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Residue calculus for c-holomorphic functions

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Abstract. In this paper we introduce Coleff–Herrera residue currents defined by systems of c-holomorphic functions and prove a Lelong–Poincaré and a Cauchy-type formula as well as the transformation law for these currents.

1. Preliminaries

In complex analysis one often comes across what is called *weakly holomorphic* functions. These functions appear in a natural way e.g. in problems related to Abel's or Lie–Griffiths' theorem – see [HP]. They are defined and holomorphic on the regular part of a (complex) analytic set and locally bounded near the singularities. However, they are not as *handy* as one would like them to be.

Among other possible notions of 'holomorphicity' for functions defined on analytic sets there is one which is of greater interest and was introduced by Remmert (see [R]). Let A be an analytic subset of an open set $\Omega\subset\mathbb{C}^m$.

Definition 1.1. ([W]) A mapping $f\colon A\to\mathbb{C}^n$ is called *c-holomorphic* if it is continuous and the restriction of f to the subset $\operatorname{Reg} A$ of regular points is holomorphic. We denote by $\mathcal{O}_c(A,\mathbb{C}^n)$ the ring of c-holomorphic mappings, and by $\mathcal{O}_c(A)$ the ring of c-holomorphic functions.

This happens to be a very good generalization of holomorphic functions on analytic sets. It is well-known that a mapping defined in an open set is holomorphic if and only if it is continuous and its graph is an analytic set (it is then a submanifold). We have a similar result for c-holomorphic mappings (cf. [W, Theorem 4.5Q]), which motivates this generalization:

Theorem 1.2. *A mapping $f\colon A\to\mathbb{C}^n$ is c-holomorphic if and only if it is continuous and its graph $\Gamma_f:=\{(x,f(x)):x\in A\}$ is an analytic subset of $\Omega\times\mathbb{C}^n$.*

It is worth noting that by a recent result of N. V. Shcherbina [S] the pluripolarity of the graph is sufficient (unlike for instance sub- or semianalyticity: $f(x):=|x|$ for $x\in\mathbb{C}$ has a semianalytic graph which is not complex analytic). By Theorem 1.2 the zero set of a c-holomorphic function is analytic.

Throughout this paper we assume that $A\subset\Omega$ is a purely k-dimensional analytic set in an open set $\Omega\subset\mathbb{C}^m$.

Note that in general we have only an inclusion $\Gamma_{f|_{\mathrm{Reg}\,A}}\subset\mathrm{Reg}\,\Gamma_f$. However, since the $(2k-1)$-dimensional Hausdorff measure of $\Gamma_{f|_{\mathrm{Sng}\,A}}$ is zero, we may always replace the set $\mathrm{Reg}\,\Gamma_f$ by $\Gamma_{f|_{\mathrm{Reg}\,A}}$ in the approximating integrals from the next section.

2. Residue currents defined by c-holomorphic functions

Let $f\in\mathcal{O}_c(A)$ be such that it does not vanish identically on any irreducible component of A. The aim of this part is to define, following an idea of A. Yger, a *residue current* which would generalize to the c-holomorphic case the *restricted residue current* of Coleff–Herrera $[A]\wedge\bar\partial[1/f]$ (see [CH] and [TY]). Were f holomorphic in Ω, we would have for any $\varphi\in\mathcal{D}_{(k,k-1)}(\Omega)$ by the definition of Coleff and Herrera:

$$\left\langle [A]\wedge\bar\partial\left[\frac{1}{f}\right],\varphi\right\rangle := \lim_{\varepsilon\to0^+}\frac{1}{2\pi i}\int_{\{z\in\mathrm{Reg}\,A:|f(z)|^2=\varepsilon\}}\frac{\varphi}{f}$$
$$= -\lim_{\varepsilon\to0^+}\frac{1}{2\pi i}\int_{\{z\in\mathrm{Reg}\,A:|f(z)|^2>\varepsilon\}}\frac{\bar\partial\varphi}{f}.$$

The current we obtain is $\bar\partial$-closed and supported by $A\cap f^{-1}(0)$. It is a deep result that such a current is well-defined – actually, we are concealing here the problem of the existence of the current $\bar\partial[1/f]$, i.e. the $\bar\partial$ of the *principal value* current $[1/f](\varphi):=(2\pi i)^{-1}\lim_{\varepsilon\to0^+}\int_{\{z:|f(z)|^2>\varepsilon\}}\varphi/f$ solving the equation $(2\pi i)f\mathfrak{t}=1$ in Ω.

We introduce the notation

$$\mathrm{Res}\begin{bmatrix}\varphi(z)\\ f(z)\end{bmatrix}_A := \left\langle [A]\wedge\bar\partial\left[\frac{1}{f}\right],\varphi\right\rangle,\quad \varphi\in\mathcal{D}_{(k,k-1)}(\Omega).$$

When $A=\Omega$ we simply omit it in the subscript since it does not interfere with anything. Note that the Lelong–Poincaré formula says in particular that if the hypersurface $X=\{z:g(z)=0\}$ is given by a reduced analytic equation, then

$$\text{(LP)}\qquad \langle [X],\varphi\rangle = \mathrm{Res}\begin{bmatrix}dg\wedge\varphi\\ g\end{bmatrix}.$$

Observe that the above equality can be rewritten as $[X]=\bar\partial[1/g]\wedge dg$ (since one has $(2\pi i)^{-1}\bar\partial\partial\log|g|^2=\bar\partial[1/g]\wedge dg$ in the sense of currents).

Now, if f is just c-holomorphic on A we define a residue current of type $(m-k,m-k+1)$ setting

$$(*)\qquad \operatorname{Res}\begin{bmatrix}\varphi(z)\\ f(z)\end{bmatrix}_A := \operatorname{Res}\begin{bmatrix}\varphi(z)\\ w\end{bmatrix}_{\Gamma_f},\quad \varphi\in\mathcal{D}_{(k,k-1)}(\Omega),$$

where $(z,w)\in\Gamma_f\subset\Omega\times\mathbb{C}$ (i.e. on the right-hand side we have $[\Gamma_f]\wedge\bar\partial[1/p]$, where $p(z,w)=w$; note that the graph is also purely k-dimensional). In other words, we use the graph to properly define the current

$$\operatorname{Res}\begin{bmatrix}\varphi(z)\\ f(z)\end{bmatrix}_A.$$

This definition makes sense in that it coincides with the usual one when f is holomorphic as we will see in the proof of the following theorem. Note that there is no problem of support relative to the vertical variable w since we may successfully replace the form $\varphi(z)$ in $(*)$ by $\varphi(z)\cdot\theta(w)$, where $\theta(w)$ is a $\mathcal{C}^\infty$ function with compact support, identically equal to 1 on a ball of radius $r>\max_{z\in\operatorname{supp}\varphi}|f(z)|$. For simplicity we will omit writing these cut-off functions (this does not affect the proofs – to illustrate this we are keeping this extra function in the first proof below).

