Ark. Mat., **47** (2009), 127–141
DOI: 10.1007/s11512-008-0086-9

Analytic continuation of residue currents

Håkan Samuelsson

Abstract. Let X be a complex manifold and let $f\colon X\to\mathbb{C}^p$ be a holomorphic mapping defining a complete intersection. We prove that the iterated Mellin transform of the residue integral associated with f has an analytic continuation to a neighborhood of the origin in $\mathbb{C}^p$.

1. Introduction

Let X be a complex manifold of complex dimension $\dim_{\mathbb{C}} X=n$ and let $f=(f_1,...,f_{p+q})\colon X\to\mathbb{C}^{p+q}$ be a holomorphic mapping defining a complete intersection. For a test form $\varphi\in\mathcal{D}_{n,n-p}(X)$ we let the residue integral of f be the integral

$$I_f^{\varphi}(\varepsilon)=\int_{T(\varepsilon)}\frac{\varphi}{f_1...f_{p+q}},$$

where $T(\varepsilon)$ is the tubular set $T(\varepsilon)=\bigcap_{j=1}^{p}\{z;|f_j(z)|^2=\varepsilon_j\}\cap\bigcap_{j=p+1}^{p+q}\{z;|f_j(z)|^2>\varepsilon_j\}$. If we let ε tend to zero along a path to the origin in the first orthant such that $\varepsilon_j/\varepsilon_{j+1}^k\to 0$ for $j=1,...,p+q-1$ and all $k\in\mathbb{N}$, a so called "admissible path", then by fundamental results of Coleff–Herrera [7] and Passare [12] the residue integral converges and the limit defines the action of a $(0,p)$-current on the test form φ. We will refer to this current as the Coleff–Herrera–Passare current and denote it suggestively by

$$\bar\partial\left[\frac{1}{f_1}\right]\wedge...\wedge\bar\partial\left[\frac{1}{f_p}\right]\left[\frac{1}{f_{p+1}}\right]...\left[\frac{1}{f_{p+q}}\right],$$

or sometimes $R^pP^q[1/f]$ for short. The current $R^p[1/f]$ is the classical Coleff–Herrera product, a current which has proven to be a good notion of a multivariable residue of f, but also the currents $R^pP^q[1/f]$, with $q\geq 1$, have turned out to be important for the theory. In particular, if $q=1$ then $R^pP^q[1/f]$ is a $\bar\partial$-potential to the Coleff–Herrera product.

Author supported by a postdoctoral fellowship from the Swedish Research Council.

The first question raised by Coleff and Herrera in the book [7] is whether the residue integral $I_f^{\varphi}(\varepsilon)$ has an unrestricted limit as ε tends to zero. This question was answered in the negative by Passare and Tsikh in [14], where they found two polynomials in $\mathbb{C}^2$, with the origin as the only common zero, such that the corresponding residue integral does not converge unrestrictedly; large classes of such examples were then found by Björk. In this sense, the definition of the Coleff–Herrera–Passare current is quite unstable. A different and, as we will see, more rigid approach is based on analytic continuation. Let $\lambda_1, ..., \lambda_{p+q}$ be complex parameters with $\operatorname{Re}\lambda_j$ large. Then the integral

$$\Gamma_f^{\varphi}(\lambda) = \int_X \frac{\bar\partial|f_1|^{2\lambda_1}\wedge...\wedge\bar\partial|f_p|^{2\lambda_p}|f_{p+1}|^{2\lambda_{p+1}}...|f_{p+q}|^{2\lambda_{p+q}}}{f_1...f_{p+q}}\wedge\varphi$$

makes sense and defines an analytic function of λ. This function is the iterated Mellin transform of the residue integral, i.e.,

$$\pm\Gamma_f^{\varphi}(\lambda) = \int_{[0,\infty)^{p+q}} I_f^{\varphi}(s)\, d(s_1^{\lambda_1})\wedge...\wedge d(s_{p+q}^{\lambda_{p+q}}).$$

It is known that $\Gamma_f^{\varphi}(\lambda)$ has a meromorphic continuation to all of $\mathbb{C}^{p+q}$ and that its only possible poles in a neighborhood of the half space $\bigcap_{j=1}^{p+q}\{\lambda;\operatorname{Re}\lambda_j\ge 0\}$ are along hyperplanes of the form $\sum_{j=1}^{p+q} a_j\lambda_j=0$, $a_j\in\mathbb{Q}_+$. Moreover, by results of Yger, the restriction of $\Gamma_f^{\varphi}(\lambda)$ to any complex line of the form $\{\lambda=(t_1z, ..., t_pz); z\in\mathbb{C}\}$, $t_j\in\mathbb{R}_+$, is analytic at the origin and the value there equals the action of the Coleff–Herrera–Passare current on φ. In the case of codimension two, i.e., when $f=(f_1, f_2)$, it is also known that the corresponding Γ-functions are analytic in some neighborhood of $\bigcap_{j=1}^{2}\{\lambda;\operatorname{Re}\lambda_j\ge 0\}$. Yger has posed the question whether this generalizes to arbitrary codimensions. The purpose of this paper is to prove the following theorem, which answers Yger's question in the affirmative.

Theorem 1. *Let X be a complex manifold of complex dimension n and let $f=(f_1, ..., f_{p+q})\colon X\to\mathbb{C}^{p+q}$ be a holomorphic mapping defining a complete intersection. If N is a positive integer and $\varphi\in\mathcal{D}_{n,n-p}(X)$ is a test form on X then the integral*

$$\Gamma_{f^N}^{\varphi}(\lambda) = \int_X \frac{\bar\partial|f_1|^{2\lambda_1}\wedge...\wedge\bar\partial|f_p|^{2\lambda_p}|f_{p+1}|^{2\lambda_{p+1}}...|f_{p+q}|^{2\lambda_{p+q}}}{f_1^N...f_{p+q}^N}\wedge\varphi, \tag{1}$$

is analytic in a half space $\{\lambda\in\mathbb{C}^{p+q};\operatorname{Re}\lambda_j>-\varepsilon, 1\le j\le p+q\}$ for some $\varepsilon\in\mathbb{Q}_+$ independent of N.

We remark that for non-complete intersections, the Γ-function still is meromorphic in $\mathbb{C}^{p+q}$ but will in general have poles along hyperplanes through the origin.

