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Values of the Euler phi function not divisible by a given odd prime

Blair K. Spearman and Kenneth S. Williams

Abstract. An asymptotic formula is given for the number of integers $n \leq x$ for which $\phi(n)$ is not divisible by a given odd prime.

1. Introduction

We denote the set of natural numbers by $\mathbf{N}$ and the set of integers by $\mathbf{Z}$. If $a \in \mathbf{Z}$ and $b \in \mathbf{Z}$ are not both 0, we denote the greatest common divisor of a and b by (a, b). We let ϕ denote Euler's phi function so that for $n \in \mathbf{N}$ we have

$$\phi(n) := \operatorname{card}\{m \in \mathbf{N} \mid 1 \leq m \leq n \text{ and } (m, n) = 1\} = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right), \tag{1}$$

where the product is taken over the distinct primes p dividing n. Throughout this paper p denotes a prime. It is well known that for $n \in \mathbf{N}$,

$$2 \nmid \phi(n) \quad \Longleftrightarrow \quad n = 1, 2.$$

We are interested in those $n \in \mathbf{N}$ for which $q \nmid \phi(n)$, where q is a fixed odd prime. We set

$$E_q(x) = \operatorname{card}\{n \leq x \mid q \nmid \phi(n)\}. \tag{2}$$

In 1990 Erdős, Granville, Pomerance and Spiro gave an upper bound for $E_q(x)$, which is valid for all sufficiently large x, see [1, Equation (4.2) with $k=1$, p. 191].

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In this paper we give an asymptotic formula for $E_q(x)$ as $x\to\infty$, see the theorem in Section 4. Let $0<\varepsilon<1$. For q a fixed odd prime, we show that

$$E_q(x)=e(q)x(\log x)^{-1/(q-1)}+O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x\to\infty$, where $e(q)$ is given in Definition 4.1 and the constant implied by the O-symbol depends only on q and ε. In 2002 Luca and Pomerance [2, Lemma 2, p. 114] proved the related result: For some constant $c>0$, for almost all n, $\phi(n)$ is divisible by all prime powers $p^a\le c\log\log n/\log\log\log n$.

2. Notation

We denote the sets of real numbers and complex numbers by $\mathbf{R}$ and $\mathbf{C}$, respectively. As usual Γ denotes the gamma function and γ is Euler's constant. If K is an algebraic number field we write $h(K)$ for the class number of K and $R(K)$ for the regulator of K, see for example [3, pp. 97, 106]. Throughout this paper q denotes a fixed odd prime. We set

$$K_q:=\mathbf{Q}(e^{2\pi i/q})\subseteq\mathbf{C}, \tag{3}$$

so that K_q is a cyclotomic field with $[K_q:\mathbf{Q}]=\phi(q)=q-1$. For brevity we set

$$h(q):=h(K_q)\quad\text{and}\quad R(q):=R(K_q). \tag{4}$$

We also let

$$\omega:=e^{2\pi i/(q-1)}\in\mathbf{C}, \tag{5}$$

so that $\omega^{q-1}=1$. The principal character $\chi_0 \pmod q$ is defined as follows: for $n\in\mathbf{Z}$ we have

$$\chi_0(n)=\begin{cases}1, & \text{if } n\not\equiv 0 \pmod q,\\ 0, & \text{if } n\equiv 0 \pmod q.\end{cases} \tag{6}$$

Let g be a primitive root $\pmod q$. For $n\in\mathbf{Z}$ with $n\not\equiv 0 \pmod q$ the index $\operatorname{ind}_g(n)$ of n with respect to g is defined modulo $q-1$ by

$$n\equiv g^{\operatorname{ind}_g(n)} \pmod q.$$

We define a character $\chi_g \pmod q$ as follows: for $n\in\mathbf{Z}$ we set

$$\chi_g(n)=\begin{cases}\omega^{\operatorname{ind}_g n}, & \text{if } n\not\equiv 0 \pmod q,\\ 0, & \text{if } n\equiv 0 \pmod q.\end{cases} \tag{7}$$

There are exactly $\phi(q)=q-1$ characters $\pmod q$. They are

$$\chi_0,\ \chi_g,\ \chi_g^2,\dots,\ \chi_g^{q-2}, \tag{8}$$

where $\chi_g^{q-1}=\chi_0$.

3. The constant $C(q)$

It is convenient to define the following constant involving χ_g.