The Coleff–Herrera residue is also defined for n functions in the proper intersection case. Namely, if $f_1,...,f_n\in\mathcal{O}(\Omega)$ are such that $A\cap\bigcap_{j=1}^n f_j^{-1}(0)$ has pure dimension $k-n$, then for any $\varphi\in\mathcal{D}_{(k,k-n)}(\Omega)$,

$$\operatorname{Res}\begin{bmatrix}\varphi\\ f_1,...,f_n\end{bmatrix}_A := \lim_{\delta\to0^+}\left(\frac{1}{2\pi i}\right)^n\int_{\operatorname{Reg}A\cap T_\delta(f)}\frac{\varphi}{f_1\cdot...\cdot f_n},$$

where $T_\delta(f)=\{z:|f_j(z)|^2=\varepsilon_j(\delta), j=1,...,n\}$ and $\varepsilon_1\ll...\ll\varepsilon_n$ are special functions tending to zero with δ (along what is called an *admissible path*:

$$\lim_{\delta\to0^+}\frac{\varepsilon_j(\delta)}{\varepsilon_{j+1}(\delta)^p}=0\quad\text{for all } p\in\mathbb{N},\ j=1,...,n-1;$$

and the limit is independent of the choice of the admissible path), is a well-defined current of type $(m-k,m-k+n)$. For more information see [CH] and [TY]. Note that the actual ordering of $\{f_1,...,f_n\}$ is important.

It is quite easy to extend this notion of residue current to the case of a c-holomorphic mapping $f=(f_1,...,f_n)\colon A\to\mathbb{C}^n_w$ defining a proper intersection on A, i.e. $f^{-1}(0)$ is of pure dimension $k-n$ (then Γ_f and $\Omega\times\{0\}$ intersect properly in $\mathbb{C}^m\times\mathbb{C}^n$). We put

$$(**)\qquad \operatorname{Res}\begin{bmatrix}\varphi(z)\\ f_1(z),...,f_n(z)\end{bmatrix}_A := \operatorname{Res}\begin{bmatrix}\varphi(z)\\ w_1,...,w_n\end{bmatrix}_{\Gamma_f},\quad \varphi\in\mathcal{D}_{(k,k-n)}(\Omega),$$

getting a current of type $(m-k, m-k+n)$. It is a natural generalization of the Coleff–Herrera restricted residue current $[A]\wedge\bar\partial[1/f_1]\wedge\ldots\wedge\bar\partial[1/f_n]$ to the case of c-holomorphic functions since we have the following result.

Theorem 2.1. *If the function f (respectively the mapping $f=(f_1,\ldots,f_n)$) is holomorphic in Ω, then equality $(*)$ (respectively $(**)$) holds.*

Proof. If we look at the approximating integrals, it is clear that the residue depends only on the values of f on $\operatorname{Reg} A$, i.e. if $g\in\mathcal{O}(\Omega,\mathbb{C}^n_w)$ is such that $g=f$ on $\operatorname{Reg} A$, then

$$\operatorname{Res}\begin{bmatrix} \cdot \\ f_1,\ldots,f_n\end{bmatrix}_A = \operatorname{Res}\begin{bmatrix} \cdot \\ g_1,\ldots,g_n\end{bmatrix}_A.$$

Now, if we consider $f_1,\ldots,f_n$ as holomorphic in $\Omega\times\mathbb{C}^n$, then for any test form $\varphi\in\mathcal{D}_{(k,k-n)}(\Omega)$,

$$\operatorname{Res}\begin{bmatrix} \varphi(z) \\ f_1(z),\ldots,f_n(z)\end{bmatrix}_A = \operatorname{Res}\begin{bmatrix} \varphi(z)\theta(w) \\ f_1(z,w),\ldots,f_n(z,w)\end{bmatrix}_{A\times\{0\}^n}$$

with some cut-off function θ equal to 1 in a neighbourhood of zero (the intersection $\Gamma_{f(z,w)}\cap\Omega\times\mathbb{C}^n\times\{0\}^n$ is still proper).

Let $\Phi\colon\Omega\times\mathbb{C}^n\to\Omega\times\mathbb{C}^n$ be the biholomorphism $\Phi(z,w)=(z,f(z)+w)$. We clearly have $\Phi(A\times\{0\}^n)=\Gamma_f$. But

$$\operatorname{Res}\begin{bmatrix} \varphi(z)\theta(w) \\ f_1,\ldots,f_n\end{bmatrix}_{A\times\{0\}^n} = \operatorname{Res}\begin{bmatrix} \varphi(z)\theta(w) \\ f_1+w_1,\ldots,f_n+w_n\end{bmatrix}_{A\times\{0\}^n}$$

since $w=0$ on $A\times\{0\}^n$. By the change of variables theorem we obtain

$$\operatorname{Res}\begin{bmatrix} \varphi(z)\theta(w) \\ f_1+w_1,\ldots,f_n+w_n\end{bmatrix}_{A\times\{0\}^n} = \operatorname{Res}\begin{bmatrix} \varphi(z)\theta(w-f(z)) \\ w_1,\ldots,w_n\end{bmatrix}_{\Gamma_f}$$

as wanted. $\square$

One more thing is perhaps worth noting. Throughout the paper we are using to a great extent a change of variables theorem for residue currents. Such a theorem is valid also in the case of c-holomorphic functions. It may be stated in the following form.

Lemma 2.2. *Let $\Phi\colon\Omega\to\Omega'$ be a biholomorphism between open subsets of $\mathbb{C}^m$. Let $A\subset\Omega$ be analytic and purely k-dimensional. If $f=(f_1,\ldots,f_n)\in\mathcal{O}_c(\Phi(A),\mathbb{C}^n)$ is such that $f^{-1}(0)$ has pure dimension $k-n$, then*

$$\operatorname{Res}\begin{bmatrix} \varphi \\ f_1,\ldots,f_n\end{bmatrix}_{\Phi(A)} = \operatorname{Res}\begin{bmatrix} \Phi^*\varphi \\ f_1\circ\Phi,\ldots,f_n\circ\Phi\end{bmatrix}_A$$

for any test form $\varphi\in\mathcal{D}_{(k,k-n)}(\Omega')$.

