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Carleson's counterexample and a scale of Lorentz-BMO spaces on the bitorus

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Abstract. We introduce a full scale of Lorentz-BMO spaces $\mathrm{BMO}_{L^{p,q}}$ on the bidisk, and show that these spaces do not coincide for different values of p and q. Our main tool is a detailed analysis of Carleson's construction in [C].

1. Introduction and notation

Throughout the paper $\mathcal{D}$ denotes the set of dyadic intervals in the unit circle $\mathbf{T}$. We write $\mathcal{R}=\mathcal{D}\times\mathcal{D}$ for the dyadic rectangles in the bitorus $\mathbf{T}^2$, $|I|$ for the length of I and $|R|$ for the area of R. We let $(h_I)_{I\in\mathcal{D}}$ stand for the Haar basis in $L^2(\mathbf{T})$ and $(h_R)_{R\in\mathcal{R}}$ for the product Haar basis of $L^2(\mathbf{T}^2)$. Here

$$h_I(t)=\frac{1}{|R|^{1/2}}(\chi_{I^+}(t)-\chi_{I^-}(t))$$

for each dyadic interval $I\in\mathcal{D}$, where I^- denotes the left half of I, and I^+ denotes the right half of I. For each dyadic rectangle $R=I\times J\in\mathcal{R}$, h_R is defined by $h_R(s,t)=h_I(s)h_J(t)$. For any $f\in L^2(\mathbf{T}^2)$, we use the notation $f_R=\langle f,h_R\rangle$ for the Haar coefficients of f, and

$$m_If(s)=\frac{1}{|I|}\int_I f(t,s)\,dt,$$
$$m_Jf(t)=\frac{1}{|J|}\int_J f(t,s)\,ds,$$
$$m_Rf(t)=\frac{1}{|R|}\int_R f(t,s)\,dt\,ds$$

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for the averages in the first, second and both variables respectively. We will use "$\approx$" to denote equivalence of expressions. Given a complex-valued measurable function $f\in L^2(\mathbf{T}^2)$, we write $\mu_f(\lambda)=|E_\lambda|$ for $\lambda>0$, where $E_\lambda=\{w\in\mathbf{T}^2:|f(w)|>\lambda\}$, for the distribution function of f, $f^*(t)=\inf\{\lambda:\mu_f(\lambda)\le t\}$ for the nonincreasing rearrangement of f, and $f^{**}(t)=(1/t)\int_0^t f^*(s)\,ds$. In this note, we introduce a scale of Lorentz-BMO spaces on the bitorus and distinguish the spaces in this scale by a detailed analysis of the Carleson counterexample. Now, given a measurable set $\Omega\subseteq\mathbf{T}^2$ and $0<p,q\le\infty$, the Lorentz space $L^{p,q}_\Omega=L^{p,q}(\Omega,\mu_\Omega)$, where $\mu_\Omega(A)=|A|/|\Omega|$ is the normalized Lebesgue measure, consists of those measurable functions f supported in Ω such that $\|f\|^*_{L^{p,q}_\Omega}<\infty$, where

$$(1)\qquad \|f\|^*_{L^{p,q}_\Omega}=\begin{cases}\left(\dfrac{q}{p}\displaystyle\int_0^1 t^{q/p}f^*(t)^q\frac{dt}{t}\right)^{1/q}, & 0<p<\infty,\ 0<q<\infty,\\ \sup\limits_{t>0} t^{1/p}f^*(t), & 0<p\le\infty,\ q=\infty.\end{cases}$$

We write $L^{p,q}$ for the Lorentz space over $L^{p,q}_{\mathbf{T}^2}$. The reader should be aware that $\|f\|^*_{L^{p,q}_\Omega}$ is in general not a norm on $L^{p,q}_\Omega$. Nevertheless, replacing f^* by f^{**} in (1) and writing $\|f\|_{L^{p,q}_\Omega}=\|f^{**}\|^*_{L^{p,q}_\Omega}$, one gets a norm on $L^{p,q}_\Omega$ for $1<p\le\infty$ and $1\le q\le\infty$, which is equivalent to $\|f\|^*_{L^{p,q}_\Omega}$ (see e.g. [SW]). The space $L^{p,p}_\Omega$, for which we will write L^p_Ω, is then the ordinary L^p space $L^p(\Omega,\mu_\Omega)$. We write $S[f]$ for the dyadic square function of an integrable function f,

$$S[f]=\left(\sum_{R\in\mathcal{R}}\frac{\chi_R}{|R|}|f_R|^2\right)^{1/2}.$$

It is well known that $\|S[f]\|_p\approx\|f\|_p$ for $1<p<\infty$. Using interpolation, one has also $\|S[f]\|_{L^{p,q}}\approx\|f\|_{L^{p,q}}$ for $1<p,q<\infty$. For each measurable set $\Omega\subseteq\mathbf{T}^2$, let P_Ω be the orthogonal projection on the subspace spanned by the Haar functions $h_{R'}$, $R'\in\mathcal{R}$, $R'\subseteq\Omega$. In particular, for each dyadic rectangle $R\in\mathcal{R}$ and for $f=\sum_{R'\in\mathcal{R}}h_{R'}f_{R'}\in L^2(\mathbf{T}^2)$, one has

$$P_Rf=\sum_{\substack{R'\in\mathcal{R}\\ R'\subseteq R}}h_{R'}f_{R'}.$$

It is easy to see that for $R=I\times J\in\mathcal{R}$,

$$(2)\qquad P_Rf=(f-m_If-m_Jf+m_{I\times J}f)\chi_{I\times J}.$$

We are now ready to introduce our scale of Lorentz-BMO spaces. Let $1\le p,q\le\infty$. We denote by $\mathrm{BMO}_{L^{p,q}}$ the space of all $\varphi\in L^2(\mathbf{T}^2)$ such that

$$\|\varphi\|_{\mathrm{BMO}_{L^{p,q}}}=\sup_{R\in\mathcal{R}}\|P_R\varphi\|_{L^{p,q}_R}<\infty.$$

