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Decomposable symmetric mappings between infinite-dimensional spaces

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Abstract. Decomposable mappings from the space of symmetric k-fold tensors over E, $\bigotimes_{s,k} E$, to the space of k-fold tensors over F, $\bigotimes_{s,k} F$, are those linear operators which map nonzero decomposable elements to nonzero decomposable elements. We prove that any decomposable mapping is induced by an injective linear operator between the spaces on which the tensors are defined. Moreover, if the decomposable mapping belongs to a given operator ideal, then so does its inducing operator. This result allows us to classify injective linear operators between spaces of homogeneous approximable polynomials and between spaces of nuclear polynomials which map rank-1 polynomials to rank-1 polynomials.

1. Introduction

Given Banach spaces E and F there are, in general, far too many linear mappings from E into F to allow a systematic classification. Extra conditions either on the spaces or the mappings can sometimes make our task realisable. Symmetric tensor products and spaces of homogeneous polynomials are places where this additional structure is available. Indeed, in [4] we were able to classify all linear mappings of the spaces of symmetric injective tensors which map the set of powers to itself. This in turn allowed us to characterise the isometries of various spaces of homogeneous polynomials. Decomposable mappings are similar to power preserving mappings and in this paper we shall look for a characterisation of decomposable mappings analogous to that obtained in [4] for power preserving mappings. This will allow us to describe mappings between spaces of homogeneous polynomials which preserve rank-1 polynomials.

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The main result of this paper is the characterisation of decomposable mappings between spaces of symmetric tensor products. The approach we shall take is based on that of Cummings [8] in his description of decomposable mappings between spaces of symmetric tensor products over finite-dimensional vector spaces. However, the infinite-dimensional setting requires a more general approach and some of Cummings definitions must be altered to suit this more general situation.

In Section 2 we present the results closely related with the finite-dimensional case, where there is no need to consider any particular tensor norm. In the final three sections we will examine decomposable mappings which belong to a given operator ideal. To facilitate this, in Section 3 we will examine families of symmetric tensor norms of different degrees on infinite-dimensional spaces defined in such a way that we have natural complementations of the space of symmetric tensors of a fixed degree in spaces of symmetric tensors of higher degrees.

In Sections 4 and 5 we characterise the decomposable mappings between symmetric tensor spaces of the same and different degrees, respectively. We prove that any decomposable linear operator T from a symmetric k-fold tensor product over a Banach space E into a symmetric l-fold tensor product over a Banach space F, with $k\leq l$, considered with suitable symmetric tensor norms, is a power of an injective operator A from E into F, multiplied by a fixed $l-k$ decomposable tensor. Moreover, if T belongs to any given ideal of operators, then so does also A.

Because of the one-to-one correspondence between symmetric injective tensor products and approximable polynomials and the one-to-one correspondence between spaces of symmetric projective tensor products and nuclear polynomials, for spaces whose dual has the approximation property, we will be also able to describe rank-1 preserving mappings between spaces of approximable polynomials and rank-1 preserving mappings between spaces of nuclear polynomials. Results of this type are discussed in Section 6.

Given a Banach space E we can form the space $\bigotimes_k E$ of all k-fold tensors in E. We consider the subspace, $\bigotimes_{s,k}E$, of $\bigotimes_k E$ consisting of all tensors of the form $\sum_{i=1}^n \lambda_i x_i\otimes\ldots\otimes x_i$, where $\lambda_i=\pm 1$. Such k-fold tensors are said to be symmetric. There are two natural symmetric norms defined on the, in general, uncompleted space $\bigotimes_{s,k}E$, the least or injective and the greatest or projective symmetric tensor norms.

Given a k-fold symmetric tensor $\sum_{i=1}^n \lambda_i x_i\otimes\ldots\otimes x_i$ on E we define its *symmetric injective* or $\varepsilon_{s,k}$*-norm* by

$$\sup_{\phi\in B_{E'}}\left|\sum_{i=1}^n \lambda_i \phi(x_i)^k\right|.$$

We denote the completion of $\bigotimes_{s,k}E$ with respect to this norm by $\widehat{\bigotimes}_{s,k,\varepsilon_{s,k}}E$.

The *symmetric projective* or $\pi_{s,k}$*-norm* is defined as follows. Suppose that θ belongs to $\bigotimes_{s,k}E$, $\theta=\sum_{i=1}^{n}\lambda_i x_i\otimes\ldots\otimes x_i$. We would like to define the norm of θ to be $\sum_{i=1}^{n}\|x_i\|^k$. Unfortunately tensors in $\bigotimes_k E$ do not have unique representation as a sum of basic symmetric tensors. Therefore the projective norm of θ is defined to be the infimum of $\sum_{i=1}^{n}\|x_i\|^k$ over all possible representations of θ as sums of basic symmetric tensors. We use $\widehat{\bigotimes}_{s,k,\pi_{s,k}}E$ to denote the completion of $\bigotimes_{s,k,\pi_{s,k}}E$ with respect to this norm.

When the underlying space is a dual space both of these spaces of symmetric tensor products have representations as spaces of homogeneous polynomials. A function $P\colon E\to\mathbf{K}$ ($\mathbf{K}=\mathbf{R}$ or $\mathbf{K}=\mathbf{C}$) is said to be a (*continuous*) k*-homogeneous polynomial* if there exists a unique symmetric (continuous) k-linear map

$$L_P\colon \underbrace{E\times\ldots\times E}_{k\text{-times}}\longrightarrow\mathbf{K}$$

such that $P(x)=L_P(x,\ldots,x)$ for all $x\in E$. Continuous k-homogeneous polynomials are bounded on the unit ball and we denote by $\mathcal{P}(^kE)$ the Banach space of all continuous k-homogeneous polynomials on E endowed with the norm:

$$P\longmapsto\|P\|:=\sup_{\|x\|\le 1}|P(x)|.$$

A k-homogeneous polynomial P in $\mathcal{P}(^kE)$ is said to be of *finite type* if there is $\{\phi_j\}_{j=1}^{n}$ in E' such that $P(x)=\sum_{j=1}^{n}\pm\phi_j(x)^k$ for all x in E. Polynomials in the closure of the finite type k-homogeneous polynomials in $\mathcal{P}(^kE)$ are called *approximable polynomials*. We use $\mathcal{P}_f(^kE)$ to denote the space of all finite type k-homogeneous polynomials and $\mathcal{P}_A(^kE)$ to denote the space of all k-homogeneous approximable polynomials. The mapping $\phi^k\mapsto\phi\otimes\ldots\otimes\phi$ induces an isometric isomorphism between the spaces $\mathcal{P}_A(^kE)$ and $\widehat{\bigotimes}_{s,k,\varepsilon_{s,k}}E'$.

In relation with the projective norm, the space of nuclear polynomials comes into play. We say that a k-homogeneous polynomial P on a Banach space E is *nuclear* if there is a bounded sequence $\{\phi_j\}_{j=1}^{\infty}\subset E'$ and a sequence $\{\lambda_j\}_{j=1}^{\infty}$ in ℓ_1 such that

$$P(x)=\sum_{j=1}^{\infty}\lambda_j\phi_j(x)^k$$

for every x in E. The space of all k-homogeneous nuclear polynomials on E is denoted by $\mathcal{P}_N(^kE)$ and becomes a Banach space when the norm of P, $\|P\|_N$, is given as the infimum of $\sum_{j=1}^{\infty}|\lambda_j|\,\|\phi_j\|^k$ taken over all representations of P of the form described above. When E' has the approximation property $(\mathcal{P}_N(^kE),\|\cdot\|_N)$ is isometrically isomorphic to $\widehat{\bigotimes}_{s,k,\pi_k}E'$ under the map induced by $\phi^k\mapsto\phi\otimes\ldots\otimes\phi$.

A k-homogeneous polynomial P is said to be a *rank*-1 *polynomial* if it has the form $P=\phi_1\phi_2...\phi_k$ for $\phi_1,\phi_2,...,\phi_k$ in E'.

We refer the reader to [11] for further information on homogeneous polynomials and symmetric tensor products and to [17] and [18] for an overview of decomposable mappings on finite-dimensional vector spaces.

2. Decomposable mappings and adjacent subspaces

The space of k-fold symmetric tensors, $\bigotimes_{s,k} E$, is a complemented subspace of the space of full k-fold tensors, $\bigotimes_k E$, with projection given by

$$x_1\otimes...\otimes x_k \longmapsto \frac{1}{k!}\sum_{\sigma\in S_k} x_{\sigma(1)}\otimes...\otimes x_{\sigma(k)},$$

where S_k is the symmetric group on $\{1,...,k\}$. Symmetric tensors in $\bigotimes_{s,k} E$ of the form $(1/k!)\sum_{\sigma\in S_k} x_{\sigma(1)}\otimes...\otimes x_{\sigma(k)}$ are called decomposable elements. To simplify the notation we let $x_1\vee...\vee x_k:=(1/k!)\sum_{\sigma\in S_k} x_{\sigma(1)}\otimes...\otimes x_{\sigma(k)}$ and $x^k:=x\vee x\vee...\vee x$.