Proof. By definition,

$$\mathrm{Res}\begin{bmatrix}\varphi\\ f_1,\dots,f_n\end{bmatrix}_{\Phi(A)}=\mathrm{Res}\begin{bmatrix}\varphi\\ w_1,\dots,w_n\end{bmatrix}_{\Gamma_f}.$$

Besides, $\Psi:=\Phi\times\mathrm{Id}_{\mathbb{C}^n}$ is a biholomorphism such that $\Psi(\Gamma_{f\circ\Phi|_A})=\Gamma_f$. Therefore, by the usual change of variables theorem (applied to the approximating integrals) we obtain

$$\mathrm{Res}\begin{bmatrix}\varphi\\ w_1,\dots,w_n\end{bmatrix}_{\Gamma_f}=\mathrm{Res}\begin{bmatrix}\Psi^*\varphi\\ w_1,\dots,w_n\end{bmatrix}_{\Gamma_{f\circ\Phi|_A}}$$

and the latter is equal to

$$\mathrm{Res}\begin{bmatrix}\Phi^*\varphi\\ f_1\circ\Phi,\dots,f_n\circ\Phi\end{bmatrix}_A. \quad \square$$

3. A Lelong–Poincaré formula

The key-point of this part is a more or less known version of the restricted Lelong–Poincaré formula mentioned above. If $f_j\in\mathcal{O}(\Omega)$, $j=1,\dots,r$, are such that $\bigcap_{j=1}^r f_j^{-1}(0)$ has pure dimension $m-r$, then by [T, p. 133] (see also [CH]),

$$[Z_f]=\mathrm{Res}\begin{bmatrix}df_1\wedge\dots\wedge df_r\wedge(\,\cdot\,)\\ f_1,\dots,f_r\end{bmatrix}=\bar\partial\left[\frac{1}{f_1}\right]\wedge\dots\wedge\bar\partial\left[\frac{1}{f_r}\right]\wedge df_1\wedge\dots\wedge df_r,$$

where Z_f is the cycle of zeroes of $f=(f_1,\dots,f_r)$ (computed as the proper intersection cycle $\Gamma_f\cdot(\Omega\times\{0\})$ following Draper [Dr]). Note that there is $Z_f=Z_{f_1}\cdot\ldots\cdot Z_{f_r}$ and the intersection being proper, the product of cycles is associative (see [Ch]). By the Lelong–Poincaré formula, for each j, $[Z_{f_j}]=\bar\partial\partial u_j$, where we put $u_j:=(2\pi i)^{-1}\log|f_j|^2$. Let A be a purely k-dimensional analytic subset of Ω such that $f^{-1}(0)\cap A$ has pure dimension $k-r$. If we put $\mathfrak{t}:=[A]$, then by the results of Bedford and Taylor, the product $\mathfrak{t}\wedge\bar\partial\partial u$ is well defined by $\bar\partial\partial(u\mathfrak{t})$, which in turn is equal to $[A\cdot Z_{f_1}]$ (by the version of Lelong–Poincaré from [Ch, p. 216]). If now we put $T_1:=A\cdot Z_{f_1}$ and $\mathfrak{t}_1:=[T_1]$, we obtain $\bar\partial\partial(u_2\mathfrak{t}_1)=[T_1\cdot Z_{f_2}]$ by the same theorem. Iterating this, we get

$$[A]\wedge[Z_{f_1}]\wedge\dots\wedge[Z_{f_r}]=[A\cdot Z_{f_1}\cdot\ldots\cdot Z_{f_r}]=[A\cdot Z_f],$$

where the left-hand side of the equality is understood in the Bedford–Taylor sense. By [De], Corollaire 5.5, we know that $[Z_{f_1}]\wedge\dots\wedge[Z_{f_r}]=[Z_f]$. Therefore,

$$\text{(LP}'\text{)}\qquad [A\cdot Z_f]=\mathrm{Res}\begin{bmatrix}df_1\wedge\dots\wedge df_r\wedge(\,\cdot\,)\\ f_1,\dots,f_r\end{bmatrix}_A.$$

The main idea of all our constructions in the c-holomorphic setting is that we replace the non-existent df by dw taken on the graph of f.

Formula (LP′) leads to three c-holomorphic results. The first one is the c-holomorphic counterpart of the Lelong–Poincaré formula. Let $f\in\mathcal{O}_c(A)$ be such that it does not vanish on any irreducible component of A. Then, by an observation made in [D], the set $f^{-1}(0)$ has pure dimension $k-1$ and so $\Gamma_f\cap(\Omega\times\{0\})$ is a proper intersection. Thus $Z_f:=\Gamma_f\cdot(\Omega\times\{0\})$ is well defined.

Theorem 3.1. *In the introduced setting,*

$$[Z_f]=\frac{1}{2\pi i}[\Gamma_f]\wedge\bar\partial\partial\log|w|^2 \quad \text{on } \mathcal{D}_{(k-1,k-1)}(\Omega),$$

where w is the variable from the target space.

Proof. It suffices to observe that $(\Omega\times\{0\})=Z_w$, whence $Z_f=\Gamma_f\cdot Z_w$, and so by (LP′),

$$[Z_f]=\mathrm{Res}\begin{bmatrix} dw\wedge(\,\cdot\,) \\ w \end{bmatrix}_{\Gamma_f}=\frac{1}{2\pi i}[\Gamma_f]\wedge\bar\partial\partial\log|w|^2,$$

which completes the proof. □

This has a straightforward generalization to the case of several functions:

Theorem 3.2. *Let $f_1,\dots,f_n\in\mathcal{O}_c(A)$ be such that $f^{-1}(0)$ has pure dimension $k-n$ for $f:=(f_1,\dots,f_n)$. Then on $\mathcal{D}_{(k-n,k-n)}(\Omega)$ there is*

$$[Z_f]=\mathrm{Res}\begin{bmatrix} dw_1\wedge\ldots\wedge dw_n\wedge(\,\cdot\,) \\ w_1,\dots,w_n \end{bmatrix}_{\Gamma_f},$$

where $Z_f:=\Gamma_f\cdot(\Omega\times\{0\}^n)$ is the proper intersection cycle of f and w_j are the variables from the target space.

Proof. It is similar to the previous one – we just observe that $[Z_f]=[\Gamma_f\cdot Z_w]$, where Z_w is the cycle of zeroes of the projection onto the target space, $w\colon\Omega\times\mathbb{C}^n\ni(z,w)\mapsto w\in\mathbb{C}^n$. □

If $n=k$, then we can compute the (geometric) multiplicity $m_0(f)$ for $f\in\mathcal{O}_c(A,\mathbb{C}^k)$ with 0 isolated in $f^{-1}(0)$ similarly to the holomorphic case. Recall first that $m_0(f)$ is by definition the number of points in the generic fibre of f which coincides with the proper intersection multiplicity at zero, denoted $i(\Gamma_f\cdot(\Omega\times\{0\}^k);0)$, of Γ_f and $\Omega\times\{0\}^k$.

Corollary 3.3. *Let $f\in\mathcal{O}_c(A,\mathbb{C}^k_w)$ be such that $f^{-1}(0)=\{0\}$. Then*

$$m_0(f)\delta_0=\mathrm{Res}\begin{bmatrix} dw_1\wedge...\wedge dw_k\wedge(\,\cdot\,) \\ w_1,...,w_k \end{bmatrix}_{\Gamma_f},$$

where δ_0 is the Dirac delta at zero.

