k-1}}^{x_{n,k}} d\mu \le \lambda_n(\mu;x_{n,k-1})+\lambda_n(\mu;x_{n,k})$$
$$\le \lambda_{[Km]}(\mu;x_{n,k-1})+\lambda_{[Km]}(\mu;x_{n,k}) \le 2\frac{2w(x_0)}{[Km]\omega_E(x_0)}.$$

But the left-hand side is at least

$$\int_{x_{n,k}}^{x_{n,k+1}} w(x)\,dx \ge \min\Big\{\int_{x_0-1/2m}^{x_0} w(x)\,dx, \int_{x_0}^{x_0+1/2m} w(x)\,dx\Big\} = \frac{w(x_0)+o(1)}{2m},$$

where, in the last step, we used that x_0 is a Lebesgue point for w. The last two inequalities contradict each other for large m (which is the same as large n) if K is a fixed number $>8/\omega_E(x_0)$, and this proves (ii).

Now suppose that $J\subset[x_0-L/n, x_0+L/n]$ contains at least three zeros, say $x_{m,k-1}$, $x_{m,k}$, $x_{m,k+1}$. Apply (4.5) to them. The right-hand side is at most

$$\int_{x_0-L/n}^{x_0+L/n} d\mu_s + \int_J w(x)\,dx \le \int_{x_0-L/n}^{x_0+L/n} d\mu_s + w(x_0)|J| + \int_{x_0-L/n}^{x_0+L/n} |w(t)-w(x_0)|\,dt,$$

where μ_s is the singular part of μ with respect to Lebesgue measure. By the existence of the Radon–Nikodym derivative in (2.3) and from the Lebesgue point property of x_0 we can conclude

$$\lambda_n(\mu; x_{n,k}) \le w(x_0)|J| + o\left(\frac{1}{n}\right).$$

Now (4.7) can be applied to the left-hand side of (4.5), and we obtain

$$\frac{w(x_0)}{2n\omega_E(x_0)} \le w(x_0)|J| + o\left(\frac{1}{n}\right),$$

which is impossible for large n if $|J| < 1/3n\omega_E(x_0)$, and this is (i). □

Next we introduce admissible polynomials that will be our aid to transform results from $[-1,1]$ to general sets. Let $T = T_N$ be a polynomial of degree $N \ge 2$ with real and simple zeros $X_1 < X_2 < ... < X_N$. Let $Y_1 < Y_2 < ... < Y_{N-1}$ be the zeros of T', and assume that $|T(Y_j)| \ge 1$, $j = 1, ..., N-1$ (note that $T(Y_j)$ are the local extrema of T). Then (see [7, Lemma 1]) there exists a unique sequence of closed intervals $F_1, ..., F_n$ such that for all $1 \le i \le N$ we have $T(F_i) = [-1,1]$, $X_i \in F_i$, and for $1 \le i \le N-1$ the set $F_i \cap F_{i+1}$ contains at most one point. We call any such polynomial *admissible*, and we are interested in the inverse image $F = T^{-1}[-1,1] = \bigcup_{i=1}^N F_i$. We denote by T_i^{-1} the branch of T^{-1} that maps $[-1,1]$ into F_i.

If ν_0 is a Borel measure on $[-1,1]$, then we set

$$\nu(H) := \frac{1}{N}\nu(T(H)) \quad \text{for } H \subset F_i, \ i = 1, ..., N.$$

This generates a Borel measure on F, which we call the pull-back of ν_0 under the polynomial mapping $y = T(x)$. For example, if

$$d\nu_0(y) = \frac{dy}{\pi\sqrt{1-y^2}}$$

is the equilibrium measure on $[-1,1]$, then its pull-back

$$d\nu(x) = \frac{1}{N\pi}\frac{|T'(x)|}{\sqrt{1-T^2(x)}} \tag{4.8}$$

is the equilibrium measure of $F = T^{-1}[-1,1]$ (see the first formula on p. 577 of [7]).

Polynomial inverse images of intervals, i.e. sets of the form $T^{-1}[-1,1]$ with admissible T, and pull-back measures on them have many interesting properties, see the paper [7] by J. Geronimo and W. Van Assche and the paper [19]. What will be relevant to us is that $p_{nN}(\nu; x) = p_n(\nu_0; T(x))$ for all $n = 0, 1, ...$

5. Proof of Theorem 2.1 for a subsequence

Below $o_\varepsilon(1)$ denotes a quantity that tends to zero together with ε. Instead of $o_{1/n}(1)$ we shall simply write the usual $o(1)$.

Select a small $\varepsilon>0$ such that $\mathrm{dist}(S,\mathbf{R}\setminus\mathrm{Int}(E))>2\varepsilon$ and such that μ is absolutely continuous in the 2ε-neighborhood of S. Hence $w(x)$ exists whenever $\mathrm{dist}(x,S)<2\varepsilon$. By the regularity of the logarithmic capacity (see [2]) there is an open set containing E with capacity arbitrarily close to $\mathrm{cap}(E)$, and by compactness of E we may assume that this open set consists of finitely many intervals. Thus, for every η_1 there is a set $E'=\bigcup_{j=1}^{l}[\alpha_j',\beta_j']$ consisting of finitely many disjoint intervals such that $E\subset\mathrm{Int}(E')$, and $\mathrm{cap}(E')\le\mathrm{cap}(E)+\eta_1$. We may assume that η_1 is so small that

$$\omega_{E'}(x)\le\omega_E(x)\le\omega_{E'}(x)+\varepsilon \tag{5.1}$$

uniformly for $\mathrm{dist}(x,S)\le\varepsilon$ (see Lemmas 4.1 and 4.2).

Let $3\eta<\mathrm{dist}(E,\mathbf{R}\setminus\mathrm{Int}(E'))$, $\eta<\eta_1$. By [20] there is an admissible polynomial $T=T_N$ of some degree N such that $F:=T^{-1}[-1,1]$ consists of l intervals: $F=\bigcup_{j=1}^{l}[\alpha_j,\beta_j]$ such that for each j we have $[\alpha_j,\beta_j]\subset[\alpha_j'+\eta,\beta_j'-\eta]$,

$$|(\alpha_j'+\eta)-\alpha_j|\le\eta \quad\text{and}\quad |(\beta_j'-\eta)-\beta_j|\le\eta.$$

Then for all $0\le s\le\eta$ the translated sets $F^s:=F-s$ satisfy $E\subseteq F^s\subseteq E'$, and hence, in view of Lemma 4.1 and (5.1),

$$\omega_{F^s}(x)\le\omega_E(x)\le\omega_{E'}(x)+\varepsilon\le\omega_{F^s}(x)+\varepsilon \tag{5.2}$$

uniformly for $\mathrm{dist}(x,S)\le\varepsilon$.

Let also $T^s(x)=T(x+s)$. Then

$$F^s=(T^s)^{-1}[-1,1] \quad\text{and}\quad (F_i)^s=(T^s)_i^{-1}[-1,1].$$

Let $Z^*\in S$ be a fixed point. Without loss of generality we may assume that for some i^* the point Z^* lies in the interior of F_{i^*}. In fact, if this is not the case, then Z^* is an endpoint of some F_{i^*}, and then consider, for some small $0<s<\eta/2$, the set $F^s=F-s$ instead of F (and the mapping T^s instead of T), and use $\eta/2$ instead of η everywhere below. Let $I\subset I'\subset\mathrm{Int}(F_{i^*})$ be two closed intervals such that $I\subset\mathrm{Int}(I')$ and Z^* lies in the interior of I. Without loss of generality we may assume that F_{i^*} lies in the ε-neighborhood of S. Indeed, let $\mathcal{T}_m(x)=\cos(m\arccos x)$ be the classical Chebyshev polynomials. Note that together with T_N the polynomials $T_N(\mathcal{T}_m)$ are also admissible and for them the inverse image of $[-1,1]$ is again F. Now for large m for this polynomial all the F_i's that intersect S lie in the ε-neighborhood of S.