Definition 3.1. Let q be an odd prime. Let g be a primitive root $\pmod{q}$. Let $r\in\{1,2,\ldots,q-2\}$. We define

$$C(q,r,\chi_g):=\prod_{\chi_g(p)=\omega^r}\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right), \tag{9}$$

where the product is taken over all primes p such that $\chi_g(p)=\omega^r$.

Note that the prime q is not included in the product as $\chi_g(q)=0$ by (7). As $1\leq(r,q-1)\leq\frac{1}{2}(q-1)$ for $r\in\{1,2,\ldots,q-2\}$ we have

$$\frac{q-1}{(r,q-1)}\geq 2 \tag{10}$$

so that the infinite product in (9) converges. Let h be another primitive root $\pmod{q}$. Then there exists an integer s such that

$$h\equiv g^s \pmod{q},\quad (s,q-1)=1.$$

Let t be an integer such that $st\equiv 1 \pmod{q-1}$. Then, for $n\in\mathbf{N}$ with $n\not\equiv 0 \pmod{q}$, we have

$$\mathrm{ind}_h(n)\equiv t\,\mathrm{ind}_g(n) \pmod{q-1}$$

so that

$$\chi_h(n)=\omega^{\mathrm{ind}_h(n)}=\omega^{t\,\mathrm{ind}_g(n)}=(\chi_g(n))^t=\chi_g^t(n),$$

that is $\chi_h=\chi_g^t$. Hence

$$\begin{aligned}\prod_{r=1}^{q-2}C(q,r,\chi_h)^{(r,q-1)}&=\prod_{r=1}^{q-2}\prod_{\chi_h(p)=\omega^r}\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{(r,q-1)}\\&=\prod_{r=1}^{q-2}\prod_{\chi_g^t(p)=\omega^r}\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{(r,q-1)}\\&=\prod_{r=1}^{q-2}\prod_{\chi_g(p)=\omega^{rs}}\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{(r,q-1)}\end{aligned}$$

$$=\prod_{r=1}^{q-2}\prod_{\chi_g(p)=\omega^{rs}}\left(1-\frac{1}{p^{(q-1)/(rs,q-1)}}\right)^{(rs,q-1)}$$
$$=\prod_{r=1}^{q-2}\prod_{\chi_g(p)=\omega^{r}}\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{(r,q-1)}$$
$$=\prod_{r=1}^{q-2}C(q,r,\chi_g)^{(r,q-1)}$$

so that the product

$$\prod_{r=1}^{q-2}C(q,r,\chi_g)^{(r,q-1)} \tag{11}$$

does not depend on the choice of primitive root g. Thus we can make the following definition.

Definition 3.2. Let q be an odd prime. We define the constant $C(q)$ by

$$C(q):=\prod_{r=1}^{q-2}C(q,r,\chi_g)^{(r,q-1)}. \tag{12}$$

We take this opportunity to determine $C(3)$. It is convenient to define the constant $k_{a,b}(m)$ by

$$k_{a,b}(m):=\prod_{p\equiv b\ (\mathrm{mod}\, a)}\left(1-\frac{1}{p^m}\right), \tag{13}$$

where $a\in\mathbf{N}$ and $b\in\mathbf{N}\cup\{0\}$ are such that $0\le b<a$ and $(a,b)=1$ and $m\in\mathbf{N}$ is such that $m\ge 2$.

Lemma 3.1. $C(3)=k_{3,2}(2)$.

Proof. Let $q=3$. Then $\omega=-1$, $r=1$, $g=2$ and $\chi_2(n)=(-3/n)$. Hence

$$C(3)=C(3,1,\chi_2)=\prod_{\chi_2(p)=-1}\left(1-\frac{1}{p^2}\right)=\prod_{p\equiv 2\ (\mathrm{mod}\, 3)}\left(1-\frac{1}{p^2}\right)=k_{3,2}(2),$$

as asserted. $\square$

4. Statement of main result

We begin with a definition.

Definition 4.1. Let q be an odd prime. We define

$$e(q) := \frac{(q+1)(q-1)^{(q-2)/(q-1)}\Gamma\left(\dfrac{1}{q-1}\right)\sin\left(\dfrac{\pi}{q-1}\right)}{2^{(q-3)/2(q-1)}q^{3(q-2)/2(q-1)}\pi^{3/2}(h(q)R(q)C(q))^{1/(q-1)}}. \tag{14}$$

Before stating our main result, we give the value of $e(3)$.