Proof. Clearly $m_0(f)\delta_0=[\Gamma_f\cdot(\Omega\times\{0\}^k)]$, since $m_0(f)=i(\Gamma_f\cdot(\Omega\times\{0\}^k);0)$. It remains to apply the previous result. □

Note. In particular we have by Corollary 3.3 the equality

$$m_0(f)=\mathrm{Res}\begin{bmatrix} dw_1\wedge...\wedge dw_k \\ w_1,...,w_k \end{bmatrix}_{\Gamma_f},$$

generalizing the well-known holomorphic formula

$$m_0(f)=\mathrm{Res}\begin{bmatrix} df_1\wedge...\wedge df_m \\ f_1,...,f_m \end{bmatrix}$$

in the case $A=\Omega$ and $\bigcap_{j=1}^m f_j^{-1}(0)=\{0\}$ (see [T, Section II.6]).

4. Residue currents with numerators

We are keeping the notation introduced so far and consider n c-holomorphic functions $f_j\colon A\to\mathbb{C}$ not vanishing identically on any irreducible component of A and such that $f^{-1}(0)\subset A$ has pure dimension $k-n$ (see [D] for considerations on the dimension of zero-sets of c-holomorphic mappings). These play the role of denominators. Let us take a 'numerator' $h\in\mathcal{O}_c(A)$. Our aim is to define a residue current which would be an analogue of the restricted Coleff–Herrera current of type $(m-k,m-k+n)$,

$$h\cdot[A]\wedge\bar\partial\left[\frac{1}{f_1}\right]\wedge...\wedge\bar\partial\left[\frac{1}{f_n}\right]$$

for $h,f_1,...,f_n$ holomorphic. Once again we follow the idea of A. Yger – we shall make use of the graph.

We introduce a new variable $t\in\mathbb{C}$ and consider the c-holomorphic mapping

$$H\colon \mathbb{C}\times A\ni(t,z)\longmapsto(t-h(z),f(z))\in\mathbb{C}_{w_0}\times\mathbb{C}^n_w.$$

Then we put by definition for $\varphi\in\mathcal{D}_{(k,k-n)}(\Omega)$,

$$(\star)\qquad h(z)\,\mathrm{Res}\begin{bmatrix} \varphi(z) \\ f_1(z),...,f_n(z) \end{bmatrix}_A:=\mathrm{Res}\begin{bmatrix} t\varphi(z)\wedge dt \\ w_0,w_1,...,w_n \end{bmatrix}_{\Gamma_H}.$$

This coincides in the holomorphic case with the usual definition as we prove in the following result.

Theorem 4.1. *If $h, f_1, ..., f_n$ are holomorphic, then* ($\star$) *holds.*

Proof. By assumptions we have $H\in\mathcal{O}(\mathbb{C}\times\Omega)$. Let Λ be the graph of $H|_{\mathbb{C}\times A}$. Consider the biholomorphism

$$\Xi\colon \mathbb{C}\times\Omega\times\mathbb{C}^n\ni(t,z,w_0,w)\longmapsto(t,z,w_0+t-h(z),w+f(z))\in\mathbb{C}\times\Omega\times\mathbb{C}^n.$$

Then $\Lambda=\Xi(\mathbb{C}\times A\times\{0\}^{1+n})$ and by Lemma 2.2,

$$\begin{aligned}\operatorname{Res}\left[\begin{matrix}t\varphi(z)\wedge dt\\ w_0,w_1,...,w_n\end{matrix}\right]_\Lambda &= \operatorname{Res}\left[\begin{matrix}\Xi^*(t\varphi(z)\wedge dt)\\ w_0+t-h, f_1+w_1,...,f_n+w_n\end{matrix}\right]_{\mathbb{C}\times A\times\{0\}^{1+n}}\\ &= \operatorname{Res}\left[\begin{matrix}t\varphi(z)\wedge dt\\ t-h, f_1,...,f_n\end{matrix}\right]_{\mathbb{C}\times A\times\{0\}^{1+n}},\end{aligned}$$

since $w_0=w_1=...=w_n=0$ on $\mathbb{C}\times A\times\{0\}^{1+n}$. The latter residue may be rewritten replacing $\mathbb{C}\times A\times\{0\}^{1+n}$ with $\mathbb{C}\times A$, since it does not depend on the variables $w_0,w_1,...,w_n$.

By Fubini's theorem (applied to the approximating integrals) and Cauchy's formula,

$$\operatorname{Res}\left[\begin{matrix}t\varphi(z)\wedge dt\\ t-h, f_1,...,f_n\end{matrix}\right]_{\mathbb{C}\times A}=\operatorname{Res}\left[\begin{matrix}h(z)\varphi(z)\\ f_1,...,f_n\end{matrix}\right]_A$$

which completes the proof. $\square$

Proposition 4.2. *If $f_1,...,f_n$ are just c-holomorphic but h is holomorphic on A, then*

$$h\operatorname{Res}\left[\begin{matrix}\varphi\\ f_1,...,f_n\end{matrix}\right]_A=\operatorname{Res}\left[\begin{matrix}h\varphi\\ f_1,...,f_n\end{matrix}\right]_A,\quad \varphi\in\mathcal{D}_{(k,k-n)}(\Omega).$$

Proof. The left-hand side of the required equality is defined by ($\star$). Consider the following biholomorphism

$$\Psi\colon \mathbb{C}\times\Omega\times\mathbb{C}\times\mathbb{C}^n\ni(t,z,w_0,w)\longmapsto(t,z,w_0+t-h(z),w)\in\mathbb{C}\times\Omega\times\mathbb{C}\times\mathbb{C}^n.$$

Put $\Gamma:=\{(t,z,0,f(z)):t\in\mathbb{C}\text{ and }z\in A\}$. We have $\Psi(\Gamma)=\Gamma_H$ and so by Lemma 2.2,

$$\operatorname{Res}\left[\begin{matrix}t\varphi(z)\wedge dt\\ w_0,w_1,...,w_n\end{matrix}\right]_{\Gamma_H}=\operatorname{Res}\left[\begin{matrix}t\varphi(z)\wedge dt\\ w_0+t-h,w_1,...,w_n\end{matrix}\right]_\Gamma=\operatorname{Res}\left[\begin{matrix}t\varphi(z)\wedge dt\\ t-h,w_1,...,w_n\end{matrix}\right]_{\mathbb{C}\times\Gamma_f},$$

since $w_0=0$ on Γ and once we got rid of w_0 we may replace Γ by $\mathbb{C}\times\Gamma_f$.