Using the continuity and positivity of w on S, with the same reasoning we may assume that there are two constants $0<M_1<M_2$ such that

$$M_1 \le w(x) \le M_2 \quad \text{for all } x\in F_{i^*}. \tag{5.3}$$

Now define a measure ν_0 on $[-1,1]$ by stipulating

$$\nu_0(H)=\mu(T_{i^*}^{-1}(H))N \tag{5.4}$$

for all Borel sets $H\subset[-1,1]$, and then define a measure ν on F by setting

$$\nu(H)=\frac{1}{N}\sum_{i=1}^{N}\nu_0(T_i(H\cap F_i)) \tag{5.5}$$

for all Borel subsets H of F (see [7]). We say that ν and ν_0 are associated with μ through T_{i^*}. Clearly, $\nu\equiv\mu$ on F_{i^*}. The crucial property that we shall use is that the orthonormal polynomials $p_{nN}(\nu;x)$ are just $p_n(\nu;T(x))$ (see [7], and for more on this see the discussion below). This is how we transfer information from $[-1,1]$ (where ν_0 and $p_n(\nu_0;\,\cdot\,)$ live) to F (where ν and $p_{nN}(\nu;\,\cdot\,)$ live).

But first we prove the following result.

Lemma 5.1. *Let $0\le s_m\le L/m$ with some fixed L. Then*

$$\lim_{m\to\infty}\frac{1}{m}\left(K_m\left(\nu;x+\frac{a}{m},x+\frac{b}{m}\right)-K_m\left(\nu;x+\frac{a}{m}-s_m,x+\frac{b}{m}-s_m\right)\right)=0$$

uniformly in $x\in S\cap I$ and a and b lying in compact subsets of $\mathbf{R}$.

Proof. First note that

$$K_m(\nu;u-s_m,v-s_m)=K_m(\nu^{s_m};u,v) \tag{5.6}$$

where ν^s is the translated measure $\nu^s(H)=\nu(H-s)$ (which is *not* the measure associated with μ and $(T^s)_{i^*}$!).

By Theorem 3.1,

$$\lim_{m\to\infty}m\lambda_n\left(\nu^{s_m};x+\frac{a}{m}\right)=\lim_{m\to\infty}m\lambda_n\left(\nu;x+\frac{a}{m}-s_m\right)=\frac{w(x)}{\omega_F(x)} \tag{5.7}$$

uniformly in $x\in S\cap I$ and bounded $a\in\mathbf{R}$.

T_N' has $N-1$ distinct zeros, which implies that T_N'' has $N-2$ distinct zeros in between them. Hence, at the endpoints τ of F_j that lie in the interior of F, and where then necessarily $T(\tau)=\pm1$, $T'(\tau)=0$, and the second derivative $T''(\tau)$ does not vanish. Thus, in this neighborhood, on F_j the polynomial $T(t)$ is of the form

$$T(t)=1-c(t-\tau)^2+o(t-\tau)^2 \quad\text{or}\quad T(t)=-1+c(t-\tau)^2+o(t-\tau)^2.$$

In particular, this is true when τ is either of the endpoints of F_{i^*}.

These easily imply that the mapping $h_j(t):=T_{i^*}^{-1}(T_j)$ is continuously differentiable with positive derivative around τ. When τ is also an endpoint of one of the subintervals of F (i.e. τ is one of the α_j or β_j), then the derivative of T at τ is not zero, so in this case $h_j(t)=T_{i^*}^{-1}(T_j)$ is of the form const $\pm c\sqrt{|t-\tau|}$ and $|h_j'(t)|$ is of the form $c/2\sqrt{|t-\tau|}$ for t lying close to τ. Since on F_j the measure ν is the pull-back of $w(x)\,dx$ on F_{i^*} under these mappings $h_j(t)$, it follows that if $v(x)$ is the density of ν, then on F_j,

$$v(t)=w(h_j(t))|h_j'(t)|.$$

Since w has a positive lower bound on F_{i^*}, it follows that the density v has a positive lower bound, say, $m^\#$ on F. Of course, $v=w$ in F_{i^*}.

With some small $\theta>0$ consider the function

$$x\longmapsto \inf_{[x-\theta,x+\theta]\cap F} v.$$

It is $\geq m^\#$, and it is close to w on $S\cap I$ because $v=w$ is continuous on $S\cap I$. Therefore, we can can find a continuous $w^\#$ on F such that $w^\#\geq m^\#$,

$$w^\#(x)\leq \inf_{[x-\theta,x+\theta]\cap F} v(x), \quad x\in F,$$

and

$$w^\#(x)=w(x)+o_\theta(1) \quad \text{on } S\cap I. \tag{5.8}$$

Let now $d\nu^\#(x)=w^\#(x)\,dx$, on $F_\theta:=\bigcup_{j=1}^l[\alpha_j+\theta,\beta_j+\theta]$ (recall that $F=\bigcup_{j=1}^l[\alpha_j,\beta_j]$). For large m the sets $F^{s_m}=F-s_m$ contain F_θ, and, by the construction, $\nu^{s_m}\geq\nu^\#$ on F_θ, hence $\nu^{s_m}\geq\nu^\#$. Theorem 3.1 gives

$$\lim_{m\to\infty} m\lambda_m\left(\nu^\#;x+\frac{a}{m}\right)=\frac{w^\#(x)}{\omega_{F_\theta}(x)} \tag{5.9}$$

uniformly in $x\in S\cap I$ and bounded $a\in\mathbf{R}$. Now we can apply Lubinsky's inequality (3.2), to $\nu^{s_m}\geq\nu^\#$, $u_m=x+a/m$ and $v_m=x+b/m$ to conclude from (5.7) and (5.9) that

$$\limsup_{m\to\infty}\frac{1}{m}|K_m(\nu^\#;u_m,v_m)-K_m(\nu^{s_m};u_m,v_m)|\leq\frac{\omega_{F_\theta}(x)}{w^\#(x)}\left(1-\frac{w^\#(x)\omega_F(x)}{w(x)\omega_{F_\theta}(x)}\right)^{1/2}$$

uniformly in $S\cap I$ and locally uniformly in $a,b\in\mathbf{R}$. This same inequality holds with ν in place of ν^{s_m} (note that if $s_m\equiv0$ then $\nu^{s_m}=\nu$), and hence

$$\limsup_{m\to\infty}\frac{1}{m}|K_m(\nu;u_m,v_m)-K_m(\nu^{s_m};u_m,v_m)|\leq2\frac{\omega_{F_\theta}(x)}{w^\#(x)}\left(1-\frac{w^\#(x)\omega_F(x)}{w(x)\omega_{F_\theta}(x)}\right)^{1/2}.$$

Here, on $S\cap I$, the right-hand side is (recall Lemma 4.2 and (5.8))

$$2(1+o_\theta(1))\frac{\omega_F(x)}{w(x)}\left(1-\frac{(w(x)+o_\theta(1))\omega_F(x)}{w(x)(\omega_F+o_\theta(1))}\right)^{1/2},$$

which tends to 0 uniformly on $S\cap I$ as $\theta\to 0$. Therefore,

$$\lim_{m\to\infty}\frac{1}{m}(K_m(\nu;u_m,v_m)-K_m(\nu^{s_m};u_m,v_m))=0,$$

which, in view of (5.6), is precisely the statement in the lemma. $\square$

Next we prove the following result.

Lemma 5.2.

$$\frac{1}{m}\left(K_m\left(\mu;x+\frac{a}{m},x+\frac{b}{m}\right)-K_m\left(\nu;x+\frac{a}{m},x+\frac{b}{m}\right)\right)=o_\varepsilon(1)+o(1)$$

uniformly in $x\in S\cap I$ and a and b in compact subsets of $\mathbf{R}$.

This is basically the localization theorem [12, Theorem 3.1] of Lubinsky, though here the bound on the right depends also on ε. Since the proof is short, we include it here.