A decomposable element $x_1\vee...\vee x_k$ is zero if and only if $x_i=0$ for at least one $i=1,...,k$. Moreover,

$$\begin{aligned}(1)\qquad & x_1\vee...\vee x_k=y_1\vee...\vee y_k\neq 0\\ &\iff \quad y_i=\lambda_i x_{\sigma(i)},\ i=1,...,k, \text{ for some } \sigma\in S_k \text{ and } \lambda_1,...,\lambda_k,\end{aligned}$$

where $\lambda_1,...,\lambda_k$ are scalars such that $\lambda_1\lambda_2...\lambda_k=1$.

A linear mapping T between spaces of symmetric tensors over E and F which contain $\bigotimes_{s,k} E$ and $\bigotimes_{s,k} F$, respectively, is said to be a *power-preserver* if given any x in E there exists y in F so that $T(x^k)=\pm y^k$, while T is said to be *decomposable* if T maps decomposable elements to decomposable elements and $\ker T\cap\{x_1\vee...\vee x_k: x_1,...,x_k\in E\}=\{0\}$. We note that if T is injective and maps decomposable elements to decomposable elements then T is decomposable.

Since decomposable mappings preserve both the decomposable and the linear structure of spaces of symmetric tensors, they will also preserve subspaces of the space of k-fold symmetric tensors where all elements are decomposable tensors. Such subspaces are called *decomposable subspaces* and their study will provide us with a method of classifying decomposable mappings. The simplest way of constructing a decomposable subspace of $\bigotimes_{s,k} E$ is to take a subspace S of E and vectors $x_1,...,x_{k-1}$ in E and to set

$$M=x_1\vee x_2\vee...\vee x_{k-1}\vee S=\{x_1\vee x_2\vee...\vee x_{k-1}\vee s: s\in S\}.$$

We call $\langle x_1\rangle, \ldots, \langle x_{k-1}\rangle$ the *factors* of M and say that M is *directed* by S. Note that the dimension of M is equal to that of S. We shall call such decomposable subspaces *type*-1 *subspaces.* We note that our definition of a type-1 subspace is more general than that given by Cummings in [7], where E is finite-dimensional and type-1 subspaces are always directed by E.

When E is a real Banach space, type-1 subspaces are the only decomposable subspaces which arise. For complex Banach spaces however we have a second family of decomposable subspaces. Suppose that x and y are two linearly independent vectors in E. Let $\langle x,y\rangle$ denote the two-dimensional subspace of E spanned by x and y. For $1\le r\le k$ and $x_1,\ldots,x_{k-r}$ in E consider the subset $x_1\vee\ldots\vee x_{k-r}\vee\langle x,y\rangle_r$ consisting of all decomposable elements of the form $x_1\vee\ldots\vee x_{k-r}\vee z_1\vee\ldots\vee z_r$ with $z_1,\ldots,z_r$ in $\langle x,y\rangle$. Each element of the set $x_1\vee\ldots\vee x_{k-r}\vee\langle x,y\rangle_r$ can be written in the form

$$x_1\vee\ldots\vee x_{k-r}\vee(\gamma_0x^r+\gamma_1x^{r-1}\vee y+\ldots+\gamma_{r-1}x\vee y^{r-1}+\gamma_ry^r).$$

This means that $x_1\vee\ldots\vee x_{k-r}\vee\langle x,y\rangle_r$ can be identified with a subset of $P_r(\alpha)$, the vector space of all polynomials of degree at most r over the complex numbers, under the mapping

$$x_1\vee\ldots\vee x_{k-r}\vee(\gamma_0x^r+\gamma_1x^{r-1}\vee y+\ldots+\gamma_ry^r)\longmapsto\gamma_0+\gamma_1\alpha+\ldots+\gamma_r\alpha^r.$$

Because each polynomial over $\mathbf{C}$ can be written as a product of linear factors, it follows from [7, Proposition 10] that this mapping is surjective. This means that for complex Banach spaces $x_1\vee\ldots\vee x_{k-r}\vee\langle x,y\rangle_r$ will also be a decomposable subspace. Such subspaces are called *type-r subspaces.* Type-r subspaces have dimension $r+1$.

There is no unique representation of a type-1 subspace. The following lemma will therefore be of use to us when we want to compare such subspaces.

Lemma 1. *Fix a positive integer $k\ge2$ and let S and $\widetilde{S}$ be subspaces of E such that*

$$x_1\vee x_2\vee\ldots\vee x_{k-1}\vee S=y_1\vee y_2\vee\ldots\vee y_{k-1}\vee\widetilde{S}.$$

Then $S=\widetilde{S}$ and $\langle x_1\vee x_2\vee\ldots\vee x_{k-1}\rangle=\langle y_1\vee y_2\vee\ldots\vee y_{k-1}\rangle$ in $\bigotimes_{s,k-1}E$.

Proof. Take $t\in\widetilde{S}$ so that $t\notin[x_1]\cup\ldots\cup[x_{k-1}]$, then there exists $s\in S$ so that

$$x_1\vee x_2\vee\ldots\vee x_{k-1}\vee s=y_1\vee y_2\vee\ldots\vee y_{k-1}\vee t.$$

By our choice of t, it follows that $t=\lambda s$ for some $\lambda\in\mathbf{K}$ and therefore $\widetilde{S}\subset S$. Also we have that $y_j=\lambda_jx_{\sigma(j)}$ for $j=1,\ldots,k-1$. By symmetry we have that $S\subset\widetilde{S}$ and the lemma is proved. $\square$

We now show that type-1 and type-r subspaces are the only decomposable subspaces.

Theorem 2. *Let k be a positive integer and E be a Banach space of dimension strictly greater than $k+1$. Then every decomposable subspace of $\bigotimes_{s,k} E$ of dimension strictly greater than $k+1$ is a type-1 subspace.*

Proof. Let V be a decomposable subspace of $\bigotimes_{s,k} E$ with $\dim(V)>k+1$. Choose a linearly independent set $\{z_1, ..., z_{k+2}\}$ in V and set $Z=\mathrm{span}\{z_1, ..., z_{k+2}\}$. For each i, write z_i as $z_i=x_{i1}\vee...\vee x_{ik}$ and consider the finite-dimensional subspace of E, given by $X=\langle x_{ij}\colon 1\le i\le k+2$ and $1\le j\le k\rangle$. It is clear that $Z\subset\bigotimes_{s,k} X$. It follows from [7, Theorem] that Z must be a type-1 subspace or a type-r subspace with $r\le k$. Since type-r subspaces have dimension $r+1$, Z is a type-1 subspace. Write Z as $Z=x_1\vee x_2\vee...\vee x_{k-1}\vee S$ for some finite-dimensional subspace S of E. If $\widetilde{Z}$ is any other finite-dimensional subspace of V with $\dim(\widetilde{Z})>k+1$ it follows from the proof of Lemma 1 that $Z+\widetilde{Z}$ can be written as $x_1\vee x_2\vee...\vee x_{k-1}\vee\widetilde{S}$ for a finite-dimensional subspace $\widetilde{S}$ of E. Hence $\widetilde{Z}=x_1\vee x_2\vee...\vee x_{k-1}\vee W$ and we see that V is a type-1 subspace. $\square$

The hypothesis on the dimension can be slightly improved in the case where the decomposable subspace is the image under a decomposable mapping of a type-1 subspace. The proof of the following result can be deduced from that of [19, Theorem 3].

Corollary 3. *Let k be a positive integer and E and F be Banach spaces of dimension at least $k+1$. Let $T\colon \bigotimes_{s,k} E\to\bigotimes_{s,k} F$ be a decomposable mapping. Then T maps any type-1 subspace of dimension greater than or equal to $k+1$ onto a type-1 subspace.*

Decomposable mappings from $\bigotimes_{s,k} E$ into $\bigotimes_{s,k} F$ which map type-1 subspaces to type-1 subspaces are called *type-1 mappings.* Of course, for real Banach spaces every decomposable mapping is a type-1 mapping, however in the complex case there may be decomposable mappings which are not type-1. For example, given any complex Banach space E of dimension strictly greater than 2, consider the projection from $\bigotimes_{s,k} E$ onto a type-k subspace. This is a decomposable mapping which is not type-1.

Suppose T is a type-1 mapping of $\bigotimes_{s,k}E$ into $\bigotimes_{s,k}F$. Let $M=x_1\vee...\vee x_{k-1}\vee S$. Choose $y_1, ..., y_{k-1}$ in F and a subspace $\widetilde{S}$ of F so that $T(M)=y_1\vee...\vee y_{k-1}\vee\widetilde{S}$. Then T induces a linear mapping A of S into $\widetilde{S}$ by $As=t$, where

$$T(x_1\vee...\vee x_{k-1}\vee s)=y_1\vee...\vee y_{k-1}\vee t.$$

Following [8] we call A an *associate mapping.* This associated mapping A depends on the choice of M and $y_1, ..., y_{k-1}$, however, it follows as in [8, Proposition 2] that any two associate mappings with respect to the same type-1 subspace are multiples of each other.