Applying now Fubini's theorem and the Cauchy formula we obtain

$$\operatorname{Res}\begin{bmatrix} t\varphi(z)\wedge dt \\ t-h, w_1, \dots, w_n \end{bmatrix}_{\mathbb{C}\times\Gamma_f} = \operatorname{Res}\begin{bmatrix} h(z)\varphi(z) \\ w_1, \dots, w_n \end{bmatrix}_{\Gamma_f},$$

which is the equality sought for. □

Added in response to the referee's remark. As observed by the referee, a simpler way to define the multiplication would be to put

$$(\star\star)\qquad h(z)\operatorname{Res}\begin{bmatrix} \varphi(z) \\ f_1, \dots, f_n \end{bmatrix}_A = \operatorname{Res}\begin{bmatrix} w_0\varphi(z) \\ w_1, \dots, w_n \end{bmatrix}_{\Gamma_{(h,f)}},$$

where $\Gamma_{(h,f)}$ is the graph of $(h,f)\colon A\to\mathbb{C}_{w_0}\times\mathbb{C}^n_w$. This would lead to even shorter proofs of Theorem 4.1 and Proposition 4.2. However, the results from the following two sections would be harder to establish (actually, formula $(\star)$ has a form leading directly to the transformation law from Section 6). Fortunately, as we will show below, it turns out that both formulae, $(\star)$ and $(\star\star)$, do coincide.

Proposition 4.3. *In the situation under consideration, for any test form $\varphi\in\mathcal{D}_{(k,k-n)}(\Omega)$,*

$$\operatorname{Res}\begin{bmatrix} w_0\varphi(z) \\ w_1, \dots, w_n \end{bmatrix}_{\Gamma_{(h,f)}} = \operatorname{Res}\begin{bmatrix} t\varphi(z)\wedge dt \\ w_0, w_1, \dots, w_n \end{bmatrix}_{\Gamma_{(t-h,f)}}$$

with $(t-h,f)\colon \mathbb{C}_t\times A\ni(t,z)\to(t-h(z),f(z))\in\mathbb{C}_{w_0}\times\mathbb{C}_w$.

Proof. Consider the biholomorphism $\Phi(t,z,w_0,w)=(t,z,t-w_0,w)$ and apply Lemma 2.2: since $\Phi(\mathbb{C}\times\Gamma_{(h,f)})=\Gamma_{(t-h,f)}$, the residue $(\star)$ becomes

$$\operatorname{Res}\begin{bmatrix} t\varphi(z)\wedge dt \\ t-w_0, w_1, \dots, w_n \end{bmatrix}_{\mathbb{C}\times\Gamma_{(h,f)}}.$$

By Fubini's theorem and Cauchy's formula this is equal to

$$\operatorname{Res}\begin{bmatrix} w_0\varphi(z) \\ w_1, \dots, w_n \end{bmatrix}_{\Gamma_{(h,f)}}$$

as required. □

5. A Cauchy-type formula

If $f\in\mathcal{O}(\Omega)$ and $0\in\Omega\subset\mathbb{C}^m$, then the usual Cauchy's formula may be expressed as follows (cf. Fubini's theorem):

$$f(0)=\left(\frac{1}{2\pi i}\right)^m \lim_{\delta\to 0^+}\int_{T_\delta(z)}\frac{f(z)dz_1\wedge\dots\wedge dz_m}{z_1\dots z_m},$$

where $T_\delta(z)$ is the tube defined earlier, taken for $z_1,...,z_m$ and an admissible path. This formula may be more generally written as

$$f(0)=(f\mathfrak{t})(\theta dz_1\wedge...\wedge dz_m),$$

where θ is a $\mathcal{C}^\infty$ function with compact support, identically equal to 1 in a neighbourhood of zero (we shall not write it any longer, it is 'cosmetics') and $\mathfrak{t}$ is the current (of type $(m,0)$) defined by

$$\mathfrak{t}(\varphi):=\operatorname{Res}\begin{bmatrix}\varphi\\ z_1,...,z_m\end{bmatrix}.$$

This approach cannot be directly transposed to the c-holomorphic case (roughly speaking, the main problem is that there are too many variables $z_1,...,z_m$ for a set of dimension $<m$). Nonetheless, we may proceed in the following way: let as earlier $A\subset\Omega$ be an analytic set containing 0 and of pure dimension k. Suppose that the natural projection π on the first k coordinates realizes the degree (Lelong number) $\deg_0 A$ as the sheet number (multiplicity) of the branched covering $\pi|_A$ (see [Ch]). Then by (LP′), for any $f\in\mathcal{O}(A)$ and all $\xi\in\mathbb{C}^k$ sufficiently small,

$$(\dagger)\qquad \sum_{\zeta\in\pi^{-1}(\xi)\cap A}\mu_\zeta(\pi|_A)f(\zeta)=\operatorname{Res}\begin{bmatrix}f(z)dz_1\wedge...\wedge dz_k\\ z_1-\xi_1,...,z_k-\xi_k\end{bmatrix}_A,$$

where $\mu_\zeta(\pi|_A)$ is the multiplicity of $\pi|_A$ at the point $\zeta\in A$ (see [Ch] for this notion). More generally, we have the following proposition.

Proposition 5.1. *In the introduced setting,*

$$\sum_{\zeta\in\pi^{-1}(\xi)\cap A}\mu_\zeta(\pi|_A)\delta_\zeta=\operatorname{Res}\begin{bmatrix}(\cdot)\wedge dz_1\wedge...\wedge dz_k\\ z_1-\xi_1,...,z_k-\xi_k\end{bmatrix}_A,$$

where δ_ζ are Dirac functions. In particular, for any function $f\in\mathcal{O}(\Omega)$,

$$\deg_0 A\cdot f(0)=\operatorname{Res}\begin{bmatrix}f(z)dz_1\wedge...\wedge dz_k\\ z_1,...,z_k\end{bmatrix}_A.$$

Proof. Fix ξ and take $h(z):=(z_1-\xi_1,...,z_k-\xi_k)$ for $z\in\mathbb{C}^m$. Then the cycle Z_h is well defined and equal to $\{\xi\}\times\mathbb{C}^{m-k}$ (with multiplicity 1). This intersects A properly at the points $\zeta\in A$ for which $\pi(\zeta)=\xi$ and the multiplicities attached to these points correspond to the multiplicities $\mu_\zeta(\pi|_A)$ (see [Ch]). Now, by (LP′) and a similar argument to the one used in the proof of Corollary 3.3 we obtain

$$[A\cdot Z_h]=\operatorname{Res}\begin{bmatrix}(\cdot)\wedge dz_1\wedge...\wedge dz_k\\ z_1-\xi_1,...,z_k-\xi_k\end{bmatrix}_A=\sum_{\zeta\in\pi^{-1}(\xi)\cap A}\mu_\zeta(\pi|_A)\delta_\zeta$$

and the proof is accomplished (to get the assertion for f holomorphic, we just replace f by a compactly supported smooth function equal to f in a small enough neighbourhood $U\times V\subset\mathbb{C}^k\times\mathbb{C}^m$ of the fibre $\pi^{-1}(\xi)\cap A$ chosen so that $\mathrm{cl}(A\cap(U\times V))$ does not meet $U\times\partial V$). $\square$