Lemma 4.1.

$$e(3) = \frac{2^{7/2}}{3^{9/4}}k_{3,1}(2)^{1/2}.$$

Proof. We have $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, $C(3)=k_{3,2}(2)$ and $h(3)=R(3)=1$, so that Definition 4.1 with $q=3$ gives

$$e(3) = \frac{2^{5/2}}{3^{3/4}\pi k_{3,2}(2)^{1/2}}.$$

As

$$\left(1-\frac{1}{3^2}\right)k_{3,1}(2)k_{3,2}(2) = \prod_p\left(1-\frac{1}{p^2}\right) = \frac{6}{\pi^2}$$

we have

$$k_{3,2}(2) = \frac{27}{4\pi^2}\frac{1}{k_{3,1}(2)} \quad \text{and} \quad e(3) = \frac{2^{7/2}}{3^{9/4}}k_{3,1}(2)^{1/2},$$

as asserted. □

Our main result is the following asymptotic formula for $E_q(x)$.

Theorem. *Let $0<\varepsilon<1$. For q an odd prime, we have*

$$E_q(x) = e(q)x(\log x)^{-1/(q-1)} + O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x\to\infty$, where the constant implied by the O-symbol depends only on q and ε, and $e(q)$ is given in Definition 4.1.

This theorem is proved in Section 7 after some preliminary results are given in Sections 5 and 6.

5. Preliminary results

The following results will be used in Sections 6 and 7.

Proposition 5.1. *Let* $n \in \mathbf{N}$ *and let* q *be an odd prime. Then*

$$q \nmid \phi(n) \iff n = \prod_{p \not\equiv 1 \,(\operatorname{mod} q)} p^{a(p)} \text{ or } n = q \prod_{p \not\equiv 1 \,(\operatorname{mod} q)} p^{a(p)},$$

where the product is taken over all primes $p \neq q$ *with* $p \not\equiv 1 \pmod q$ *and the* $a(p)$ *are non-negative integers.*

Proof. If

$$n = q^a \prod_{j=1}^{t} p_j^{a_j},$$

where a and t are non-negative integers, the p_j are distinct primes $\neq q$, and the a_j are non-negative integers, then by (1)

$$\phi(n) = \begin{cases} \displaystyle\prod_{j=1}^{t} p_j^{a_j-1}(p_j-1), & \text{if } a=0, \\ \displaystyle q^{a-1}(q-1)\prod_{j=1}^{t} p_j^{a_j-1}(p_j-1), & \text{if } a \geq 1. \end{cases}$$

Hence $q \nmid \phi(n) \Leftrightarrow a \in \{0,1\}$ and $q \nmid p_j - 1$ $(j=1,\dots,t)$, which proves Proposition 5.1. □

Next we define the set A by

(15) $$A = \{m \in \mathbf{N} \mid p\,(\text{prime}) \mid m \Rightarrow p \neq q \text{ and } p \not\equiv 1 \,(\operatorname{mod}\, q)\}.$$

The function $A(x)$ is defined for $x \in \mathbf{R}$ by

(16) $$A(x) = \sum_{\substack{m \leq x \\ m \in A}} 1.$$

Proposition 5.2. *For* $x \in \mathbf{R}$ *and* q *an odd prime we have*

$$E_q(x) = A(x) + A\left(\frac{x}{q}\right).$$

Proof. This follows immediately from Proposition 5.1. □

Proposition 5.3. (Wirsing's theorem) *Let* $f\colon \mathbf{N}\to\mathbf{R}$ *be multiplicative with* $f(n)\geq 0$ *for all* $n\in\mathbf{N}$*. Suppose that there exist constants* c_1 *and* c_2 *with* $c_1>0$ *and* $0<c_2<2$ *such that*

$$0\leq f(p^k)\leq c_1 c_2^k,$$

for all primes p *and all* $k\in\mathbf{N}$*, and also that there is a constant* τ *with* $\tau>0$ *such that*

$$\sum_{p\leq x} f(p)=\tau\frac{x}{\log x}+o\left(\frac{x}{\log x}\right),$$

as $x\to\infty$*, then*

$$\sum_{n\leq x} f(n)=\left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)}+o(1)\right)\frac{x}{\log x}\prod_{p\leq x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right),$$

