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Boundedness for pseudodifferential operators on multivariate α-modulation spaces

Lasse Borup and Morten Nielsen

Abstract. The α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$, $\alpha\in[0,1]$, form a family of spaces that contain the Besov and modulation spaces as special cases. In this paper we prove that a pseudodifferential operator $\sigma(x,D)$ with symbol in the Hörmander class $S^b_{\rho,0}$ extends to a bounded operator $\sigma(x,D)\colon M^{s,\alpha}_{p,q}(\mathbf{R}^d)\to M^{s-b,\alpha}_{p,q}(\mathbf{R}^d)$ provided $0\le\alpha\le\rho\le 1$, and $1<p,q<\infty$. The result extends the well-known result that pseudodifferential operators with symbol in the class $S^b_{1,0}$ maps the Besov space $B^s_{p,q}(\mathbf{R}^d)$ into $B^{s-b}_{p,q}(\mathbf{R}^d)$.

1. Introduction

In this paper we study pseudodifferential operators on the so-called α-modulation spaces. It was proved by Yamazaki [23] that any pseudodifferential operator $\sigma(x,D)$ in the Hörmander class $\mathrm{Op}(S^b_{1,0})$ extends uniquely to a bounded operator from the Besov space $B^s_{p,q}$ to $B^{s-b}_{p,q}$ when $s\in\mathbf{R}$ and $1<p,q<\infty$. The main result of the present paper is to generalize this result to the full scale of α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$. We prove that for $\alpha\in[0,1]$, any pseudodifferential operator $\sigma(x,D)$ in the class $\mathrm{Op}(S^b_{\rho,0}(\mathbf{R}^d\times\mathbf{R}^d))$, with $1\ge\rho\ge\alpha$, extends to a bounded operator

$$\sigma(x,D)\colon M^{s,\alpha}_{p,q}(\mathbf{R}^d)\longrightarrow M^{s-b,\alpha}_{p,q}(\mathbf{R}^d),\quad 1<p,q<\infty, \tag{1.1}$$

where $\sigma(x,D)$ is defined in terms of the symbol $\sigma(x,\xi)$ by

$$\sigma(x,D)f(x):=\frac{1}{(2\pi)^{d/2}}\int_{\mathbf{R}^d}\sigma(x,\xi)\hat{f}(\xi)e^{ix\xi}\,d\xi,\quad f\in\mathcal{S}(\mathbf{R}^d).$$

The precise definition of the Hörmander class $S^b_{\rho,\delta}(\mathbf{R}^d\times\mathbf{R}^d)$ is given in Section 4. For $\rho<1$, we have a strict inclusion $S^b_{1,0}(\mathbf{R}^d\times\mathbf{R}^d)\subset S^b_{\rho,0}(\mathbf{R}^d\times\mathbf{R}^d)$ so the

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estimate (1.1) holds for symbols not covered by the corresponding result for Besov spaces. An example of a symbol $\sigma\in S^b_{1/2,0}(\mathbf{R}\times\mathbf{R})\setminus S^b_{1,0}(\mathbf{R}\times\mathbf{R})$ is the symbol associated with the convolution kernel $K(x)=e^{i/|x|}|x|^{-\gamma}$, $\gamma>0$. It can be shown that $\widehat{K}(\xi)\in S^{\gamma/2-3/4}_{1/2,0}(\mathbf{R}^2)$, see [18, Chapter VII].

The family of α-modulation spaces was introduced by Gröbner [9]. Gröbner used the general framework of decomposition type Banach spaces considered by Feichtinger and Gröbner in [7] and [8] to build the α-modulation spaces. The parameter α determines a specific type of decomposition of the frequency space $\mathbf{R}^d$ used to define the space $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$, the precise definition will be given in Section 2. The α-modulation spaces contain the Besov spaces and the modulation spaces, introduced by Feichtinger [6], as special cases. The choice $\alpha=0$ corresponds to the classical modulation spaces $M^s_{p,q}(\mathbf{R}^d)$, and $\alpha=1$ corresponds to the Besov scale of spaces. The family of coverings used to construct the α-modulation spaces was considered independently by Päivärinta and Somersalo in [15]. Päivärinta and Somersalo used the partitions to extend the Calderón–Vaillancourt boundedness result for pseudodifferential operators to the local Hardy spaces.

Pseudodifferential operators on α-modulation spaces has been considered by Nazaret and Holschneider in [14]. Their results are based on a continuous wavelet-type decomposition and can be seen as an extension of the fundamental results by Córdoba and Fefferman [5]. Pseudodifferential operators on α-modulation spaces have also been studied by one of the present authors in [4]. The results in [4] are weaker than the corresponding results in the present paper and they apply only to the univariate case. It is used in [4] that nice orthonormal brushlet bases can be found for $M^{s,\alpha}_{p,q}(\mathbf{R})$. At present, there is no construction of nice bases for the multivariate α-modulation spaces. Pseudodifferential operators on modulation spaces were first studied by Tachizawa [19], and later by a number of authors, see e.g. [1], [2], [10], [11], [13], [20] and [21]. Extensions of Tachizawa's results to the case of ultramodulation spaces and pseudodifferential operators with symbols which might grow faster than polynomials have been done by Pilipović and Teofanov [16] and [17].

The structure of the paper is as follows. In Section 2 we give the precise definition of the α-modulation spaces based on a so-called bounded admissible partition of unity (BAPU). The spaces are independent of the specific choice of BAPU, which we exploit to construct a partition with "nice" functions from the Schwartz space $\mathcal{S}(\mathbf{R}^d)$. The construction of the BAPU is done in Section 2.1. In Section 3 we make preparations for the main result in Section 4 by proving boundedness results for multiplier operators on α-modulation spaces. The main result is proved in Section 4 using the multiplier result from Section 3. Since we do not have an atomic decomposition of $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$, $d>1$, the idea of the proof is to expand the symbol $\sigma(x,\xi)$ in a Taylor series in x, and then estimate each contributing factor. Hypoelliptic pseudodifferential operators on the α-modulation spaces are considered in Section 5. Finally, there is an appendix where we prove certain facts about α-coverings needed for the construction of the BAPU in Section 2.1.

2. Modulation spaces

In this section we define the α-modulation spaces. The α-modulation spaces, first introduced by Gröbner in [9], are a family of spaces that contain the classical modulation and Besov spaces as special "extremal" cases. The spaces are defined by a parameter α, belonging to the interval $[0,1]$. This parameter determines a segmentation of the frequency domain from which the spaces are built.

Definition 2.1. A countable set $\mathcal{Q}$ of subsets $Q\subset\mathbf{R}^d$ is called an admissible covering if $\mathbf{R}^d=\bigcup_{Q\in\mathcal{Q}}Q$ and there exists $n_0<\infty$ such that $\#\{Q'\in\mathcal{Q}:Q\cap Q'\neq\varnothing\}\leq n_0$ for all $Q\in\mathcal{Q}$. An admissible covering is called an α-covering, $0\leq\alpha\leq1$, of $\mathbf{R}^d$ if $|Q|\asymp\langle x\rangle^{\alpha d}$ (uniformly) for all $x\in Q$ and for all $Q\in\mathcal{Q}$ where $\langle x\rangle:=(1+|x|^2)^{1/2}$ for $x\in\mathbf{R}^d$.

We let $\mathcal{F}(f)(\xi):=\hat{f}(\xi):=(2\pi)^{-d/2}\int_{\mathbf{R}^d}f(x)e^{-ix\xi}\,dx$, $f\in L_1(\mathbf{R}^d)$, denote the Fourier transform.