By the way, observe that since $\deg_0 A=i(A\cdot\pi^{-1}(0);0)=m_0(\pi|_A)$ (π is seen as a function $\Omega\to\mathbb{C}^k$), by Corollary 3.3 and in view of the fact that $\pi|_A$ is holomorphic, there is

$$\deg_0 A=\mathrm{Res}\begin{bmatrix} dw_1\wedge\ldots\wedge dw_k \\ w_1,\ldots,w_k\end{bmatrix}_{\Gamma_{\pi|A}}=\mathrm{Res}\begin{bmatrix} dz_1\wedge\ldots\wedge dz_k \\ z_1,\ldots,z_k\end{bmatrix}_A.$$

On the right-hand side of (†) we have the residue

$$\mathfrak{s}:=\mathrm{Res}\begin{bmatrix} \cdot \\ z_1-\xi_1,\ldots,z_k-\xi_k\end{bmatrix}_A$$

(a current of type $(m-k,m)$) multiplied by f and computed on the test form $dz_1\wedge\ldots\wedge dz_k$. This we may try to transpose to the c-holomorphic case. Let now $f\in\mathcal{O}_c(A)$ and $\xi\in\mathbb{C}^k$, then we set

$$\mathfrak{r}_\xi(\varphi):=\mathrm{Res}\begin{bmatrix} t\varphi(z)\wedge dt \\ w_0,\ldots,w_k\end{bmatrix}_{T_\xi},\quad \varphi\in\mathcal{D}_{(k,0)}(\Omega),$$

where T_ξ is the graph of the c-holomorphic mapping

$$g_\xi\colon \mathbb{C}\times A\ni(t,z)\longmapsto(t-f(z),z_1-\xi_1,\ldots,z_k-\xi_k)\in\mathbb{C}_{w_0}\times\mathbb{C}^k_w.$$

We obtain a current of type $(m+1,m+2+k)$ and we have to compute $\mathfrak{r}_\xi(dz_1\wedge\ldots\wedge dz_k)$. It is easy to see that we may replace this current by

$$\tilde{\mathfrak{r}}_\xi(\varphi):=\mathrm{Res}\begin{bmatrix} t\varphi(z)\wedge dt \\ w_0,z_1-\xi_1,\ldots,z_k-\xi_k\end{bmatrix}_{\Gamma},\quad \varphi\in\mathcal{D}_{(k,0)}(\Omega),$$

where Γ is the graph of $\mathbb{C}\times A\ni(t,z)\mapsto t-f(z)\in\mathbb{C}_{w_0}$.

Theorem 5.2. *In the introduced setting, Proposition* 5.1 *holds true for c-holomorphic functions and so in particular for* $\xi=0$ *we have*

$$\tilde{\mathfrak{r}}_0(dz_1\wedge\ldots\wedge dz_k)=\mathfrak{r}_0(dz_1\wedge\ldots\wedge dz_k)=\deg_0 A\cdot f(0).$$

Proof. We shall use $\tilde{\mathfrak{r}}_\xi$ and (LP′). Fix ξ and let

$$H(t,z,w_0):=(w_0,z_1-\xi_1,\ldots,z_k-\xi_k),\quad (t,z,w_0)\in\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}.$$

Clearly, the proper cycle of zeroes $Z_H=\mathbb{C}\times(\{\xi\}\times\mathbb{C}^{m-k})\times\{0\}$. Observe now that

$$\Gamma\cdot Z_H=\sum_{\zeta\in\pi^{-1}(\xi)\cap A}\mu_\zeta(\pi|_A)\{(f(\zeta),\zeta,0)\}.$$

If now p denotes the projection $(t,z,w_0)\mapsto t$, then obviously

$$\langle[\Gamma\cdot Z_H],p\rangle=\sum_{\zeta\in\pi^{-1}(\xi)\cap A}\mu_\zeta(\pi|_A)\delta_{(f(\zeta),\zeta,0)}(p)=\sum_{\zeta\in\pi^{-1}(\xi)\cap A}\mu_\zeta(\pi|_A)f(\zeta),$$

and since by (LP′),

$$\langle[\Gamma\cdot Z_H],p\rangle=\tilde{\mathfrak{r}}_\xi(dz_1\wedge...\wedge dz_k)$$

the proof is completed. □

Remark 5.3. What also may be treated as a c-holomorphic counterpart of a Cauchy-type formula for c-holomorphic functions is the integral dependence relation established in the following easy lemma (cf. [W]).

Lemma 5.4. *Suppose that A has pure dimension k. Then a continuous function $f\colon A\to\mathbb{C}$ is c-holomorphic if and only if for any point $a\in A$ there is a neighbourhood $U\ni a$ and a polynomial $P\in\mathcal{O}(U)[t]$ monic in t (i.e. unitary) and such that $P(x,f(x))=0$ for $x\in U\cap A$.*

Proof. If f is c-holomorphic, then for any point $a\in A$ we may choose coordinates in $\mathbb{C}^m$ in such a way that the projection π onto the first k coordinates is a branched covering on A in a neighbourhood $V\times W\subset\mathbb{C}^k\times\mathbb{C}^{m-k}$ of a and $\pi^{-1}(\pi(a))\cap A\cap(V\times W)=\{a\}$. Then for any point $v\in V$ outside the critical locus σ of $\pi|_A$ there are exactly d different points w^j such that $(v,w^j)\in A$. Then setting $P(v,w,t):=\prod_{j=1}^d(t-f(v,w^j))$ and extending its coefficients analytically through σ by the Riemann extension theorem we obtain the required $P\in\mathcal{O}(V\times W)[t]$.

On the other hand, if such a polynomial exists in a neighbourhood U of $a\in\operatorname{Reg}A$, then shrinking U if necessary, we may assume that $U\cap\operatorname{Reg}A$ is biholomorphic to the unit polydisc in $\mathbb{E}^k\subset\mathbb{C}^k$. Thus in fact we reduce ourselves to the case of a continuous function $f\colon\mathbb{E}^k\to\mathbb{C}$ such that $P(x,f(x))=0$, $x\in\mathbb{E}^k$, for some monic $P\in\mathcal{O}(\mathbb{E}^k)[t]$. It is well known that f must be holomorphic. □

Therefore, in the situation under consideration,

$$f(z)^d+a_1(\xi)f(z)^{d-1}+...+a_d(\xi)\equiv 0,$$

in a neighbourhood of zero, with $\pi(z)=\xi$, $d:=\deg_0 A$ and $a_j(\xi)$ being the symmetric functions (taking account of the sign) of $f(z^{(j)})$ for $\pi^{-1}(\xi)\cap A=\{z^{(1)},...,z^{(d)}\}$.

6. Transformation law

The aim of this part is to prove the transformation law in the c-holomorphic case. Assume, as earlier, that A is a purely k-dimensional analytic set in an open set $\Omega\subset\mathbb{C}^m$.