Proof. We follow the argument of [12, Theorem 3.1] of Lubinsky that was also used in the preceding proof. In fact, we can apply Theorem 3.1 to both μ and ν, and then use the comparison inequality (3.2). Let μ^* be equal to $\mu=\nu$ on I' and equal to $\mu+\nu$ on $\mathbf{R}\setminus I'$. By the localization theorem [18, Theorem 5.3.3] (applied to some K_1 an K_2, the interiors of which intersect the real line in $\operatorname{Int}(I')$ and in $\operatorname{Int}(F)\setminus I$) it follows that μ^* is regular. Thus, we get from Lubinsky's inequality (3.2) (applied to $\sigma=\mu$ and $\sigma^*=\mu^*$) and from

$$m\lambda_m\left(\mu^*;x+\frac{a}{m}\right)\to\frac{w(x)}{\omega_F(x)}\quad\text{and}\quad m\lambda_m\left(\mu;x+\frac{a}{m}\right)\to\frac{w(x)}{\omega_E(x)}$$

(see Theorem 3.1) that

$$\limsup_{m\to\infty}\frac{1}{m}\left|K_m\left(\mu^*;x+\frac{a}{m},x+\frac{b}{m}\right)-K_m\left(\mu;x+\frac{a}{m},x+\frac{b}{m}\right)\right|$$
$$\le\frac{\omega_E(x)}{w(x)}\left(1-\frac{\omega_F(x)}{\omega_E(x)}\right)^{1/2}$$

uniformly on $S\cap I$ and locally uniformly in $a,b\in\mathbf{R}$. We can here replace μ by ν and the right-hand side then becomes 0 (the common support of ν^* and ν is F), hence the statement in Lemma 5.2 follows if we note that $1-\omega_F(x)/\omega_E(x)=o_\varepsilon(1)$ uniformly on I. $\square$

After this let us return to the proof of Theorem 2.1. First we prove the asymptotics for the zero spacing only for indices of the form nN.

Lemmas 5.1 and 5.2 give that if $0\le s_m\le L/m$ with some fixed L, then

(5.10)
$$\frac{1}{m}\left(K_m\left(\mu;x+\frac{a}{m},x+\frac{b}{m}\right)-K_m\left(\nu;x+\frac{a}{m}-s_m,x+\frac{b}{m}-s_m\right)\right)=o_\varepsilon(1)+o(1)$$

uniformly in $x\in S\cap I$ and a and b in compact subsets of $\mathbf{R}$. We here set $m=nN$ and choose $a=a_m$ and s_m in such a way that $x+a/n$ is a zero of $p_{nN}(\mu;z)$, while $x+a/nN-s_{nN}$ is a zero of $p_{nN}(\nu;z)=p_n(\nu_0;T(z))$. By Lemma 4.3 this is possible with $0\le a\le L/n$ and $0\le s_n\le L/n$ for some L (and large n).

Using the Christoffel–Darboux formula (4.1), after multiplication by $a-b$ the preceding relation goes into:

$$\begin{aligned}(5.11)\quad &-a_{nN}(\mu)p_{nN-1}\left(\mu;x+\frac{a}{nN}\right)p_{nN}\left(\mu;x+\frac{b}{nN}\right)\\ &+a_{nN}(\nu)p_{nN-1}\left(\nu;x+\frac{a}{nN}-s_{nN}\right)p_{nN}\left(\nu;x+\frac{b}{nN}-s_{nN}\right)=o_\varepsilon(1)+o(1).\end{aligned}$$

Here, according to [7, (2.13)], on the right

$$\begin{aligned}p_{nN-1}(\nu;z)=\frac{N}{T'(z)}\frac{a_n(\nu_0)}{a_{(n-1)N+1}(\nu)}\biggl(&\frac{a_{(n-1)N+1}(\nu)}{a_{nN}(\nu)}p_0^{(nN)}(\nu_0;z)p_{n-1}(\nu_0;T(z))\\ &+p_{N-2}^{((n-1)N+1)}(\nu;z)p_n(\nu_0;T(z))\biggr),\end{aligned}$$

where $p_j^{(k)}(\nu;z)$ are the associated polynomials for which the recurrence relation is

$$xp_j(\nu;x)=a_{j+k+1}(\nu)p_{j+1}^{(k)}(\nu;x)+b_{j+k}(\nu)p_j^{(k)}(\nu;x)+a_{j+k}(\nu)p_{j-1}^{(k)}(\nu;x),$$

and $p_0^{(k)}\equiv 1$ and $p_{-1}^{(k)}\equiv 0$. Recalling that $x+a/nN-s_{nN}$ is a zero of $p_{nN}(\nu;z)=p_n(\nu_0;T(z))$, for $z=x+a/nN-s_{nN}$ the formula for $p_{nN-1}(\nu;z)$ takes the form

$$p_{nN-1}(\nu;z)=\frac{N}{T'(z)}\frac{a_n(\nu_0)}{a_{nN}(\nu)}p_{(n-1)N}(\nu;z).$$

Plugging this into (5.11) we obtain

$$\begin{aligned}(5.12)\qquad &-a_{nN}(\mu)p_{nN-1}\left(\mu;x+\frac{a}{nN}\right)p_{nN}\left(\mu;x+\frac{b}{nN}\right)\\ &+a_n(\nu_0)\frac{N}{T'(z)}p_{(n-1)N}(\nu;z)p_{nN}(\nu;z')=o_\varepsilon(1)+o(1),\end{aligned}$$

where $z=x+a/nN-s_{nN}$ and $z'=x+b/nN-s_{nN}$.

We set $S_0=T(S\cap I)$ and $I_0=T(I)$, which are compact subsets of $(-1,1)$, $T(I)$ being an interval, and let w_0 be the density of ν_0 (on $[-1,1]$). For simpler notation we also set $\omega_0=\omega_{[-1,1]}$. Thus, the subscript 0 indicates that the corresponding object lives on $[-1,1]$. Let us recall Lubinsky's theorem [12, Therorem 1.1] according to which

$$\lambda_n(\nu_0;y)K_n\left(\nu_0;y+\frac{\alpha\lambda_n(\nu_0;y)}{w_0(y)},y+\frac{\beta\lambda_n(\nu_0;y)}{w_0(y)}\right)-\frac{\sin\pi(\alpha-\beta)}{\pi(\alpha-\beta)}=o(1)$$

uniformly in $y\in S_0$ and locally uniformly in $\alpha,\beta\in\mathbf{R}$. Theorem 3.1 gives that

$$n\lambda_n(\nu_0;y)\to\frac{w_0(y)}{\omega_0(y)},\quad \text{as } n\to\infty,$$

hence the preceding relation can be written as

$$\frac{1}{n}K_n\left(\nu_0;y+\frac{\alpha}{n\omega_0(y)},y+\frac{\beta}{n\omega_0(y)}\right)-\frac{\sin\pi(\alpha-\beta)}{\pi(\alpha-\beta)}\frac{\omega_0(y)}{w_0(y)}=o(1) \tag{5.13}$$

uniformly in $y\in S_0$ and locally uniformly in $\alpha,\beta\in\mathbf{R}$. Let

$$T(x)=y,\quad T\left(x+\frac{a}{nN}-s_{nN}\right)=y+\frac{\alpha}{n\omega_0(y)}$$

and

$$T(x+b/nN-s_{nN})=y+\frac{\beta}{n\omega_0(y)}.$$

Since

$$T\left(x+\frac{a}{nN}-s_{nN}\right)=T(x)+T'(x)\left(\frac{a}{nN}-s_{nN}\right)+O(n^{-2}),$$

it follows from

$$\omega_F(x)=\frac{|T'(x)|}{N}\omega_0(T(x))=\frac{|T'(x)|}{N}\omega_0(y)$$

(see the discussion of (4.8)) that

$$\alpha=\pm\omega_F(x)(a-nNs_{nN})+O(n^{-1}), \tag{5.14}$$

and a similar reasoning gives

$$\beta=\pm\omega_F(x)(b-nNs_{nN})+O(n^{-1}). \tag{5.15}$$

Here the sign $\pm=T'(x)/|T'(x)|$ is independent of $x\in S\cap I$.