Definition 2.2. Let $\mathcal{Q}$ be an α-covering of $\mathbf{R}^d$. A corresponding bounded admissible partition of unity (BAPU) $\{\psi_Q\}_{Q\in\mathcal{Q}}$ is a family of functions satisfying

$$\operatorname{supp}(\psi_Q)\subset Q,\quad \sum_{Q\in\mathcal{Q}}\psi_Q(\xi)=1\quad\text{and}\quad\sup_Q\|\mathcal{F}^{-1}\psi_Q\|_{L_1}<\infty.$$

Definition 2.3. Given $1\leq p,q\leq\infty$, $s\in\mathbf{R}$, and $0\leq\alpha\leq1$, let $\mathcal{Q}$ be an α-covering of $\mathbf{R}^d$ and let $\{\psi_Q\}_{Q\in\mathcal{Q}}$ be a BAPU. Then we define the *α-modulation space* $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$ as the set of distributions $f\in S'(\mathbf{R}^d)$ satisfying

$$\|f\|_{M^{s,\alpha}_{p,q}}:=\left(\sum_{Q\in\mathcal{Q}}\langle\xi_Q\rangle^{qs}\big\|\mathcal{F}^{-1}(\psi_Q\mathcal{F}f)\big\|^q_{L_p}\right)^{1/q}<\infty,\tag{2.1}$$

with $\{\xi_Q\}_{Q\in\mathcal{Q}}$ being a sequence satisfying $\xi_Q\in Q$. For $q=\infty$ we have the usual change of the sum to sup over $Q\in\mathcal{Q}$.

It is proved in [9] that the definition of $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$ is independent of the α-covering and of the BAPU, see also [8, Theorem 2.3]. In Section 2.1 below we construct a BAPU $\{\psi_k\}_{k\in\mathbf{Z}^d\setminus\{0\}}\subset\mathcal{S}(\mathbf{R}^d)$, satisfying $|\partial^\beta\psi_k(\xi)|\leq C_\beta\langle\xi\rangle^{-\alpha|\beta|}$ for every multi-index $\beta\in\mathbf{N}_0^d$. We will use this particular BAPU to simplify the proof of our main result in Section 4.

2.1. Bounded admissible partitions of unity and their properties

The α-modulation spaces are defined using a bounded admissible partition of unity, but the spaces are actually independent of the specific choice. The results in Sections 3 and 4 rely on the fact that it is possible to construct a smooth BAPU with certain "nice" properties. We have the following construction.

Proposition 2.4. *For $\alpha\in[0,1)$, there exists an α-covering of $\mathbf{R}^d$ with a corresponding BAPU $\{\psi_k\}_{k\in\mathbf{Z}^d\setminus\{0\}}\subset\mathcal{S}(\mathbf{R}^d)$ satisfying*

$$|\partial^\beta\psi_k(\xi)|\le C_\beta\langle\xi\rangle^{-|\beta|\alpha},$$

for every multi-index β and $k\in\mathbf{Z}^d\setminus\{0\}$.

Proof. For $r>0$, and $k\in\mathbf{Z}^d\setminus\{0\}$ we define the ball

$$B_k^r:=\big\{\xi\in\mathbf{R}^d:\big|\,\xi-|k|^{\alpha/(1-\alpha)}k\,\big|<r|k|^{\alpha/(1-\alpha)}\big\}.$$

By Lemma A.1, there exists $r_1>0$ such that $\{B_k^{r_1}\}_{k\in\mathbf{Z}^d\setminus\{0\}}$ is an α-covering of $\mathbf{R}^d$. There also exists $0<r_2<r_1$, such that $\{B_k^{r_2}\}_{k\in\mathbf{Z}^d\setminus\{0\}}$ are pairwise disjoint.

Fix $r>r_1$. We now take $\Phi\in C^\infty(\mathbf{R}^d)$ satisfying $\inf_{\xi\in B(0,r_1)}|\Phi(\xi)|:=c>0$ and $\operatorname{supp}(\Phi)\subset B(0,r)$. Let

$$g_k(\xi):=\Phi(|c_k|^{-\alpha}(\xi-c_k)),\quad k\in\mathbf{Z}^d\setminus\{0\},$$

where $c_k:=|k|^{\alpha/(1-\alpha)}k$. Clearly, we have $g_k\in C^\infty(\mathbf{R}^d)$ with $\operatorname{supp}(g_k)\subset B_k^r$. In fact, $\{\operatorname{supp}(g_k)\}_{k\in\mathbf{Z}^d\setminus\{0\}}$ is an α-covering of $\mathbf{R}^d$. The covering is admissible (see Lemma A.1) since $\{B_k^{r_2}\}_{k\in\mathbf{Z}^d\setminus\{0\}}$, with $B_k^{r_2}\subset\operatorname{supp}(g_k)$, are pairwise disjoint. It is easy to see that the partition has "finite height", i.e., $\sum_{k\in\mathbf{Z}^d\setminus\{0\}}\chi_{\operatorname{supp}(g_k)}(\xi)\le n_1$ for some uniform constant n_1.

Notice that

$$|\partial^\beta g_k(\xi)|=|c_k|^{-\alpha|\beta|}\big|(\partial^\beta\Phi)(|c_k|^{-\alpha}(\xi-c_k))\big|\le C_\beta|c_k|^{-\alpha|\beta|},$$

and since $|c_k|\ge1$ for all $k\in\mathbf{Z}^d\setminus\{0\}$, we have

$$|\partial^\beta g_k(\xi)|\le C'_\beta\langle c_k\rangle^{-\alpha|\beta|}\asymp\langle\xi\rangle^{-\alpha|\beta|}\quad\text{for all }\xi\in B_k^r.$$

Since we want a BAPU, we consider the sum $g(\xi):=\sum_{k\in\mathbf{Z}^d\setminus\{0\}}g_k(\xi)$. Now, $\{\operatorname{supp}(g_k)\}_{k\in\mathbf{Z}^d\setminus\{0\}}$ has finite height, so g is well defined, and the finite overlap ensures that $|\partial^\beta g(\xi)|\le C'_\beta\langle\xi\rangle^{-|\beta|\alpha}$. Recall that $g_k(\xi)\ge c$ for all $\xi\in B_k^{r_1}$, and since $\{B_k^{r_1}\}_{k\in\mathbf{Z}^d\setminus\{0\}}$ covers $\mathbf{R}^d$, we have $g(\xi)\ge c$. Thus, we can define

$$\psi_n(\xi):=\frac{g_n(\xi)}{\sum_{k\in\mathbf{Z}^d\setminus\{0\}}g_k(\xi)}.$$

By Lemma B.2 in Appendix B, $|\partial^\beta \psi_k(\xi)| \le C_\beta \langle \xi \rangle^{-|\beta|\alpha}$. In order to conclude, we need to verify that $\sup_{k\in \mathbf{Z}^d\setminus\{0\}} \|\mathcal{F}^{-1}\psi_k\|_{L_1} < \infty$. Let $\tilde{\psi}_k(\xi) = \psi_k(|c_k|^\alpha \xi + c_k)$. By a simple substitution in each of the following integrals, we obtain

$$\begin{aligned}\|\mathcal{F}^{-1}\psi_k\|_{L_1} &= (2\pi)^{-d/2} \int_{\mathbf{R}^d} \left| \int_{\mathbf{R}^d} \psi_k(\xi) e^{ix\xi}\, d\xi \right| dx \\ &= (2\pi)^{-d/2} \int_{\mathbf{R}^d} \left| \int_{\mathbf{R}^d} \tilde{\psi}_k(\xi) e^{ix\xi}\, d\xi \right| dx \\ &\le C_d \biggl(\sum_{|\beta|\le d+1} \|\partial^\beta \tilde{\psi}_k\|_{L_1} \biggr) \int_{\mathbf{R}^d} \langle x \rangle^{-d-1}\, dx \\ &\le C'_d,\end{aligned}$$

where we have used Lemmas B.1 and B.3 for the last estimate. We conclude that $\{\psi_k\}_{k\in\mathbf{Z}^d\setminus\{0\}}$ is a BAPU corresponding to the α-covering $\{\operatorname{supp}(g_k)\}_{k\in\mathbf{Z}^d\setminus\{0\}}$. □

Let us briefly return to Definition 2.3. We rewrite (2.1) in terms of the BAPU from Proposition 2.4, using the multiplier operators $\psi_k(D)$,

$$\|f\|_{M^{s,\alpha}_{p,q}} = \bigl\| \langle |k|^{1/(1-\alpha)} \rangle^s \|\psi_k(D) f\|^q_{L_p} \bigr\|_{l_q(\mathbf{Z}^d)}.$$

We need the following result proved in [8, Theorem 2.3]. Define $\Psi_k := \sum_{k'} \psi_{k'}$, where the sum is taken over all $k' \in \mathbf{Z}^d\setminus\{0\}$ with $B^r_{k'} \cap B^r_k \neq \varnothing$. Then

$$\|f\|_{M^{s,\alpha}_{p,q}} \asymp \bigl\| \langle |k|^{1/(1-\alpha)} \rangle^s \|\Psi_k(D) f\|^q_{L_p} \bigr\|_{l_q}. \tag{2.2}$$

Recall that the definition of $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$ does not depend on the particular choice of BAPU, see [9]. It is easy to see, using the BAPU above, that $\mathcal{S}(\mathbf{R}^d)$ is dense in $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$, $1 \le p, q < \infty$.