To characterise decomposable mappings we generalise the concept of adjacent type-1 subspaces of [8] to our setting.

Definition 4. Let k be a positive integer and E be a Banach space. We say that two elements of $\bigotimes_{s,k} E$ are *adjacent* if they share precisely $k-1$ factors (counting multiplicity). That is, they can be written in the form $x_1 \vee ... \vee x_{k-1} \vee z_1$ and $x_1 \vee ... \vee x_{k-1} \vee z_2$, with z_1 and z_2 linearly independent. We say that two type-1 subspaces M and N of $\bigotimes_{s,k} E$ are *adjacent* if they are directed by the same subspace and share exactly $k-2$ factors (counting multiplicity). This means that they can be written in the form $M = x_1 \vee ... \vee x_{k-2} \vee z_1 \vee S$ and $N = x_1 \vee ... \vee x_{k-2} \vee z_2 \vee S$ for some subspace S of E with z_1 and z_2 linearly independent. When $k=2$ we say that any two type-1 subspaces directed by the same subspace are *adjacent.*

When M and N are adjacent type-1 subspaces we write $M \curlywedge N$. Any two type-1 subspaces of $\bigotimes_{s,k} E$ which are directed by the same subspace of E can be 'linked' to each other through a chain of at most $k-1$ pairwise adjacent subspaces.

Decomposable mappings and adjacency can also be defined for the space of full k-fold tensors (see [20]). Westwick [20] shows that decomposable mappings between spaces of k-fold tensors map adjacent decomposable elements to adjacent decomposable elements. When E has dimension at least $k+1$ this result is also true for spaces of symmetric k-fold tensors. The proof, as we shall see, is completely different.

Proposition 5. *Let k be a positive integer, E and F be Banach spaces of dimension at least $k+1$ and let $T\colon \bigotimes_{s,k} E \to \bigotimes_{s,k} F$ be a decomposable mapping. Then T maps adjacent decomposable elements to adjacent decomposable elements.*

Proof. Consider any two adjacent elements $v = x_1 \vee ... \vee x_{k-2} \vee x_{k-1} \vee z_1$ and $\bar{v} = x_1 \vee ... \vee x_{k-2} \vee x_{k-1} \vee z_2$ in $\bigotimes_{s,k} E$. Take $M = x_1 \vee ... \vee x_{k-2} \vee x_{k-1} \vee S$, where S is a subspace of E containing z_1 and z_2 with $\dim(S) \geq k+1$. As $v, \bar{v} \in M$ we know that $T(v)$ and $T(\bar{v})$ belong to $T(M)$ which, by Corollary 3, is a type-1 subspace. Therefore $T(v)$ and $T(\bar{v})$ are adjacent. $\square$

The following proposition gives a necessary and sufficient condition for two type-1 subspaces directed by the same subspace to be adjacent. The proof is very similar to that given in [8, Proposition 4]. Since the definition of adjacent type-1 subspaces in [8] and here are slightly different, we include a proof for the sake of completeness.

Proposition 6. *Let k be a positive integer and E be a Banach space. Let M and N be a pair of type-1 subspaces of $\bigotimes_{s,k} E$ directed by the same subspace. If $\dim(M\cap N)=1$ then $M \curlywedge N$. Conversely, with the additional assumption that N and M are directed by E then $M \curlywedge N$ implies that $\dim(M\cap N)=1$.*

Proof. Suppose that

$$M = x_1\vee\ldots\vee x_{k-1}\vee S, \quad N = y_1\vee\ldots\vee y_{k-1}\vee S \quad \text{and} \quad \dim(M\cap N) = 1.$$

Then $M\cap N\neq 0$. Take $s,t\in S$ so that $x_1\vee\ldots\vee x_{k-1}\vee s=y_1\vee\ldots\vee y_{k-1}\vee t$. If $\langle s\rangle=\langle t\rangle$ then, $x_i=\lambda_i y_{\sigma(i)}$ for $i=1,\ldots,k-1$, which means that $M=N$. Otherwise, we may assume without loss of generality that $\langle x_1\rangle=\langle t\rangle$ and $\langle s\rangle=\langle y_1\rangle$. Then $y_i=\lambda_i x_{\sigma(i)}$ for $i=2,\ldots,k-1$ and we get $M \curlywedge N$.

Conversely, suppose that M and N are both directed by E and that $M \curlywedge N$. Then $M=x_1\vee\ldots\vee x_{k-2}\vee z_1\vee E$ and $N=x_1\vee\ldots\vee x_{k-2}\vee z_2\vee E$, with z_1 and z_2 linearly independent. We proceed as in [8, Proposition 4]. It follows that $M\cap N\neq 0$. Consider s and t in E with

$$x_1\vee\ldots\vee x_{k-2}\vee z_1\vee s = x_1\vee\ldots\vee x_{k-2}\vee z_2\vee t. \tag{2}$$

For each $j=1,\ldots,k-1$ and $x\in E$ let $g_j(x)\colon E^j\to\bigotimes_{s,j+1} E$ be the symmetric multilinear mapping defined by

$$(u_1,\ldots,u_j)\longmapsto x\vee u_1\vee\ldots\vee u_j.$$

The function $g_j(x)$ induces a map $h_j(x)\colon \bigotimes_{s,j} E\to\bigotimes_{s,j+1} E$. If x is nonzero then each $h_j(x)$ is injective and so is the composition

$$h = h_{k-1}(x_1)\circ\ldots\circ h_{k-i}(x_i)\circ\ldots\circ h_2(x_{k-2}).$$

We can write (2) as $h(z_1\vee s)=h(z_2\vee t)$ and so $z_1\vee s=z_2\vee t$. Since z_1 and z_2 are linearly independent, z_1 must be a multiple of t. Thus

$$M\cap N = \langle x_1\vee\ldots\vee x_{k-2}\vee z_1\vee z_2\rangle. \quad \square$$

Proposition 7. *Let k be a positive integer, S be a finite-dimensional space of dimension at least $k+1$ and F be a Banach space. Let $T\colon \bigotimes_{s,k} S\to\bigotimes_{s,k} F$ be a type-1 mapping. Let M and N be adjacent type-1 subspaces directed by S. Then $T(M)$ and $T(N)$ are also adjacent.*

Proof. Let us write $M=x_1\vee\ldots\vee x_{k-2}\vee z_1\vee S$ and $N=x_1\vee\ldots\vee x_{k-2}\vee z_2\vee S$ for linearly independent z_1 and z_2 in S. Then there are injective linear operators A

and B from S into F, such that $A(S)$ directs $T(M)$ and $B(S)$ directs $T(N)$. Since $M \curlywedge N$, Proposition 6 implies that $\dim(M\cap N)=1$ and therefore $T(M)\cap T(N)\neq 0$. We claim that $T(M)$ and $T(N)$ share at least $k-2$ factors. To see this let $T(M)=y_1\vee\ldots\vee y_{k-2}\vee y_{k-1}\vee A(S)$ and $T(N)=\tilde{y}_1\vee\ldots\vee\tilde{y}_{k-2}\vee\tilde{y}_{k-1}\vee B(S)$ and take $u\neq 0$, $u\in T(M)\cap T(N)$. As

$$u=y_1\vee\ldots\vee y_{k-2}\vee y_{k-1}\vee y=\tilde{y}_1\vee\ldots\vee\tilde{y}_{k-2}\vee\tilde{y}_{k-1}\vee\tilde{y}$$

our assertion follows. We can now write $T(M)=y_1\vee\ldots\vee y_{k-2}\vee w_1\vee A(S)$ and $T(N)=y_1\vee\ldots\vee y_{k-2}\vee w_2\vee B(S)$, for some $w_1,w_2\in F$. If we show that $A(S)=B(S)$ then, provided $T(M)$ and $T(N)$ are distinct, Proposition 6 will ensure that $T(M)\curlywedge T(N)$. Let $p=\dim(S)$. If $A(S)\subset B(S)$ then they are equal. Suppose that there is s_1 in S so that $As_1\notin B(S)$. Complete $\{s_1\}$ to form a basis $\{s_1,s_2,\ldots,s_p\}$ for S. Now consider the adjacent pairs $v_i=x_1\vee\ldots\vee x_{k-2}\vee z_1\vee s_i$ and $\bar{v}_i=x_1\vee\ldots\vee x_{k-2}\vee z_2\vee s_i$, $i=1,\ldots,p$. By definition, type-1 mappings are decomposable. Then, by Proposition 5, v_i and $\bar{v}_i$ have adjacent images $T(v_i)=y_1\vee\ldots\vee y_{k-2}\vee w_1\vee As_i$ and $T(\bar{v}_i)=y_1\vee\ldots\vee y_{k-2}\vee w_2\vee Bs_i$ for all $i=1,\ldots,p$. Since $As_1\notin B(S)$, we have $\langle As_1\rangle=\langle w_2\rangle$. The injectivity of A now implies that no other As_i is a multiple of w_2, i.e. $As_i\in B(S)$, for all $i=2,\ldots,p$. Also $\langle A(s_1+s_2)\rangle=\langle w_2\rangle$ would imply that $\langle As_2\rangle=\langle w_2\rangle$ which is impossible. Thus $A(s_1+s_2)$ belongs to $B(S)$ and therefore so does As_1. From this contradiction we get that $A(S)\subset B(S)$. So we have just proved that $T(M)$ and $T(N)$ are directed by the same subspace. The proof given in [8, Proposition 5] assures that $T(M)$ and $T(N)$ are distinct and therefore they are adjacent. □

Following the proof of [8, Theorem 1] we obtain the following result.