Recall now that $u=y+\alpha/n\omega_0(y)=T(x+a/nN-s_{nN})$ is a zero of the polynomial $p_n(\nu_0;z)$ (indeed this amounts to the same as $x+a/nN-s_{nN}$ being a zero of $p_{nN}(\nu;z)=p_n(\nu_0;T(z)))$. Hence (5.13) and the Christoffel–Darboux formula (4.1) for $u=y+\alpha/n\omega_0(y)$ and $v=y+\beta/n\omega_0(y)$ gives (after multiplication through by $\alpha-\beta$)

$$a_n(\nu_0)p_{n-1}(\nu_0;u)p_n(\nu_0;v)+\frac{\sin\pi(\alpha-\beta)}{\pi w_0(y)}=o(1). \tag{5.16}$$

In view of the fact that $p_{nN}(\nu;x)=p_n(\nu_0;T(x))$, and $u=T(x+a/nN-s_{nN})$, $v=T(x+b/nN-s_{nN})$ and $y=T(x)$, and the expressions for α and β from (5.14) and (5.15), this takes the form

$$\begin{aligned}a_n(\nu_0)p_{(n-1)N}\Big(\nu;x+\frac{a}{nN}-s_{nN}\Big)p_{nN}\left(\nu;x+\frac{b}{nN}-s_{nN}\right)&\\ +\frac{\sin(\pm\pi\omega_F(x)(a-b))}{\pi w_0(T(x))}&=o(1)\end{aligned} \tag{5.17}$$

uniformly in $x\in S\cap I$ and locally uniformly in $b\in\mathbf{R}$.

Substituting this into (5.12) and recalling that with $z=x+a/nN-s_{nN}$ we have $T'(z)=T'(x)+O(n^{-1})$ and

$$w(x)=\pm\frac{1}{N}w_0(T(x))T'(x).$$

Finally we obtain

$$a_{nN}(\mu)p_{nN-1}\Big(\mu;x+\frac{a}{nN}\Big)p_{nN}\left(\mu;x+\frac{b}{nN}\right)+\frac{\sin\pi\omega_E(x)(a-b)}{\pi w(x)}=o_\varepsilon(1)+o(1) \tag{5.18}$$

uniformly for $x\in S\cap I$ and b lying in compact subsets of $\mathbf{R}$. This formula was deduced under the assumption that $x+a/nN$ is a zero of $p_{nN}(\mu;z)$, and that $\operatorname{dist}(x,x+a/nN)\le L/n$ with some fixed L. The $o_\varepsilon(1)$ term is related to the choice of $F=T^{-1}[-1,1]$, and also N depends on ε.

Let now $\delta>0$ be a fixed small number, and set $b=a+1/\omega_F(x)\pm\delta$ into (5.18). There is a $\Delta>0$ such that

$$\frac{\sin(-\pi-\delta\pi\omega_E(x))}{\pi w(x)}>\Delta,$$

while

$$\frac{\sin(-\pi+\delta\pi\omega_E(x))}{\pi w(x)}<-\Delta$$

uniformly in $x\in S\cap I$. We may assume that $\varepsilon>0$ is so small and n is so large that the $(o_\varepsilon(1)+o(1))$-term in (5.18) is, in absolute value, less than $\Delta/2$. Thus, the first term on the left-hand side must change sign as b runs through the interval $[a+1/\omega_E(x)-\delta, a+1/\omega_E(x)+\delta]$, which means that $p_{nN}(\mu;z)$ has a zero in

$$x+\frac{1}{nN}\left[a+\frac{1}{\omega_E(x)}-\delta, a+\frac{1}{\omega_E(x)}+\delta\right]. \tag{5.19}$$

It also follows from the same argument that it cannot have a zero in

$$x+\frac{1}{nN}\left[a+\delta, a+\frac{1}{\omega_E(x)}-\delta\right].$$

Furthermore, if $p_{nN}(\mu;z)$ has an additional zero on the interval (5.19) then it must have at least three zeros there (it is of different signs at the endpoints), which is impossible for small δ by Lemma 4.3.

The same argument gives that $p_{nN}(\mu;z)$ has a single zero on

$$x+\frac{1}{nN}\left[a-\frac{1}{\omega_E(x)}-\delta, a-\frac{1}{\omega_E(x)}+\delta\right].$$

Finally, choosing this single zero instead of $x+a/nN$ above, it follows that $p_{nN}(\mu;z)$ has a single zero on $x+(1/nN)[a-\delta, a+\delta]$ (which then must be $x+a/nN$). We can conclude that the smallest zero of $p_{nN}(\mu;z)$ that is larger than $x+a/nN$ lies in the interval (5.19), and for large n this property uniformly holds in $x\in S\cap I$ and a lying in a compact subset of $\mathbf{R}$ (assuming still that $x+a/nN$ is a zero of $p_{nN}(\mu;z)$).

Here I was a small neighborhood of any point Z^* of S. Therefore, by compactness, we can conclude the same uniformly on S.

In summary, so far we have proved that if $\delta>0$ is arbitrary, then there is an N such that

$$|nN(x_{nN,k+1}-x_{nN,k})\omega_E(x)-1|\le 2\delta \tag{5.20}$$

for all large n, and this relation is uniform in $x\in S$ and $|x_{Nn,k}-x|\le L/nN$ for any fixed L.

6. Proof of Theorem 2.1 for the full sequence

Let F and $T=T_N$ be selected as before for an $\varepsilon>0$. In the preceding section we proved (5.20); and now we show that for each $j=0,...,N-1$ the analogous relation

$$|nN(x_{nN+j,k+1}-x_{nN+j,k})\omega_E(x)-1|\le 2\delta \tag{6.1}$$

holds for all large n, and this relation is uniform in $x\in S$ and $|x_{Nn+j,k}-x|\le L/nN$. This proves Theorem 2.1.

Let z_0 be the smallest accumulation point of $E{=}\mathrm{supp}(\mu)$. It is known that any part of the support attract zeros of the orthogonal polynomials for large n (see e.g. [15]), hence in any neighborhood of z_0 the number of zeros of $p_{m+j}(\mu;z)$ tends to infinity as $m\to\infty$. Thus, for large m, there are at least j zeros, say $z_{m+j,1},\dots,z_{m+j,j}$ in that neighborhood. If $Z_m:=(z_{m+j,1},\dots,z_{m+j,j})$ and $Z_0=(z_0,\dots,z_0)$, then the vectors Z_m can be selected so that $Z_m\to Z_0$ as $m\to\infty$. Our proof of (6.1) is based on the following simple folklore fact: if

$$d\mu_{Z_m}(x)=\prod_{k=1}^{j}(x-z_{m+j,k})^2\,d\mu(x), \tag{6.2}$$

and $q_m(\mu;z)=p_m(\mu;z)/\gamma_m(\mu)$ are the monic orthogonal polynomials, then

$$q_{m+j}(\mu;z)=\left(\prod_{k=1}^{j}(x-z_{m+j,k})\right)q_m(\mu_{Z_m};z). \tag{6.3}$$

In fact, the polynomial $q_m(\mu_{Z_m};z)$ on the right is the unique polynomial that minimizes the integral

$$\int_{\mathbf{R}}P_m(x)^2\,d\mu_{Z_m}(x)=\int_{\mathbf{R}}P_m(x)^2\prod_{k=1}^{j}(x-z_{m,k})^2\,d\mu(x) \tag{6.4}$$

among all monic polynomials P_m of degree m (see (4.2)). The right-hand side is clearly at least as large as the infimum of

$$\int_{\mathbf{R}}Q_{m+j}(x)^2\,d\mu(x)$$

for all monic polynomials of degree $m+j$, for which the infimum is attained for $Q_{m+j}=q_{m+j}(\mu;\,\cdot\,)$. Finally, the right-hand side of (6.4) equals $\int_{\mathbf{R}}q_{m+j}(\mu;\,\cdot\,)^2\,d\mu$ for $P_m(x)=q_{m+j}(\mu;x)/\prod_{k=1}^{j}(x-z_{m+j,k})$, therefore (6.3) follows.