3. Differential operators on α-modulation spaces

In this section we consider a special class of pseudodifferential operators, namely Fourier multipliers, and show that this class is well behaved on $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$. One important example of such an operator is the Bessel potential $J^b := (I-\Delta)^{b/2}$ defined by $\widehat{J^b f}(\xi) = \langle \xi \rangle^b \hat{f}(\xi)$. It is well known that for the Besov spaces $J^b B^s_{p,q}(\mathbf{R}^d) = B^{s-b}_{p,q}(\mathbf{R}^d)$, and it is perhaps surprising that J^b has exactly the same lifting property when considered on $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$, $\alpha > 0$.

Proposition 3.1. *Let $L=\lceil d/2\rceil$. Suppose that the function $\sigma\in C^L(\mathbf{R}^d)$ satisfies $|\partial^\beta\sigma(\xi)|\le C\langle\xi\rangle^{b-|\beta|\rho}$ for $|\beta|\le L$, $b\in\mathbf{R}$ and $0\le\rho\le 1$. Let T be the Fourier multiplier given by $\widehat{Tf}:=\sigma\hat{f}$. Then T extends to a bounded operator $T\colon M^{s,\alpha}_{p,q}(\mathbf{R}^d)\to M^{s-b,\alpha}_{p,q}(\mathbf{R}^d)$ for $0\le\alpha\le\rho$, $s\in\mathbf{R}$, $1<p<\infty$, and $1\le q\le\infty$, i.e.,*

$$\|Tf\|_{M^{s-b,\alpha}_{p,q}}\le C\|f\|_{M^{s,\alpha}_{p,q}}\quad \textit{for all } f\in M^{s,\alpha}_{p,q}(\mathbf{R}^d).$$

Proof. For $\alpha=1$ (i.e., in the Besov space case) the result is well known, see e.g. [22, Chapter 2]. Suppose $\alpha<1$. Let $\{\psi_k\}_{k\in\mathbf{Z}^d\setminus\{0\}}$ be the BAPU from Proposition 2.4, and let $c_k=k|k|^{\alpha/(1-\alpha)}$ be the center of the ball B^r_k, see Section 2.1. Define $\Psi_k:=\sum_{k'}\psi_{k'}$, where the sum is taken over all $k'\in\mathbf{Z}^d\setminus\{0\}$ with $B^r_{k'}\cap B^r_k\neq\varnothing$. By Proposition 2.4,

$$|\partial^\beta\Psi_k(\xi)|\le C\langle\xi\rangle^{-\alpha|\beta|},$$

with C independent of $k\in\mathbf{Z}^d\setminus\{0\}$. Define

$$\sigma_k(\xi):=\langle c_k\rangle^{-b}\sigma(\xi)\Psi_k(\xi).$$

Since $\alpha\le\rho$, we have

$$\begin{aligned}|\partial^\beta\sigma_k(\xi)|&\le\langle c_k\rangle^{-b}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}|\partial^\gamma\sigma(\xi)|\,|\partial^{\beta-\gamma}\Psi_k(\xi)|\\ &\le C\langle c_k\rangle^{-b}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\langle\xi\rangle^{b-|\gamma|\rho}\langle\xi\rangle^{-\alpha(|\beta|-|\gamma|)}\le C_\beta\langle c_k\rangle^{-b}\langle\xi\rangle^{b-\alpha|\beta|}.\end{aligned}$$

Moreover, for $\xi\in\operatorname{supp}(\Psi_k)$ we have $\langle c_k\rangle\asymp\langle\xi\rangle$, and $|\xi-c_k|^d\le C|B^{r_1}_k|\asymp\langle\xi\rangle^{\alpha d}$, which implies that $\langle\xi\rangle^{-\alpha|\beta|}\le C|\xi-c_k|^{-|\beta|}$. Therefore,

$$|\partial^\beta\sigma_k(\xi)|\le C'\langle\xi\rangle^{-|\beta|\alpha}\le C''|\xi-c_k|^{-|\beta|}.$$

Now, by the Hörmander–Mikhlin multiplier theorem (applied to the multiplier $\tilde\sigma_k(\xi):=\sigma_k(\xi+c_k)$) we deduce that σ_k extends to a bounded multiplier on $L_p(\mathbf{R}^d)$, $1<p<\infty$, with bound independent of $k\in\mathbf{Z}^d\setminus\{0\}$. Since $\Psi_k(\xi)=1$ for $\xi\in\operatorname{supp}(\psi_k)$, this implies that

$$\|\mathcal{F}^{-1}(\psi_k\sigma\hat{f})\|_{L_p}\le C\langle c_k\rangle^b\|\mathcal{F}^{-1}(\psi_k\hat{f})\|_{L_p},$$

with C independent of k. The result now follows from Definition 2.3,

$$\begin{aligned}\|Tf\|^q_{M^{s,\alpha}_{p,q}}&\asymp\sum_{k\in\mathbf{Z}^d\setminus\{0\}}\langle c_k\rangle^{qs}\|\mathcal{F}^{-1}(\psi_k\sigma\hat{f})\|^q_{L_p}\\ &\le C\sum_{k\in\mathbf{Z}^d\setminus\{0\}}\langle c_k\rangle^{q(s+b)}\|\mathcal{F}^{-1}(\psi_k\hat{f})\|^q_{L_p}\asymp\|f\|^q_{M^{s+b,\alpha}_{p,q}}.\quad\square\end{aligned}$$

Remark 3.2. Related results in the case of modulation spaces have been obtained by Tachizawa [19] and by Toft [21].

As a corollary, we deduce the following result about $J^b=(1-\Delta)^{b/2}$, which will be used to simplify the proof of our main result, Theorem 4.1.

Corollary 3.3. *Given $b\in\mathbf{R}$, let $J^b=(1-\Delta)^{b/2}$. Then for $0\le\alpha\le1$, $s\in\mathbf{R}$, $1<p<\infty$, and $1\le q\le\infty$ we have $J^bM^{s,\alpha}_{p,q}(\mathbf{R}^d)=M^{s-b,\alpha}_{p,q}(\mathbf{R}^d)$, in the sense that*

$$\|f\|_{M^{s,\alpha}_{p,q}} \asymp \|J^b f\|_{M^{s-b,\alpha}_{p,q}} \quad \textit{for all } f\in M^{s,\alpha}_{p,q}(\mathbf{R}^d).$$

Proof. The result follows by Proposition 3.1 using the identity $(J^b)^{-1}=J^{-b}$. □

4. Pseudodifferential operators on α-modulation spaces

This section contains the main result. Recall that the Hörmander class $S^b_{\rho,\delta}(\mathbf{R}^d\times\mathbf{R}^d)$ is the family of functions $\sigma\in C^\infty(\mathbf{R}^d\times\mathbf{R}^d)$ satisfying

$$|\sigma|^{(b)}_{N,M} := \max_{\substack{|\alpha|\le N\\ |\beta|\le M}} \sup_{x,\xi\in\mathbf{R}^d} \langle\xi\rangle^{\rho|\alpha|-\delta|\beta|-b} |\partial^\alpha_\xi\partial^\beta_x\sigma(\xi,x)| < \infty$$

for $M,N\in\mathbf{N}$.