Theorem 8. *Let k be a positive integer, S be a finite-dimensional space of dimension at least $k+1$ and F be a Banach space. Let $T\colon \bigotimes_{s,k} S\to\bigotimes_{s,k} F$ be a type-1 mapping. Then the associated mappings to any type-1 subspace of $\bigotimes_{s,k} S$, directed by S, form a one-dimensional space of $\mathcal{L}(S;F)$.*

Note that for a type-1 mapping any associated mapping to a type-1 subspace must be injective.

Theorem 9. *Let k be a positive integer, S be a finite-dimensional space of dimension at least $k+1$ and F be a Banach space. Let $T\colon \bigotimes_{s,k} S\to\bigotimes_{s,k} F$ be a type-1 mapping. Then there exists an injective operator $A\in\mathcal{L}(S;F)$ such that for any $x=x_1\vee\ldots\vee x_k$ in $\bigotimes_{s,k} S$,*

$$T(x)=\pm Ax_1\vee\ldots\vee Ax_k.$$

Proof. First we show that there exists an injective operator $A\in\mathcal{L}(S;F)$ so that given $x=x_1\vee\ldots\vee x_k$ nonzero, there exists $\lambda_x\in\mathbf{K}$ such that

$$T(x)=\lambda_x Ax_1\vee\ldots\vee Ax_k. \tag{3}$$

For each $i=1,...,k$, consider the type-1 subspaces of dimension at least $k+1$,

$$V_i = x_1 \vee ... \vee \hat{x}_i \vee ... \vee x_k \vee S,$$

where $\hat{x}_i$ means that the ith factor is omitted. Then $\langle x\rangle = \bigcap_{i=1}^{k} V_i$ and by Theorem 8, the mappings associated with any of these type-1 subspaces are all multiples of an operator in $\mathcal{L}(S;F)$. Let us fix one of these associated mappings and call it A. Since

$$T(x_1 \vee x_2 \vee ... \vee x_{k-1} \vee s) = y_1 \vee y_2 \vee ... \vee y_{k-1} \vee As$$

we know that $\langle Ax_i\rangle$ is a factor of $T(x)$ for each $i=1,...,k$. If x has k distinct factors then (3) holds.

Next, suppose that for $m<k$, $\langle x_1\rangle,...,\langle x_m\rangle$ are all the distinct factors in x, and p_i is the multiplicity of $\langle x_i\rangle$, $i=1,...,m$. Then $x=x_1\vee...\vee x_1\vee...\vee x_m\vee...\vee x_m$ and for at least one i, $p_i\geq 2$. Choose $\langle x_{m+1}\rangle,...,\langle x_{k-1}\rangle$ so that together with $\langle x_1\rangle,...,\langle x_m\rangle$ we have $k-1$ distinct factors. Without loss of generality we may assume that x_1 has multiplicity $p_1\geq 2$. Take the adjacent type-1 subspaces M and N, the first one has all its factors distinct and the second has precisely one of its factors $\langle x_1\rangle$ repeating twice

$$M = x_1 \vee ... \vee x_{k-2} \vee x_{k-1} \vee S,$$
$$N = x_1 \vee ... \vee x_{k-2} \vee x_1 \vee S.$$

As $M \curlywedge N$, by Proposition 7 we have that $T(M) \curlywedge T(N)$. Then, as

$$T(x_1 \vee x_2 \vee ... \vee x_{k-1} \vee s) = y_1 \vee y_2 \vee ... \vee y_{k-1} \vee As,$$

Theorem 8 implies that $\langle Ax_j\rangle$, for $j=1,...,m$, are distinct factors of $T(M)$ and $T(N)$ and

$$T(M) = Ax_1 \vee ... \vee Ax_m \vee q_{m+1} \vee ... \vee q_{k-2} \vee z \vee A(S), \tag{4}$$
$$T(N) = Ax_1 \vee ... \vee Ax_m \vee q_{m+1} \vee ... \vee q_{k-2} \vee w \vee A(S). \tag{5}$$

Since $v=x_1\vee...\vee x_{k-2}\vee x_{k-1}\vee x_1\in M\cap N$, taking $s=x_1$ and $s=x_{k-1}$, respectively, and comparing (4) and (5), we have that

$$\begin{aligned} T(v) &= Ax_1 \vee ... \vee Ax_m \vee q_{m+1} \vee ... \vee q_{k-2} \vee z \vee Ax_1 \\ &= Ax_1 \vee ... \vee Ax_m \vee q_{m+1} \vee ... \vee q_{k-2} \vee w \vee Ax_{k-1}. \end{aligned}$$

It follows from (1) that the factors in both expressions are equal (counting multiplicity). By the choice of $x_{m+1},...,x_{k-1}$, $\langle Ax_1\rangle \neq \langle Ax_{k-1}\rangle$, whence $\langle Ax_1\rangle$ must be $\langle w\rangle$ or one of the factors $\langle q_i\rangle$, for $i=m+1,...,k-2$. We may assume without loss of generality that $q_{m+1}=\lambda_1 Ax_1$. Then by the symmetry of symmetric tensor products we can write, using the expression for $T(N)$,

$$T(x_1\vee\ldots\vee x_{k-2}\vee x_1\vee S)=\lambda_1 Ax_1\vee Ax_1\vee\ldots\vee Ax_m\vee q_{m+2}\vee\ldots\vee q_{k-2}\vee w\vee A(S). \tag{6}$$

If $p_1\geq 3$ take the pair of adjacent type-1 subspaces $M=x_1\vee x_1\vee\ldots\vee x_{k-2}\vee S$ and $N=x_1\vee x_1\vee\ldots\vee x_{k-3}\vee x_1\vee S$. The first is the former N for which (6) holds. In the second $\langle x_1\rangle$ appears 3 times. Reasoning as above we now obtain that $\langle Ax_1\rangle$ is a factor of $T(N)$ with multiplicity at least 3. Repeating this construction p_1-1 times we get that Ax_1 is a factor of $T(x_1\vee\ldots\vee x_1\vee\ldots\vee x_m\vee\ldots\vee x_m)$ with multiplicity p_1. Completing this procedure with all the x_i's such that $p_i\geq 2$ we obtain a chain of pairs of type-1 subspaces, the last of which is

$$x_1\vee\ldots\vee x_1\vee\ldots\vee x_m\vee\ldots\vee x_m\vee S,$$

where x_i appears p_i times except for the last one which appears p_m-1 times (of course this could be zero). We also have that

$$T(x_1\vee\ldots\vee x_1\vee\ldots\vee x_m\vee\ldots\vee x_m\vee S)=\lambda_x Ax_1\vee\ldots\vee Ax_1\vee\ldots\vee Ax_m\vee\ldots\vee Ax_m\vee A(S)$$

and taking $s=x_m$ we get (3).

The last expression above also shows that the value of λ_x is independent of s in S. For decomposables x and y in $\bigotimes_{s,k} S$ which are not in the same type-1 subspace consider type-1 subspaces M_x and M_y containing x and y and a chain of pairs of adjacent type-1 subspaces from M_x to M_y. Since the intersection of any two consecutive subspaces is nonempty we have that $\lambda_x=\lambda_{M_x}=\lambda_{M_y}=\lambda_y$. Then $T(x)=\lambda Ax_1\vee\ldots\vee Ax_k$ or equivalently if $\tilde{A}=\lambda^{1/k}A$, $T(x)=\pm\tilde{A}x_1\vee\ldots\vee\tilde{A}x_k$ which completes the proof of the theorem. □

Note that the sign $\pm$ in the statement of Theorem 9 may be avoided if the spaces are complex or if they are real and k is odd.

3. Families of complemented symmetric seminorms

The purpose of this section is to describe a method of obtaining families of seminorms on spaces of symmetric tensors, $\{\alpha_k\}_{k=1}^{\infty}$ such that for each k and l with $k<l$ and each Banach space E we have 'natural' identifications of $\bigotimes_{s,k,\alpha_k} E$ with a complemented subspace of $\bigotimes_{s,l,\alpha_l} E$. These families will be useful in the next section when we study decomposable mappings between Banach spaces. Recall that ε_k and π_k denote the injective and projective tensor norms on $\bigotimes_k E$, respectively, while $\varepsilon_{s,k}$ and $\pi_{s,k}$ denote the symmetric injective and symmetric projective tensor

norms on $\bigotimes_{s,k}E$, respectively. It is worthwhile to note that although the restriction of π_k to the symmetric k-tensors, usually denoted by $\pi_k|_s$, does not coincide with $\pi_{s,k}$, both are equivalent norms on $\bigotimes_{s,k} E$. The same holds for $\varepsilon_k|_s$, the restriction of ε_k to the symmetric k-tensors and the symmetric injective tensor norm $\varepsilon_{s,k}$.