Now we insert $m=nN$ into (6.3), hence our task is to find asymptotics for the zeros of $q_{nN}(\mu_{Z_{nN}};z)$. So we are back in the case considered in the preceding section, but now the measure μ is replaced by the varying measure $\mu_{Z_{nN}}$. Still, we follow the same proof with μ replaced everywhere by $\mu_{Z_{nN}}$; in particular, in the present case the measure ν_{Z_m} associated with $\mu_{Z_{nN}}$ through T_{i^*} also changes with n. Note however, that $Z_m\to Z_0$ and Z_0 is outside of S, therefore, the densities

$$\frac{d\mu_{Z_m}(x)}{dx}=w_{Z_m}(x)=\left(\prod_{k=1}^{j}(x-z_{m+j,k})\right)^2w(x) \tag{6.5}$$

are uniformly equicontinuous on S and are uniformly bounded away from 0 and infinity in a neighborhood of S. Furthermore, in this neighborhood they uniformly converge to

$$w_{Z_0}(x)=(x-z_0)^{2j}w(x), \tag{6.6}$$

as $m\to\infty$.

The proof in the preceding section was based on (5.13) and on (5.10). If we can show that these are valid uniformly when μ is replaced by μ_{Z_m}, then the rest of the proof goes through word by word.

For clearer discussion, let ν_{Z_m} and ν_{0,Z_m} be the measures ν and ν_0 associated with μ_{Z_m} via T_{i^*} (see (5.4) and (5.5)). The fixed measure μ_{Z_0} is just like μ, so we certainly have (5.13) when μ is replaced by μ_{Z_0} and ν_0 is replaced by ν_{0,Z_0}. Therefore, for the validity of (5.13) for μ_{Z_m} it is enough to show that with $u=y+\alpha/m\omega_0(y)$ and $v=y+\beta/m\omega_0(y)$,

$$\frac{1}{m}|K_m(\nu_{0,Z_m};u,v)-K_m(\nu_{0,Z_0};u,v)|=o(1). \tag{6.7}$$

With some small $\theta>0$ consider the measure

$$d\mu^*_{Z_0}(x)=((x-z_0)^{2j}+\theta)\,d\mu(x), \tag{6.8}$$

and let $\nu^*_{Z_0}$ and ν^*_{0,Z_0} be the associated measures (through T_{i^*}). This $\mu^*_{Z_0}$ is larger than μ_{Z_m} for large m, and is larger than μ_{Z_0}, hence ν^*_{0,Z_0} is larger than ν_{0,Z_m} for large m, and is larger than ν_{0,Z_0}. We can apply Theorem 3.1 to the measures ν^*_{0,Z_0} and ν_{0,Z_0}. Since, as $m\to\infty$, we have for the densities on $[-1,1]$ the limit relation $w_{0,Z_m}(y)\to w_{0,Z_0}(y)$ uniformly in $y\in[-1,1]$, it easily follows from Theorem 3.1 and from the monotonicity (in the measure) of the Christoffel functions that

$$\lim_{m\to\infty} m\lambda_m\Big(\nu_{0,Z_m};y+\frac{\alpha}{m}\Big)=\lim_{m\to\infty} m\lambda_m\Big(\nu_{0,Z_0};y+\frac{\alpha}{m}\Big)=\frac{w_{0,Z_0}(y)}{\omega_0(y)}$$

(with $\omega_0(y)=\omega_{[-1,1]}(y)=1/\pi\sqrt{1-y^2}$) and

$$\lim_{m\to\infty} m\lambda_m\Big(\nu^*_{0,Z_0};y+\frac{\alpha}{m}\Big)=\frac{w^*_{0,Z_0}(y)}{\omega_0(y)}$$

uniformly in $y\in T(S\cap I)$ and α lying in some compact subset of $\mathbf{R}$. Now insert into Lubinsky's inequality (3.2) first $\nu_{0,Z_m}\le\nu^*_{0,Z_0}$ and then $\nu_{0,Z_0}\le\nu^*_{0,Z_0}$ to conclude for $u=y+\alpha/m\omega_0(y)$ and $v=y+\beta/m\omega_0(y)$ that

$$\frac{1}{m}|K_m(\nu_{0,Z_m};u,v)-K_m(\nu^*_{0,Z_0};u,v)|=o_\theta(1)+o(1)$$

and

$$\frac{1}{n}|K_n(\nu_{0,Z_0};u,v)-K_n(\nu^*_{0,Z_0};u,v)|=o_\theta(1)+o(1),$$

and these prove (6.7) since $\theta>0$ is arbitrarily small.

As for the analogue of (5.10) for the measures μ_{Z_m}, i.e. for

$$\frac{1}{m}(K_m(\mu_{Z_m};u,v)-K_m(\nu_{Z_m};u-s_m,v-s_m))=o_\varepsilon(1)+\omega(1) \tag{6.9}$$

with $u=x+a/m$ and $v=x+b/m$, we use the same argument. In fact,

$$\frac{1}{m}(K_m(\mu_{Z_m};u,v)-K_m(\mu_{Z_0};u,v))=o(1)$$

is obtained by comparing both terms on the left with $K_m(\mu^*_{Z_0};u,v)$ (see (6.8) for the definition of $\mu^*_{Z_0}$) via (3.2) (this produces an error term $o_\theta(1)$ on the right but $\theta>0$ is arbitrarily small and the left-hand side is independent of θ), and

$$\frac{1}{m}(K_m(\nu_{Z_m};u-s_m,v-s_m)-K_m(\nu_{Z_0};u-s_m,v-s_m))=o_\varepsilon(1)+o(1)$$

is obtained by comparing the two terms on the left with $K_m(\nu^*_{Z_0};u-s_m,v-s_m)$ via (3.2). This is possible because all the appearing Christoffel functions converge to the appropriate limits. Indeed, for the convergence of the Christoffel functions with fixed weight just apply Theorem 3.1, and for $\lambda_m(\mu_{Z_m};x+a/m)$ and $\lambda_m(\nu_{Z_m};x+a/m)$ this is the content of Lemma 6.1 below. Finally, (6.9) is obtained from these and from (5.10) applied to the fixed measures μ_{Z_0} and ν_{Z_0}. This completes the proof of Theorem 2.1 pending the proof of Lemma 6.1 below.

Lemma 6.1.

$$\lim_{m\to\infty} m\lambda_m\Big(\mu_{Z_m};x+\frac{a}{m}\Big)=\frac{w(x)(x-z_0)^{2k}}{\omega_E(x)} \tag{6.10}$$

and

$$\lim_{m\to\infty} m\lambda_m\Big(\nu_{Z_m};x+\frac{a}{m}\Big)=\frac{w(x)(x-z_0)^{2k}}{\omega_F(x)} \tag{6.11}$$

uniformly in $S\cap I$.

Proof. Let E_1 be what we obtain from E by omitting a small interval around z_0. If this interval is small enough, then

$$\omega_E(x)\le\omega_{E_1}(x)\le\omega_E(x)+\theta,\quad x\in I,$$

whatever $\theta>0$ is given (see Lemmas 4.1 and 4.2). Let μ_1 be the restriction of $(1-\theta)\mu_{Z_0}$ to E_1 (see (6.2) and (6.6) for the definition of the measure μ_{Z_0}). Then, for large m, we have $\mu_{Z_m}\ge\mu_1$, so

$$\liminf_{m\to\infty} m\lambda_m\Big(\mu_{Z_m};x+\frac{a}{m}\Big)\ge\lim_{m\to\infty} m\lambda_m\Big(\mu_1;x+\frac{a}{m}\Big)=\frac{(1-\theta)w_{Z_0}(x)}{\omega_{E_1}(x)}$$

uniformly in $x\in S$ and locally uniformly in $a\in\mathbf{R}$, where we used Theorem 3.1 for μ_1 at the equality (the measure μ_1 is automatically regular). Now for $\theta\to 0$ we obtain that

$$\liminf_{m\to\infty} m\lambda_m\left(\mu_{Z_m};x+\frac{a}{m}\right)\geq\frac{w_{Z_0}(x)}{\omega_E(x)}.$$

Next let

$$d\mu^*_{Z_0}(x)=d\mu_{Z_0}(x)+\theta\,d\mu(z)=((x-z_0)^{2j}+\theta)\,d\mu(x)$$

be the measure (6.8). For large m we have $\mu_{Z_m}\leq\mu^*_{Z_0}$, hence

$$\limsup_{m\to\infty} m\lambda_m\left(\mu_{Z_m};x+\frac{a}{m}\right)\leq\frac{w_{Z_0}(x)}{\omega_E(x)}$$

follows if we apply Theorem 3.1 to $\mu^*_{Z_0}$ and let $\theta\to 0$. All these relations are uniform in $x\in S$ and locally uniform in $a\in\mathbf{R}$, and therefore (6.10) follows.