Theorem 4.1. *Suppose $b\in\mathbf{R}$, $\alpha\in[0,1]$, $\sigma\in S^b_{\rho,0}(\mathbf{R}^d\times\mathbf{R}^d)$, $\alpha\le\rho\le1$, $s\in\mathbf{R}$, $p\in(1,\infty)$, and $q\in[1,\infty)$. Then*

$$\sigma(x,D)\colon M^{s,\alpha}_{p,q}(\mathbf{R}^d)\longrightarrow M^{s-b,\alpha}_{p,q}(\mathbf{R}^d).$$

The proof of Theorem 4.1 in the case $\alpha=1$ [i.e., $M^{s,\alpha}_{p,q}(\mathbf{R}^d)=B^s_{p,q}(\mathbf{R}^d)$] is relatively easy since it is possible to use wavelet bases to reduce the proof to a matrix estimate. However, we do not have this option in the general case since no nice bases are known for $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$ when $d>1$. In the case $d=1$, so-called brushlet bases for $M^{s,\alpha}_{p,q}(\mathbf{R})$ are available and it is indeed possible to use discrete methods as demonstrated by one of the authors in [4]. Therefore, our proof of Theorem 4.1 is more in the spirit of the analytic methods used to prove the Besov space case before wavelets and other atomic decompositions became available (see [23]).

Before we give the proof of Theorem 4.1, let us state and prove a technical lemma. We let $\check{f}$ denote the inverse Fourier transform of f.

Lemma 4.2. *Suppose $\sigma\in S^0_{\rho,0}$, $\alpha\le\rho\le1$. Then for $|\gamma|\le K$ and $m\ge0$, we have*

$$I := \int_{\mathbf{R}^d} \sup_{z\in\mathbf{R}^d} \big|(\partial^\gamma_x\sigma(z,\cdot\,)\partial^\nu_\xi\psi_k)^\vee(x)\big|\langle x\rangle^m\,dx \le C|\sigma|^{(0)}_{L,K},$$

where $L\in\mathbf{N}$ satisfies $L>m+d$, and C does not depend on $k\in\mathbf{Z}^d\setminus\{0\}$.

Proof. Let $\sigma_\eta^\gamma(x,\xi):=\partial_x^\gamma\partial_\xi^\eta\sigma(x,\xi)$.
We have the equality

$$I=\int_{\mathbf{R}^d}\sup_{z\in\mathbf{R}^d}\left|\int_{\mathbf{R}^d} e^{ix\xi}\sigma^\gamma(z,\xi)\partial_\xi^\nu\psi_k(\xi)\,d\xi\right|\langle x\rangle^m\,dx.$$

Let $\tilde\psi_k(\xi):=\psi_k(|c_k|^\alpha\xi+c_k)$. Then a substitution in each integral gives

$$I=|c_k|^{-\alpha|\nu|}\int_{\mathbf{R}^d}\sup_{z\in\mathbf{R}^d}\left|\int_{\mathbf{R}^d} e^{ix\xi}\sigma^\gamma(z,|c_k|^\alpha\xi+c_k)\partial_\xi^\nu\tilde\psi_k(\xi)\,d\xi\right|\langle x\rangle^m\,dx.$$

By using Lemma B.1 in the inner integral we get

$$I\le C\int_{\mathbf{R}^d}\sup_{z\in\mathbf{R}^d}\sum_{|\beta|\le L}\int_{\mathbf{R}^d}\left|\partial_\xi^\beta[\sigma^\gamma(z,|c_k|^\alpha\xi+c_k)\partial_\xi^\nu\tilde\psi_k(\xi)]\right|d\xi\langle x\rangle^{-L+m}\,dx$$

and by Leibniz's rule, we obtain

$$\begin{aligned}I&\le C'\sum_{\substack{|\beta|\le L\\0\le\eta\le\beta}}\sup_{z\in\mathbf{R}^d}\int_{\mathbf{R}^d}|c_k|^{\alpha|\eta|}|\sigma_\eta^\gamma(z,|c_k|^\alpha\xi+c_k)|\,|\partial_\xi^{\nu+\beta-\eta}\tilde\psi_k(\xi)|\,d\xi\int_{\mathbf{R}^d}\langle x\rangle^{-L+m}\,dx\\&\le C''\sum_{\substack{|\beta|\le L\\0\le\eta\le\beta}}|\sigma|_{|\eta|,K}^{(0)}\int_{\mathbf{R}^d}|\partial_\xi^{\nu+\beta-\eta}\tilde\psi_k(\xi)|\,d\xi\le C'''|\sigma|_{L,K}^{(0)},\end{aligned}$$

where we have used Lemma B.3, $\alpha\le\delta$, and the fact that for $\xi\in\operatorname{supp}(\tilde\psi_k)$,

$$|\sigma_\eta^\gamma(z,|c_k|^\alpha\xi+c_k)|\le|\sigma|_{|\eta|,K}^{(0)}\langle|c_k|^\alpha\xi+c_k\rangle^{-\rho|\eta|}\le C|\sigma|_{|\eta|,K}^{(0)}\langle c_k\rangle^{-\rho|\eta|}.\quad\square$$

Proof of Theorem 4.1. From the facts that $J^{-a}M_{p,q}^{s,\alpha}=M_{p,q}^{s+a,\alpha}$, $\sigma(x,D)J^a\in\operatorname{Op}S_{\rho,0}^{b+a}$, and $J^a\sigma(x,D)\in\operatorname{Op}S_{\rho,0}^{b+a}$ when $\sigma\in S_{\rho,0}^b$, it follows that it is no restriction to assume that $s>3d$ and $b=0$. Moreover, it suffices to prove that $\|\sigma(x,D)f\|_{M_{p,q}^{s,\alpha}}\le C\|f\|_{M_{p,q}^{s,\alpha}}$ for $f\in\mathcal{S}(\mathbf{R}^d)$ since $\mathcal{S}(\mathbf{R}^d)$ is dense in $M_{p,q}^{s,\alpha}(\mathbf{R}^d)$.

Fix $f\in\mathcal{S}(\mathbf{R}^d)$. We need to estimate the L_p-norm of $\psi_k(D)\sigma(x,D)f$. Notice that for $g\in\mathcal{S}(\mathbf{R}^d)$,

$$(4.1)\quad[\psi_k(D)g](x)=(2\pi)^{-d/2}\int_{\mathbf{R}^d}e^{ixy}\psi_k(y)\hat g(y)\,dy=(2\pi)^{-d/2}\int_{\mathbf{R}^d}\hat\psi_k(y)g(x+y)\,dy.$$

Letting $\sigma_\eta^\gamma(x,\xi):=\partial_x^\gamma\partial_\xi^\eta\sigma(x,\xi)$, we obtain

$$
\begin{aligned}
\sigma(x+y,D)f(x+y)& \\
&=(2\pi)^{-d/2}\int_{\mathbf{R}^d} e^{i(x+y)\xi}\sigma(x+y,\xi)\hat f(\xi)\,d\xi\\
&=(2\pi)^{-d/2}\sum_{|\gamma|\le K-1}\frac{y^\gamma}{\gamma!}\int_{\mathbf{R}^d} e^{i(x+y)\xi}\sigma^\gamma(x,\xi)\hat f(\xi)\,d\xi\\
&\quad+(2\pi)^{-d/2}\sum_{|\gamma|=K}K\frac{y^\gamma}{\gamma!}\int_{\mathbf{R}^d} e^{i(x+y)\xi}\int_0^1(1-\tau)^{K-1}\sigma^\gamma(x+\tau y,\xi)\hat f(\xi)\,d\tau\,d\xi\\
&:=T(x,y)+R(x,y),
\end{aligned}
\tag{4.2}
$$

where we have expanded $\sigma(x+y,\xi)$ in a Taylor series around x. The order K is chosen such that $K\alpha>\min\{0,s+(1-\alpha)(1+3d)\}$. Using (4.2) in (4.1), we obtain

$$
\psi_k(D)\sigma(x,D)f(x)=(2\pi)^{-d/2}\int_{\mathbf{R}^d}\hat\psi_k(y)T(x,y)\,dy+(2\pi)^{-d/2}\int_{\mathbf{R}^d}\hat\psi_k(y)R(x,y)\,dy.
\tag{4.3}
$$