Our proof of Theorem 1 uses Hironaka's theorem on resolutions of singularities, [10], to reduce to the case when $\{z; f_1(z)...f_{p+q}(z)=0\}$ has normal crossings, i.e., in local coordinates on a blow-up manifold lying above X, the pull-back, $\hat{f}_j$, of the f_j are monomials times invertible holomorphic functions. (For our proof it is actually enough to use the weaker version of Hironaka's theorem where the projection from the blow-up manifold to X is allowed to be "finite-to-one" outside the exceptional divisor.) In general, the $\hat{f}_j$ do not define a complete intersection on the blow-up manifold but the information that the f_j do on the base manifold is coded in the pull-back, $\widehat{\varphi}$, of the test form φ. We are able to recover this information using a Whitney-type division lemma for (anti)-holomorphic forms. It is also worth noticing that for $p=1$, the problem of analytic continuation of $\Gamma^{\varphi}_{f^N}(\lambda)$ is of local nature on the blow-up manifold, i.e., it suffices to consider one chart on the blow-up at a time. This is not the case if $p\geq 2$ and $q\geq 1$, all charts on the blow-up have to be considered simultaneously. We give a simple example showing this in Section 3. In [16] we were able to overcome this problem in the special case when $p=2$ and $q=1$ by quite involved integrations by parts on the blow-up manifold. Very rewarding discussions with Jan-Erik Björk have resulted in a much simpler and more transparent argument based on induction over p.

We continue and give a short historical account of analytic continuation of residue currents. The case $p=0$ and $q=1$ is the most studied one and the analytic continuation was in this case proved by Atiyah in [2] using Hironaka's theorem; see also [5]. The main point was to get a multiplicative inverse of f in the space of currents, and indeed, the value at $\lambda=0$ gives a current U such that $fU=1$ in the sense of currents. At the same time, Dolbeault and Herrera–Lieberman proved, also using Hironaka's theorem, that the principal value current of $1/f$, defined by

$$\mathcal{D}_{n,n}(X) \ni \varphi \longmapsto \lim_{\varepsilon\to 0} \int_{|f|^2>\varepsilon} \frac{\varphi}{f},$$

and denoted $[1/f]$, exists, cf. [8] and [9]. It is elementary to see that this current coincides with the current defined by Atiyah if f is a monomial and for general f it then follows from Hironaka's theorem. A perhaps more conceptual explanation for this equality is that the two definitions are linked via the Mellin transform; recall from above that $\int_X |f|^{2\lambda}\varphi/f$ is the Mellin transform of $\varepsilon\mapsto\int_{|f|^2>\varepsilon}\varphi/f$. The poles of the current-valued function $\lambda\mapsto|f|^{2\lambda}/f$ are closely related to the roots of the Bernstein–Sato polynomial, $b(\lambda)$, associated with f. By Bernstein–Sato theory, see, e.g., [6], f satisfies some functional equation

$$b(\lambda)\bar{f}^{\lambda} = \sum_{j=1}^{p+q} \lambda^j Q_j(\bar{f}^{\lambda+1}),$$

where Q_j are anti-holomorphic differential operators. By iterating m times and multiplying with f^λ/f^N it follows that

$$b(\lambda+m)...b(\lambda)\frac{|f|^{2\lambda}}{f^N}=\sum_{j=1}^{p+q}\lambda^j R_j\left(\bar{f}^m\frac{|f|^{2\lambda}}{f^N}\right)$$

for some anti-holomorphic differential operators R_j. If $\varphi\in\mathcal{D}_{n,n}(X)$ and R_j^* is the adjoint operator of R_j it thus follows that

$$\int_X |f|^{2\lambda}\frac{\varphi}{f^N}=\frac{1}{b(\lambda+m)...b(\lambda)}\sum_{j=1}^{p+q}\lambda^j\int_X |f|^{2\lambda}\frac{\bar{f}^m}{f^N}R_j^*(\varphi). \tag{2}$$

Now, from Kashiwara's result, [11], we know that $b(\lambda)$ has all of its roots contained in the set of negative rational numbers. Hence, we can read off from (2) that the current-valued function $\lambda\mapsto|f|^{2\lambda}/f^N$ has a meromorphic continuation to all of $\mathbb{C}$ and that its poles are contained in arithmetic progressions of the form $\{-s-\mathbb{N}\}$ with $s\in\mathbb{Q}_+$. In particular, $\lambda\mapsto\int_X|f|^{2\lambda}\varphi/f^N$ is holomorphic in some half space $\operatorname{Re}\lambda>-\varepsilon$ for some $\varepsilon\in\mathbb{Q}_+$ independent of N. A detailed study of the poles that actually appear was done by Barlet in [3]. Consider now instead the current-valued function $\lambda\mapsto\bar\partial|f|^{2\lambda}/f$. It is the $\bar\partial$-image of $\lambda\mapsto|f|^{2\lambda}/f$ and has thus also a meromorphic continuation to all of $\mathbb{C}$ with poles contained in arithmetic progressions of the form $\{-s-\mathbb{N}\}$. The value at $\lambda=0$ is now the residue current $\bar\partial[1/f]$, i.e., the $\bar\partial$-image of $[1/f]$. The case when f is one function, i.e., $p+q=1$, is thus well understood. When $p+q>1$, the picture is not that coherent. We have seen that $\Gamma_f^\varphi(\lambda)$ is the iterated Mellin transform of the residue integral but from the examples by Passare–Tsikh and Björk mentioned above, we know that $\Gamma_f^\varphi(\lambda)$ is *not* the Mellin transform of a continuous function in general. A multivariable Bernstein–Sato approach has been considered, but by results of Sabbah [15] the zero set of the multivariable Bernstein–Sato polynomial will in general intersect $\bigcap_{j=1}^{p+q}\{\lambda;\operatorname{Re}\lambda_j\geq 0\}$, and so this method cannot be used to prove our result. On the other hand, it shows that $\Gamma_f^\varphi(\lambda)$ has a meromorphic continuation to all of $\mathbb{C}^{p+q}$. More direct approaches have been considered by, e.g., Berenstein, Gay, Passare, Tsikh, and Yger and the case $q=0$ has got the most attention. For instance, a direct proof of the meromorphic continuation of $\Gamma_f^\varphi(\lambda)$ to all of $\mathbb{C}^p$ can be found in [13]. Also, as mentioned above, it is proved in [17] that the restriction of $\lambda\mapsto\Gamma_f^\varphi(\lambda)$ to any complex line of the form $\{\lambda=(t_1z,...,t_pz);z\in\mathbb{C}\}$, where $t_j\in\mathbb{R}_+$, is analytic at the origin and that the value there equals the Coleff–Herrera product. The first analyticity result in several variables was obtained by Berenstein and Yger. They proved that if $p+q=2$, then $\Gamma_f^\varphi(\lambda)$ is in fact analytic in a half space in $\mathbb{C}^2$ containing the origin, see, e.g., [4] and [13] for proofs. In view of these positive results it has been believed that this holds in general, but to our knowledge no complete proof has appeared.