In a completely similar way, from the uniform convergence of w_{Z_m} to w_{Z_0} on F_{i^*} it follows that $(1-\theta)\nu_{Z_0}\leq\nu_{Z_m}\leq(1+\theta)\nu_{Z_0}$, for large m, and so (6.11) is a consequence of

$$\lim_{m\to\infty} m\lambda_m\left(\nu_{Z_0};x+\frac{a}{m}\right)=\frac{w_{Z_0}(x)}{\omega_F(x)}$$

(see Theorem 3.1) by comparison. □

7. Proof of Theorem 2.2

For $x\in S$ it follows from (5.18) that

$$p_{nN}\left(\mu;x+\frac{b}{nN}\right)$$

has the form

$$c_{nN}(x)[\sin(\pi\omega_E(x)b+\sigma_{nN}(x))+o_\varepsilon(1)+o(1)]$$

with some $c_{nN}(x)=c_{nN,\varepsilon}(x)$ and $\sigma_{nN}(x)=\sigma_{nN,\varepsilon}(x)$ and with $o_\varepsilon(1)+o(1)$ uniform in $x\in S$ and $|b|\leq B$ for any fixed B. For a fixed $j=0,1,...,N-1$ using the analogous argument from the preceding section with $\mu_{Z_{nN}}$ we similarly obtain that

$$p_{nN+j}\left(\mu;x+\frac{b}{nN}\right)$$

is

$$\left(\prod_{k=1}^{j}(x-z_{nN+j,k})\right)c_{nN}^{(j)}(x)[\sin(\pi\omega_E(x)b+\sigma_{nN}^{(j)}(x))+o_\varepsilon(1)+o(1)]$$

with some $c_{nN}^{(j)}(x)=c_{nN,\varepsilon}^{(j)}(x)$ and $\sigma_{nN}^{(j)}(x)=\sigma_{nN,\varepsilon}^{(j)}(x)$, and with $o_\varepsilon(1)+o(1)$ uniform in $x\in S$ and $|b|\le B$. We can simply write all these as

$$p_m\left(\mu;x+\frac{b}{m}\right)=c_m(x)[\sin(\pi\omega_E(x)b+\sigma_m(x))+o_\varepsilon(1)+o(1)]$$

with the same uniformity range for $o_\varepsilon(1)+o(1)$ as before. Note however, that here $c_m(x)$ and $\sigma_m(x)$ depend on $\varepsilon>0$ (via the choice $T=T_N$). Simple trigonometry shows that then for any $a,b\in[-B,B]$,

$$\begin{aligned}\frac{1}{m}K_m&\left(\mu;x+\frac{a}{m},x+\frac{b}{m}\right)\\&=a_m(\mu)\frac{p_m\left(x+\frac{a}{m}\right)p_{m-1}\left(x+\frac{b}{m}\right)-p_{m-1}\left(x+\frac{a}{m}\right)p_m\left(x+\frac{b}{m}\right)}{a-b}\end{aligned}$$

with $p_m(z)=p_m(\mu;z)$ of the form

$$a_m(\mu)c_m(x)c_{m-1}(x)\left[\frac{\sin\pi\omega_E(x)(a-b)}{a-b}\sin(\sigma_{m-1}(x)-\sigma_m(x))+\frac{o_\varepsilon(1)+o(1)}{a-b}\right].$$

Let here $m=nN$. When $x+a/m$ is a zero of $p_m(\mu;z)$, then

$$\frac{(b-a)K_m\left(\mu;x+\frac{a}{m},x+\frac{b}{m}\right)}{m}=a_m(\mu)p_{m-1}\left(\mu;x+\frac{a}{m}\right)p_m\left(\mu;x+\frac{b}{m}\right),$$

for which we have another asymptotic formula in (5.18). These two asymptotic formulae give (note that b is still a free variable)

$$a_m(\mu)c_m(x)c_{m-1}(x)\sin(\sigma_{m-1}(x)-\sigma_m(x))=\frac{1}{\pi w(x)}+o_\varepsilon(1)+o(1),$$

and hence

$$\frac{1}{nN}K_{nN}\left(\mu;x+\frac{a}{nN},x+\frac{b}{nN}\right)-\frac{\sin\pi\omega_E(x)(a-b)}{\pi(a-b)}\frac{1}{w(x)}=\frac{o_\varepsilon(1)+o(1)}{a-b},\tag{7.1}$$

which is essentially what we want to prove but only for the indices nN.

Next we show that (7.1) implies the same for all other indices. Using the Cauchy–Schwarz inequality we get for $1\le j<N$ from the definition of K_n and λ_n in (1.2) and (3.1) that

$$\begin{aligned}
|K_{nN+j}(\mu;u,v)-K_{nN}(\mu;u,v)| \\
&=\Bigl|\sum_{k=nN+1}^{nN+j} p_k(\mu;u)p_k(\mu;v)\Bigr| \\
&\le\Bigl(\sum_{k=nN+1}^{nN+j} p_k^2(\mu;u)\Bigr)^{1/2}\Bigl(\sum_{k=nN+1}^{nN+j} p_k^2(\mu;uv)\Bigr)^{1/2} \\
&=\bigl(\lambda_{nN+j+1}^{-1}(\mu;u)-\lambda_{nN+1}^{-1}(\mu;u)\bigr)^{1/2}\bigl(\lambda_{nN+j+1}^{-1}(\mu;v)-\lambda_{nN+1}^{-1}(\mu;v)\bigr)^{1/2}.
\end{aligned}$$

Setting here $u=x+a/(nN+j)$ and $v=x+b/(nN+j)$, it follows from the limit relations (see Theorem 3.1) that

$$\lim_{n\to\infty} nN\lambda_{nN+j+1}(\mu;u)=\lim_{n\to\infty} nN\lambda_{nN+1}(\mu;u)=\frac{w(x)}{\omega_E(x)}$$

and from similar relations for v that the difference

$$\frac{1}{nN}K_{nN+j}\Bigl(\mu;x+\frac{a}{nN+j},x+\frac{b}{nN+j}\Bigr)-\frac{1}{nN}K_{nN}\Bigl(\mu;x+\frac{a}{nN+j},x+\frac{b}{nN+j}\Bigr)$$

is $o(1)$ (as $n\to\infty$) uniformly in $x\in S$ and a,b lying in compact subsets of $\mathbf{R}$. Here we can apply for the second term on the right the asymptotic formula in (7.1) (with a and b there replaced by a' and b' for which $a'/nN=a/(nN+j)$ and $b'/nN=b/(nN+j)$) and we can conclude (7.1) for the indices $nN+j$ instead of nN.

In summary, we have for all indices n as $n\to\infty$,

$$\frac{1}{n}K_n\Bigl(\mu;x+\frac{a}{n},x+\frac{b}{n}\Bigr)-\frac{\sin\pi\omega_E(x)(a-b)}{\pi(a-b)}\frac{1}{w(x)}=\frac{o_\varepsilon(1)+o(1)}{a-b}. \tag{7.2}$$

Here $o_\varepsilon(1)+o(1)$ is uniform in $x\in S$ and a and b lying in compact subsets of $\mathbf{R}$. The left-hand side is independent of ε, so the $(o_\varepsilon(1)+o(1))$-term is simply $o(1)$. This is Theorem 2.2 when $a-b$ stays away from 0.