We estimate each of the two terms separately. For the first term we have

$$
\begin{aligned}
\int_{\mathbf{R}^d}\hat\psi_k(y)T(x,y)\,dy& \\
&=(2\pi)^{-d/2}\int_{\mathbf{R}^d}\hat\psi_k(y)\sum_{|\gamma|\le K-1}\frac{y^\gamma}{\gamma!}\int_{\mathbf{R}^d} e^{i(x+y)\xi}\sigma^\gamma(x,\xi)\hat f(\xi)\,d\xi\,dy\\
&=(2\pi)^{-d/2}\sum_{|\gamma|\le K-1}\frac{1}{\gamma!}\int_{\mathbf{R}^d} e^{ix\xi}\sigma^\gamma(x,\xi)\hat f(\xi)\int_{\mathbf{R}^d} e^{iy\xi}\hat\psi_k(y)y^\gamma\,dy\,d\xi\\
&=\sum_{|\gamma|\le K-1}\frac{1}{\gamma!}\int_{\mathbf{R}^d} e^{ix\xi}\sigma^\gamma(x,\xi)\partial_\xi^\gamma\psi_k(\xi)\hat f(\xi)\,d\xi.
\end{aligned}
\tag{4.4}
$$

We apply the L_p norm to (4.4). Define $\Psi_k:=\sum_{k'}\psi_{k'}$, where the sum is taken over all $k'\in\mathbf{Z}^d\setminus\{0\}$ with $B_{k'}^r\cap B_k^r\neq\varnothing$. Using Minkowski's inequality and the fact that $\Psi_k(\xi)=1$ on $\operatorname{supp}(\psi_k)$, we get

$$
\begin{aligned}
\biggl(\int_{\mathbf{R}^d}\biggl|\int_{\mathbf{R}^d}\hat\psi_k(y)T(x,y)\,dy\biggr|^p\,dx\biggr)^{1/p}& \\
&\le\sum_{|\gamma|\le K-1}\frac{1}{\gamma!}\biggl(\int_{\mathbf{R}^d}\biggl|\int_{\mathbf{R}^d} e^{ix\xi}\sigma^\gamma(x,\xi)\partial_\xi^\gamma\psi_k(\xi)\hat f(\xi)\,d\xi\biggr|^p\,dx\biggr)^{1/p}
\end{aligned}
$$

$$\leq \sum_{|\gamma|\leq K-1} \frac{1}{\gamma!}\bigg(\int_{\mathbf{R}^d} \sup_{z\in\mathbf{R}^d}\bigg|\int_{\mathbf{R}^d} e^{ix\xi}\sigma^\gamma(z,\xi)\partial_\xi^\gamma\psi_k(\xi)\hat{f}(\xi)\,d\xi\bigg|^p\,dx\bigg)^{1/p}$$
$$= \sum_{|\gamma|\leq K-1} \frac{1}{\gamma!}\bigg(\int_{\mathbf{R}^d} \sup_{z\in\mathbf{R}^d}\bigg|\int_{\mathbf{R}^d} e^{ix\xi}\sigma^\gamma(z,\xi)\partial_\xi^\gamma\psi_k(\xi)\Psi_k(\xi)\hat{f}(\xi)\,d\xi\bigg|^p\,dx\bigg)^{1/p}.$$

We now use the relation $(\hat{f}\hat{g})^\vee=(2\pi)^{-d/2}f*g$ to estimate the right-hand side with

$$\sum_{|\gamma|\leq K-1} \frac{1}{\gamma!}\bigg(\int_{\mathbf{R}^d} \sup_{z\in\mathbf{R}^d}\bigg|\int_{\mathbf{R}^d} |(\sigma^\gamma(z,\xi)\partial_\xi^\gamma\psi_k(\xi))^\vee(y)|\,|\Psi_k(D)f(x-y)|\,dy\bigg|^p\,dx\bigg)^{1/p}$$
$$\leq \sum_{|\gamma|\leq K-1} \frac{1}{\gamma!}\bigg(\int_{\mathbf{R}^d} \bigg|\int_{\mathbf{R}^d} \sup_{z\in\mathbf{R}^d}|(\sigma^\gamma(z,\xi)\partial_\xi^\gamma\psi_k(\xi))^\vee(y)|\,|\Psi_k(D)f(x-y)|\,dy\bigg|^p\,dx\bigg)^{1/p}$$
$$\leq \sum_{|\gamma|\leq K-1} \frac{1}{\gamma!}\Big\|\sup_{z\in\mathbf{R}^d}(\sigma^\gamma(z,\xi)\partial_\xi^\gamma\psi_k(\xi))^\vee\Big\|_{L_1}\|\Psi_k(D)f\|_{L_p},$$

where we used standard norm estimates for convolutions in the last estimate. Hence, by Lemma 4.2 we may conclude that

$$\bigg(\int_{\mathbf{R}^d}\bigg|\int_{\mathbf{R}^d}\hat{\psi}_k(y)T(x,y)\,dy\bigg|^p\,dx\bigg)^{1/p} \leq C|\sigma|^{(0)}_{d+1,K}\|\Psi_k(D)f\|_{L_p}. \tag{4.5}$$

Now we turn to the second term in (4.3). We let $\mu_k(\xi)=\psi_k(a_k\xi)$, where $a_k:=\langle|k|^{1/(1-\alpha)}\rangle$. Notice that

$$\int_{\mathbf{R}^d}\hat{\psi}_k(y)R(x,y)\,dy=\int_{\mathbf{R}^d}\hat{\mu}_k(y)R(x,a_k^{-1}y)\,dy.$$

We have,

$$\bigg|\sum_{|\gamma|=K}\frac{a_k^{-K}}{\gamma!}\int_{\mathbf{R}^d}y^\gamma\widehat{\mu}_k(y)\int_{\mathbf{R}^d}e^{i(x+a_k^{-1}y)\xi}\int_0^1(1-\tau)^{K-1}\sigma^\gamma(x+a_k^{-1}\tau y,\xi)\hat{f}(\xi)\,d\tau\,d\xi\,dy\bigg|$$
$$\leq Ca_k^{-K}\sum_{|\gamma|=K}\int_{\mathbf{R}^d}\langle y\rangle^K|\widehat{\mu}_k(y)|\bigg|\int_0^1(1-\tau)^{K-1}\int_{\mathbf{R}^d}e^{i(x+a_k^{-1}y)\xi}\sigma^\gamma(x+a_k^{-1}\tau y,\xi)\hat{f}(\xi)\,d\xi\,d\tau\bigg|\,dy.$$

Using Lemma B.4 with $m=K+d+1+\theta d$ for a fixed $1<\theta<2$ we obtain the following estimate for the right-hand side,

$$C'a_k^{-\widetilde{K}}\sum_{|\gamma|=K}\int_{\mathbf{R}^d}\frac{\langle y\rangle^{-d-1}}{\langle y\rangle^{\theta d}}\sup_{z\in\mathbf{R}^d}|[\sigma^\gamma(z,D)f](x+a_k^{-1}y)|\,dy$$
$$=C'a_k^{-\widetilde{K}}\sum_{|\gamma|=K}\int_{\mathbf{R}^d}\langle y\rangle^{-d-1}\sup_{z\in\mathbf{R}^d}\frac{|[\sigma^\gamma(z,D)f](x+a_k^{-1}y)|}{\langle y\rangle^{\theta d}}\,dy$$

$$
\begin{aligned}
&\le C' a_k^{-\widetilde{K}} \sum_{|\gamma|=K} \int_{\mathbf{R}^d} \langle y\rangle^{-d-1} \sup_{z,\eta\in\mathbf{R}^d} \frac{|[\sigma^\gamma(z,D)f](x+\eta)|}{\langle a_k\eta\rangle^{\theta d}}\,dy \\
&\le C' a_k^{-\widetilde{K}} \sum_{|\gamma|=K} \int_{\mathbf{R}^d} \langle y\rangle^{-d-1} \sup_{z,\eta\in\mathbf{R}^d} \frac{|[\sigma^\gamma(z,D)f](x+\eta)|}{\langle \eta\rangle^{\theta d}}\,dy, \\
&\le C' a_k^{-\widetilde{K}} \sum_{|\gamma|=K} \sup_{z,\eta\in\mathbf{R}^d} \frac{|[\sigma^\gamma(z,D)f](x+\eta)|}{\langle \eta\rangle^{\theta d}},
\end{aligned}
$$