as $x\to\infty$*.*

Proof. See [7, Satz 1, p. 76]. □

Proposition 5.4. (Odoni's theorem) *Let* $f\colon \mathbf{N}\to\mathbf{R}$ *be multiplicative with* $f(n)\geq 0$ *for all* $n\in\mathbf{N}$*. Suppose that there exist constants* $a_1>1$ *and* $a_2>1$ *such that*

$$0\leq f(p^k)\leq a_1 k^{a_2},$$

for all primes p *and all* $k\in\mathbf{N}$*, and also that there are constants* τ *and* β *with* $\tau>0$ *and* $0<\beta<1$ *such that*

$$\sum_{p\leq x} f(p)=\tau\frac{x}{\log x}+O\left(\frac{x}{(\log x)^{1+\beta}}\right),$$

as $x\to\infty$*, then there is a constant* $B>0$ *such that*

$$\sum_{n\leq x} f(n)n^{-1}=B(\log x)^{\tau}+O((\log x)^{\tau-\beta}),$$

as $x\to\infty$*. Further, for each fixed* $\lambda>0$*, we have*

$$\sum_{n\leq x} f(n)n^{\lambda-1}=\lambda^{-1}Bx^{\lambda}\tau(\log x)^{\tau-1}+O(x^{\lambda}(\log x)^{\tau-1-\beta}), \tag{17}$$

as $x\to\infty$*.*

Proof. See [4, Theorem II, p. 205; Theorem III, p. 206; Note added in proof, p. 216]. □

From Propositions 5.3 and 5.4 we obtain the following corollary.

Proposition 5.5. *Let* $f\colon \mathbf{N}\to\mathbf{R}$ *be multiplicative with* $0\le f(n)\le 1$ *for all* $n\in\mathbf{N}$. *Suppose that there are constants* τ *and* β *with* $\tau>0$ *and* $0<\beta<1$ *such that*

$$\sum_{p\le x} f(p)=\tau\frac{x}{\log x}+O\left(\frac{x}{(\log x)^{1+\beta}}\right).$$

Then

$$\lim_{x\to\infty}\frac{1}{(\log x)^{\tau}}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right)$$

exists, and

$$\sum_{n\le x} f(n)=Ex(\log x)^{\tau-1}+O(x(\log x)^{\tau-1-\beta}),$$

with

$$E=\frac{e^{-\gamma\tau}}{\Gamma(\tau)}\lim_{x\to\infty}\frac{1}{(\log x)^{\tau}}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right).$$

Proof. The conditions of Odoni's theorem are met (with $a_1=a_2=2$) so by (17) with $\lambda=1$ there is a constant $B>0$ such that

$$\sum_{n\le x} f(n)=Bx\tau(\log x)^{\tau-1}+O(x(\log x)^{\tau-1-\beta}).$$

The conditions of Wirsing's theorem are also met (with $c_1=c_2=1$) so that

$$\sum_{n\le x} f(n)=\left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)}+o(1)\right)\frac{x}{\log x}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right).$$

Equating the two expressions for $\sum_{n\le x} f(n)$, and dividing by $x(\log x)^{\tau-1}$, we obtain

$$B\tau+O((\log x)^{-\beta})=\left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)}+o(1)\right)(\log x)^{-\tau}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right).$$

Letting $x\to\infty$ we have

$$\lim_{x\to\infty}(\log x)^{-\tau}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\ldots\right)=B\tau\Gamma(\tau)e^{\gamma\tau}.$$

Thus

$$\sum_{n\le x} f(n) = Ex(\log x)^{\tau-1} + O(x(\log x)^{\tau-1-\beta}),$$

with

$$E = B\tau = \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \lim_{x\to\infty} (\log x)^{-\tau} \prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right),$$

as asserted. □

Proposition 5.6. *Let $k\in\mathbf{N}$ and $l\in\mathbf{N}$ be such that $1\le l\le k$ and $(k,l)=1$. Then*

$$\sum_{\substack{p\le x\\ p\equiv l \,(\mathrm{mod}\, k)}} 1 = \frac{1}{\phi(k)}\frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right), \quad \textit{as } x\to\infty.$$

Proof. This is the prime number theorem for the arithmetic progression $\{kr+l \mid r=0,1,2,\dots\}$, see for example [5, p. 139]. □