Next we deduce the statement in Theorem 2.2 from Theorem 3.1 when $a-b$ is close to zero. Indeed, consider the polynomials

$$Q_n(u)=Q_{n,x}(u)=\frac{1}{n}K_n\Bigl(\mu;x+\frac{a}{n},x+u\Bigr).$$

From the definition of K_n and λ_n in (1.2) and (3.1) and from Cauchy's inequality we can infer that

$$|Q_n(u)| \le \frac{1}{n}\lambda_n\Big(\mu; x+\frac{a}{n}\Big)^{-1/2}\lambda_n(\mu; x+u)^{-1/2},$$

so there are a $d, D>0$ such that if $x\in S$ then $|Q_n(u)|\le D$ for all $u\in[-d,d]$ (recall that in a neighborhood of S the density w is bounded away from 0, hence the claimed boundedness follows from the monotonicity in the weight of the Christoffel functions and from Theorem 3.1 – applied to some minorizing measure that is constant on the neighborhood in question). By Bernstein's inequality on the derivative of polynomials (see e.g. [5, Corollary 4.1.2]) we then have $|Q_n'(u)|\le 2nD/d$ for $|u|\le d/2$, hence

$$\frac{1}{n}\Big|K_n\Big(\mu; x+\frac{a}{n}, x+\frac{b}{n}\Big)-K_n\Big(\mu; x+\frac{a}{n}, x+\frac{a}{n}\Big)\Big| \le \frac{2D}{d}|b-a| \tag{7.3}$$

uniformly in $x\in S$ and locally uniformly in $a, b\in\mathbf{R}$ and for large n for which we have $|b-a|/n<d/2$.

Given $\varepsilon>0$ we have by Theorem 3.1,

$$\Big|\frac{1}{n}K_n\Big(\mu; x+\frac{a}{n}, x+\frac{a}{n}\Big)-\frac{\omega_E(x)}{w(x)}\Big| < \varepsilon \tag{7.4}$$

for $n\ge n_0$ with some n_0, uniformly in the aforementioned range. Finally, there is a $\delta>0$ such that for $|b-a|<\delta$ we have

$$\Big|\frac{\omega_E(x)}{w(x)}-\frac{\sin\pi\omega_E(x)(a-b)}{\pi(a-b)}\frac{1}{w(x)}\Big| < \varepsilon \tag{7.5}$$

uniformly in $x\in S$. The inequalities (7.3)–(7.5) show that if $|b-a|<\min\{\delta, \varepsilon d/2D\}$ and $n\ge n_0$ then

$$\Big|\frac{1}{n}K_n\Big(\mu; x+\frac{a}{n}, x+\frac{b}{n}\Big)-\frac{\sin\pi\omega_E(x)(a-b)}{\pi(a-b)}\frac{1}{w(x)}\Big| < 3\varepsilon$$

provided $x\in S$ and a is lying in some compact subset of the real line.

For $|b-a|\ge\min\{\delta, \varepsilon d/2D\}$ we can use (7.2) to deduce for all large n, say $n\ge n_1$, the same inequality with 3ε replaced by $o_\varepsilon(1)+\varepsilon$. Thus,

$$\frac{1}{n}K_n\Big(\mu; x+\frac{a}{n}, x+\frac{b}{n}\Big) \to \frac{\sin\pi\omega_E(x)(a-b)}{\pi(a-b)}\frac{1}{w(x)}$$

uniformly in the specified range.

8. Proof of Theorem 3.1

It was proved in [19] that if ν is a regular measure with support $E\subset\mathbf{R}$, E is regular with respect to the Dirichlet problem (in $\mathbf{R}\setminus E$) and O is an open subset of E on which μ is absolutely continuous and its density w is continuous, then

$$\lim_{n\to\infty} n\lambda_n(\nu;x)=\frac{w(x)}{\omega_E(x)} \tag{8.1}$$

locally uniformly in O. In [19] the emphasis was on proving (8.1) a.e. in an interval I, provided $\log w\in L^1(I)$ (and μ is regular), but the proof gives the (easier) limit (8.1) under a continuity condition, and also the local uniformity of this limit within O.

Next we get rid of the regularity of E. Let $\varepsilon>0$ and let H,H', $H\subset\operatorname{Int}(H')$ be compact subsets of O consisting of finitely many intervals. By Ancona's theorem [1] for every $\varepsilon>0$ there is an $E_1\subset E$ which is regular with respect to the Dirichlet problem and $\operatorname{cap}(E_1)>\operatorname{cap}(E)-\eta$. We may clearly assume that $H'\subset\operatorname{Int}(E_1)$. This implies for small η that (see Lemmas 4.1 and 4.2)

$$\omega_E(x)\le\omega_{E_1}(x)\le\omega_E(x)+\varepsilon$$

uniformly on H'. If ν_1 is the restriction of ν to E_1, then we can apply (8.1) to E_1, and hence

$$\liminf_{n\to\infty} n\lambda_n(\nu;x)\ge\lim_{n\to\infty} n\lambda_n(\nu_1;x)=\frac{w(x)}{\omega_{E_1}(x)}\ge\frac{w(x)}{\omega_E(x)+\varepsilon}$$

uniformly on H, which, for $\varepsilon\to0$, gives

$$\liminf_{n\to\infty} n\lambda_n(\nu;x)\ge\frac{w(x)}{\omega_E(x)}. \tag{8.2}$$

Next, let $E\subset E_2$, where E_2 consists of finitely many intervals and is such that $\operatorname{cap}(E_2)\le\operatorname{cap}(E)+\eta$. Then for small η (see Lemmas 4.1 and 4.2)

$$\omega_E(x)-\varepsilon\le\omega_{E_2}(x)\le\omega_E(x)$$

uniformly in H'. Extend w from H' to a continuous function on E_2 such that it is positive on $E_2\setminus H'$. Then $w_2(x)\,dx$ is a regular measure, hence

$$d\nu_2(x)=w_2(x)\,dx+d\nu|_{E_2\setminus H'}$$

is also regular, and bigger than ν. On applying (8.1) to ν_2 it follows that

$$\limsup_{n\to\infty} n\lambda_n(\nu;x)\le\lim_{n\to\infty} n\lambda_n(\nu_2;x)=\frac{w(x)}{\omega_{E_2}(x)}\le\frac{w(x)}{\omega_E(x)-\varepsilon},$$

and on letting $\varepsilon\to 0$ we get

$$\limsup_{n\to\infty} n\lambda_n(\nu;x)\le \frac{w(x)}{\omega_E(x)}$$

uniformly on H. Since here $H\subset O$ is arbitrary, this and (8.2) show that, indeed, in (8.1), E need not be regular.

Finally, we turn to the proof of the theorem. By the continuity of w on S there are positive continuous functions w_1 and w_2 on some open neighborhood O of S such that $w=w_1=w_2$ on S and $w_1\le w\le w_2$ on O. We may also suppose that O consists of finitely many intervals and its closure $\overline{O}$ is part of an open set where μ is absolutely continuous. By the localization theorem [18, Theorem 5.3.3] the measures $\mu|_{\overline{O}}$ and $\mu|_{E\setminus O}$ are regular, and of course $w_1(x)\,dx$ and $w_2(x)\,dx$ are also regular on $\overline{O}$. Hence, again by the localization theorem of [18], $\mu_1=w_1(x)\,dx+\mu|_{E\setminus O}$ and $\mu_2=w_2(x)\,dx+\mu|_{E\setminus O}$ are regular measures with support E, and $\mu_1\le\mu\le\mu_2$. According to what we have shown above, as $n\to\infty$,

$$n\lambda_n(\mu_1;y)\to\frac{w_1(y)}{\omega_E(y)}\quad\text{and}\quad n\lambda_n(\mu_2;y)\to\frac{w_2(y)}{\omega_E(y)}$$

uniformly on any H, where $S\subset\operatorname{Int}(H)$ and H is a compact subset of O. If $a\in[-A,A]$ with some A and $x\in S$, then for large n the point $y=x+a/n$ belongs to H. Therefore, with some n_0, for $n\ge n_0$ we have

$$n\lambda_n\Big(\mu_1;x+\frac{a}{n}\Big)\ge\frac{w_1\Big(x+\dfrac{a}{n}\Big)}{\omega_E\Big(x+\dfrac{a}{n}\Big)}-\varepsilon\ge\frac{w_1(x)}{\omega_E(x)}-2\varepsilon,$$

and similarly

$$n\lambda_n\Big(\mu_2;x+\frac{a}{n}\Big)\le\frac{w_2\Big(x+\dfrac{a}{n}\Big)}{\omega_E\Big(x+\dfrac{a}{n}\Big)}+\varepsilon\le\frac{w_2(x)}{\omega_E(x)}+2\varepsilon.$$