where $\widetilde{K}=K\alpha-(1+\theta d)(1-\alpha)\ge K\alpha-(1+2d)(1-\alpha)>s+d(1-\alpha)$ and we have used that $\alpha_k\ge 1$ in the last but one inequality. Now,

$$
\begin{aligned}
&\biggl(\sum_{k\in\mathbf{Z}^d\setminus\{0\}} a_k^{sq}\biggl\|\int_{\mathbf{R}^d}\hat{\psi}_k(y)R(x,y)\,dy\biggr\|_{L_p}^q\biggr)^{1/q} \\
&\qquad\le \biggl(C\sum_{k\in\mathbf{Z}^d\setminus\{0\}} a_k^{(s-\widetilde{K})q}\biggl(\sum_{|\gamma|=K}\biggl\|\sup_{\eta,z\in\mathbf{R}^d}\frac{|[\sigma^\gamma(z,D)f](x+\eta)|}{\langle\eta\rangle^{\theta d}}\biggr\|_{L_p}\biggr)^q\biggr)^{1/q}.
\end{aligned}
$$

Since $L^q:=C\sum_{k\in\mathbf{Z}^d\setminus\{0\}} a_k^{(s-\widetilde{K})q}\le C\sum_{k\in\mathbf{Z}^d\setminus\{0\}}|k|^{-d-1}$ is finite, we estimate the right-hand side with

$$
\begin{aligned}
&L\sum_{|\gamma|=K}\biggl\|\sup_{\eta,z\in\mathbf{R}^d}\frac{|[\sigma^\gamma(z,D)f](x+\eta)|}{\langle\eta\rangle^{\theta d}}\biggr\|_{L_p(dx)} \\
&\qquad= L\sum_{|\gamma|=K}\biggl\|\sup_{\eta,z\in\mathbf{R}^d}\frac{\bigl|\bigl[\sigma^\gamma(z,D)\sum_{k\in\mathbf{Z}^d\setminus\{0\}}\psi_k(D)f\bigr](x+\eta)\bigr|}{\langle\eta\rangle^{\theta d}}\biggr\|_{L_p(dx)} \\
&\qquad\le L\sum_{|\gamma|=K}\sum_{k\in\mathbf{Z}^d\setminus\{0\}}\biggl\|\sup_{\eta,z\in\mathbf{R}^d}\frac{|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)|}{\langle\eta\rangle^{\theta d}}\biggr\|_{L_p(dx)}.
\end{aligned}
$$

We estimate the term $A_k:=|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)|$. Let $f_k(x):=[\Psi_k(D)f](x)$. We have

$$
\begin{aligned}
A_k &= \biggl|\int_{\mathbf{R}^d}(\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(x+\eta-y)f_k(y)\,dy\biggr| \\
&\le \int_{\mathbf{R}^d}|(\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(x+\eta-y)|\,|f_k(y)|\,dy \\
&\le \sup_{u\in\mathbf{R}^d}\frac{|f_k(u)|}{\langle x-u\rangle^{\theta d}}\int_{\mathbf{R}^d}|(\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(x+\eta-y)|\langle x-y\rangle^{\theta d}\,dy.
\end{aligned}
$$

Now, by Petree's inequality, $\langle x-y\rangle^{\theta d}\le 2^{1/2\theta}\langle x-y+\eta\rangle^{\theta d}\langle\eta\rangle^{\theta d}$, so

$$\sup_{z,\eta\in\mathbf{R}^d}\frac{A_k}{\langle\eta\rangle^{\theta d}}\le C\sup_{\eta\in\mathbf{R}^d}\frac{|f_k(x-\eta)|}{\langle\eta\rangle^{\theta d}}\sup_{z\in\mathbf{R}^d}\int_{\mathbf{R}^d}|(\sigma^\gamma(z,\xi)\psi_k(\xi))^\vee(u)|\langle u\rangle^{\theta d}\,du$$
$$\le C'\sup_{\eta\in\mathbf{R}^d}\frac{|f_k(x-\eta)|}{\langle\eta\rangle^{\theta d}}|\sigma|^{(0)}_{3d+1,K},$$

where we used Lemma 4.2 and the fact that $\theta<2$. Hence,

$$\sum_{|\gamma|=K}\sum_{k\in\mathbf{Z}^d\setminus\{0\}}\left\|\sup_{\eta,z\in\mathbf{R}^d}\frac{|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)|}{\langle\eta\rangle^{\theta d}}\right\|_{L_p(dx)}$$
$$=\sum_{|\gamma|=K}\sum_{k\in\mathbf{Z}^d\setminus\{0\}}a_k^{\theta d}\left\|\sup_{\eta,z\in\mathbf{R}^d}\frac{|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)|}{a_k^{\theta d}\langle\eta\rangle^{\theta d}}\right\|_{L_p(dx)}$$
$$\le C'|\sigma|^{(0)}_{3d+1,K}\sum_{k\in\mathbf{Z}^d\setminus\{0\}}a_k^{\theta d}\left\|\sup_{\eta\in\mathbf{R}^d}\frac{|f_k(x-\eta)|}{\langle a_k\eta\rangle^{\theta d}}\right\|_{L_p}.$$

Let

$$\hat g_k(\xi):=a_k^d\hat f_k(a_k\xi)=a_k^d\Psi_k(a_k\xi)\hat f(a_k\xi),$$

and notice that $\operatorname{supp}(\hat g_k)\subset B(0,c)$ for some $c>0$ independent of k. The following maximal inequality is proved in Triebel [22, p. 16]

$$\sup_{z\in\mathbf{R}^d}\frac{|g_k(x-z)|}{\langle z\rangle^{\theta d}}\le CM[|g_k|^{1/\theta}(x)]^\theta.$$

Expressing this in terms of f_k, we get

$$\sup_{z\in\mathbf{R}^d}\frac{|f_k(x-z)|}{\langle a_kz\rangle^{\theta d}}\le C[(M|f_k|^{1/\theta})(x)]^\theta,$$

where C does not depend on k. We apply L_p-norms and use the maximal inequality to obtain

$$\left\|\frac{|f_k(x-z)|}{\langle a_kz\rangle^{\theta d}}\right\|_{L_p(dx)}\le C\|[(M|f_k|^{1/\theta})(x)]^\theta\|_{L_p(dx)}=C\|[(M|f_k|^{1/\theta})(x)]\|^\theta_{L_{p\theta}(dx)}$$
$$\le C'\||f_k|^{1/\theta}\|^\theta_{L_{p\theta}(dx)}=C'\|f_k\|_{L_p}.$$

Putting these estimates together yields

$$\sum_{|\gamma|=K}\sum_{k\in\mathbf{Z}^d\setminus\{0\}}\left\|\sup_{\eta,z\in\mathbf{R}^d}\frac{|[\sigma^\gamma(z,D)\psi_k(D)f](x+\eta)|}{\langle\eta\rangle^{\theta d}}\right\|_{L_p(dx)}$$
$$\le C''|\sigma|^{(0)}_{3d+1,K}\sum_{k\in\mathbf{Z}^d\setminus\{0\}}a_k^{\theta d}\|\Psi_k(D)f\|_{L_p},$$

and consequently

$$\Bigg(\sum_{k\in\mathbf{Z}^d\setminus\{0\}} a_k^{sq}\bigg\|\int_{\mathbf{R}^d}\hat{\psi}_k(y)R(x,y)\,dy\bigg\|_{L_p}^q\Bigg)^{1/q} \le C''|\sigma|_{3d+1,K}^{(0)}\sum_{k\in\mathbf{Z}^d\setminus\{0\}} a_k^{\theta d}\|\Psi_k(D)f\|_{L_p}.$$
$$\le C'''|\sigma|_{3d+1,K}^{(0)}\big\|a_k^s\|\Psi_k(D)f\|_{L_p}\big\|_{l_q},$$