In contrast to the one-dimensional situation, functions in these spaces are not necessarily in the so-called product BMO space $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$, the dual of the dyadic Hardy space $H^1_d(\mathbf{T}^2):=\{f\in L^1(\mathbf{T}^2):S[f]\in L^1(\mathbf{T}^2)\}$. For $p=q=2$, a continuous version of this fact was shown in [F]. In [BP], this was shown for all $1\le p=q<\infty$. (For an overview of the theory of BMO spaces in two variables and characterizations of the duals of $H^1_d(\mathbf{T}^2)$ and $H^1(\mathbf{T}^2)$ in terms of the projections P_Ω, see [Be], [Ch], [CF1] and [CF2]). Recall that in the one-variable case, due to the John–Nirenberg lemma, the spaces $\mathrm{BMO}_{L^{p,q}}$ coincide for all values of p and q. We shall see that this is not the case in the two variables situation. Certainly $\mathrm{BMO}_{L^{p,q}}\subseteq L^{p,q}(\mathbf{T}^2)$, since $m_I(f)=m_I(P_{I\times\mathbf{T}}f)$ and $m_J(f)=m_I(P_{\mathbf{T}\times J}f)$. Note that for $f\ge0$ and $\operatorname{supp} f\subseteq\Omega_1\subseteq\Omega_2$,

$$\|f\|_{L^{p,q}_{\Omega_1}}=\left(\frac{|\Omega_2|}{|\Omega_1|}\right)^{1/p}\|f\|_{L^{p,q}_{\Omega_2}}. \tag{3}$$

It is well known that for $1\le p_1<p_2\le\infty$ and $1\le q_1,q_2\le\infty$, we have $L^{p_2,q_2}\subseteq L^{p_1,q_1}$, and the embedding is continuous. For $1<p<\infty$ and $1\le q_1<q_2\le\infty$, $L^{p,q_1}\subseteq L^{p_1,q_2}$, and the embedding is contractive. Therefore for $f\ge0$, $\operatorname{supp} f\subseteq\Omega$, $1\le p_1<p_2\le\infty$ and $1\le q_1,q_2\le\infty$,

$$\|f\|_{L^{p_1,q_1}}\le C_{p_1,p_2,q_1,q_2}|\Omega|^{1/p_1-1/p_2}\|f\|_{L^{p_2,q_2}}. \tag{4}$$

Hence, we have

$$\|P_Rf\|_{L^{p,q_1}_R}\le C_{p,q_1,q_2}\|P_Rf\|_{L^{p,q_2}_R}\quad\text{for } 1\le p_1<p_2<\infty \text{ and } 1\le q_1,q_2\le\infty,$$

and

$$\|P_Rf\|_{L^{p_1,q_1}_R}\le C_{p_1,p_2,q_1,q_2}\|P_Rf\|_{L^{p_2,q_2}_R}\quad\text{for } 1<p<\infty \text{ and } 1\le q_1<q_2\le\infty.$$

This shows that $\mathrm{BMO}_{L^{p_2,q_2}}\subseteq\mathrm{BMO}_{L^{p_1,q_1}}$, if $1\le p_1<p_2<\infty$ and $1\le q_1,q_2\le\infty$, and $\mathrm{BMO}_{L^{p,q_1}}\subseteq\mathrm{BMO}_{L^{p,q_2}}$, if $1\le p<\infty$ and $1\le q_1<q_2\le\infty$. We shall see later that they are actually different. L. Carleson showed in [C] in an ingenious geometric construction that for each $N\in\mathbf{N}$, there exists a finite collection Φ_N of dyadic rectangles in $[0,1]^2$ such that

(1) the total area of all rectangles is 1, i.e. $\sum_{R\in\Phi_N}|R|=1$,

(2) the rectangles "intersect heavily", $|\bigcup_{R\in\Phi_N}R|<C_1/N$,

(3) the rectangles are evenly distributed over the unit square in the sense that a localized version of (1) holds, i.e. for each dyadic rectangle R, we have $\sum_{R'\in\Phi_N,R'\subseteq R}|R'|\le C_2|R|$.

Here, C_1 and C_2 are absolute constants independent of N. Let us write, as in [C],

$$\phi_N = \sum_{R\in\Phi_N} h_R|R|^{1/2}.$$

Hence we have that

$$\|\phi_N\|_{L^2}=1 \text{ and } |\operatorname{supp}\phi_N|\le C_1\frac{1}{N} \quad \text{for all } N\in\mathbf{N}.$$

This construction was originally devised to show that the naive generalization of the Carleson embedding theorem to two variables is not true. However, it contains much more information. R. Fefferman used the construction to show that the dual of the Hardy space $H^1(\mathbf{T}^2)$ does not coincide with the so-called rectangular BMO space in two variables [F]. C. Sadosky and the second author used the Carleson construction to give a new proof of the fact that the Carleson embedding theorem also does not extend to operator-valued measures (first proved in [NTV]), and to show that Bonsall's theorem does not hold for little Hankel operators on the bidisk [PS]. Here comes our key result, which makes use of a "localization property" of ϕ_N. It establishes the following behaviour of the functions ϕ_N.

Theorem 1.1. *Let* $1<p<\infty$ *and* $1\le q\le\infty$. *There exists a constant* A_p' *such that*

$$\|\phi_N\|_{L^{p,q}}\le\|\phi_N\|_{\mathrm{BMO}_{L^{p,q}}}\le A_p'\|\phi_N\|_{L^{p,q}}$$

for all $N\in\mathbf{N}$.

It was shown in [BP] by the authors that the spaces $\mathrm{BMO}_{\mathrm{rect},p_1}$ and $\mathrm{BMO}_{\mathrm{rect},p_2}$, again in contrast to the one-dimensional situation, are different for different values of p_1 and p_2 (for $p_1=2$ and $p_2=4$, this is contained in [F]). Here we recover and improve the results given in [BP] as a consequence of Theorem 1.1.

Theorem 1.2. *If* $1<p_1<p_2<\infty$ *and* $1\le q_1,q_2\le\infty$ *then we have* $\mathrm{BMO}_{L^{p_1,q_1}}\not\subseteq L^{p_2,q_2}(\mathbf{T}^2)$. *In particular,* $\mathrm{BMO}_{L^{p_1,q_1}}\ne\mathrm{BMO}_{L^{p_2,q_2}}$.