This paper is organized as follows. Section 2 contains an outline of the proof in the case $p=2$ and $q=1$. This is to show the essential steps without confronting technical and notational difficulties. In Section 3 we give a simple example showing that global effects on the blow-up manifold have to be taken into account when $p\geq 2$ and $q\geq 1$. The detailed proof of Theorem 1 is contained in Section 4.

Acknowledgements. I am grateful to Jan-Erik Björk for his help and support during the preparation of this paper. His insightful comments and suggestions have substantially improved and simplified many arguments as well as the presentation.

2. The main elements of the proof

In this section we illustrate the main new ideas in our proof by considering the case when $p=2$ and $q=1$. Let f_1, f_2, and f_3 be holomorphic functions in $\mathbb{C}^3$ (for simplicity) and assume that the origin is the only common zero. Using the techniques of, e.g., [4] or [13] it is not hard to prove that the current-valued function $\lambda\mapsto(f_1^{-1}\bar\partial|f_1|^{2\lambda_1})f_2^{-1}|f_2|^{2\lambda_2}f_3^{-1}|f_3|^{2\lambda_3}$ can be analytically continued to a neighborhood of the origin; see also Proposition 4 below. Assume now that we can prove that there is a polynomial $P_{12}(\lambda_1,\lambda_2)$, which is a product of linear factors $a\lambda_1+b\lambda_2$ in λ_1 and λ_2, such that the current-valued function

$$\lambda\longmapsto P_{12}(\lambda_1,\lambda_2)\frac{\bar\partial|f_1|^{2\lambda_1}\wedge\bar\partial|f_2|^{2\lambda_2}|f_3|^{2\lambda_3}}{f_1f_2f_3} \tag{3}$$

can be analytically continued to a neighborhood of the origin. That is, we assume for the moment that the only possible poles (close to the origin) of the meromorphic current-valued function $(f_1^{-1}\bar\partial|f_1|^{2\lambda_1})\wedge(f_2^{-1}\bar\partial|f_2|^{2\lambda_2})f_3^{-1}|f_3|^{2\lambda_3}$ are along hyperplanes of the form $a\lambda_1+b\lambda_2=0$. Consider the equality of currents

$$\bar\partial\frac{\bar\partial|f_1|^{2\lambda_1}|f_2|^{2\lambda_2}|f_3|^{2\lambda_3}}{f_1f_2f_3}=-\frac{\bar\partial|f_1|^{2\lambda_1}\wedge\bar\partial|f_2|^{2\lambda_2}|f_3|^{2\lambda_3}}{f_1f_2f_3}-\frac{\bar\partial|f_1|^{2\lambda_1}|f_2|^{2\lambda_2}\wedge\bar\partial|f_3|^{2\lambda_3}}{f_1f_2f_3}, \tag{4}$$

which holds for $\operatorname{Re}\lambda_1,\operatorname{Re}\lambda_2,\operatorname{Re}\lambda_3\gg 1$. We know that the left-hand side can be analytically continued to a neighborhood of the origin and we have assumed that we can prove that the first term on the right-hand side only has poles (close to the origin) along hyperplanes $a\lambda_1+b\lambda_2=0$. The last term on the right-hand side therefore also has only such poles. But, by permuting the indices, we can, assumingly, prove that the last term on the right-hand side only has poles along hyperplanes of the form $a'\lambda_1+b'\lambda_3=0$. We can thus conclude that the only possible pole that the last term on the right-hand side can have is along $\lambda_1=0$. On the other hand, if we switch the indices 1 and 3 in (4) we similarly get that the last term in (4) only has poles along hyperplanes $a''\lambda_2+b''\lambda_3=0$. (The last term is unaffected by the switch modulo a sign.) Its possible pole along $\lambda_1=0$ is thus not present. In conclusion, the last term in (4) has an analytic continuation to a neighborhood of the origin if we can prove the existence of a polynomial $P_{12}(\lambda_1,\lambda_2)$ such that (3) can be analytically continued to a neighborhood of the origin. To do this, we use Hironaka's theorem to compute $\Gamma_f^\varphi(\lambda)$ on a blow-up manifold. More precisely, for some neighborhood U of an arbitrary point in $\mathbb{C}^3$ one can find a blow-up manifold $\mathcal{U}$, lying properly above the base U, such that the preimage $\mathcal{Z}$ of $Z=\{x;f_1(x)f_2(x)f_3(x)=0\}$ has normal crossings and $\mathcal{U}\setminus\mathcal{Z}$ is biholomorphic to $U\setminus Z$. By a partition of unity we may assume that φ has support in such a U and we pull our integral $\Gamma_f^\varphi(\lambda)$ back to $\mathcal{U}$. In local charts on $\mathcal{U}$, we then have that the pullback, $\hat{f}_j$, of the f_j are monomials, $x^{\alpha(j)}$, times invertible holomorphic functions. Let us consider a generic chart where the multiindices $\alpha(1)$, $\alpha(2)$, and $\alpha(3)$ are linearly independent. It is then possible to define new coordinates, still denoted x, such that the invertible holomorphic functions are $\equiv 1$; see, e.g., [12]. We note that, in general, there are also so called charts of resonance where one cannot choose coordinates so that the invertible functions are $\equiv 1$. These charts are responsible for the discontinuity of the residue integral, $I_f^\varphi(\varepsilon)$, but do not cause any problems in our situation. We shall thus consider the integral

$$\int_X \frac{\bar\partial|x^{\alpha(1)}|^{2\lambda_1}\wedge\bar\partial|x^{\alpha(2)}|^{2\lambda_2}|x^{\alpha(3)}|^{2\lambda_3}}{x^{\alpha(1)}x^{\alpha(2)}x^{\alpha(3)}}\wedge\rho\widehat{\varphi}, \tag{5}$$