since $s>3d>(1+\theta)d$. Finally, we can put the estimates together to close the case $b=0$ and $s>3d$. We have

$$\begin{aligned}
&\|\sigma(x,D)f\|_{M_{p,q}^{s,\alpha}}\\
&\quad\asymp \big\|a_k^s\|\psi_k(D)\sigma(x,D)f\|_{L_p}\big\|_{l^q(\mathbf{Z}^d\setminus\{0\})}\\
&\quad\le C\Bigg(\bigg\|a_k^s\bigg\|\int_{\mathbf{R}^d}\hat{\psi}_k(y)T(x,y)\,dy\bigg\|_{L_p(dx)}\bigg\|_{l_q}+\bigg\|a_k^s\bigg\|\int_{\mathbf{R}^d}\hat{\psi}_k(y)R(x,y)\,dy\bigg\|_{L_p(dx)}\bigg\|_{l_q}\Bigg)\\
&\quad\le C'\big(|\sigma|_{d+1,K}^{(0)}\big\|a_k^s\|\Psi_k(D)f\|_{L_p}\big\|_{l_q}+|\sigma|_{3d+1,K}^{(0)}\big\|a_k^s\|\Psi_k(D)f\|_{L_p}\big\|_{l_q}\big)\\
&\quad\le C''|\sigma|_{3d+1,K}^{(0)}\|f\|_{M_{p,q}^{s,\alpha}}.
\end{aligned}$$

This concludes the proof of the theorem. □

Remark 4.3. A closer examination of the arguments used in the proof reveals that there exist $M,N>0$ (depending on s,q, and ρ) such that the norm of the operator

$$\sigma(x,D)\colon M_{p,q}^{s,\alpha}(\mathbf{R}^d)\longrightarrow M_{p,q}^{s-b,\alpha}(\mathbf{R}^d)$$

is bounded by $C|\sigma|_{M,N}^{(b)}$, with C a constant.

5. Hypoelliptic pseudodifferential operators

In this final section we consider an application of the result in the previous section to hypoelliptic pseudodifferential operators, see [3] and [12]. Let us introduce some notation. Let

$$S_{\rho,\delta}^{\infty} := \bigcup_{m\in\mathbf{R}} S_{\rho,\delta}^b \quad\text{and}\quad S_{\rho,\delta}^{-\infty} := \bigcap_{m\in\mathbf{R}} S_{\rho,\delta}^b.$$

Assume that $b_0,b\in\mathbf{R}$ such that $b_0\le b$. An element $\sigma\in S_{\rho,\delta}^b(\mathbf{R}^d\times\mathbf{R}^d)$ is called *hypoelliptic* with parameters b_0 and b if there are positive constants c and a such that

$$a\langle\xi\rangle^{b_0}\le|\sigma(x,\xi)|,\quad \langle\xi\rangle\ge c,$$

and

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}|\sigma(x,\xi)|\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}, \quad \langle\xi\rangle \ge c.$$

Let $HS^{b,b_0}_{\rho,\delta}(\mathbf{R}^d\times\mathbf{R}^d)$ the family of all such symbols. We have the following result, see [12, Theorem 22.1.3].

Theorem 5.1. *Suppose* $\sigma\in HS^{b,b_0}_{\rho,\delta}$, *with* $\delta<\rho$. *Then there exists* $\tau\in HS^{-b_0,-b}_{\rho,\delta}$ *such that* $I-\sigma(x,D)\tau(x,D)$ *and* $I-\tau(x,D)\sigma(x,D)$ *are both in* $\mathrm{Op}(S^{-\infty}_{\rho,\delta})$.

Let $M^{-\infty,\alpha}_{p,q}(\mathbf{R}^d)=\bigcup_{s\in\mathbf{R}} M^{s,\alpha}_{p,q}(\mathbf{R}^d)$. Using Theorem 5.1 and the result from the previous section we have the following result.

Theorem 5.2. *Suppose* $\sigma\in HS^{b,b_0}_{\rho,\delta}$, *with* $\delta<\rho$ *and* $\rho\ge\alpha$, *and* $f\in M^{-\infty,\alpha}_{p,q}(\mathbf{R}^d)$. *If* $\sigma(\,\cdot\,,D)f\in M^{s,\alpha}_{p,q}(\mathbf{R}^d)$ *for some* $s\in\mathbf{R}$, *then* $f\in M^{s+b_0,\alpha}_{p,q}(\mathbf{R}^d)$.

Proof. Let $S=\sigma(\,\cdot\,,D)$, and let $T=\tau(\,\cdot\,,D)$ be as in Theorem 5.1. Notice that $f=T(Sf)+(I-TS)f$. By Theorem 4.1, T maps $M^{s,\alpha}_{p,q}(\mathbf{R}^d)$ to $M^{s+b,\alpha}_{p,q}(\mathbf{R}^d)$ and $(I-TS)$ maps $M^{-\infty,\alpha}_{p,q}(\mathbf{R}^d)$ to $M^{s+b,\alpha}_{p,q}(\mathbf{R}^d)$. □

The following example will conclude the paper.

Example 5.3. Consider the heat operator L given by

$$L(u):=\frac{\partial u}{\partial t}-\sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2}.$$

The symbol of L is given by

$$l(\tau,\xi)=(i\tau+|\xi|^2), \quad (\tau,\xi)\in\mathbf{R}\times\mathbf{R}^d,$$

and one can easily verify that $l\in HS^{2,1}_{1,0}$. We consider an approximate inverse P to L with symbol

$$a(\tau,\xi)=(i\tau+|\xi|^2)^{-1}\eta(\tau,\xi), \quad (\tau,\xi)\in\mathbf{R}\times\mathbf{R}^d,$$

where η is a smooth cut-off function that vanishes near the origin and is equal to 1 for large (τ,ξ). It is easy to check that $a(\tau,\xi)\in HS^{-1,-2}_{1,0}(\mathbf{R}^{d+1}\times\mathbf{R}^{d+1})$. Therefore, if $u\in M^{-\infty,\alpha}_{p,q}(\mathbf{R}^{d+1})$, $1<p,q<\infty$, $\alpha\in[0,1]$, and $P(u)\in M^{s,\alpha}_{p,q}(\mathbf{R}^{d+1})$, then $u\in M^{s-1,\alpha}_{p,q}(\mathbf{R}^{d+1})$.

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A. Admissible coverings

In this section we discuss a general construction of an α-covering of $\mathbf{R}^d$. This type of covering was considered in [9] and in [15]. A proof of Lemma A.1 below can be found in [9], but since Gröbner's work has never been published, we have included a proof for the sake of completeness. Construction of α-coverings are also considered (from another perspective) in [15].

Notice that the set of balls $\{B(z,\sqrt{d})\}_{z\in\mathbf{Z}^d\setminus\{0\}}$ is an admissible 0-covering of $\mathbf{R}^d$. Define for some $\beta\in(-1,\infty)$, the bijection δ_β on $\mathbf{R}^d$ by $\delta_\beta(\xi):=\xi\,|\xi|^\beta$ (with inverse $\sigma_{\beta'}$, $\beta'=-\beta/(1+\beta)$). Since the set $\{B(z,R)\}_{z\in\mathbf{Z}^d\setminus\{0\}}$ is admissible for $R\geq\sqrt{d}$, so is $\{\delta_\beta(B(z,R))\}_{z\in\mathbf{Z}^d\setminus\{0\}}$. Moreover, we have the following result.