Proof. Let $\Omega_N=\bigcup_{R\in\Phi_N}R$. Since $p_1<p_2$ and ϕ_N is supported on Ω_N, one has

$$\|\phi_N\|_{L^{p_1,q_1}}\le C_{p_1,p_2,q_1,q_2}|\Omega_N|^{1/p_1-1/p_2}\|\phi_N\|_{L^{p_2,q_2}}.$$

by (4). Therefore, Theorem 1.1 yields

$$\begin{aligned}\|\phi_N\|_{L^{p_2,q_2}}&\ge C^{-1}_{p_1,p_2,q_1,q_2}\|\phi_N\|_{L^{p_1,q_1}}|\Omega_N|^{1/p_2-1/p_1}\\&\ge(A_p')^{-1}C_{p_1,p_2,q_1,q_2}C_1^{1/p_2-1/p_1}N^{1/p_1-1/p_2}\|\phi_N\|_{\mathrm{BMO}_{L^{p_1,q_1}}}.\end{aligned}$$

Since $\|\phi_N\|_{L^{p_2,q_2}}\le\|\phi_N\|_{\mathrm{BMO}_{L^{p_2,q_2}}}$, it also follows that $\mathrm{BMO}_{L^{p_1,q_1}}\not\subseteq\mathrm{BMO}_{L^{p_2,q_2}}$. □

Moreover, the $\mathrm{BMO}_{L^{p,q}}$ spaces can also be separated in the second index.

Theorem 1.3. *If* $1<p<\infty$, *and* $1\le q_1<q_2\le\infty$ *then* $\mathrm{BMO}_{L^{p,q_1}}$ *does not embed continuously into* $L^{p,q_2}(\mathbf{T}^2)$. *In particular,* $\mathrm{BMO}_{L^{p,q_1}}\ne\mathrm{BMO}_{L^{p,q_2}}$.

2. The Carleson construction

Before we can turn to the proof of Theorem 1.1, we need some more details of the construction of Φ_N in [C]. (For a nice description of the Carleson counterexample, see also [T].) The function Φ_N is obtained by the following process. We first identify $\mathbf{T}^2$ with the unit square $[0,1]^2$. Take a sufficiently fast decreasing $(N+1)$-tuple $(A_N, \dots, A_0)$ (for our purposes, we want to assume that this is the tuple $2^{2^N}, 2^{2^{N-1}}, \dots, 2^{2^0}$). Now cut the unit square into A_N vertical rectangles with sides parallel to the axes, of sidelength $A_N^{-1} \times 1$. Discard every second of these rectangles, and denote the collection of the remaining rectangles by $\Phi_{N,y}^{(1)}$. Then cut the unit square into A_N horizontal rectangles with sides parallel to the axes, of sidelength $1 \times A_N^{-1}$. Discard every second of these rectangles, and denote the remaining collection by $\Phi_{N,x}^{(1)}$. The collection of the thus kept horizontal and vertical rectangles, $\Phi_{N,x}^{(1)} \cup \Phi_{N,y}^{(1)}$, is denoted by $\Phi_N^{(1)}$. Now we repeat the process and slice each rectangle in $\Phi_N^{(1)}$ vertically and horizontally into A_{N-1} rectangles with sides parallel to the boundary and again discard every second of them to obtain the collection $\Phi_N^{(2)}$. This process is iterated, until we get $\Phi_N := \Phi_N^{(N+1)}$.

Since the tuple $(A_N, \dots, A_0)$ decreases very fast, each rectangle in Φ_N has a unique "history" in the sense that it is generated from the unit square by a unique sequence of vertical and horizontal slicings. In particular, writing $\Phi_{N,x}$ for the collection of those $R \in \Phi_N$ which are generated from a rectangle in $\Phi_{N,x}^{(1)}$, and $\Phi_{N,y}$ for the collection of those $R \in \Phi_N$ which are generated from a rectangle in $\Phi_{N,y}^{(1)}$, we find that $\Phi_{N,x} \cap \Phi_{N,y} = \emptyset$ and of course $\Phi_{N,x} \cup \Phi_{N,y} = \Phi_N$. Moreover, for $R \in \Phi_N$, we have that $R \in \Phi_{N,x}$ if and only if there exists $R' \in \Phi_{N,x}^{(1)}$ with $R \subseteq R'$. One direction of this equivalence is clear, since each $R \in \Phi_{N,x}$ is generated from some $R' \in \Phi_{N,x}^{(1)}$ and therefore contained in this R'. Conversely, if $R \in \Phi_{N,y}$, then its width in the y-direction is greater than or equal to $A_{N-1}^{-1} \dots A_0^{-1} > A_N^{-1}$. Therefore, R cannot be contained in any $R' \in \Phi_{N,x}^{(1)}$. A corresponding statement holds for $\Phi_{N,y}$.

Another property of the construction we shall frequently use is that for each $R' \in \Phi_N^{(1)}$, the collection $\{R \in \Phi_N : R \subseteq R'\}$ is up to translation and dilation equal to the collection Φ_{N-1}. For each $R \in \Phi_N^{(1)}$, we write $\tau_R^{(N)}$ for the composition of the translation and dilation which transform R into the unit square, $\tau_R^{(N)}(R) = [0,1] \times [0,1]$. Given a dyadic rectangle $Q = I \times J \subseteq R \in \Phi_N^{(1)}$, we have that $\tau_R^{(N)}(Q)$ is a dyadic rectangle, and $|Q| = |R|\,|\tau_R^{(N)}(Q)|$. With this notation, our statement above means that for each $R \in \Phi_N^{(1)}$,

$$\{\tau_R^{(N)}(Q) : Q \in \Phi_N,\ Q \subseteq R\} = \Phi_{N-1}, \tag{5}$$

and consequently

$$\phi_{N-1}\circ\tau_R^{(N)}=P_R\phi_N|_R. \tag{6}$$

3. Proof of Theorem 1.1

Lemma 3.1. *For $Q=I\times J\subseteq R\in\Phi_N^{(1)}$, write $Q'=\tau_R^{(N)}(Q)$. Then*

$$\|S[P_Q\phi_N]\|_{L_Q^{p,q}}=\|S[P_{Q'}\phi_{N-1}]\|_{L_{Q'}^{p,q}}.$$

In particular

$$\|S[P_R\phi_N]\|_{L_R^{p,q}}=\|S[\phi_{N-1}]\|_{L^{p,q}}$$

for any $R\in\Phi_N^{(1)}$.