Let $k\in\mathbf{N}$. Let χ be a character $(\mathrm{mod}\, k)$. Let χ_0 be the principal character $(\mathrm{mod}\, k)$. The Dirichlet L-series corresponding to χ is given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \tag{18}$$

where $s=\sigma+it\in\mathbf{C}$. For $\chi\ne\chi_0$, the series in (18) converges for $\sigma>0$ and

$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \prod_p \left(1-\frac{\chi(p)}{p}\right)^{-1} \ne 0. \tag{19}$$

For each character χ (mod k) we define a completely multiplicative function $k_\chi(n)$ ($n\in\mathbf{N}$) by setting, for primes p,

$$k_\chi(p) = p\left[1-\left(1-\frac{\chi(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\chi(p)}\right]. \tag{20}$$

The Dirichlet series corresponding to k_χ is given by

$$K(s,\chi) = \sum_{n=1}^{\infty} \frac{k_\chi(n)}{n^s}, \tag{21}$$

where $s=\sigma+it\in\mathbf{C}$. It is shown in [6] that the series in (21) converges absolutely for $\sigma>0$ and that

$$K(1,\chi) = \sum_{n=1}^{\infty} \frac{k_\chi(n)}{n} = \prod_p \left(1-\frac{k_\chi(p)}{p}\right)^{-1} = \prod_p \left(1-\frac{\chi(p)}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{\chi(p)} \ne 0.$$

Proposition 5.7. *Let $k\in\mathbf{N}$ and $l\in\mathbf{N}$ be such that $1\le l\le k$ and $(l,k)=1$. Then*

$$\prod_{\substack{p\le x\\ p\equiv l\,(\mathrm{mod}\,k)}}\left(1-\frac{1}{p}\right)=A(l,k)(\log x)^{-1/\phi(k)}+O((\log x)^{-1/\phi(k)-1}),$$

as $x\to\infty$, where

$$A(l,k)=\left(e^{-\gamma}\frac{k}{\phi(k)}\prod_{\chi\ne\chi_0}\left(\frac{K(1,\chi)}{L(1,\chi)}\right)^{\bar{\chi}(l)}\right)^{1/\phi(k)}.$$

Proof. This proposition is Mertens' theorem for the arithmetic progression $\{kr+l\,|\,r=0,1,2,...\}$, which was first proved by Williams [6] in 1974. $\square$

Proposition 5.8. *Let $k,m,r\in\mathbf{N}$. Let ω_k be a primitive k-th root of unity. Then*

$$\prod_{j=0}^{k-1}\left(1-\frac{\omega_k^{jr}}{m}\right)=\left(1-\frac{1}{m^{k/(k,r)}}\right)^{(k,r)}.$$

Proof. Let $k,r\in\mathbf{N}$. Set

$$h=\frac{k}{(k,r)}\quad\text{and}\quad s=\frac{r}{(k,r)}.$$

As $(h,s)=1$ the h-th roots of unity are ω_h^{js}, $j=0,1,...,h-1$. Thus ω_h^{js}, $j=0,1,...,k-1$ comprise the h-th roots of unity each repeated k/h times. Hence

$$(x^h-1)^{k/h}=\prod_{j=0}^{k-1}\left(x-\omega_h^{js}\right).$$

Taking $x=m\in\mathbf{N}$, and dividing both sides by m^k, we obtain

$$\left(1-\frac{1}{m^h}\right)^{k/h}=\prod_{j=0}^{k-1}\left(1-\frac{\omega_h^{js}}{m}\right)=\prod_{j=0}^{k-1}\left(1-\frac{\omega_k^{jr}}{m}\right),$$

which is the asserted result. $\square$

6. Estimation of $\prod_{\substack{p\le x\\ p\equiv 1\,(\mathrm{mod}\,q)}}(1-1/p)$

We begin with the following result.