Lemma A.1. *Suppose $\beta\geq0$. Given $R>0$, there exists an $r>0$, such that*

$$\delta_\beta(B(z,R))\subseteq B(\delta_\beta(z),r|z|^\beta)\quad \textit{for all } z\in\mathbf{R}^d,\ \textit{with } |z|\geq1. \tag{A.1}$$

Likewise, given $r>0$ there exists an $R>0$, such that

$$B(\delta_\beta(z),r|z|^\beta)\subseteq\delta_\beta(B(z,R))\quad \textit{for all } z\in\mathbf{R}^d. \tag{A.2}$$

Proof. The proof is based on the following observation. For two points $x,z\in\mathbf{R}^d$ and $\beta\in(-1,\infty)$, we have

$$\begin{aligned}|\delta_\beta(x)-\delta_\beta(z)| &= \big|x|x|^\beta-z|z|^\beta\big|\leq\big|x|x|^\beta-x|z|^\beta\big|+\big|x|z|^\beta-z|z|^\beta\big|\\ &= |x|\,\big||x|^\beta-|z|^\beta\big|+|z|^\beta|x-z| = (|\beta|\,|x|\,|\tilde{x}|^{\beta-1}+|z|^\beta)|x-z|\end{aligned} \tag{A.3}$$

for some $\tilde{x}\in L(x,z)$, by the mean-value theorem.

Given $R>0$, suppose $x\in B(z,R)$. Then (A.3) yields $|\delta_\beta(x)-\delta_\beta(z)|\leq r|z|^\beta$ for some $r>0$ depending only on β and R. Now, take any $y\in\delta_\beta(B(z,R))$, i.e., $y=\delta_\beta(x)$ for some $x\in B(z,R)$. Then $|y-\delta_\beta(z)|\leq r|z|^\beta$, which proves (A.1).

We turn to (A.2). Suppose first that $|z|\leq K$ for some $K>r^{1+\beta}$. Then it is easy to verify that there exists a radius $P>0$ such that $B(\delta_\beta(z),r|z|^\beta)\subset B(0,P)$ for all z. Likewise, there exists a radius R such that $B(0,P)\subset\delta_\beta(B(z,R))$ for all z. This proves (A.2) for $|z|\leq K$.

Suppose now that $|z|>r^{1+\beta}$. Recall that $\delta_\beta^{-1}=\sigma_{\beta'}$, where $\beta':=-\beta/(\beta+1)$. Thus, to show the inclusion (A.2) is equivalent to show that

$$\sigma_{\beta'}(B(z,r|z|^{-\beta'}))\subseteq B(\sigma_{\beta'}(z),R). \tag{A.4}$$

Suppose $x\in B(z,r|z|^{-\beta'})$ for some $\beta'>-1$, then

$$(1-r|z|^{-(1+\beta')})|z|\leq|x|\leq(1+r|z|^{-(1+\beta')})|z|.$$

Since $1+\beta=(1+\beta')^{-1}$, (A.3) yields

$$|\delta_{\beta'}(x)-\delta_{\beta'}(z)|\le R|z|^{\beta'}|z|^{-\beta'}=R$$

for some $R>0$ depending only on r and β'. Now, take any $y\in\delta_{\beta'}(B(z,r|z|^{-\beta'}))$, i.e., $y=\delta_{\beta'}(x)$ for some $x\in B(z,r|z|^{-\beta'})$. Then, $|y-\delta_{\beta'}(z)|\le R$, which proves (A.4). □

Remark A.2. By (A.1) there exists a radius r_1 such that

$$\mathbf{R}^d\subset\bigcup_{z\in\mathbf{Z}^d\setminus\{0\}}B(\delta_\beta(z),r|z|^\beta)$$

for all $r\ge r_1$. Fix such an r and let $R:=R(r)$ be given such that (A.2) holds. Then, since $\{\delta_\beta(B(z,R(r)))\}_{z\in\mathbf{Z}^d\setminus\{0\}}$ is an admissible covering of $\mathbf{R}^d$, so is

$$\{B(\delta_\beta(z),r|z|^\beta)\}_{k\in\mathbf{Z}^d\setminus\{0\}}.$$

Suppose $\beta\ge0$, and let $\alpha=\beta/(\beta+1)$. Then it is easy to see that $|B(\delta_\beta(z),r|z|^\beta)|\asymp\langle y\rangle^{d\alpha}$ for all $y\in B(\delta_\beta(z),r|z|^\beta)$ independent of $z\in\mathbf{Z}^d\setminus\{0\}$. Therefore, by Remark A.2,

$$\{B(\delta_\beta(z),r|z|^\beta)\}_{k\in\mathbf{Z}^d\setminus\{0\}} \tag{A.5}$$

is an α-covering for any $r>r_1$.

B. Some technical lemmas

In this brief section, we have included some of the technical lemmas used in Section 2.1. The first two lemmas follow by standard computations. We let $W^{K,1}(\mathbf{R}^d)$ denote the Sobolev space of functions with derivatives of order up to K in $L_1(\mathbf{R}^d)$.

Lemma B.1. *Let $K\in\mathbf{N}$ and suppose $h\in W^{K,1}(\mathbf{R}^d)$. Then there exists a constant $C_K<\infty$ such that*

$$\langle\xi\rangle^K|\hat{h}(\xi)|\le C\sum_{|\beta|\le K}\|\partial^\beta h\|_{L_1}\le C\|h\|_{W^{K,1}}.$$

Lemma B.2. *Let $f,g\in C^\infty(\mathbf{R}^d)$ and $\gamma\in\mathbf{R}$. Suppose that for each multi-index β there exists a constant $C_\beta<\infty$ such that $|\partial^\beta f(x)|,|\partial^\beta g(x)|\le C_\beta\langle x\rangle^{\gamma|\beta|}$. If $0<c\le|g(x)|\le C<\infty$, then there exists constants C'_β such that*

$$\left|\partial^\beta\left(\frac{f}{g}\right)(x)\right|\le C'_\beta\langle x\rangle^{\gamma|\beta|}.$$

The final two lemmas give estimates on the BAPU after each function has been dilated to have support near the origin. The estimates are used in the proof of Theorem 4.1.

Lemma B.3. *Define* $\tilde{\psi}_k(\xi)=\psi_k(|c_k|^\alpha\xi+c_k)$. *Then for every* $\beta\in\mathbf{N}^d$ *there exists a constant* C_β *independent of* $k\in\mathbf{Z}^d\setminus\{0\}$ *such that*

$$|\partial_\xi^\beta\tilde{\psi}_k(\xi)|\le C_\beta\chi_{B(0,r)}(\xi).$$

Proof. Notice that

$$\tilde{\psi}_k(\xi)=\frac{\Phi(\xi)}{\sum_{k'}\Phi(|c_k|^\alpha|c_{k'}|^{-\alpha}(\xi-c_{k'})+c_k)}.$$

Thus, the result follows by Lemma B.2 □

Lemma B.4. *Let* $c_k:=k|k|^{\alpha/(1-\alpha)}$, $k\in\mathbf{Z}^d\setminus\{0\}$, *and define* $\mu_k(\xi)=\psi_k(a_k\xi)$, *where* $a_k:=\langle c_k\rangle$. *Then for every* $m\in\mathbf{N}$ *there exists a constant* C_m *independent of* k *such that*

$$|\hat{\mu}_k(y)|\le C_m a_k^{(m-d)(1-\alpha)}\langle y\rangle^{-m}.$$

Proof. By Proposition 2.4 we have for any $\beta\in\mathbf{N}^d$,

$$|\partial^\beta\mu_k(\xi)|=a_k^{|\beta|}|(\partial^\beta\psi_k)(a_k\xi)|\le Ca_k^{(1-\alpha)|\beta|}\chi_{B(1,a_k^{-(1-\alpha)})}(\xi),$$

since $\chi_{B_k^r}(a_k\xi)=\chi_{B(1,a_k^{-(1-\alpha)})}(\xi)$. By Lemma B.1 we get

$$\begin{aligned}\langle y\rangle^m|\hat{\mu}_k(y)|&\le C_m\sum_{|\beta|\le m}\|\partial^\beta\mu_k\|_{L_1}\\&\le C'_m a_k^{-(1-\alpha)d}\sum_{|\beta|\le m}a_k^{(1-\alpha)|\beta|}\le C''_m a_k^{-(1-\alpha)d}a_k^{(1-\alpha)m}.\quad\square\end{aligned}$$

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Lasse Borup
Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7G
DK-9220 Aalborg East
Denmark
lasse@math.aau.dk

Morten Nielsen
Department of Mathematics
Washington University
Campus Box 1146
St. Louis, MO 63130
U.S.A.
mnielsen@math.wustl.edu

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