Proof. Observe that

$$|\{x\in Q:S[P_Q\phi_N](x)>\lambda\}|=|R|\,|\{x\in Q':S[P_{Q'}\phi_{N-1}](x)>\lambda\}|.$$

Therefore

$$\mu_Q(\{x\in Q:S[P_Q\phi_N](x)>\lambda\})=\mu_{Q'}(\{x\in Q':S[P_{Q'}\phi_{N-1}](x)>\lambda\}). \tag{7}$$

This gives the result. □

Lemma 3.2. *If $Q\in[0,1]\times\mathcal{D}$, say $Q=[0,1]\times J$, and $|J|>A_N^{-1}$, then*

$$\|S[P_Q\phi_{N,x}]\|_{L_Q^{p,q}}=2^{-1/p}\|S[\phi_{N-1}]\|_{L^{p,q}} \tag{8}$$

and

$$\|S[P_Q\phi_{N,y}]\|_{L_Q^{p,q}}=2^{-1/p}\|S[P_Q\phi_{N-1}]\|_{L_Q^{p,q}}. \tag{9}$$

In particular

$$\|S[\phi_{N,y}]\|_{L^{p,q}}=\|S[\phi_{N,x}]\|_{L^{p,q}}=2^{-1/p}\|S[\phi_{N-1}]\|_{L^{p,q}}. \tag{10}$$

A corresponding statement holds for $Q\in\mathcal{D}\times[0,1]$, $Q=I\times[0,1]$ with $|I|>A_N^{-1}$.

Proof. We write $\phi_{N,x}=\sum_{R\in\Phi_{N,x}}h_R|R|^{1/2}$ and $\phi_{N,y}=\sum_{R\in\Phi_{N,y}}h_R|R|^{1/2}$. Note that $S[\phi_{N,x}]=(\sum_{R\in\Phi_{N,x}}\chi_R)^{1/2}$ and $S[\phi_{N,y}]=(\sum_{R\in\Phi_{N,y}}\chi_R)^{1/2}$ are supported on the union of the disjoint collection of rectangles $\Phi^{(1)}_{N,x}$ and $\Phi^{(1)}_{N,y}$, respectively. In the first situation, we have

$$S^2[P_Q\phi_{N,x}] = \sum_{\substack{R\in\Phi^{(1)}_{N,x}\\ R\subseteq Q}} S^2[P_R\phi_N],$$

where the $S[P_R\phi_N]$ are equimeasurable for different R and disjointly supported. Then we have, for any $R\subseteq Q$, $R\in\Phi^{(1)}_{N,x}$,

$$|\{x\in Q: S^2[P_Q\phi_{N,x}](x)>\lambda\}| = \frac{|Q|}{2|R|}|\{x\in R: S^2[P_R\phi_N](x)>\lambda\}|.$$

Therefore by (7), for any $\lambda>0$,

$$\mu_Q(\{x: S[P_Q\phi_{N,x}](x)>\lambda\}) = \tfrac{1}{2}|\{x: S[\phi_{N-1}](x)>\lambda\}|.$$

This gives $S[P_Q\phi_{N,x}]^{**}(t)=S[\phi_{N-1}]^{**}(2t)$. Therefore we get (8).

For the second situation we have

$$S^2[P_Q\phi_{N,y}] = \sum_{R\in\Phi^{(1)}_{N,y}} S^2[P_{R\cap Q}\phi_N],$$

where $S^2[P_{R\cap Q}\phi_N]$ are equimeasurable for different R and disjointly supported. Then we have, for any $R\in\Phi^{(1)}_{N,y}$,

$$|\{x\in Q: S^2[P_Q\phi_{N,y}](x)>\lambda\}| = \sum_{R\in\Phi^{(1)}_{N,y}} |\{x\in R\cap Q: S^2[P_{R\cap Q}\phi_N](x)>\lambda\}|.$$

Therefore, using (7) and observing that $\tau^{(N)}_R(Q\cap R)=Q$ for any $R\in\Phi^{(1)}_{N,y}$ we get

$$\begin{aligned}|\{x: S[P_Q\phi_{N,y}](x)>\lambda\}| &= \Bigg(\sum_{R\in\Phi^{(1)}_{N,y}} |R|\Bigg)|\{x: S[P_Q\phi_{N-1}](x)>\lambda\}|\\ &= \frac{1}{2}|\{x: S[P_Q\phi_{N-1}](x)>\lambda\}|.\end{aligned}$$

This gives (9). $\square$

Before we can prove the main technical result Theorem 1.1, we need to collect some more facts.

Lemma 3.3. *Let* $1<p<\infty$ *and* $1\le q\le\infty$ *. There exists* $C_p\ge(2^{1/p}-1)^{-1}$ *such that*

$$\|S[P_Q\phi_N]\|_{L^{p,q}_Q}\le C_p\max_{1\le k\le N-1}\|S[\phi_k]\|_{L^{p,q}} \tag{11}$$

for any $Q\in[0,1]\times\mathcal{D}$*. A corresponding statement holds for* $Q\in\mathcal{D}\times[0,1]$.

Proof. We shall prove this statement by induction. It is obvious for $N=1$. Assume it holds true for $N-1$. We first consider the case $Q=[0,1]\times J$, where $|J|>A_N^{-1}$. Using the inequality

$$S[P_Q\phi_N]=(S^2[P_Q\phi_{N,x}]+S^2[P_Q\phi_{N,y}])^{1/2}\le S[P_Q\phi_{N,x}]+S[P_Q\phi_{N,y}]$$

and Lemma 3.2, we obtain

$$\begin{aligned}\|S[P_Q\phi_N]\|_{L^{p,q}_Q}&\le\|S[P_Q\phi_{N,x}]+S[P_Q\phi_{N,y}]\|_{L^{p,q}_Q}\\&\le\|S[P_Q\phi_{N,x}]\|_{L^{p,q}_Q}+\|S[P_Q\phi_{N,y}]\|_{L^{p,q}_Q}\\&=2^{-1/p}(\|S[\phi_{N-1}]\|_{L^{p,q}}+\|S[P_Q\phi_{N-1}]\|_{L^{p,q}_Q})\\&\le2^{-1/p}\Big(\|S[\phi_{N-1}]\|_{L^{p,q}}+C_p\max_{1\le k\le N-2}\|S[\phi_k]\|_{L^{p,q}}\Big)\\&\le2^{-1/p}(C_p+1)\max_{1\le k\le N-1}\|S[\phi_k]\|_{L^{p,q}}\\&\le C_p\max_{1\le k\le N-1}\|S[\phi_k]\|_{L^{p,q}}.\end{aligned}\tag{12}$$

In the case $Q=[0,1]\times J$ and $A_{N-1}^{-1}A_N^{-1}<|J|\le A_N^{-1}$, we either have $P_Q\phi_N=0$, or $Q\subseteq R$ for some $R\in\Phi^{(1)}_{N,x}$ and $Q'=\tau_R^{(N)}(Q)=[0,1]\times J'$ with $|J'|>A_{N-1}^{-1}$. Now Lemma 3.1 and the previous case give

$$\|S[P_Q\phi_N]\|_{L^{p,q}_Q}=\|S[P_{Q'}\phi_{N-1}]\|_{L^{p,q}_{Q'}}\le C_p\max_{1\le k\le N-2}\|S[\phi_k]\|_{L^{p,q}}.$$

Similarly, we get the result for any $Q=[0,1]\times J$ with $|J|\le A_N^{-1}$ and $S[P_Q\phi_N]\ne0$. □

Now we can give the proof of the main theorem of the paper.