In what follows, we will work with the following definition.

Definition 10. Let E be a Banach space. A family of complemented symmetric seminorms on E is a function α which assigns to each positive integer k a norm α_k on $\bigotimes_{s,k}E$ such that

(i) for each positive integer k, α_k is a reasonable symmetric norm on $\bigotimes_{s,k}E$ in the sense that $\varepsilon_{s,k}\le\alpha_k\le\pi_{s,k}$,

(ii) for any pair of positive integers k and l there are constants $C_{k,l}$ and $D_{k,l}$ such that

$$\alpha_{k+l}(\theta\vee\xi)\le C_{k,l}\alpha_k(\theta)\alpha_l(\xi)\le D_{k,l}\alpha_{k+l}(\theta\vee\xi)$$

for all θ in $\bigotimes_{s,k}E$ and ξ in $\bigotimes_{s,l}E$.

By a family of complemented symmetric seminorms we shall understand a function α which assigns to each Banach space E a family of complemented symmetric seminorms on E.

Condition (ii) tells us that for any fixed nonzero θ in $\bigotimes_{s,l}E$ the mapping $\xi\mapsto\theta\vee\xi$ is an isomorphism of $\bigotimes_{s,k,\alpha_k}E$ onto the subspace $\{\theta\vee\xi:\xi\in\bigotimes_{s,k}E\}$ of $\bigotimes_{s,k+l}E$ with the topology induced from $\bigotimes_{s,k+l,\alpha_{k+l}}E$. We denote the completion of $\bigotimes_{s,k,\alpha_k}E$ with respect to the norm α_k by $\widehat{\bigotimes}_{s,k,\alpha_k}E$.

Let us give some examples of families of complemented symmetric seminorms. One of the methods at our disposal is via the concepts of tensor norms of order k and s-tensor norms of order k introduced by Floret in [12].

According to Floret [12] a *tensor norm* of order k is an assignment, to each k-tuple $(E_1,...,E_k)$ of normed spaces, of a norm $\beta(E_1,E_2,...,E_k)$ on $E_1\otimes E_2\otimes...\otimes E_k$ (denoted $\bigotimes_{\beta,j=1}^{k} E_j$) such that

(i) $\varepsilon_k\le\beta\le\pi_k$;

(ii) given continuous linear mappings $A_j: E_j\to F_j$, $j=1,...,k$,

$$\left\|\bigotimes_{j=1}^{k} A_j: \bigotimes_{\beta,j=1}^{k} E_j \longrightarrow \bigotimes_{\beta,j=1}^{k} F_j\right\| = \|A_1\|...\|A_k\|.$$

In an analogous way, in [12] an *s-tensor norm* of order k is defined as an assignment, to each normed space E, of a norm α on $\bigotimes_{s,k} E$ (denoted $\bigotimes_{s,k,\alpha} E$) such that

(i) $\varepsilon_{s,k} \leq \alpha \leq \pi_{s,k}$;
(ii) for all continuous linear mappings $A\colon E \to F$ we have

$$\Big\| \bigotimes_{s,k,\alpha} A\colon \bigotimes_{s,k,\alpha} E \longrightarrow \bigotimes_{s,k,\alpha} F \Big\| = \|A\|^k.$$

Properties (ii) in the definitions of a tensor norm of order k and an s-tensor norm of order k are known as the metric mapping property. Each tensor norm of order k, by restriction to the space of symmetric tensors, defines an s-tensor norm of order k. Floret, [12, Norm extension theorem] shows that the converse is also true in that given any s-tensor norm α of order k there is a tensor norm β of the same order such that for every Banach space E the norm α on $\bigotimes_{s,k,\alpha} E$ is the restriction of β to $\bigotimes_{s,k} E$.

Let $\beta = \{\beta_k\}_{k=1}^{\infty}$ be a family of tensor norms with β_k of order k. We shall say that β is an associative family of tensor norms if for every pair of positive integers k and l and for every $(k+l)$-tuple of normed spaces $E_1, \dots, E_k, E_{k+1}, \dots, E_{k+l}$ we have that $\beta_{k+l}(E_1, \dots, E_{k+l})$ is isomorphic to $\beta_2(\bigotimes_{\beta_k, i=1}^{k} E_i, \bigotimes_{\beta_l, j=1}^{l} E_{k+j})$. It can be shown that if $\{\beta_k\}_{k=1}^{\infty}$ is an associative family of tensor norms then its restriction to the spaces of symmetric tensor products is a family of complemented symmetric seminorms.

It is readily checked that both the families of injective and projective tensor norms are associative families of tensor norms. Therefore, $\varepsilon_k|_s$ and $\pi_k|_s$, their respective restrictions to the symmetric k-tensors, are families of complemented symmetric seminorms and so are both $\varepsilon_s = \{\varepsilon_{s,k}\}_{k=1}^{\infty}$ and $\pi_s = \{\pi_{s,k}\}_{k=1}^{\infty}$ by the equivalence mentioned above.

Let us give another example of families of complemented symmetric seminorms, which are related to the class of extendible polynomials. There is, in general, no Hahn–Banach theorem for k-homogeneous polynomials. However, Kirwan and Ryan [15] examine which k-homogeneous polynomials on a Banach space E have continuous extensions to every superspace. These polynomials are called *extendible*. Every algebraic k-homogeneous polynomial on a Banach space E determines a unique linear map from $\bigotimes_{s,k} E$ into $\mathbf{K}$ and the extendible polynomials are those polynomials whose linearization are continuous with respect to a specific norm on $\bigotimes_{s,k} E$, which we now proceed to describe.

First, fix a Banach space F containing E and an injective continuous operator $i\colon E \to F$. Let $i^k = \bigotimes_{s,k} i$. Then, i^k is an injection of $\bigotimes_{s,k} E$ into $\bigotimes_{s,k} F$. The η^F norm on $\bigotimes_{s,k} E$, defined by Carando in [5], is given by $\eta_{s,k}^F(z) = \pi_{s,k}(i^k(z))$. Here, by $\pi_{s,k}$ we mean the symmetric projective norm on $\bigotimes_{s,k} F$. The dual of $\bigotimes_{s,k,\eta_F} E$ is the space of all k-homogeneous polynomials on E which have a continuous extension to F. Once a k-homogeneous polynomial on E can be extended to $\ell_\infty(B_{E'})$ it can be extended to every superspace of E and we simply denote $\eta_{\ell_\infty(B_{E'})}$ by η, see [5].

Let $C_{k,l}$ and $D_{k,l}$ denote the smallest constants such that

$$\pi_{s,k}(\theta)\pi_{s,l}(\xi) \leq C_{k,l}\pi_{s,k+l}(\theta\vee\xi) \quad \text{and} \quad \pi_{s,k+l}(\theta\vee\xi) \leq D_{k,l}\pi_{s,k}(\theta)\pi_{s,l}(\xi)$$

for all θ in $\bigotimes_{s,k} E$, all ξ in $\bigotimes_{s,l} E$ and all Banach spaces E. Suppose we are given θ in $\bigotimes_{s,k} E$ and ξ in $\bigotimes_{s,l} E$. Then, clearly we have that $i^k(\theta)\vee i^l(\xi) = i^{k+l}(\theta\vee\xi)$. Thus we have that

$$\begin{aligned}\eta^F_{s,k}(\theta)\eta^F_{s,l}(\xi) &= \pi_{s,k}(i^k(\theta))\pi_{s,l}(i^l(\xi)) \\ &\leq C_{k,l}\pi_{s,k+l}(i^k(\theta)\vee i^l(\xi)) = C_{k,l}\pi_{s,k+l}(i^{k+l}(\theta\vee\xi)) = C_{k,l}\eta^F_{s,k+l}(\theta\vee\xi).\end{aligned}$$

Similarly we get that

$$\eta^F_{s,k+l}(\theta\vee\xi) \leq D_{k,l}\eta^F_{s,k}(\theta)\eta^F_{s,l}(\xi)$$

for all θ in $\bigotimes_{s,k} E$ and all ξ in $\bigotimes_{s,l} E$. In particular, when we take $F = \ell_\infty(B_{E'})$ we get constants $C_{k,l}$ and $D_{k,l}$ such that

$$\eta_{s,k}(\theta)\eta_{s,l}(\xi) \leq C_{k,l}\eta_{s,k+l}(\theta\vee\xi) \quad \text{and} \quad \eta_{s,k+l}(\theta\vee\xi) \leq D_{k,l}\eta_{s,k}(\theta)\eta_{s,l}(\xi)$$

for all θ in $\bigotimes_{s,k} E$ and all ξ in $\bigotimes_{s,l} E$. Therefore η is a family of complemented seminorms. In addition, given a Banach space F, for each Banach space E which is a subspace of F, η^F will be a family of complemented seminorms on E. We note that since η^F is not an s-tensor norm of order k, the fact that it is a family of complemented seminorms on a given Banach space E must be established with a different method to the one used for ε_s and π_s above.