Proposition 6.1.

$$\prod_{j=1}^{q-2}K\left(1,\chi_g^j\right)=\frac{1}{C(q)}.$$

Proof. By Definition 3.1 we have

$$\prod_{r=1}^{q-2} C(q,r,\chi_g)^{-(r,q-1)} = \lim_{x\to\infty} \prod_{r=1}^{q-2} \prod_{\substack{p\leq x \\ \chi_g(p)=\omega^r}} \left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{-(r,q-1)}.$$

Next, as

$$\sum_{j=0}^{q-2} \omega^{jr} = \begin{cases} q-1, & \text{if } r=0, \\ 0, & \text{if } r=1,2,\dots,q-2, \end{cases}$$

we have

$$\prod_{r=1}^{q-2} \prod_{\substack{p\leq x \\ \chi_g(p)=\omega^r}} \left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{-(r,q-1)}$$

$$= \prod_{r=0}^{q-2} \prod_{\substack{p\leq x \\ \chi_g(p)=\omega^r}} \left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{-(r,q-1)} \left(1-\frac{1}{p}\right)^{\sum_{j=0}^{q-2} \omega^{jr}}.$$

By Proposition 5.8 with $m=p$, $k=q-1$ and $\omega=\omega_{q-1}$ we have

$$\left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{(r,q-1)} = \prod_{j=0}^{q-2}\left(1-\frac{\omega^{jr}}{p}\right)$$

so that

$$\prod_{r=0}^{q-2} \prod_{\substack{p\leq x \\ \chi_g(p)=\omega^r}} \left(1-\frac{1}{p^{(q-1)/(r,q-1)}}\right)^{-(r,q-1)} \left(1-\frac{1}{p}\right)^{\sum_{j=0}^{q-2} \omega^{jr}}$$

$$= \prod_{r=0}^{q-2} \prod_{\substack{p\leq x \\ \chi_g(p)=\omega^r}} \prod_{j=0}^{q-2}\left(1-\frac{\omega^{jr}}{p}\right)^{-1} \left(1-\frac{1}{p}\right)^{\omega^{jr}}$$

$$= \prod_{j=1}^{q-2}\prod_{r=0}^{q-2} \prod_{\substack{p\leq x \\ \chi_g(p)=\omega^r}} \left(1-\frac{\omega^{jr}}{p}\right)^{-1} \left(1-\frac{1}{p}\right)^{\omega^{jr}}$$

$$= \prod_{j=1}^{q-2}\prod_{p\leq x}\left(1-\frac{\chi_g^j(p)}{p}\right)^{-1} \left(1-\frac{1}{p}\right)^{\chi_g^j(p)}.$$

Finally by Definition 3.2 we obtain

$$\frac{1}{C(q)} = \prod_{r=1}^{q-2} C(q,r,\chi_g)^{-(r,q-1)} = \prod_{j=1}^{q-2} \lim_{x\to\infty} \prod_{p\le x}\left(1-\frac{\chi_g^j(p)}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{\chi_g^j(p)}$$
$$= \prod_{j=1}^{q-2}\prod_{p}\left(1-\frac{\chi_g^j(p)}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{\chi_g^j(p)} = \prod_{j=1}^{q-2} K(1,\chi_g^j),$$

as asserted. □

Proposition 6.2.

$$\prod_{j=1}^{q-2} L\big(1,\chi_g^j\big) = 2^{(q-3)/2}q^{-q/2}\pi^{(q-1)/2}h(q)R(q).$$

Proof. The cyclotomic field K_q is a totally complex field which contains exactly $2q$ roots of unity, namely $\{\pm 1, \pm\omega_q, \pm\omega_q^2, ..., \pm\omega_q^{q-1}\}$. Hence, by the class number formula for abelian fields applied to the cyclotomic field K_q, we have

$$h(q)R(q) = 2q|d(K_q)|^{1/2}2^{-(q-1)/2}\pi^{-(q-1)/2}\prod_{j=1}^{q-2} L\big(1,\chi_g^j\big),$$

where $d(K_q)$ is the discriminant of K_q, see for example [3, Theorem 8.4, p. 436]. Now the discriminant of K_q is given by

$$d(K_q) = (-1)^{q(q-1)/2}q^{q-2},$$

see for example [3, Theorem 2.9, p. 63]. Hence

$$\prod_{j=1}^{q-2} L(1,\chi_g^j) = 2^{(q-3)/2}q^{-q/2}\pi^{(q-1)/2}h(q)R(q),$$

as asserted. □

Proposition 6.3. *Let q be an odd prime. Then*

$$\prod_{\substack{p\le x\\ p\equiv 1\,(\mathrm{mod}\, q)}}\left(1-\frac{1}{p}\right) = \lambda(q)(\log x)^{-1/(q-1)} + O((\log x)^{-q/(q-1)}),$$

as $x\to\infty$, where

$$\lambda(q) = \left(\frac{e^{-\gamma}2^{-(q-3)/2}q^{(q+2)/2}\pi^{-(q-1)/2}}{(q-1)h(q)R(q)C(q)}\right)^{1/(q-1)}.$$