Proof of Theorem 1.1. We shall first prove that there exists a constant A_p

$$\|\phi_N\|_{L^{p,q}}\le\|\phi_N\|_{\mathrm{BMO}_{L^{p,q}}}\le A_p\max_{1\le k\le N-1}\|\phi_k\|_{L^{p,q}} \tag{13}$$

for all $N\in\mathbf{N}$. For such a purpose we will show that

$$\|S[P_Q\phi_N]\|_{L^{p,q}_Q}\le B_p\max_{1\le k\le N-1}\|S[\phi_k]\|_{L^{p,q}} \tag{14}$$

for all $N\in\mathbf{N}$ and all $Q\in\mathcal{R}$. Now from Littlewood–Paley theory (albeit with a different value of the constant B_p) the result will follow. We shall use induction again. The case $N=1$ is trivial. Assume the statement holds true for $N-1$. First consider the case when Q is contained in some rectangle R in $\Phi_N^{(1)}$. Using Lemma 3.1, we obtain

$$\|S[P_Q\phi_N]\|_{L_Q^{p,q}} \le B_p \max_{1\le k\le N-2} \|S[\phi_k]\|_{L^{p,q}}.$$

Consider now the case when Q is not contained in any $R\in\Phi_N^{(1)}$. Note that if $Q=I\times J$, with $|J|\le A_N^{-1}$, then $P_Q\phi_N=P_Q\phi_{N,x}$. This is due to the fact that $|J'|>A_N^{-1}$ for any $R'=I'\times J'\in\Phi_{N,y}^{(1)}$. So either $Q\subseteq R$ for some $R\in\Phi_{N,x}^{(1)}$, or $P_Q\phi_N=0$. Similarly, one can deal with the case $Q=I\times J$, where $|I|\le A_N^{-1}$. Hence it remains to consider the case that $Q=I\times J$, where $|I|,|J|\ge A_N^{-1}$. Let us write

$$S^2[P_Q\phi_N]=S^2[P_Q\phi_{N,x}]+S^2[P_Q\phi_{N,y}]=\sum_{R\in\Phi_{N,x}^{(1)}} S^2[P_{Q\cap R}\phi_N]+\sum_{R\in\Phi_{N,y}^{(1)}} S^2[P_{Q\cap R}\phi_N].$$

Observe now that

$$S^2[P_Q\phi_{N,x}]=\sum_{\substack{R\in\Phi_{N,x}^{(1)}\\ R\cap Q\neq\emptyset}} S^2[P_{R\cap Q}\phi_N],$$

where the $S[P_{R\cap Q}\phi_N]$ are equimeasurable for different R, and disjointly supported. Then we have

$$|\{x\in Q: S^2[P_Q\phi_{N,x}](x)>\lambda\}|=\sum_{\substack{R\in\Phi_{N,x}^{(1)}\\ R\cap Q\neq\emptyset}} |\{x\in R\cap Q: S^2[P_{R\cap Q}\phi_N](x)>\lambda\}|.$$

Hence, for any $R\in\Phi_{N,x}^{(1)}$ with $R\cap Q\neq\emptyset$, we can write

$$|\{x\in Q: S^2[P_Q\phi_{N,x}](x)>\lambda\}|=\frac{|J|}{2|R|}|\{x\in R\cap Q: S^2[P_{R\cap Q}\phi_N](x)>\lambda\}|.$$

This gives that for any $\lambda>0$,

$$\begin{aligned}\mu_Q(\{x: S[P_Q\phi_{N,x}](x)>\lambda\}) &= \frac{|J|\,|R\cap Q|}{2|R|\,|Q|}\mu_{R\cap Q}(\{x: S[P_{R\cap Q}\phi_N](x)>\lambda\})\\ &= \tfrac{1}{2}\mu_{R\cap Q}(\{x: S[P_{R\cap Q}\phi_N](x)>\lambda\}).\end{aligned}$$

Hence $S[P_Q\phi_{N,x}]^{**}(t)=S[P_{R\cap Q}\phi_N]^{**}(2t)$, and consequently

$$\|S[P_Q\phi_{N,x}]\|_{L_Q^{p,q}}=2^{-1/p}\|S[P_{Q\cap R}\phi_N]\|_{L_{R\cap Q}^{p,q}}.$$

Notice that $R\cap Q\subseteq R\in\Phi_{N,x}^{(1)}$, and that $\tau_R^{(N)}(R\cap Q)=I\times[0,1]$. Applying Lemmas 3.1 and 3.3, we have

$$
\begin{aligned}
\|S[P_Q\phi_{N,x}]\|_{L_Q^{p,q}} &= 2^{-1/p}\|S[P_{Q\cap R}\phi_N]\|_{L_{R\cap Q}^{p,q}} \\
&\le 2^{-1/p}\|S[P_{I\times[0,1]}\phi_{N-1}]\|_{L_{I\times[0,1]}^{p,q}} \qquad (15)\\
&\le C_p 2^{-1/p}\max_{1\le k\le N-1}\|S[\phi_k]\|_{L^{p,q}}.
\end{aligned}
$$

A similar argument shows that

$$
\|S[P_Q\phi_{N,y}]\|_{L_Q^{p,q}} \le C_p 2^{-1/p}\max_{1\le k\le N-1}\|S[\phi_k]\|_{L^{p,q}}.
$$

Finally, since $S[P_Q\phi_N]\le S[P_Q\phi_{N,x}]+S[P_Q\phi_{N,y}]$, we get

$$
\begin{aligned}
\|S[P_Q\phi_N]\|_{L_Q^{p,q}} &\le \|S[P_Q\phi_{N,x}]\|_{L_Q^{p,q}}+\|S[P_Q\phi_{N,y}]\|_{L_Q^{p,q}} \\
&\le C_p 2^{1-1/p}\max_{1\le k\le N-1}\|S[\phi_k]\|_{L^{p,q}}.
\end{aligned}
$$