Let E and F be Banach spaces and $A\colon E \to F$ be a linear operator. Given a positive k we use A^k to denote the linear operator from $\bigotimes_{s,k} E$ into $\bigotimes_{s,k} F$ defined by

$$A^k(x_1\vee x_2\vee\ldots\vee x_k) = Ax_1\vee Ax_2\vee\ldots\vee Ax_k$$

for all $x_1,\ldots,x_k$ in E and extended by linearity to $\bigotimes_{s,k} E$. Let α_k be a symmetric tensor norm of order k. Since α_k satisfies the metric mapping property, A^k is continuous. An operator ideal $\mathcal{A}$ is said to be *α-tensor stable* for the tensor norm α if $A\in\mathcal{A}(E_1;F_1)$ and $B\in\mathcal{A}(E_2;F_2)$ implies that $A\otimes B\in\mathcal{A}(E_1\bigotimes_\alpha E_2;F_1\bigotimes_\alpha F_2)$. See [10, Section 34] for a discussion of tensor stable ideals. The following proposition investigates the converse of this result for families of complemented symmetric seminorms.

Proposition 11. *Let E and F be Banach spaces and α and β be families of complemented symmetric seminorms on E and F, respectively. Let furthermore $A\colon E\to F$ be a continuous linear operator and $\mathcal{A}$ be an operator ideal. If $A^k\in\mathcal{A}(\widehat{\bigotimes}_{s,k,\alpha_k}E;\widehat{\bigotimes}_{s,k,\beta_k}F)$ then $A\in\mathcal{A}(E;F)$.*

Proof. We adapt some ideas from [3]. Fix e in E and choose ϕ in F' with $\phi(Ae)=1$. Given a positive integer m we define $j_m\colon \bigotimes_{s,m}E\to\bigotimes_{s,m+1}E$ by

$$j_m(x^m)=\sum_{i=1}^{m+1}\binom{m+1}{i}(-1)^{i+1}\phi(Ax)^{i-1}\underbrace{e\vee\ldots\vee e}_{i\text{ times}}\vee\underbrace{x\vee\ldots\vee x}_{m-i+1\text{ times}},$$

on the elementary tensors, and extended by linearity to $\bigotimes_{s,m}E$. It is easily checked that j_m satisfies

$$\phi(Ax)j_m(x^m)=x^{m+1}-(x-\phi(Ax)e)^{m+1}.$$

We also define $p_m\colon \bigotimes_{s,m+1}F\to\bigotimes_{s,m}F$ on the elementary tensors by

$$p_m(y^{m+1})=\phi(y)y^m,$$

and extended by linearity to $\bigotimes_{s,m+1}F$.

We claim that for each integer m we have

$$p_m\circ A^{m+1}\circ j_m=A^m.$$

To see this consider $\phi(Ax)x^m$ in $\bigotimes_{s,m}E$. Then

$$\begin{aligned}p_m\circ A^{m+1}\circ j_m\phi(Ax)(x^m)&=p_m\circ A^{m+1}(x^{m+1}-(x-\phi(Ax)e)^{m+1})\\&=p_m((Ax)^{m+1}-(Ax-\phi(Ax)Ae)^{m+1})\\&=\phi(Ax)(Ax)^m-\phi(Ax-\phi(Ax)Ae)(Ax-\phi(Ax)Ae)^m\\&=\phi(Ax)(Ax)^m.\end{aligned}$$

By [3, Lemma 2], it is possible to write each θ in $\bigotimes_{s,k}E$ as $\theta=\sum_{r=1}^{l}\lambda_r y_r^k$ with $\phi(Ay_r)\neq 0$. It follows that

$$p_1\ldots p_{k-1}\circ A^k\circ j_{k-1}\ldots j_2 j_1=A.$$

As $j_1,\ldots,j_{k-1}$ and the composition $p_1\ldots p_{k-1}$, induced by $p_1\ldots p_{k-1}(x^k)=\phi^k(x)x$, are continuous linear mappings and A^k is in $\mathcal{A}(\widehat{\bigotimes}_{s,k,\alpha_k}E;\widehat{\bigotimes}_{s,k,\beta_k}F)$ it follows that A belongs to $\mathcal{A}(E;F)$. $\square$

Before we finish this section we show that for any family of complemented symmetric seminorms α on a Banach space E there is a 'natural' way in which the space of symmetric k-fold tensors $\bigotimes_{s,k,\alpha_k} E$ can be identified with a complemented subspace of the space of $\bigotimes_{s,k+1,\alpha_{k+1}} E$. Fix a nonzero element e in E and choose ϕ in E' with $\phi(e)=1$. Define $S_{k+1,e}$ by

$$S_{k+1,e} := e \vee \bigotimes_{s,k} E = \left\{ e \vee \theta : \theta \in \bigotimes_{s,k} E \right\}$$

and endow $S_{k+1,e}$ with the topology induced from $\bigotimes_{s,k+1,\alpha_{k+1}} E$. The mapping $\xi \mapsto e \vee \xi$ is an isomorphism of $\bigotimes_{s,k,\alpha_k} E$ onto $S_{k+1,e}$. Let us see that $S_{k+1,e}$ is a complemented subspace of $\bigotimes_{s,k+1} E$. Define $\Pi_e : \bigotimes_{s,k+1} E \to \bigotimes_{s,k+1} E$ by

$$\Pi_e(x_1 \vee \ldots \vee x_{k+1}) = x_1 \vee \ldots \vee x_{k+1} - (x_1 - \phi(x_1)e) \vee \ldots \vee (x_{k+1} - \phi(x_{k+1})e),$$

on basic symmetric tensors, and extended to $\bigotimes_{s,k+1} E$ by linearity. To see that Π_e is well defined suppose that $\theta = \sum_{i=1}^m \lambda_i x_i^{k+1} = \sum_{j=1}^n \delta_j y_j^{k+1}$. Given any $(k+1)$-homogeneous polynomial P on E define another $(k+1)$-homogeneous polynomial Q by

$$Q(x) = \langle P, \Pi_e(x^{k+1}) \rangle = P(x) - P(x - \phi(x)e).$$

If L_P is the symmetric k-linear form associated with P we have that

$$Q(x) = \sum_{i=1}^{k+1} \binom{k+1}{i} (-1)^{i+1} \phi(x)^i L_P(\underbrace{e, \ldots, e}_{i \text{ times}}, \underbrace{x, \ldots, x}_{k-i+1 \text{ times}}).$$

Then for each $(k+1)$-homogeneous polynomial P on E we have

$$\begin{aligned} \left\langle P, \Pi_e\left(\sum_{i=1}^m \lambda_i x_i^{k+1}\right)\right\rangle &= \left\langle Q, \sum_{i=1}^m \lambda_i x_i^{k+1}\right\rangle \\ &= \left\langle Q, \sum_{j=1}^n \delta_j y_j^{k+1}\right\rangle = \left\langle P, \Pi_e\left(\sum_{j=1}^n \delta_j y_j^{k+1}\right)\right\rangle \end{aligned}$$

and therefore $\Pi_e(\sum_{i=1}^m \lambda_i x_i^{k+1}) = \Pi_e(\sum_{j=1}^n \delta_j y_j^{k+1})$.

Clearly Π_e maps $\bigotimes_{s,k+1} E$ into $S_{k+1,e}$. To see that Π_e maps $\bigotimes_{s,k+1} E$ onto $S_{k+1,e}$ we note that

$$\begin{aligned} \Pi_e(e \vee x_2 \vee \ldots \vee x_k) &= e \vee x_2 \vee \ldots \vee x_k - (e-e) \vee (x_2 - \phi(x_2)e) \vee \ldots \vee (x_k - \phi(x_k)e) \\ &= e \vee x_2 \vee \ldots \vee x_k - 0 \\ &= e \vee x_2 \vee \ldots \vee x_k. \end{aligned}$$

Since $\Pi_e(\Pi_e(x_1 \vee x_2 \vee \ldots \vee x_k)) = \Pi_e(x_1 \vee x_2 \vee \ldots \vee x_k)$ we see that Π_e is a projection of $\bigotimes_{s,k+1} E$ onto $S_{k+1,e}$.

4. Decomposable mappings between symmetric tensor spaces of the same degree

Our main result of this section is as follows.

Theorem 12. *Let E and F be Banach spaces of dimension at least $k+1$, and let α and β be families of complemented symmetric seminorms on E and F, respectively. Suppose that $T\colon \widehat{\bigotimes}_{s,k,\alpha_k} E \to \widehat{\bigotimes}_{s,k,\beta_k} F$ is a continuous decomposable linear operator. Then there is a continuous injective linear operator $A\colon E\to F$ such that $T=\pm A^k$. Moreover, given any operator ideal $\mathcal{A}$, if $T\in\mathcal{A}(\widehat{\bigotimes}_{s,k,\alpha_k} E;\widehat{\bigotimes}_{s,k,\beta_k} F)$ then $A\in\mathcal{A}(E;F)$.*

Proof. Fix x_0 in E. Let S be a finite-dimensional subspace of E containing x_0 which has dimension at least $k+1$. Consider $T|_S\colon \bigotimes_{s,k,\alpha_k} S\to\widehat{\bigotimes}_{s,k,\beta_k} F$. By Theorem 9 we know that there is an injective linear operator $A_S\colon S\to F$ such that

(7) $\quad T|_S(x_1\vee x_2\vee\ldots\vee x_k)=\pm A_S x_1\vee A_S x_2\vee\ldots\vee A_S x_k \quad$ for all $x_1,x_2,\ldots,x_k$ in S.