where ρ is some cut-off function on $\mathcal{U}$. In $\mathbb{C}^3$, we have $\varphi(z)=\sum_{j=1}^3\varphi_j(z)\,dz\wedge d\bar z_j$, and so, by linearity we may assume that φ has a decomposition $\varphi=\phi\wedge\bar\psi$, where $\phi\in\mathcal{D}_{3,0}(\mathbb{C}^3)$ and $\bar\psi$ is the conjugate of a holomorphic 1-form. The pullback $\widehat{\varphi}=\hat\phi\wedge\overline{\hat\psi}$ thus also has such a decomposition. For simplicity we assume that $\hat\psi=h(x)\,dx_3$ for a holomorphic function h on $\mathcal{U}$. Then (5) equals

$$\begin{aligned}\lambda_1\lambda_2\int_X &\frac{|x^{\alpha(1)}|^{2\lambda_1}|x^{\alpha(2)}|^{2\lambda_2}|x^{\alpha(3)}|^{2\lambda_3}}{x^{\alpha(1)}x^{\alpha(2)}x^{\alpha(3)}}\frac{d\bar x^{\alpha(1)}\wedge d\bar x^{\alpha(2)}}{\bar x^{\alpha(1)}\bar x^{\alpha(2)}}\wedge\rho\widehat{\varphi} \\ &=\lambda_1\lambda_2\int_X\frac{|x^{\alpha(1)}|^{2\lambda_1}|x^{\alpha(2)}|^{2\lambda_2}|x^{\alpha(3)}|^{2\lambda_3}}{x^{\alpha(1)}x^{\alpha(2)}x^{\alpha(3)}}A_{12}\frac{d\bar x_1\wedge d\bar x_2}{\bar x_1\bar x_2}\wedge\rho\widehat{\varphi},\end{aligned} \tag{6}$$

where $A_{12}=\alpha(1)_1\alpha(2)_2-\alpha(1)_2\alpha(2)_1$. We may assume that $A_{12}>0$ and so, in particular, $\alpha(1)_1>0$ and $\alpha(2)_2>0$. To avoid having to consider so many cases we also assume that $\alpha(1)_2=\alpha(2)_1=0$. The cases when this is not fulfilled do not cause any additional difficulties and can be treated similarly. Three cases can occur:

(i) Neither x_1 nor x_2 divides $x^{\alpha(3)}$.

(ii) Precisely one of x_1 and x_2 divides $x^{\alpha(3)}$.

(iii) Both x_1 and x_2 divide $x^{\alpha(3)}$.

Consider the case (ii) and assume that x_1 divides $x^{\alpha(3)}$. The variety $V=\{x; f_1(x)=f_3(x)=0\}$ in $\mathbb{C}^3$ has codimension 2 since f_1, f_2, and f_3 define a complete intersection, and so the holomorphic 2-form $df_2\wedge\psi$ has a vanishing pullback to V. Since x_1 divides both $\hat{f}_1=x^{\alpha(1)}$ and $\hat{f}_3=x^{\alpha(3)}$ we see that $d\hat{f}_2\wedge\hat{\psi}=dx^{\alpha(2)}\wedge\hat{\psi}$ must have a vanishing pullback to $\{x;x_1=0\}\subseteq\{x;\hat{f}_1(x)=\hat{f}_3(x)=0\}$. But x_1 does not divide $x^{\alpha(2)}$ and hence, $dx^{\alpha(2)}\wedge\hat{\psi}|_{x_1=0}=0$ in $\mathcal{U}$, where $\hat{\psi}|_{x_1=0}$ means the pullback of $\hat{\psi}$ to $\{x;x_1=0\}$ extended constantly to $\mathcal{U}$. This implies that we may replace $\widehat{\varphi}=\hat{\phi}\wedge\overline{\hat{\psi}}$ in (6) by $\hat{\phi}\wedge(\overline{\hat{\psi}}-\overline{\hat{\psi}}|_{x_1=0})$ without affecting the integral. Now, x_1 divides $\hat{\psi}-\hat{\psi}|_{x_1=0}$ so we may in fact assume that $\widehat{\varphi}$ in (6) is divisible by $\bar{x}_1$, or formulated differently, that $(d\bar{x}_1/\bar{x}_1)\wedge\widehat{\varphi}$ is a smooth form. If we instead consider the case (iii), similar degree arguments give that $dx^{\alpha(1)}\wedge\hat{\psi}|_{x_2=0}=dx^{\alpha(2)}\wedge\hat{\psi}|_{x_1=0}=0$ in $\mathcal{U}$. We may then replace $\widehat{\varphi}$ in (6) by

$$\hat{\phi}\wedge\big(\overline{\hat{\psi}}-\overline{\hat{\psi}}|_{x_1=0}-\overline{\hat{\psi}}|_{x_2=0}+\overline{\hat{\psi}}|_{x_1=x_2=0}\big) \tag{7}$$

without affecting the integral. But (7) is divisible by $\bar{x}_1\bar{x}_2$, and so, in this case, we may assume that $(d\bar{x}_1/\bar{x}_1)\wedge(d\bar{x}_2/\bar{x}_2)\wedge\widehat{\varphi}$ is a smooth form. It is now easy to see that (6) has a meromorphic continuation and that its possible poles close to the origin are along hyperplanes $a\lambda_1+b\lambda_2=0$. In case (i), we write (6) as

$$A_{12}\frac{\lambda_1\lambda_2}{\mu_1\mu_2}\int_X \frac{\bar{\partial}|x_1|^{2\mu_1}\wedge\bar{\partial}|x_2|^{2\mu_2}|x_3|^{2\mu_3}}{x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}}\wedge\rho\widehat{\varphi},$$

where $\mu_j=\sum_{i=1}^3\lambda_i\alpha(i)_j$ and $\alpha_j=\sum_{i=1}^3\alpha(i)_j$. It is an easy one-variable problem to see that this integral (without the coefficient) has an analytic continuation to a neighborhood of the origin; cf., e.g., Lemma 2.1 in [1]. Since neither x_1 nor x_2 divides $x^{\alpha(3)}$, i.e., $\alpha(3)_1=\alpha(3)_2=0$, we have $\mu_1=\alpha(1)_1\lambda_1+\alpha(2)_1\lambda_2$ and $\mu_2=\alpha(1)_2\lambda_1+\alpha(2)_2\lambda_2$ and it follows that (6) only has poles of the allowed type in the case (i). In the case (ii) (with x_1 dividing $x^{\alpha(3)}$) we write (6) as

$$-A_{12}\frac{\lambda_1\lambda_2}{\mu_2}\int_X \frac{|x_1|^{2\mu_1}\bar{\partial}|x_2|^{2\mu_2}|x_3|^{2\mu_3}}{x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}}\wedge\frac{d\bar{x}_1}{\bar{x}_1}\wedge\rho\widehat{\varphi}.$$

Now $\mu_2=\alpha(1)_2\lambda_1+\alpha(2)_2\lambda_2$, since x_2 does not divide $x^{\alpha(3)}$, and from our considerations above we may assume that $(d\bar{x}_1/d\bar{x}_1)\wedge\widehat{\varphi}$ is smooth. It follows that (6) only has the allowed type of poles in the case (ii) as well. The case (iii) is easier; then we may assume that $(d\bar{x}_1/\bar{x}_1)\wedge(d\bar{x}_2/\bar{x}_2)\wedge\widehat{\varphi}$ is a smooth form and (6) is in this case even analytic at the origin.