Letting $B_p=C_p2^{1-1/p}$, we finish the proof of (13). To finish the proof, by (13), it suffices to prove that there exists a constant D_p such that $\|S[\phi_{N-1}]\|_{L^{p,q}}\le D_p\|S[\phi_N]\|_{L^{p,q}}$ for all $N\in\mathbf{N}$. Since $S^2[\phi_N]=S^2[\phi_{N,x}]+S^2[\phi_{N,y}]$, we have, by Lemma 3.2, that

$$
\begin{aligned}
\|S[\phi_N]\|_{L^{p,q}} &= \|(S^2[\phi_{N,x}]+S^2[\phi_{N,y}])^{1/2}\|_{L^{p,q}} \\
&\ge \|S[\phi_{N,x}]\|_{L^{p,q}} = 2^{-1/p}\|S[\phi_{N-1}]\|_{L^{p,q}}.
\end{aligned}
$$

This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.3

For $N\in\mathbf{N}$, let $\varphi_N=S^2[\phi_N]=\sum_{R\in\Phi_N}\chi_R$. Using that $P_Q(S^2\varphi)=P_Q(S^2P_Q\varphi)$ for any $Q\in\mathcal{R}$ and $\varphi\in L^2(\mathbf{T}^2)$, one easily gets the following as an immediate consequence of Theorem 1.1.

Remark 4.1. For $1\le p<\infty$ and $1\le q\le\infty$, there exist constants $a_{p,q}, A_{p,q}>0$ such that

$$
\tilde{a}_{p,q}\|\varphi_N\|_{L^{p,q}} \le \|\varphi_N\|_{\mathrm{BMO}_{L^{p,q}}} \le A_p'\|\varphi_N\|_{L^{p,q}}
$$

for all $N\in\mathbf{N}$.

Here comes the key lemma for the proof of Theorem 1.3.

Lemma 4.2. *Let $1<p<\infty$, $1\le q_1,q_2\le\infty$ and $\varepsilon>0$. Then there exists an increasing sequence $(N_j)_{j\in\mathbf{N}}$ and a constant B_{p,q_1,q_2} such that for each $M\in\mathbf{N}$ and $q\in\{q_1,q_2\}$,*

$$M-(1-2^{-M})\varepsilon<\left\|\sum_{j=1}^{M}\frac{\varphi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}}\right\|_{L^{p,q}}^{q}$$

and

$$\left\|\sum_{j=1}^{M}\frac{\varphi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}}\right\|_{\mathrm{BMO}_{L^{p,q}}}^{q}<B_{p,q_1,q_2}(M+(1-2^{-M})\varepsilon).$$

Proof. We prove this by induction. The result trivially follows for $M=1$ choosing $N_1=1$. Suppose we have already found $N_1,\dots,N_{M-1}$. For $q\in\{q_1,q_2\}$ let $\widetilde{\varphi}_N=\varphi_N/\|\varphi_N\|_{L^{p,q}}$ and $f_{M-1}^{(q)}=\sum_{j=1}^{M-1}\varphi_{N_j}/\|\varphi_{N_j}\|_{L^{p,q}}$. Let $C=C(M,p,q_1,q_2)=2\sup_{q\in\{q_1,q_2\}}\|f_{M-1}^{(q)}\|_\infty$, and define

$$\Omega_>=\{(t,s)\in\mathbf{T}^2:\widetilde{\varphi}_N(t,s)>C\}\quad\text{and}\quad\Omega_\le=\mathbf{T}^2\setminus\Omega_>=\{(t,s)\in\mathbf{T}^2:\widetilde{\varphi}_N(t,s)\le C\},$$

which depend on N, M, p, q_1 and q_2. We first observe that

$$(f_{M-1}^{(q)}+\widetilde{\varphi}_N|_{\Omega_>})^*(t)=\begin{cases}(\widetilde{\varphi}_N|_{\Omega_>})^*(t) & \text{for } 0<t\le|\Omega_>|,\\ (f_{M-1}^{(q)})^*(t-|\Omega_>|) & \text{for } t>|\Omega_>|,\end{cases}$$

and therefore

$$(f_{M-1}^{(q)}+\widetilde{\varphi}_N|_{\Omega_>})^{**}(t)\begin{cases}=(\widetilde{\varphi}_N|_{\Omega_>})^{**}(t) & \text{for } 0<t\le|\Omega_>|,\\ \ge(f_{M-1}^{(q)})^{**}(t) & \text{for } t>|\Omega_>|.\end{cases}$$

Altogether,

$$\begin{aligned}\|f_{M-1}^{(q)}+\widetilde{\varphi}_N\|_{L^{p,q}}^q&\ge\|f_{M-1}^{(q)}+\widetilde{\varphi}_N|_{\Omega_>}\|_{L^{p,q}}^q\\&\ge\|\widetilde{\varphi}_N|_{\Omega_>}\|_{L^{p,q}}^q+\int_{|\Omega_>|}^1(f_{M-1}^{(q)})^{**}(t)^qt^{p/q-1}\,dt\\&\ge\|\widetilde{\varphi}_N|_{\Omega_>}\|_{L^{p,q}}^q+\|f_{M-1}^{(q)}\|_{L^{p,q}}^q-\int_0^{|\Omega_>|}(f_{M-1}^{(q)})^{**}(t)^qt^{p/q-1}\,dt\\&>(1-\|\widetilde{\varphi}_N|_{\Omega_\le}\|_{L^{p,q}})^q+\|f_{M-1}^{(q)}\|_{L^{p,q}}^q-\int_0^{|\Omega_N|}(f_{M-1}^{(q)})^{**}(t)^qt^{p/q-1}\,dt.\end{aligned}$$

Using that $\|\widetilde{\varphi}_N|_{\Omega_\le}\|_{L^{p,q}}\le\|C\chi_{\Omega_\le}\|_{L^{p,q}}\le C|\Omega_N|^{1/p}$ and that $|\Omega_N|\to0$, as $N\to\infty$, we can then find $N\in\mathbf{N}$, $N>N_{M-1}$, such that

$$(1-\|\varphi_N|_{\Omega_\le}\|_{L^{p,q}})^q>1-\frac{\varepsilon}{2^{M+1}}\quad\text{for } q\in\{q_1,q_2\}$$

and
$$\int_{(0,|\Omega_N|]} (f_{M-1}^{(q)})^{**}(t)^q t^{q/p-1}\,dt < \frac{\varepsilon}{2^{M+1}} \quad \text{for } q\in\{q_1,q_2\}.$$
Hence choose such an N and apply the induction assumption to obtain
$$\|f_{M-1}^{(q)}+\widetilde{\varphi}_N\|_{L^{p,q}}^q \geq 1-\frac{\varepsilon}{2^{M+1}}+(M-1)-(1-2^{-(M-1)})\varepsilon-\frac{\varepsilon}{2^{M+1}}=M-(1-2^{-M})\varepsilon.$$