This means that T is a power-preserver and since $\|A_S(x)\|^k=\beta_k(A_S(x)^k)=\|T|_S(x^k)\|$ we have $\|A_S\|\le\|T|_S\|^{1/k}\le\|T\|^{1/k}$. We note that the sign $\pm$ can easily be shown to be independent of the subspace S. Choose y_0 in F so that $T(x_0^k)=\pm y_0^k$. Using [4, Lemma 4] we may suppose that in addition to A_S satisfying (7) we have that $A_S(x_0)=y_0$. These two conditions uniquely determine A_S. For finite-dimensional subspaces S and $\widetilde{S}$ of E containing x_0 with $S\subset\widetilde{S}$ we have that $A_{\widetilde{S}}|_S=A_S$. Since E is the union of its finite-dimensional subspaces and $\|A_S\|\le\|T\|^{1/k}$ we have that $A(x)=A_S(x)$, where S is any finite-dimensional subspace, of dimension at least $k+1$, of E which contains x_0, defines an injective continuous linear mapping from E into F such that

$$T(x_1\vee x_2\vee\ldots\vee x_k)=\pm Ax_1\vee Ax_2\vee\ldots\vee Ax_k$$

for all $x_1,\ldots,x_k$ in E. Since T maps $\widehat{\bigotimes}_{s,k,\alpha_k} E$ into $\widehat{\bigotimes}_{s,k,\beta_k} F$ it follows that A maps E into F.

If T is in $\mathcal{A}(\widehat{\bigotimes}_{s,k,\alpha_k} E;\widehat{\bigotimes}_{s,k,\beta_k} F)$ then it follows from Proposition 11 that A belongs to $\mathcal{A}(E;F)$. □

In the definition of a decomposable linear mapping T we require that $\ker T\cap\{x_1\vee\ldots\vee x_k : x_1,\ldots,x_k\in E\}=\{0\}$. To see that this condition is necessary let E be a Banach space of dimension at least 2 and $k\ge2$ be a positive integer. Consider k nonzero vectors $z_1,z_2,\ldots,z_k$ in E with z_1 and z_2 linearly independent. Let T be a projection of $\widehat{\bigotimes}_{s,k,\pi_{s,k}} E$ onto the subspace spanned by $z_1\vee z_2\vee\ldots\vee z_k$. Then T

maps decomposable elements of $\widehat{\bigotimes}_{s,k,\pi_{s,k}}E$ to decomposable elements of $\widehat{\bigotimes}_{s,k,\pi_{s,k}}E$. Suppose that there is a continuous linear mapping $A\colon E\to E$ such that

$$(8)\qquad T(x_1\vee x_2\vee\ldots\vee x_k)=\pm Ax_1\vee Ax_2\vee\ldots\vee Ax_k\quad\text{for all } x_1,\ldots,x_k \text{ in } E.$$

As the range of T has dimension 1, the range of A must also be one-dimensional. However, if that is the case, the image of T has the form $\{\lambda y\vee\ldots\vee y:\lambda\in\mathbf{K}\}$ which is clearly not possible since z_1 and z_2 were chosen to be linearly independent.

Taking α or β equal to ε or π in Theorem 12 we get that if E and F are Banach spaces of dimension at least $k+1$ and $T\colon\widehat{\bigotimes}_{s,k,\pi_{s,k}}E\to\widehat{\bigotimes}_{s,k,\pi_{s,k}}F$ (resp. $T\colon\widehat{\bigotimes}_{s,k,\varepsilon_{s,k}}E\to\widehat{\bigotimes}_{s,k,\varepsilon_{s,k}}F$ or $T\colon\widehat{\bigotimes}_{s,k,\pi_{s,k}}E\to\widehat{\bigotimes}_{s,k,\varepsilon_{s,k}}F$) is a continuous decomposable linear operator, then there is an injective operator $A\in\mathcal{L}(E;F)$ satisfying (8). Moreover, given any operator ideal $\mathcal{A}$, when T belongs to $\mathcal{A}$ then $A\in\mathcal{A}(E;F)$.

The analogous result is also true when we consider continuous linear operators from $\widehat{\bigotimes}_{s,k,\varepsilon_{s,k}}E$ to $\widehat{\bigotimes}_{s,k,\pi_{s,k}}F$. It is still an open question if the set of all continuous linear operators $A\colon E\to F$, such that the mapping

$$A^k\colon\widehat{\bigotimes}_{s,k,\pi_{s,k}}E\longrightarrow\widehat{\bigotimes}_{s,k,\varepsilon_{s,k}}F,$$
$$x_1\vee x_2\vee\ldots\vee x_k\longmapsto Ax_1\vee Ax_2\vee\ldots\vee Ax_k,$$

is well defined, is a subspace of $\mathcal{L}(E;F)$, see [10, Section 24] and [14]. It is known that all integral operators are 'ε-π'-continuous, see [6], on the other hand John [13] shows that every 'ε-π'-continuous operator of degree k is (k,k,k)-absolutely summing. These two ideals provide a lower and an upper bound for the set of 'ε-π'-continuous operators.

5. Mappings between spaces of different degrees

In this section we consider decomposable mappings from the space $\widehat{\bigotimes}_{s,k,\alpha_k}E$ into the space $\widehat{\bigotimes}_{s,l,\beta_l}F$, where $k\le l$.

The following lemma is proved in much the same way as [19, Lemma 5].

Lemma 13. *Let S be a finite-dimensional space and F be a Banach space. Let $k\le l$, let α be a reasonable crossnorm of order l and let $T\colon\bigotimes_{s,k}S\to\widehat{\bigotimes}_{s,l,\alpha}F$ be a decomposable mapping. Then the images of all type-1 subspaces in $\bigotimes_{s,k}S$ under T have $l-k$ common factors.*

Proposition 14. *Let S be a finite-dimensional space of dimension at least $k+1$ and F be a Banach space. Let $k\le l$, let α be a family of complemented symmetric* *seminorms and let* $T\colon \bigotimes_{s,k} S \to \widehat{\bigotimes}_{s,l,\alpha_l} F$ *be a decomposable mapping. Then there is a decomposable tensor* $w = w_1 \vee \ldots \vee w_{l-k}$ *of length* $l-k$*, and an injective linear operator* $A \in \mathcal{L}(S;F)$ *such that for any* $x = x_1 \vee \ldots \vee x_k$ *in* $\bigotimes_{s,k} S$*,*

$$T(x) = w \vee A^k(x).$$

Proof. By Lemma 13 there is a decomposable tensor w of length $l-k$ such that the image of $\bigotimes_{s,k} S$ under T is contained in $w \vee (\widehat{\bigotimes}_{s,k,\alpha_k} F)$. Therefore T induces a decomposable mapping $T_1\colon \bigotimes_{s,k} S \to \widehat{\bigotimes}_{s,k,\alpha_k} F$ such that $T(x) = w \vee T_1(x)$ for all x in $\bigotimes_{s,k} S$. By Theorem 9, there exists a continuous injective linear operator $A\colon S \to F$ such that $T_1(x_1 \vee \ldots \vee x_k) = \pm A x_1 \vee \ldots \vee A x_k$ for all $x_1, \ldots, x_k$ in S. Replacing w by $\pm w$ if necessary, this means that

$$T(x) = w \vee A x_1 \vee \ldots \vee A x_k$$

for all $x = x_1 \vee \ldots \vee x_k$ in $\bigotimes_{s,k} S$. $\square$

Note that using condition (ii) in the definition of a family of complemented symmetric seminorms, we obtain that $\|A\| \le M_{k,l} \|T\|^{1/k}$ is independent of the space S.