Remark 2. As mentioned in the introduction, and shown in the next section, it is necessary to take global effects on the blow-up manifold into account when proving analyticity of (1) beyond the origin. However, as indicated by the above argument, the problem of showing that (1) only has poles along hyperplanes of the form $\sum_{j=1}^{p} a_j\lambda_j=0$ is of a local nature on the blow-up; cf. [16].

3. An example

We present a simple example showing that global effects on the blow-up manifold have to be taken into account when proving our result for $p\geq 2$ and $q\geq 1$. Consider the integral

$$\int_X \frac{|x_1|^{2\lambda_1}\bar{\partial}|x_2|^{2\lambda_2}\wedge\bar{\partial}|x_3|^{2\lambda_3}}{x_1x_2x_3}\wedge\varphi(x)dx\wedge d\bar{x}_1 \tag{8}$$

in $\mathbb{C}^3$, where φ is a function defined as follows. Let ϕ, φ_2 and φ_3 be smooth functions on $\mathbb{C}$ with support close to the origin but non-vanishing there, and put $\varphi_1=\partial\phi/\partial\bar{z}$. We define $\varphi(x)$ to be the function $\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)$ in $\mathbb{C}^3$. Note that (8) equals

$$\int_X \frac{|x_1|^{2\lambda_1}|x_2|^{2\lambda_2}|x_3|^{2\lambda_3}}{x_1x_2x_3}\varphi_1\frac{\partial\varphi_2}{\partial\bar{x}_2}\frac{\partial\varphi_3}{\partial\bar{x}_3}\,dx\wedge d\bar{x}$$

after two integrations by parts, from which we see that (8) is analytic at $\lambda=0$. Now we blow up $\mathbb{C}^3$ along the x_1-axis and look at the pullback of (8) to this manifold. Let $\pi\colon\mathbb{C}\times\mathcal{B}_0\mathbb{C}^2\to\mathbb{C}^3$ be the blow-up map. In the natural coordinates z and ζ on $\mathbb{C}\times\mathcal{B}_0\mathbb{C}^2$ it then looks like

$$\pi(z_1,z_2,z_3)=(z_1,z_2,z_2z_3),$$
$$\pi(\zeta_1,\zeta_2,\zeta_3)=(\zeta_1,\zeta_2\zeta_3,\zeta_2).$$

Since φ has support close to the origin, $\pi^*\varphi$ has support close to $\pi^{-1}(0)=\{z;z_1=z_2=0\}\cup\{\zeta;\zeta_1=\zeta_2=0\}\cong\mathbb{CP}^1$. Note that z_3 and ζ_3 are natural coordinates on this $\mathbb{CP}^1$ and choose a partition of unity, $\{\rho_1,\rho_2\}$ on $\operatorname{supp}(\pi^*\varphi)$ such that $\operatorname{supp}(\rho_1)\subset\{z;|z_3|<2\}$ and $\operatorname{supp}(\rho_2)\subset\{\zeta;|\zeta_3|<2\}$. The pullback of (8) under π now equals

$$\int_X \frac{|z_1|^{2\lambda_1}\bar{\partial}|z_2|^{2\lambda_2}\wedge\bar{\partial}|z_2z_3|^{2\lambda_3}}{z_1z_2^2z_3}\wedge\rho_1(z)\varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_2z_3)z_2\,dz\wedge d\bar{z}_1$$
$$-\int_X \frac{|\zeta_1|^{2\lambda_1}\bar{\partial}|\zeta_2\zeta_3|^{2\lambda_2}\wedge\bar{\partial}|\zeta_2|^{2\lambda_3}}{\zeta_1\zeta_2^2\zeta_3}\wedge\rho_2(\zeta)\varphi_1(\zeta_1)\varphi_2(\zeta_2\zeta_3)\varphi_3(\zeta_2)\zeta_2\,d\zeta\wedge d\bar{\zeta}_1.$$

We know that this sum (difference) is analytic at $\lambda=0$ but we will see that none of the terms are. Consider the first term. It is easily verified that it can be written as

$$\frac{\lambda_2}{\lambda_2+\lambda_3}\int_X \frac{|z_1|^{2\lambda_1}\bar\partial|z_2|^{2(\lambda_2+\lambda_3)}\wedge\bar\partial|z_3|^{2\lambda_3}}{z_1z_2z_3}\rho_1(z)\varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_2z_3)\,dz\wedge d\bar z_1.$$

We denote this integral, with the coefficient $\lambda_2/(\lambda_2+\lambda_3)$ removed, by $I(\lambda)$. After two integrations by parts one sees that $I(\lambda)$ is analytic at the origin, and so $\lambda_2 I(\lambda)/(\lambda_2+\lambda_3)$ is analytic at the origin if and only if $I(\lambda)$ vanishes on the hyperplane $\lambda_2+\lambda_3=0$. In particular we must have that $I(0)=0$. But $I(0)$ can be computed using Cauchy's formula, and one obtains $I(0)=-(2\pi i)^3\phi(0)\varphi_2(0)\varphi_3(0)\neq 0$, where i here, but not elsewhere, denotes the imaginary unit.