Theorem 6.1. *Assume that $a,f\in\mathcal{O}_c(A)$ are such that neither of them vanishes identically on any irreducible component of A. Then*

$$\operatorname{Res}\begin{bmatrix}\cdot\\ f\end{bmatrix}_A = a\operatorname{Res}\begin{bmatrix}\cdot\\ af\end{bmatrix}_A.$$

Proof. On the left-hand side of the required equality we have by definition

$$\text{(L)}\qquad \operatorname{Res}\begin{bmatrix}\varphi\\ f\end{bmatrix}_A := \operatorname{Res}\begin{bmatrix}\varphi(z)\\ w\end{bmatrix}_{\Gamma_f},\quad \varphi\in\mathcal{D}_{(k,k-1)}(\Omega),\ (z,w)\in\Omega\times\mathbb{C},$$

while on the right-hand side

$$\text{(R)}\qquad a\operatorname{Res}\begin{bmatrix}\varphi\\ af\end{bmatrix}_A := \operatorname{Res}\begin{bmatrix}t\varphi(z)\wedge dt\\ v,w\end{bmatrix}_{\Gamma},\quad \varphi\in\mathcal{D}_{(k,k-1)}(\Omega),$$

where Γ denotes the graph of the c-holomorphic mapping

$$h\colon \mathbb{C}\times A\ni(t,z)\longmapsto(t-a(z),a(z)f(z))\in\mathbb{C}_v\times\mathbb{C}_w.$$

Take now the mapping

$$\Xi\colon \mathbb{C}\times\Omega\times\mathbb{C}\times\mathbb{C}\ni(t,z,v,w)\longmapsto(t,z,t-v,vw)\in\mathbb{C}\times\Omega\times\mathbb{C}\times\mathbb{C}$$

whose Jacobian is equal to $-v$. Thus Ξ is a biholomorphism when restricted to the open set $\mathbb{C}\times(\Omega\setminus a^{-1}(0))\times\mathbb{C}_*\times\mathbb{C}$. Note that we may restrict the approximating integrals defining (L) and (R) to graphs taken over the set $\operatorname{Reg}A\setminus a^{-1}(0)$ since we forget only zero-measure sets. Keeping the same notation for the restricted graphs we have

$$\Xi(\mathbb{C}\times\Gamma_{(a,f)})=\Gamma_{(t-a,af)}=\Gamma.$$

Applying now the change of variables formula to (R) (to its approximating integrals, actually) we will obtain

$$\operatorname{Res}\begin{bmatrix}t\varphi(z)\wedge dt\\ v,w\end{bmatrix}_{\Gamma}=\operatorname{Res}\begin{bmatrix}t\varphi(z)\wedge dt\\ t-h,vw\end{bmatrix}_{\mathbb{C}\times\Gamma_{(a,f)}}=\operatorname{Res}\begin{bmatrix}v\varphi(z)\\ vw\end{bmatrix}_{\Gamma_{(a,f)}},$$

the latter being a consequence of Fubini's theorem and Cauchy's formula.

By the restricted holomorphic transformation law (see [BVY]),

$$\operatorname{Res}\begin{bmatrix}v\varphi(z)\\ vw\end{bmatrix}_{\Gamma_{(a,f)}}=\operatorname{Res}\begin{bmatrix}\varphi(z)\\ w\end{bmatrix}_{\Gamma_{(a,f)}}.$$

Since in the integrals from the right-hand side the variable v is now a ‘phantom’ one, we may forget it getting just (L) as required. □

We turn now to proving a more general version of this theorem. To achieve this aim we shall need the following lemma proposed by A. Yger.

Lemma 6.2. *Assume that* $f=(f_1,\dots,f_n)\in\mathcal{O}_c(A,\mathbb{C}^n)$ *is such that* $f^{-1}(0)$ *has pure dimension* $k-n$*. Let* $a_1,\dots,a_l\in\mathcal{O}_c(A)$*. Then for any polynomial* $Q\in\mathbb{C}[t_1,\dots,t_l]$ *we have the following equality between currents of type* $(m-k,m-k+n)$*: for any test form* $\varphi(z)$*,*

$$Q(a_1,\dots,a_l)\,\mathrm{Res}\begin{bmatrix}\varphi(z)\\ f_1,\dots,f_n\end{bmatrix}_A=\mathrm{Res}\begin{bmatrix}Q(t_1,\dots,t_l)\varphi(z)\wedge dt_1\wedge\dots\wedge dt_l\\ v_1,\dots,v_l,w_1,\dots,w_n\end{bmatrix}_\Gamma,$$

where Γ *is the graph of* $\gamma(t_1,\dots,t_l,z)=(t_1-a_1(z),\dots,t_l-a_l(z),f(z))$ *defined and c-holomorphic on* $\mathbb{C}^l_t\times A$ *with values in* $\mathbb{C}^l_v\times\mathbb{C}^n_w$*.*

Proof. By definition, for $\varphi\in\mathcal{D}_{(k,k-n)}(\Omega)$,

$$Q(a_1,\dots,a_l)\,\mathrm{Res}\begin{bmatrix}\varphi(z)\\ f_1,\dots,f_n\end{bmatrix}_A=\mathrm{Res}\begin{bmatrix}t_0\varphi(z)\wedge dt_0\\ w_0,w_1,\dots,w_n\end{bmatrix}_\Lambda,$$

where Λ is the graph of $(t_0,z)\mapsto(t_0-Q(a_1(z),\dots,a_l(z)),f(z))$. To prove the assertion we will compute, in two different ways, the residue

$$\mathfrak{t}(\varphi):=\mathrm{Res}\begin{bmatrix}Q(t_1,\dots,t_l)\varphi(z)\wedge dt_1\wedge\dots\wedge dt_l\wedge dt_0\\ v_1,\dots,v_l,w_0,w_1,\dots,w_n\end{bmatrix}_\Upsilon,$$

where Υ is the graph of

$$(t_0,t_1,\dots,t_l,z)\longmapsto(t_1-a_1(z),\dots,t_l-a_l(z),t_0-Q(a_1(z),\dots,a_l(z)),f(z)).$$

Put $a(z)=(a_1(z),\dots,a_l(z))$ and $dt:=dt_1\wedge\dots\wedge dt_l$. The integrals appearing in the definition of $\mathfrak{t}(\varphi)$ are computed over the set

$$E:=\{(t_0,t,z,v,w_0,w):v=t-a(z),w_0=t_0-Q(a(z)),w=f(z),\\ |v_0|^2=\eta_0,|v_\iota|^2=\eta_\iota,|w_0|^2=\varepsilon_0\text{ and }|w_j|^2=\varepsilon_j\}$$