Now the claim in the theorem follows from these inequalities, the fact that $w_1(x)=w_2(x)=w(x)$, $x\in S$, and from

$$\lambda_n(\mu_1;z)\le\lambda_n(\mu;z)\le\lambda_n(\mu_2;z).$$

9. Proof of Theorem 3.2

This is much like the proof of Theorem 3.1 if one uses the result of Findley [6, Theorem 11] on asymptotics of Christoffel functions for regular measures on $[-1,1]$

under local Szegő condition. One uses polynomial mappings and appproximation exactly as in [19] (together with the fact that the polynomial maps – actually C^1-maps – in question preserve Lebesgue points). We skip the details.

10. Proof of Theorem 2.3

Let μ_0 be a measure that is $w(x_0)\,dx$ in a small neighborhood I of x_0 and agrees with μ outside this neighborhood. This is a regular measure in view of the localization theorem of [18, Theorem 5.3.3].

Then, by Theorems 2.1 and 2.2 the statements in the theorem hold with μ_0 in place of μ (note that μ_0 has constant density in I). We shall use this μ_0 and Lubinsky's comparison method to deduce the conclusion for μ. First we deal with (2.2).

Let $\mu^*=\max\{w(x_0),w(x)\}\,dx+d\mu_s$ in the aforementioned neighborhood I of x_0 and let $\mu^*=\mu$ outside this neighborhood. Then $\mu\le\mu^*$ and $\mu_0\le\mu^*$, and clearly x_0 is a Lebesgue point for the Radon–Nikodym derivative $w^*(x)=\max\{w(x_0),w(x)\}$ of μ^*. This μ^* is automatically regular since $\mu\le\mu^*$ and μ and μ^* have equal support.

For either of $\sigma=\mu,\mu_0,\mu^*$ we can apply Theorem 3.2:

$$\lim_{n\to\infty} n\lambda_n\Big(\sigma;x_0+\frac{a}{n}\Big)=\frac{w(x_0)}{\omega_E(x_0)}.$$

If we substitute this for $\mu_0\le\mu^*$ and $\mu\le\mu^*$ into Lubinsky's inequality (3.2), then we get

$$\frac{1}{n}K_n\left(\mu;x_0+\frac{a}{n},x_0+\frac{b}{n}\right)-\frac{1}{n}K_n\left(\mu_0;x_0+\frac{a}{n},x_0+\frac{b}{n}\right)=o(1)$$

uniformly in $a\in[-L/n,L/n]$ for any fixed L. Now we can deduce (2.2) since, as we have just mentioned, it holds for μ_0.

Once (2.2) is established, (2.1) is easy to deduce if we set $a=a_n$ into (2.2) so that x_0+a/n is a zero of $p_n(\mu;z)$ (which is the original argument of [10]). Lemma 4.4 tells us that this is possible for some $|a_n|\le 9/\omega_E(x_0)$ provided n is sufficiently large. In this case (2.2) takes the form

$$a_n(\mu)\frac{p_{\mu;n-1}\Big(x_0+\dfrac{a}{n}\Big)p_n\left(\mu;x_0+\dfrac{b}{n}\right)}{a-b}+\frac{\sin\pi\omega_E(x_0)(a-b)}{\pi(a-b)w(x_0)}=o(1),$$

so with the argument applied after (5.18) we obtain that for any fixed $\delta>0$ there is a zero of $p_n(\mu;z)$ in

$$x_0+\frac{1}{n}\left[a+\frac{1}{\omega_E(x_0)}-\delta,\,a+\frac{1}{\omega_E(x_0)}+\delta\right].$$

Lemma 4.4(i) gives that there is only one zero in this interval for small δ and then this is the smallest zero lying to the right of x_0+a/n. In a similar fashion, the largest zero lying to the left of x_0+a/n lies in the interval

$$x_0+\frac{1}{n}\left[a-\frac{1}{\omega_E(x_0)}-\delta, a-\frac{1}{\omega_E(x_0)}+\delta\right].$$

Repeating the same argument with any zero x_0+a/n of $p_n(\mu; z)$ lying in the interval $[x_0-L/n, x_0+L/n]$, the limit (2.1) follows, since $\delta>0$ is arbitrary here.

References

1. Ancona, A., Démonstration d'une conjecture sur la capacité et l'effilement, *C. R. Acad. Sci. Paris Sér. I Math.* **297** (1983), 393–395.
2. Carleson, L., *Selected Problems on Exceptional Sets*, Van Nostrand Mathematical Studies **13**, Van Nostrand, Princeton, NJ, 1967.
3. Deift, P. A., *Orthogonal Polynomials and Random Matrices: a Riemann–Hilbert Approach*, Courant Lecture Notes in Mathematics **3**, Courant Institute of Mathematical Sciences, New York University, New York, 1999.
4. Deift, P. A., Kriecherbauer, T., McLaughlin, K. T. R., Venakides, S. and Zhou, X., Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* **52** (1999), 1335–1425.
5. DeVore, R. A. and Lorentz, G. G., *Constructive Approximation*, Grundlehren der Mathematischen Wissenschaften **303**, Springer, Berlin–Heidelberg, 1993.
6. Findley, M., Universality for locally Szegő measures, to appear in *J. Approx. Theory.*
7. Geronimo, J. S. and Van Assche, W., Orthogonal polynomials on several intervals via a polynomial mapping, *Trans. Amer. Math. Soc.* **308** (1988), 559–581.
8. Hoffman, K., *Banach Spaces of Analytic Functions*, Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.
9. Kuijlaars, A. B. J. and Vanlessen, M., Universality for eigenvalue correlations from the modified Jacobi unitary ensemble, *Int. Math. Res. Not.* **2002**:30 (2002), 1575–1600.
10. Levin, A. L. and Lubinsky, D. S., Applications of universality limits to zeros and reproducing kernels of orthogonal polynomials, *J. Approx. Theory* **150** (2008), 69–95.
11. Levin, A. L. and Lubinsky, D. S., Universality limits for exponential weights, to appear in *Constr. Approx.* doi:10.107/s00365-008-9020-4
12. Lubinsky, D. S., A new approach to universality limits involving orthogonal polynomials, to appear in *Annals of Math.*
13. Mehta, M. L., *Random Matrices*, Academic Press, Boston, MA, 1991.
14. Rudin, W., *Real and Complex Analysis*, McGraw-Hill, New York, 1987.
15. Saff, E. B. and Totik, V., What parts of a measure's support attract zeros of the corresponding orthogonal polynomials?, *Proc. Amer. Math. Soc.* **114** (1992), 185–190.

16. Saff, E. B. and Totik, V., *Logarithmic Potentials with External Fields*, Grundlehren der Mathematischen Wissenschaften **316**, Springer, Berlin–Heidelberg, 1997.
17. Simon, B., Two extensions of Lubinsky's universality theorem, to appear in *J. Anal. Math.*
18. Stahl, H. and Totik, V., *General Orthogonal Polynomials*, Encyclopedia of Mathematics and its Applications **43**, Cambridge University Press, Cambridge, 1992.
19. Totik, V., Asymptotics for Christoffel functions for general measures on the real line, *J. Anal. Math.* **81** (2000), 283–303.
20. Totik, V., Polynomial inverse images and polynomial inequalities, *Acta Math.* **187** (2001), 139–160.

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