Proof. By Propositions 6.1 and 6.2 we obtain

$$\prod_{j=1}^{q-2} \frac{K(1,\chi_g^j)}{L(1,\chi_g^j)} = \frac{2^{-(q-3)/2} q^{q/2} \pi^{-(q-1)/2}}{h(q)R(q)C(q)}.$$

By Proposition 5.7 with $k=q$ and $l=1$, we have

$$\prod_{\substack{p\leq x \\ p\equiv 1 \,(\operatorname{mod} q)}} \left(1-\frac{1}{p}\right) = \lambda(q)(\log x)^{-1/(q-1)} + O((\log x)^{-q/(q-1)}),$$

where

$$\begin{aligned}\lambda(q) = A(1,q) &= \left(e^{-\gamma}\frac{q}{q-1}\prod_{j=1}^{q-2}\frac{K(1,\chi_g^j)}{L(1,\chi_g^j)}\right)^{1/(q-1)} \\ &= \left(e^{-\gamma}\frac{q}{q-1}\frac{2^{-(q-3)/2}q^{q/2}\pi^{-(q-1)/2}}{h(q)R(q)C(q)}\right)^{1/(q-1)} \\ &= \left(\frac{e^{-\gamma}2^{-(q-3)/2}q^{(q+2)/2}\pi^{-(q-1)/2}}{(q-1)h(q)R(q)C(q)}\right)^{1/(q-1)},\end{aligned}$$

as asserted. □

Proposition 6.4. *Let* $0<\varepsilon<1$. *Then*

$$A(x) = \alpha(q)x(\log x)^{-1/(q-1)} + O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x\to\infty$, *where*

$$\alpha(q) = \frac{(q-1)^{(q-2)/(q-1)}\Gamma\left(\dfrac{1}{q-1}\right)\sin\left(\dfrac{\pi}{q-1}\right)}{2^{(q-3)/2(q-1)}q^{(q-4)/2(q-1)}\pi^{3/2}(h(q)R(q)C(q))^{1/(q-1)}}.$$

(The constant implied by the O-symbol depends only on q *and* ε.*)*

Proof. By (16) we have

$$A(x) = \sum_{\substack{n\leq x \\ n\in A}} 1 = \sum_{n\leq x} f(n),$$

where

$$f(n) = \begin{cases} 1, & \text{if } n\in A, \\ 0, & \text{if } n\notin A. \end{cases}$$

Clearly $f(n)$ is a multiplicative function by (15). Moreover $0 \le f(n) \le 1$ for all $n \in \mathbf{N}$. By Proposition 5.6 we have

$$\sum_{p \le x} f(p) = \sum_{\substack{p \le x \\ p \in A}} 1 = \sum_{\substack{p \le x \\ p \not\equiv 1 \,(\mathrm{mod}\, q)}} 1 + O(1) = \frac{q-2}{q-1} \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

as $x \to \infty$. Hence, by Proposition 5.5 with $\tau = (q-2)/(q-1)$ and $\beta = 1-\varepsilon$, the limit

$$\lim_{x \to \infty} \frac{1}{(\log x)^{(q-2)/(q-1)}} \prod_{\substack{p \le x \\ p \ne q \\ p \not\equiv 1 \,(\mathrm{mod}\, q)}} \left(1 - \frac{1}{p}\right)^{-1}$$

exists, say equal to $M(q)$, and

$$A(x) = \frac{e^{-\gamma(q-2)/(q-1)}}{\Gamma\left(\dfrac{q-2}{q-1}\right)} M(q) x (\log x)^{-1/(q-1)} + O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x \to \infty$. Now for $x \ge q$

$$\prod_{\substack{p \le x \\ p \ne q \\ p \not\equiv 1 \,(\mathrm{mod}\, q)}} \left(1 - \frac{1}{p}\right) = \frac{\displaystyle\prod_{p \le x} \left(1 - \frac{1}{p}\right)}{\left(1 - \dfrac{1}{q}\right) \displaystyle\prod_{\substack{p \le x \\ p \equiv 1 \,(\mathrm{mod}\, q)}} \left(1 - \frac{1}{p}\right)}.$$