(given by an admissible path $\eta_0\ll\dots\ll\eta_n\ll\varepsilon_0\ll\dots\ll\varepsilon_n$), and are of the form

$$\int_E\frac{Q(t)\varphi(z)\wedge dt\wedge dt_0}{v_1\dots v_lw_1\dots w_nw_0}=\int_{E_1}\left(\int_{E_2^z}\frac{dt_0}{t_0-Q(a(z))}\right)\frac{Q(t)\varphi(z)\wedge dt}{v_1\dots v_lw_1\dots w_n}\\ =2\pi i\int_{E_1}\frac{Q(t)\varphi(z)\wedge dt}{v_1\dots v_lw_1\dots w_n},$$

where $E_1 := \{(t,z,v,w): v=t-a(z), w=f(z), |v_\iota|^2=\eta_\iota \text{ and } |w_j|^2=\varepsilon_j\}$ and on $E_2^z := \{(t_0,w_0): w_0=t_0-Q(a(z)) \text{ and } |w_0|^2=\varepsilon_0\}$ we computed the index (independent of z). Therefore

$$\mathfrak{t}(\varphi)=\operatorname{Res}\begin{bmatrix} Q(t_1,...,t_l)\varphi(z)dt_1\wedge...\wedge dt_l \\ v_1,...,v_l,w_1,...,w_n \end{bmatrix}_\Gamma.$$

Let us find another expression for this current. First observe that in the expression of $\mathfrak{t}(\varphi)$ we may write t_0 instead of $Q(t_1,...,t_l)$. Indeed, on Υ we have $t_0-w_0=Q(a(z))$ and $t-w=a(z)$. Remember that the residue is annihilated by the ideal of the functions defining it. Thus, since Q is a polynomial, we may first replace the factor $Q(t)$ in $\mathfrak{t}(\varphi)$ by $Q(t-w)$. This in turn is equal to t_0-w_0 on Υ and since w_0 is in the ideal, we get the assertion.

If we repeat now the above argument extracting this time, by means of Fubini's theorem ($Q(t_1,...,t_l)$ does not bother us any longer), all the integrals

$$\int_{\{(t_j,v_j):v_j=t_j-a_j(z),|v_j|^2=\eta_j\}}\frac{dt_j}{t_j-a_j(z)}=2\pi i,$$

then we get

$$\mathfrak{t}(\varphi)=\operatorname{Res}\begin{bmatrix} t_0\varphi(z)\wedge dt_0 \\ w_0,w_1,...,w_n \end{bmatrix}_\Lambda.$$

This completes the proof. □

Theorem 6.3. *Assume that* $f_1,...,f_n,g_1,...,g_n\in\mathcal{O}_c(A)$ *are such that* $\bigcap_{j=1}^n f_j^{-1}(0)$ *and* $\bigcap_{j=1}^n g_j^{-1}(0)$ *have pure dimension* $k-n$. *If there exist functions* $a_{\iota j}\in\mathcal{O}_c(A)$, $\iota,j=1,...,n$ *such that* $g_j=\sum_{\iota=1}^n a_{\iota j}f_\iota$ *for all* j, *then*

$$\operatorname{Res}\begin{bmatrix} \varphi \\ f_1,...,f_n \end{bmatrix}_A=\Delta\operatorname{Res}\begin{bmatrix} \varphi \\ g_1,...,g_n \end{bmatrix}_A,\quad \varphi\in\mathcal{D}_{(k,k-n)}(\Omega),$$

where $\Delta:=\det[a_{\iota j}]_{\iota,j}\in\mathcal{O}_c(A)$.

Proof. For the sake of simplicity we shall restrict ourselves to the case $n=2$, the main idea being the same in the general case. Due to the preceding lemma we only need to show that for any $\varphi\in\mathcal{D}_{(k,k-2)}(\Omega)$,

$$\operatorname{Res}\begin{bmatrix} \varphi \\ f_1,f_2 \end{bmatrix}_A=\operatorname{Res}\begin{bmatrix} (t_{11}t_{22}-t_{12}t_{21})\varphi(z)\wedge dt_{11}\wedge dt_{12}\wedge dt_{21}\wedge dt_{22} \\ v_{11},v_{12},v_{21},v_{22},w_1,w_2 \end{bmatrix}_\Gamma,$$

where Γ is the graph of $\gamma(t_{11},t_{12},t_{21},t_{22},z)=((t_{\iota j}-a_{\iota j}(z))_{\iota j},g_1(z),g_2(z))$.

In the integrals approximating the residue on the right-hand side we change the variables in the following way: we leave the z_j, the $t_{\iota j}$ and the $v_{\iota j}$ untouched changing only

$$(w_1, w_2) \text{ to } (u_1, u_2) \quad \text{such that } \begin{cases} w_1 = u_1 t_{11} + u_2 t_{12}, \\ w_2 = u_1 t_{21} + u_2 t_{22}. \end{cases}$$

The integrals become (we forget only a zero-measure set not affecting them)

$$\int_E \frac{(t_{11}t_{22} - t_{12}t_{21})\varphi(z) \wedge dt_{11} \wedge dt_{12} \wedge dt_{21} \wedge dt_{22}}{v_{11}v_{12}v_{21}v_{22}(u_1t_{11} + u_2t_{12})(u_1t_{21} + u_2t_{22})}$$

computed over

$$E := \{((t_{\iota j})_{\iota j}, z, (v_{\iota j})_{\iota j}, u_1, u_2) : |v_{\iota j}|^2 = \varepsilon_{\iota j}, |u_1t_{11} + u_2t_{12}|^2 = \varepsilon_1, \text{ and } |u_1t_{21} + u_2t_{22}|^2 = \varepsilon_2\}.$$

Note that this is a subset of the graph Γ' of $((t_{\iota j} - a_{\iota j}(z))_{\iota j}, f_1(z), f_2(z))$. Applying now the restricted transformation law to the residue obtained in this way, we have

$$\operatorname{Res}\begin{bmatrix} (t_{11}t_{22} - t_{12}t_{21})\varphi(z) \wedge dt_{11} \wedge dt_{12} \wedge dt_{21} \wedge dt_{22} \\ v_{11}, v_{12}, v_{21}, v_{22}, (u_1t_{11} + u_2t_{12}), (u_1t_{21} + u_2t_{22}) \end{bmatrix}_{\Gamma'} = \operatorname{Res}\begin{bmatrix} \varphi(z) \wedge dt_{11} \wedge dt_{12} \wedge dt_{21} \wedge dt_{22} \\ v_{11}, v_{12}, v_{21}, v_{22}, u_1, u_2 \end{bmatrix}_{\Gamma'}.$$

Finally, applying Fubini's theorem and the index formula we easily check (as in the previous theorem) that the latter is equal to

$$\operatorname{Res}\begin{bmatrix} \varphi(z) \\ u_1, u_2 \end{bmatrix}_{\Gamma_{(f_1, f_2)}} = \operatorname{Res}\begin{bmatrix} \varphi(z) \\ f_1, f_2 \end{bmatrix}_A$$

which ends the proof. □

Final remark. The idea of using the graph and the coordinate functions on it to compute the residue could be perhaps useful when looking for a desingularization-free proof of the existence of the Coleff–Herrera residue currents. At least, the approach involving the graphs carries over the problem of desingularization from functions to sets. This may turn out to be simpler in use, in some sense.

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