Remark 3. This example could be a little confusing. The variable z_1 just appears as a "dummy variable" in the computations above, to which nothing interesting happens. This indicates that global effects appear already in the case $p=2$ and $q=0$. It is in fact so, but in this case the analyticity follows, simply by applying to $\bar\partial$-exact test forms, if we can prove analyticity of

$$(\lambda_1,\lambda_2)\longmapsto\int_X \frac{|f_1|^{2\lambda_1}\bar\partial|f_2|^{2\lambda_2}}{f_1\cdot f_2}\wedge\varphi,\quad \varphi\in\mathcal{D}_{n,n-1}(X).$$

This can actually be done using only local arguments, see, e.g., Proposition 4 below. The case $p=2$, $q=0$ can therefore be *reduced* to a case where only local arguments are needed, but for $p\geq 2$ and $q\geq 1$ this is in general not possible.

4. The proof

We give here the detailed proof of Theorem 1. We begin with the following proposition, whose proof relies on the Whitney-type division lemma (Lemma 5) below.

Proposition 4. *Let $f=(f_1,...,f_p)\colon X\to\mathbb{C}^p$ be a holomorphic mapping defining a complete intersection and let $g_1,...,g_q$ be holomorphic functions on X such that $(f_1,...,f_p,g_j)$ defines a complete intersection for each $j=1,...,q$. Let also $\varphi\in\mathcal{D}_{n,n-p}(X)$ be a test form and N a positive integer. Then, for some $\varepsilon\in\mathbb{Q}_+$ independent of N, the function*

$$\Gamma^{\varphi}_{f,g}(\lambda)=\int_X \frac{\bar\partial|f_1|^{2\lambda_1}\wedge...\wedge\bar\partial|f_p|^{2\lambda_p}|g_1|^{2\lambda_{p+1}}...|g_q|^{2\lambda_{p+q}}}{f_1^N...f_p^N g_1^N...g_q^N}\wedge\varphi,$$

originally defined when all $\operatorname{Re}\lambda_j$ are large, has a meromorphic continuation to all of $\mathbb{C}^{p+q}$ and its only possible poles in the half space $H=\{\lambda\in\mathbb{C}^{p+q};\operatorname{Re}\lambda_j>-\varepsilon,$ $1\leq j\leq p+q\}$ are along hyperplanes of the form $\sum_{j=1}^p a_j\lambda_j=0$, where $a_j\in\mathbb{N}$ and at *least two of the a_j are non-zero. In particular, if $p=1$ then $\Gamma^{\varphi}_{f,g}(\lambda)$ is analytic in H.*

Proof. It is well known that $\Gamma^{\varphi}_{f,g}(\lambda)$ has a meromorphic continuation to all of $\mathbb{C}^{p+q}$ so we only check that its possible poles in H are of the prescribed form. We will compute $\Gamma^{\varphi}_{f,g}(\lambda)$ by pulling the integral back to a blow-up manifold, $\mathcal{X}$, given by Hironaka's theorem, where the variety $\{x;\hat{f}_1(x)...\hat{f}_p(x)\cdot\hat{g}_1(x)...\hat{g}_q(x)=0\}$ has normal crossings; cf. Section 2. (The hat, $\hat{}$, means pullback to the blow-up.) We can thus write

$$\Gamma^{\varphi}_{f,g}(\lambda)=\sum_{\rho}\int_{\mathcal{X}}\frac{\bar{\partial}|\hat{f}_1|^{2\lambda_1}\wedge...\wedge\bar{\partial}|\hat{f}_p|^{2\lambda_p}|\hat{g}_1|^{2\lambda_{p+1}}...|\hat{g}_q|^{2\lambda_{p+q}}}{\hat{f}_1^N...\hat{f}_p^N\hat{g}_1^N...\hat{g}_q^N}\wedge\rho\widehat{\varphi},$$

where $\{\rho\}$ is a partition of unity of $\operatorname{supp}(\widehat{\varphi})$ and each ρ has support in a coordinate chart where $\hat{f}_i$ and $\hat{g}_j$ are monomials times invertible holomorphic functions. Let us consider a chart with holomorphic coordinates x in which $\hat{f}_1=u_1x^{\alpha(1)},...,\hat{f}_p=u_px^{\alpha(p)}$ and $\hat{g}_1=v_1x^{\beta(1)},...,\hat{g}_q=v_qx^{\beta(q)}$, where the u_i and the v_j are invertible and holomorphic. Denote by m the number of vectors in a maximal linearly independent subset of $\{\alpha(1),...,\alpha(p)\}$ and assume for simplicity that $\alpha(1),...,\alpha(m)$ are linearly independent. It is then possible to define new coordinates, still denoted by x, such that $u_1=...=u_m\equiv1$ in the new coordinates; see, e.g., [12, p. 46]. Now, for each $j=m+1,...,p$, $\alpha(j)$ is a linear combination of $\alpha(1),...,\alpha(m)$ and it follows from exterior algebra that $dx^{\alpha(j)}\wedge dx^{\alpha(1)}\wedge...\wedge dx^{\alpha(m)}=0$. In the x-chart, the term we are looking at can therefore be written

$$\int_X\frac{\bar{\partial}|x^{\alpha(1)}|^{2\lambda_1}\wedge...\wedge\bar{\partial}|x^{\alpha(m)}|^{2\lambda_m}|x^{\lambda\gamma}|^2}{x^{N\alpha}x^{N\beta}}\rho V^{\lambda}U^{\lambda}\wedge d\bar{u}_{m+1}\wedge...\wedge d\bar{u}_p\wedge\widehat{\varphi}, \tag{9}$$

where we have introduced the notation:

$$\alpha=\sum_{j=1}^{p}\alpha(j),$$

$$\beta=\sum_{j=1}^{q}\beta(j),$$

$$\lambda\gamma=\sum_{j=m+1}^{p}\lambda_j\alpha(j)+\sum_{j=1}^{q}\lambda_{p+j}\beta(j),$$

$$V^{\lambda}=\frac{|v_1|^{2\lambda_{p+1}}...|v_q|^{2\lambda_{p+q}}}{(v_1...v_q)^N},$$

$$U^{\lambda}=\lambda_{m+1}...\lambda_p\frac{|u_{m+1}|^{2\lambda_{m+1}-2}...|u_p|^{2\lambda_p-2}}{(u_{m+1}...u_p)^{N-1}}.$$

Let $K\subseteq\{1,...,n\}$ be the set of indices i such that x_i divides at least some $\hat{g}_j$. We will use the following division lemma, proved below, to replace the form $d\bar{u}_{m+1}\wedge ...\wedge d\bar{u}_p\wedge\widehat{\varphi}$ in (9) by another one, which vanishes on the variety $\{x;\prod_{i\in K}x_i=0\}$.