Now we will show that $(N_j)_{j\in\mathbf{N}}$ can be chosen in such a way that there exists a constant A_{p,q_1,q_2} such that for each $M\in\mathbf{N}$,

(16)
$$\left\|S^2\left[P_Q\sum_{j=1}^{M}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}^{1/2}}\right]\right\|_{L_*^{p,q}}^q \leq A_{p,q_1,q_2}|Q|^{q/p}(M+(1-2^{-M})\varepsilon) \quad \text{for } q\in\{q_1,q_2\}.$$

Let us take $Q\in\mathcal{R}$ and write $\varphi_{Q,N}=S^2(P_Q\varphi_N)$. We first show that for a suitable constant A_{p,q_1,q_2} independent of N and Q,

$$\left\|\frac{\varphi_{Q,N}}{\|\varphi_N\|_{L^{p,q}}}\right\|_{L_*^{p,q}}^q \leq A_{p,q_1,q_2}|Q|^{q/p}. \tag{17}$$

Indeed, using the boundedness of the dyadic square function and Theorem 1.1, one gets

$$\begin{aligned}\left\|\frac{\varphi_{Q,N}}{\|\varphi_N\|_{L^{p,q}}}\right\|_{L_*^{p,q}}^q &= \left\|\frac{S(P_Q\phi_N)}{\|\varphi_N\|_{L^{p,q}}^{1/2}}\right\|_{L_*^{2p,2q}}^{2q}\\ &\leq C_{p,q}\left\|\frac{P_Q\phi_N}{\|\varphi_N\|_{L^{p,q}}^{1/2}}\right\|_{L^{2p,2q}}^{2q}\\ &\leq |Q|^{q/p}C_{p,q}\left\|\frac{\phi_N}{\|\varphi_N\|_{L^{p,q}}^{1/2}}\right\|_{\mathrm{BMO}_{L^{2p,2q}}}^{2q}\\ &\leq |Q|^{q/p}(A'_{2p})^{2q}C_{p,q}\left\|\frac{\phi_N}{\|\varphi_N\|_{L^{p,q}}^{1/2}}\right\|_{L^{2p,2q}}^{2q}\\ &\leq |Q|^{q/p}(A'_{2p})^{2q}C'_{p,q}\left\|\frac{S^2\phi_N}{\|\varphi_N\|_{L^{p,q}}}\right\|_{L^{p,q}}^{q}\\ &= A_{p,q_1,q_2}|Q|^{q/p},\end{aligned}$$

where $A_{p,q_1,q_2}=\sup_{q\in\{q_1,q_2\}}(A'_{2p})^{2q}C'_{p,q}$ and the constant $C'_{p,q}$ contains the bounds appearing from the square function norm and the equivalence norm between $L^{2p,2q}$ and $L_*^{2p,2q}$. The inequality (16) is now easily checked for $M=1$ and $N_1=1$. As before, fix $q\in\{q_1,q_2\}$ and assume that $N_1,\dots,N_{M-1}$ satisfying (16) have already been found. By the Carleson construction, we know that

$$|R|=\prod_{i=1}^{N_j} A_i^{-1} > A_{N_j}^{-2} \quad \text{for each } R\in\Phi_{N_j}.$$

Assume first that $|Q|\le A_{N_{M-1}}^{-2}$. Observe that in this case

$$P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}^{1/2}}=0,$$

hence, applying (17) for any $N>N_{M-1}$, we obtain

$$\left\|S^2\left[P_Q\left(\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}^{1/2}}+\frac{\phi_N}{\|\varphi_N\|_{L^{p,q}}^{1/2}}\right)\right]\right\|_{L^{p,q}_*}^q=\left\|\frac{\varphi_{Q,N}}{\|\varphi_N\|_{L^{p,q}}}\right\|_{L^{p,q}_*}^q\le A_{p,q_1,q_2}|Q|^{q/p}.$$

Assume now that $|Q|>A_{N_{M-1}}^{-2}$. Let

$$C=C(q_1,q_2,M)=2\sup_{q\in\{q_1,q_2\}}\|f_{M-1}^{(q)}\|_\infty A_{N_{M-1}}^2.$$

Then

$$S^2\left[P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}^{1/2}}\right](t,s)\le f_{M-1}^{(q)}(t,s)\le\frac{C}{2}\quad\text{for all } Q\in\mathcal{R}.$$

As before, we write $\widetilde{\varphi}_N=\varphi_N/\|\varphi_N\|_{L^{p,q}}$,

$$\Omega_>=\{(t,s)\in\mathbf{T}^2:\widetilde{\varphi}_N(t,s)>C\}\quad\text{and}\quad\Omega_\le=\{(t,s)\in\mathbf{T}^2:\widetilde{\varphi}_N(t,s)\le C\}.$$

We have

$$S^2\left[P_Q\left(\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}^{1/2}}+\frac{\phi_N}{\|\varphi_N\|_{L^{p,q}}^{1/2}}\right)\right]=S^2\left[P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}^{1/2}}\right]+\frac{\varphi_{N,Q}}{\|\varphi_N\|_{L^{p,q}}}$$

and

$$\left(S^2\left[P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}^{1/2}}\right]+\frac{\varphi_{N,Q}}{\|\varphi_N\|_{L^{p,q}}}\bigg|_{\Omega_>}\right)^*(t)$$
$$=\begin{cases}\left(\dfrac{\varphi_{N,Q}}{\|\varphi_N\|_{L^{p,q}}}\bigg|_{\Omega_>}\right)^*(t) & \text{for } 0<t\le|\Omega_>|,\\[2ex] \left(S^2\left[P_Q\displaystyle\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}^{1/2}}\right]\right)^*(t-|\Omega_>|) & \text{for } t>|\Omega_>|.\end{cases}$$