Theorem 15. *Let* E *and* F *be Banach spaces of dimension at least* $k+1$*, let* $k \le l$ *and let* α *and* β *be families of complemented seminorms on* E *and* F*, respectively. Let* $T\colon \widehat{\bigotimes}_{s,k,\alpha_k} E \to \widehat{\bigotimes}_{s,l,\beta_l} F$ *be a continuous decomposable linear operator. Then there is a decomposable tensor* w *of length* $l-k$ *in* $\bigotimes_{s,l-k} F$ *and an injective operator* $A \in \mathcal{L}(E;F)$ *such that for any* $x = x_1 \vee \ldots \vee x_k$ *in* $\bigotimes_{s,k} E$

$$T(x) = w \vee A^k(x).$$

Moreover, for any operator ideal $\mathcal{A}$*, if* $T \in \mathcal{A}(\widehat{\bigotimes}_{s,k,\alpha_k} E; \widehat{\bigotimes}_{s,l,\beta_l} F)$*, then* $A \in \mathcal{A}(E;F)$*.*

Proof. Fix x_0 in E. For each finite-dimensional subspace S of E containing x_0 which has dimension at least $k+1$, Proposition 14 gives a decomposable tensor w_S of length $l-k$, an injective linear operator $A_S\colon S \to F$ and a constant $M_{k,l}$ independent of S with $\|A_S\| \le M_{k,l} \|T|_S\|^{1/k} \le M_{k,l} \|T\|^{1/k}$ such that

$$T|_S(x_1 \vee x_2 \vee \ldots \vee x_k) = w_S \vee A_S x_1 \vee A_S x_2 \vee \ldots \vee A_S x_k$$

for all $x_1, \ldots, x_k$ in S.

Now consider two finite-dimensional subspaces S and $\widetilde{S}$ of E which have dimension strictly greater than $k+1$ with $S \subset \widetilde{S}$. Suppose that

$$w_{\widetilde{S}} = \widetilde{w}_1 \vee \widetilde{w}_2 \vee \ldots \vee \widetilde{w}_{l-k}.$$

Choose $x_1, ..., x_k$ in S so that none of $A_S x_j$, $j=1, ..., k$, lie in $\langle\widetilde{w}_1\rangle\cup\langle\widetilde{w}_2\rangle\cup...\cup\langle\widetilde{w}_{l-k}\rangle$. Since

$$w_S\vee A_S x_1\vee A_S x_2\vee...\vee A_S x_k = w_{\widetilde{S}}\vee A_{\widetilde{S}} x_1\vee A_{\widetilde{S}} x_2\vee...\vee A_{\widetilde{S}} x_k$$

we see that each $A_S x_j$ is a factor of $w_{\widetilde{S}}\vee A_{\widetilde{S}} x_1\vee A_{\widetilde{S}} x_2\vee...\vee A_{\widetilde{S}} x_k$. However, by our choice of x_j, $A_S x_j$ cannot be a factor of $w_{\widetilde{S}}$ and hence must be a multiple of one of the $A_{\widetilde{S}} x_i$'s. It follows from (1) that w_S is a scalar multiple of $w_{\widetilde{S}}$.

Let $w=w_S$, where S is any finite-dimensional subspace of E of dimension at least $k+1$. Then for each x in E we have $T(x^k)=w\vee y^k$ for some y in F. Choose y_0 in F such that $T(x_0^k)=w\vee y_0^k$. Another application of [4, Lemma 4] tells us that our choice of A_S can be made in such a way that in addition to having

$$T|_S(x_1\vee x_2\vee...\vee x_k) = w\vee A_S x_1\vee A_S x_2\vee...\vee A_S x_k$$

for all $x_1, x_2, ..., x_k$ in S we have $A_S(x_0)=y_0$ and therefore each A_S is uniquely determined. As in Theorem 12 we construct a continuous injective linear operator $A\colon E\to F$ with $A|_S=A_S$ for every finite-dimensional subspace S of E. An argument similar to that given in Theorem 12 shows that

$$T(x_1\vee x_2\vee...\vee x_k) = w\vee Ax_1\vee Ax_2\vee...\vee Ax_k$$

for all $x_1, ..., x_k$ in E.

If T belongs to $\mathcal{A}(\widehat{\bigotimes}_{s,k,\alpha_k}E; \widehat{\bigotimes}_{s,l,\beta_l}F)$ then it follows from the discussion at the end of Section 3 that A^k will belong to $\mathcal{A}(\widehat{\bigotimes}_{s,k,\alpha_k}E; \widehat{\bigotimes}_{s,k,\beta_k}F)$. It now follows from Proposition 11 that A belongs to $\mathcal{A}(E;F)$. $\square$

In particular the above result is true for mappings from spaces of injective tensor products into spaces of injective tensor products, for mappings from spaces of projective tensor products into spaces of projective tensor products and when we consider mappings from the space of symmetric projective tensors into the space of symmetric injective tensors.

6. Rank-1 preserving mappings between spaces of homogeneous polynomials

Given a Banach space E we use J_E to denote the canonical embedding of E into its bidual E''. As we have mentioned above, there is no Hahn–Banach theorem for homogeneous polynomials of degree 2 or greater. However, Aron and Berner [1] and Davie and Gamelin [9] show that for every $P\in\mathcal{P}(^kE)$ there is a norm-preserving extension of P to $\overline{P}\in\mathcal{P}(^kE'')$ such that $\overline{P}\circ J_E(x)=P(x)$ for all $x\in E$.

Given Banach spaces E and F and an isomorphism $s\colon E'\to F'$ it is shown in [16] that the mapping $\bar{s}$, defined by $\bar{s}(P)=\overline{P}\circ s'\circ J_F$, induces an isomorphism of $\mathcal{P}_A({}^kE)$ onto $\mathcal{P}_A({}^kF)$ and of $\mathcal{P}_N({}^kE)$ onto $\mathcal{P}_N({}^kF)$. We will see that mappings of this form are precisely those operators between spaces of approximable and nuclear homogeneous polynomials which preserve rank-1 polynomials.

Aron and Schottenloher [2] show that given any nonzero linear functional ϕ on a Banach space E and any positive integer k, the mapping $r\colon P\mapsto \phi P$ is an injective linear mapping of $\mathcal{P}({}^kE)$ into $\mathcal{P}({}^{k+1}E)$. Moreover, when both spaces are given the compact open topology of uniform convergence on compact subspace of E, this mapping identifies $\mathcal{P}({}^kE)$ with a complemented subspace of $\mathcal{P}({}^{k+1}E)$. The mapping r maps rank-1 polynomials to rank-1 polynomials. The following results show that for the subspaces of approximable and nuclear polynomials a composition of the types of mappings obtained in [2] and in [16] give all linear mappings which map rank-1 polynomials to rank-1 polynomials. To obtain these results we use the one-to-one correspondence between spaces of injective symmetric tensor products, $\widehat{\bigotimes}_{s,k,\varepsilon_{s,k}}E'$, and approximable polynomials, $\mathcal{P}_A({}^kE)$, and the one-to-one correspondence between spaces of projective symmetric tensor products, $\widehat{\bigotimes}_{s,k,\pi_{s,k}}E'$, and nuclear polynomials, $\mathcal{P}_N({}^kE)$, for spaces whose dual have the approximation property, together with Theorems 12 and 15.

Theorem 16. *Let E and F be Banach spaces of dimension at least $k+1$. Let $k\le l$ and let $T\colon \mathcal{P}_A({}^kE)\to\mathcal{P}_A({}^lF)$ be a continuous injective linear operator which maps rank-1 polynomials to rank-1 polynomials. Then there are $\psi_1,...,\psi_{l-k}$ in F' and a continuous injective linear operator $s\colon E'\to F'$ such that*

$$T(P)=\psi_1...\psi_{l-k}\overline{P}\circ s'\circ J_F$$

for all $P\in\mathcal{P}_A({}^kE)$. Moreover, given any operator ideal $\mathcal{A}$, if $T\in\mathcal{A}(\mathcal{P}_A({}^kE);\mathcal{P}_A({}^lF))$ then $s\in\mathcal{A}(E';F')$.

From Theorem 15 we obtain the following result.

Theorem 17. *Let E and F be Banach spaces of dimension at least $k+1$ whose duals have the approximation property. Let $k\le l$ and let $T\colon \mathcal{P}_N({}^kE)\to\mathcal{P}_N({}^lF)$ be a continuous injective linear operator which maps rank-1 polynomials to rank-1 polynomials. Then there are $\psi_1,...,\psi_{l-k}$ in F' and a continuous injective linear operator $s\colon E'\to F'$ such that*

$$T(P)=\psi_1...\psi_{l-k}\overline{P}\circ s'\circ J_F$$

for all $P\in\mathcal{P}_N({}^kE)$. Moreover, given any operator ideal $\mathcal{A}$, if $T\in\mathcal{A}(\mathcal{P}_N({}^kE);\mathcal{P}_N({}^lF))$ then $s\in\mathcal{A}(E';F')$.

Theorem 18. *Let E and F be Banach spaces of dimension at least $k+1$ such that E' has the approximation property. Let $k \le l$ and let $T\colon \mathcal{P}_N({}^kE) \to \mathcal{P}_A({}^lF)$ be a continuous injective linear operator which maps rank-1 polynomials to rank-1 polyno-
mials. Then there are $\psi_1, ..., \psi_{l-k}$ in F' and a continuous injective linear operator $s\colon E' \to F'$ such that*

$$T(P) = \psi_1 ... \psi_{l-k} \overline{P} \circ s' \circ J_F$$

for all $P \in \mathcal{P}_N({}^kE)$. Moreover, given any operator ideal $\mathcal{A}$, if $T \in \mathcal{A}(\mathcal{P}_N({}^kE); \mathcal{P}_A({}^lE))$ then $s \in \mathcal{A}(E'; F')$.

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