Lemma 5. *If ψ is a holomorphic $(n-p)$-form on the base manifold X, then one can find explicitly a holomorphic $(n-m)$-form ω in the x-chart on $\mathcal{X}$ such that*

(i) $(dx_j/x_j)\wedge(du_{m+1}\wedge...\wedge du_p\wedge\hat{\psi}-\omega)$ *is non-singular for all $j\in K$, and*

(ii) $dx^{\alpha(1)}\wedge...\wedge dx^{\alpha(m)}\wedge\omega=0$.

By linearity, we may assume that φ is decomposable and write $\varphi=\phi_1\wedge\bar{\phi}_2$, where $\phi_1\in\mathcal{D}_{n,0}(X)$ and ϕ_2 is a holomorphic $(n-p)$-form. With ϕ_2 as in-data to Lemma 5 we thus see that we may replace $d\bar{u}_{m+1}\wedge...\wedge d\bar{u}_p\wedge\widehat{\varphi}$ in (9) by an $(n,n-m)$-form ξ, without affecting the integral, such that $(d\bar{x}_j/\bar{x}_j)\wedge\xi$ is smooth for all $j\in K$. It follows that for any $L\subseteq K$, $\bigwedge_{j\in L}(d\bar{x}_j/\bar{x}_j)\wedge\xi$ is a smooth form. Using Leibniz' rule to expand the expressions $\bar{\partial}|x^{\alpha(j)}|^{2\lambda_j}$, $1\leq j\leq m$, the integral (9) can be written

$$\text{(10)}\qquad \sum_{i_1<...<i_m}\det(A(i_1,...,i_m))\int_{\mathcal{X}}\frac{|x^{\lambda(\alpha+\beta)}|^2}{x^{N(\alpha+\beta)}}\frac{d\bar{x}_{i_1}\wedge...\wedge d\bar{x}_{i_m}}{\bar{x}_{i_1}...\bar{x}_{i_m}}\wedge\Phi(x;\lambda),$$

where $A(i_1,...,i_m)$ is the matrix $(\alpha(i_k)_{i_l})_{k,l}$,

$$\lambda(\alpha+\beta)=\sum_{j=1}^{p}\lambda_j\alpha(j)+\sum_{j=1}^{q}\lambda_{p+j}\beta(j),$$

and

$$\Phi(x;\lambda)=\lambda_1...\lambda_m\rho V^{\lambda}U^{\lambda}\wedge\xi.$$

We emphasize that Φ is a smooth compactly supported form depending analytically on λ and that $\bigwedge_{j\in L}(d\bar{x}_j/\bar{x}_j)\wedge\Phi$, $L\subseteq K$, also is. For notational convenience, we consider the term of (10) with $i_j=j$ and to make our considerations non-trivial we then assume that $A:=A(1,...,m)$ is non-singular. Furthermore, we assume, also for simplicity, that $1,...,k\notin K$ and that $k+1,...,m\in K$. If we put $\mu=\lambda(\alpha+\beta)$ we can write the term under consideration as

$$\text{(11)}\qquad \frac{\det(A)}{\mu_1...\mu_k}\int_{\mathcal{X}}\frac{\bar{\partial}|x_1|^{2\mu_1}\wedge...\wedge\bar{\partial}|x_k|^{2\mu_k}|x_{k+1}|^{2\mu_{k+1}}...|x_n|^{2\mu_n}}{x^{N(\alpha+\beta)}}$$
$$\wedge\frac{d\bar{x}_{k+1}\wedge...\wedge d\bar{x}_m}{\bar{x}_{k+1}...\bar{x}_m}\wedge\Phi(x;\lambda).$$

Here, the expression on the second row is a smooth compactly supported form depending analytically on λ. After this observation it is a one-variable problem to see that the integral, without the coefficient in front, has an analytic continuation to some half space H independent of N; see, e.g., Lemma 2.1 in [1]. The possible poles are therefore only along hyperplanes of the form $\mu_j=0$. Fix a j with $1\le j\le k$. Then $j\notin K$, which means that x_j does not divide any of the $\hat{g}$-functions. Hence, $\beta(1)_j=...=\beta(q)_j=0$, and consequently, $\mu_j=\sum_{i=1}^p \alpha(i)_j\lambda_i$. Moreover, if μ_j happens to be proportional to some λ_i then, first of all, x_j must divide $x^{\alpha(i)}$ but no other $x^{\alpha(l)}$ (or any $x^{\beta(l)}$). Secondly, the term (11) of (10) that we are considering must have arisen from the term in the Leibniz expansion of (9) when the $\bar\partial$ in front of $|x^{\alpha(i)}|^{2\lambda_i}$ has fallen on $|x_j^{\alpha(i)_j}|^{2\lambda_i}$. Thus, $i\le m$ and no other μ_ν with $1\le\nu\le k$ can be proportional to λ_i. Since $\Phi(x;\lambda)$ is divisible by λ_i we can therefore cancel poles along hyperplanes $\mu_j=0$ if μ_j is proportional to some λ_i. In conclusion, $\Gamma^{\varphi}_{f,g}(\lambda)$ has a meromorphic continuation to some half space H with possible poles only along hyperplanes of the form $\sum_{j=1}^p a_j\lambda_j=0$, where $a_j\in\mathbb{N}$ and at least two a_j must be non-zero. □

Proof of Lemma 5. Put $\Psi=du_{m+1}\wedge...\wedge du_p\wedge\hat\psi$ and define

$$\omega=\sum_{j\in K}\Psi_j-\sum_{\substack{i,j\in K\\ i<j}}\Psi_{ij}+...+(-1)^{|K|-1}\Psi_{i_1...i_{|K|}},$$

where $\Psi_{i_1...i_l}$ means the pullback of Ψ to $\{x;x_{i_1}=...=x_{i_l}=0\}$ extended constantly to $\mathbb{C}^n$. A straightforward induction over $|K|$ shows that ω so defined satisfies (i). (See also [16].) To see that ω satisfies (ii), consider a $\Psi_{i_1...i_l}$. Let L be the set of indices j such that no x_{i_k}, $1\le k\le l$, divides $\hat{f}_j$ and write $L=L'\cup L''$, where $L'=\{j\in L;j\le m\}$ and $L''=\{j\in L;m+1\le j\le p\}$. For each x_{i_k}, with $1\le k\le l$, we know that x_{i_k} divides some $\hat{g}$-function, say $\hat{g}_{j_k}$. The variety $\{x;x_{i_1}=...=x_{i_l}=0\}$ is then contained in $\{x;\hat{g}_{j_1}(x)=...=\hat{g}_{j_l}(x)=0\}\cap\bigcap_{i\notin L}\{x;\hat{f}_i(x)=0\}$, i.e., in the preimage of $V:=\{x;g_{j_1}(x)=...=g_{j_l}(x)=0\}\cap\bigcap_{i\notin L}\{x;f_i(x)=0\}$. Since (f,g_j) defines a complete intersection for any j, the variety V has codimension at least $p-|L|+1$. Now, the form $\bigwedge_{j\in L}df_j\wedge\psi$ has degree $n-p+|L|$ and thus, has a vanishing pullback to V. Hence, we get that