By Mertens' theorem we have

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) = e^{-\gamma}(1 + o(1)) \frac{1}{\log x},$$

as $x \to \infty$, so appealing to Proposition 6.3, we obtain

$$\begin{aligned} \prod_{\substack{p \le x \\ p \ne q \\ p \not\equiv 1 \,(\mathrm{mod}\, q)}} \left(1 - \frac{1}{p}\right) &= \frac{e^{-\gamma}(1 + o(1))(\log x)^{-1}}{\left(1 - \dfrac{1}{q}\right) \lambda(q)(1 + o(1))(\log x)^{-1/(q-1)}} \\ &= \frac{q e^{-\gamma}}{(q-1)\lambda(q)} (1 + o(1))(\log x)^{-(q-2)/(q-1)}, \end{aligned}$$

so that

$$\frac{1}{(\log x)^{(q-2)/(q-1)}} \prod_{\substack{p\le x \\ p\ne q \\ p\not\equiv 1 \pmod q}} \left(1-\frac{1}{p}\right)^{-1} = \frac{(q-1)e^{\gamma}\lambda(q)}{q}(1+o(1)).$$

Hence

$$M(q)=\frac{(q-1)e^{\gamma}\lambda(q)}{q}.$$

Finally

$$A(x)=\frac{e^{\gamma/(q-1)}}{\Gamma\left(\dfrac{q-2}{q-1}\right)}\frac{(q-1)}{q}\lambda(q)x(\log x)^{-1/(q-1)}+O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x\to\infty$, so that as

$$\Gamma\left(\frac{1}{q-1}\right)\Gamma\left(\frac{q-2}{q-1}\right)=\frac{\pi}{\sin\dfrac{\pi}{q-1}}$$

we have

$$\alpha(q)=\frac{e^{\gamma/(q-1)}}{\Gamma\left(\dfrac{q-2}{q-1}\right)}\frac{(q-1)}{q}\lambda(q)=\frac{(q-1)^{(q-2)/(q-1)}\Gamma\left(\dfrac{1}{q-1}\right)\sin\left(\dfrac{\pi}{q-1}\right)}{2^{(q-3)/2(q-1)}q^{(q-4)/2(q-1)}\pi^{3/2}(h(q)R(q)C(q))^{1/(q-1)}}.$$

This completes the proof of Proposition 6.4. $\square$

7. Proof of the theorem

By Propositions 5.2 and 6.4 we have

$$\begin{aligned}
E_q(x) &= A(x)+A\left(\frac{x}{q}\right)\\
&= \alpha(q)x(\log x)^{-1/(q-1)}+O(x(\log x)^{-q/(q-1)+\varepsilon})\\
&\quad +\alpha(q)\frac{x}{q}\left(\log\frac{x}{q}\right)^{-1/(q-1)}+O\left(\frac{x}{q}\left(\log\frac{x}{q}\right)^{-q/(q-1)+\varepsilon}\right)\\
&= \alpha(q)x(\log x)^{-1/(q-1)}+O(x(\log x)^{-q/(q-1)+\varepsilon})\\
&\quad +\frac{\alpha(q)}{q}x((\log x)^{-1/(q-1)}+O((\log x)^{-q/(q-1)}))+O(x(\log x)^{-q/(q-1)+\varepsilon})
\end{aligned}$$

$$= \alpha(q)\left(1+\frac{1}{q}\right)x(\log x)^{-1/(q-1)} + O(x(\log x)^{-q/(q-1)+\varepsilon})$$
$$= e(q)x(\log x)^{-1/(q-1)} + O(x(\log x)^{-q/(q-1)+\varepsilon}),$$

as $x \to \infty$. $\square$

By Lemma 4.1 and the theorem (with $q=3$), the number of $n \leq x$ for which $3 \nmid \phi(n)$ is

$$\frac{2^{7/2}}{3^{9/4}}\left(\prod_{p\equiv 1 \pmod 3}\left(1-\frac{1}{p^2}\right)\right)^{1/2} x(\log x)^{-1/2} + O_{\varepsilon}(x(\log x)^{-3/2+\varepsilon}),$$

as $x \to \infty$, for any $\varepsilon > 0$.

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Blair K. Spearman
Department of Mathematics and Statistics
University of British Columbia Okanagan
Kelowna, BC
Canada V1V 1V7

Kenneth S. Williams
School of Mathematics and Statistics
Carleton University
Ottawa, Ontario
Canada K1S 5B6

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