$$\Big(\bigwedge_{j\in L}df_j\wedge\psi\Big)^{\wedge}=\bigwedge_{j\in L}d\hat{f}_j\wedge\hat\psi=\bigwedge_{i\in L'}dx^{\alpha(i)}\wedge\bigwedge_{j\in L''}d(u_jx^{\alpha(j)})\wedge\hat\psi$$

has a vanishing pullback to $\{x;x_{i_1}=...=x_{i_l}=0\}$. But this means that

$$x^{\sum_{i\in L''}\alpha(i)}\bigwedge_{j\in L'}dx^{\alpha(j)}\wedge\Big(\bigwedge_{k\in L''}du_k\wedge\hat\psi\Big)_{i_1...i_l}+\bigwedge_{\iota\in L'}dx^{\alpha(\iota)}\wedge\sum_{\nu\in L''}dx^{\alpha(\nu)}\wedge\xi_\nu=0$$

for some forms ξ_ν, where the first term arises when no differential hits any $x^{\alpha(j)}$, $j\in L''$. Taking the exterior product with $\bigwedge_{j\notin L''}(du_j)_{i_1\dots i_l}$ we obtain

$$x^{\sum_{i\in L''}\alpha(i)}\bigwedge_{j\in L'}dx^{\alpha(j)}\wedge\Psi_{i_1\dots i_l}+\bigwedge_{\iota\in L'}dx^{\alpha(\iota)}\wedge\sum_{\nu\in L''}dx^{\alpha(\nu)}\wedge\tilde{\xi}_\nu=0.$$

We now multiply this equation with the exterior product of all $dx^{\alpha(j)}$ with $j\le m$ and $j\notin L'$. Then we get $dx^{\alpha(1)}\wedge\dots\wedge dx^{\alpha(m)}$ in front of the sum and this makes all terms under the summation sign disappear since every $\alpha(\nu)$, with $\nu\in L''$, is a linear combination of $\alpha(1),\dots,\alpha(m)$. It thus follows that

$$x^{\sum_{i\in L''}}dx^{\alpha(1)}\wedge\dots\wedge dx^{\alpha(m)}\wedge\Psi_{i_1\dots i_l}=0,$$

and since this holds everywhere we may remove the factor $x^{\sum_{i\in L''}}$ and conclude that ω has the property (ii). □

Proof of Theorem 1. The proof is based on induction over p. The induction start, $p=1$, follows from Proposition 4. Assume therefore that the theorem is proved for $p=k$. We introduce the notation $\gamma(\lambda_{i_1},\dots,\lambda_{i_p};\lambda_*)$ for the current-valued function

$$\frac{\bar\partial|f_{i_1}|^{2\lambda_{i_1}}\wedge\dots\wedge\bar\partial|f_{i_p}|^{2\lambda_{i_p}}|f_{j_1}|^{2\lambda_{j_1}}\dots|f_{j_q}|^{2\lambda_{j_q}}}{f_1^N\dots f_{p+q}^N}.$$

When all $\operatorname{Re}\lambda_j$ are large we have the equality of currents

$$\bar\partial\gamma(\lambda_1,\dots,\lambda_k;\lambda_*)=(-1)^k\sum_{j=1}^{q}\gamma(\lambda_1,\dots,\lambda_k,\lambda_{k+j};\lambda_*),$$

and by our assumption, the left-hand side is analytic in some half space H independent of N. Moreover, by Proposition 4, the term on the right-hand side corresponding to j has only poles along hyperplanes of the form $a_{k+j}\lambda_{k+j}+\sum_{i=1}^k a_i\lambda_i=0$. It thus follows that $\gamma(\lambda_1,\dots,\lambda_{k+1};\lambda_*)$, on one hand, only has poles along hyperplanes $\sum_{i=1}^{k+1}a_i\lambda_i=0$ and, on the other, only has poles along hyperplanes $a_{k+j}\lambda_{k+j}+\sum_{i=1}^k a_i\lambda_i=0$ with $j>1$. But then $\gamma(\lambda_1,\dots,\lambda_{k+1};\lambda_*)$ can only have poles along hyperplanes of the form $\sum_{i=1}^k a_i\lambda_i=0$. Consider now the current equality

$$\begin{aligned}\bar\partial\gamma(\lambda_1,\dots,\lambda_{k-1},\lambda_{k+1};\lambda_*)&=(-1)^{k+1}\gamma(\lambda_1,\dots,\lambda_{k+1};\lambda_*)\\&\quad+(-1)^k\sum_{i=2}^{q}\gamma(\lambda_1,\dots,\lambda_{k-1},\lambda_{k+1},\lambda_{k+i};\lambda_*).\end{aligned}$$

From this it follows similarly that $\gamma(\lambda_1,\dots,\lambda_{k+1};\lambda_*)$ can only have poles along hyperplanes of the form $a_{k+1}\lambda_{k+1}+a_{k+j}\lambda_{k+j}+\sum_{i=1}^{k-1}a_i\lambda_i=0$ with $j>1$. Since we know that its only poles are along $\sum_{i=1}^{k} a_i\lambda_i=0$, we see that it in fact only can have poles along $\sum_{i=1}^{k-1} a_i\lambda_i=0$. Continuing in this way, looking at appropriate current equalities and using the induction hypothesis and Proposition 4, we eventually see that $\gamma(\lambda_1, ..., \lambda_{k+1}; \lambda_*)$ cannot have any poles at all in H. This concludes the induction step and consequently the proof of Theorem 1. □

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Håkan Samuelsson
Mathematical Sciences
Chalmers University of Technology and
the University of Gothenburg
SE-412 96 Göteborg Sweden
hasam@chalmers.se

Received September 26, 2006
in revised form September 11, 2007
published online August 27, 2008