Thus, using continuity of addition in the $L^{p,q}_*$ quasinorm and that $\|\widetilde{\varphi}_N|_{\Omega_\le}\|_{L^{p,q}}\to 0$, as $N\to\infty$, for $q\in\{q_1,q_2\}$, one has for sufficiently large N,

$$\left\|S^2\left[P_Q\left(\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|^{1/2}_{L^{p,q}}}+\frac{\phi_N}{\|\varphi_N\|^{1/2}_{L^{p,q}}}\right)\right]\right\|^q_{L^{p,q}_*}$$

$$(18)\qquad =\left\|S^2\left[P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|^{1/2}_{L^{p,q}}}\right]+\frac{\varphi_{N,Q}}{\|\varphi_N\|_{L^{p,q}}}\right\|^q_{L^{p,q}_*}$$

$$=\left\|\frac{\varphi_{N,Q}}{\|\varphi_N\|_{L^{p,q}}}\Big|_{\Omega_\le}+S^2\left[P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|^{1/2}_{L^{p,q}}}\right]+\frac{\varphi_{N,Q}}{\|\varphi_N\|_{L^{p,q}}}\Big|_{\Omega_>}\right\|^q_{L^{p,q}_*}$$

$$\le\left\|S^2\left[P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|^{1/2}_{L^{p,q}}}\right]+\frac{\varphi_{N,Q}}{\|\varphi_N\|_{L^{p,q}}}\Big|_{\Omega_>}\right\|^q_{L^{p,q}_*}+A^{-2q/p}_{N_{M-1}}\frac{\varepsilon}{2^{M+2}}.$$

For sufficiently large N and $q\in\{q_1,q_2\}$,

$$(19)\qquad \int_0^1\left(\left(S^2\left[P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|^{1/2}_{L^{p,q}}}\right]\right)^*(t)\right)^q(t+|\Omega_>|)^{q/p-1}\,dt$$

$$\le\left\|S^2\left[P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|^{1/2}_{L^{p,q}}}\right]\right\|^q_{L^{p,q}_*}+\frac{\varepsilon}{2^{M+2}}A^{-2q/p}_{N_{M-1}}.$$

Indeed, if $q\le p$, then $(t+|\Omega_>|)^{q/p-1}\le t^{q/p-1}$ and the estimate holds trivially. For $q>p$, one uses $(t+|\Omega_>|)^{q/p-1}\le(t+|\Omega_N|)^{q/p-1}$ and then the Lebesgue monotone convergence theorem. Therefore, from the previous estimates, $|Q|>A^{-2}_{N_{M-1}}$, (17), (18), and (19), we obtain

$$\left\|S^2\left[P_Q\left(\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|^{1/2}_{L^{p,q}}}+\frac{\phi_N}{\|\varphi_N\|^{1/2}_{L^{p,q}}}\right)\right]\right\|^q_{L^{p,q}_*}$$

$$\le\left\|\frac{\varphi_{N,Q}}{\|\varphi_N\|_{L^{p,q}}}\Big|_{\Omega_>}\right\|^q_{L^{p,q}_*}+A^{-2q/p}_{N_{M-1}}\frac{\varepsilon}{2^{M+2}}$$

$$+\int_0^1\left(\left(S^2\left[P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|^{1/2}_{L^{p,q}}}\right]\right)^*(t)\right)^q(t+|\Omega_>|)^{q/p-1}\,dt$$

$$\le\left\|\frac{\varphi_{N,Q}}{\|\varphi_N\|_{L^{p,q}}}\right\|^q_{L^{p,q}_*}+2A^{-2q/p}_{N_{M-1}}\frac{\varepsilon}{2^{M+2}}+\left\|S^2\left[P_Q\sum_{j=1}^{M-1}\frac{\phi_{N_j}}{\|\varphi_{N_j}\|^{1/2}_{L^{p,q}}}\right]\right\|^q_{L^{p,q}_*}$$

$$\leq A_{p,q_1,q_2}|Q|^{q/p}+A_{p,q_1,q_2}|Q|^{q/p}\Big(M-1+\varepsilon-\frac{\varepsilon}{2^{M-1}}\Big)+2A_{N_{M-1}}^{-2q/p}\frac{\varepsilon}{2^{M+2}}$$
$$=A_{p,q_1,q_2}|Q|^{q/p}\Big(M+\Big(\varepsilon-\frac{\varepsilon}{2^M}\Big)\Big).$$

Thus by choosing $N_M>N_{M-1}$ sufficiently large, the induction proceeds. To finish the proof of Lemma 4.2, remark that

$$\left\|P_Q\sum_{j=1}^{M}\frac{\varphi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}}\right\|_{L^{p,q}}^q=\left\|P_Q S^2\left[\sum_{j=1}^{M}\frac{P_Q\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}^{1/2}}\right]\right\|_{L^{p,q}}^q$$
$$\leq A_{p,q_1,q_2}^q 2^q\left\|S^2\left[\sum_{j=1}^{M}\frac{P_Q\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q}}^{1/2}}\right]\right\|_{L^{p,q}_*}^q. \qquad \square$$

Proof of Theorem 1.3. It suffices to show the theorem for the case $q_2<\infty$. So let $1<p<\infty$ and $1\leq q_1<q_2<\infty$. Assume towards a contradiction that $\mathrm{BMO}_{L^{p,q_2}}$ embeds continuously into L^{p,q_1}. Since by Remark 4.1, $\|\varphi_N\|_{L^{p,q}}\approx\|\varphi_N\|_{\mathrm{BMO}_{L^{p,q}}}$ for all p, q and N, it follows in particular that $\|\varphi_N\|_{L^{p,q_1}}\approx\|\varphi_N\|_{L^{p,q_2}}$. Let $\varepsilon>0$. For $M\in\mathbf{N}$, let

$$f^{(M)}=\sum_{j=1}^{M}\frac{\varphi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q_1}}},$$

where $(N_j)_{j\in\mathbf{N}}$ is the sequence from Lemma 4.2. Then

$$\begin{aligned}
M-\varepsilon&\leq\|f^{(M)}\|_{L^{p,q_1}}^{q_1}\\
&\lesssim\|f^{(M)}\|_{\mathrm{BMO}_{L^{p,q_2}}}^{q_1}\\
&=\sup_{Q\in\mathcal{R}}\|P_Qf^{(M)}\|_{L^{p,q_2}}^{q_1}|Q|^{-q_1/p}\\
&=\sup_{Q\in\mathcal{R}}\left\|P_QS^2\left[\sum_{j=1}^{M}\frac{P_Q\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q_1}}^{1/2}}\right]\right\|_{L^{p,q_2}}^{q_1}|Q|^{-q_1/p}\\
&\leq 2^{q_1}\sup_{Q\in\mathcal{R}}\left\|S^2\left[\sum_{j=1}^{M}\frac{P_Q\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q_1}}^{1/2}}\right]\right\|_{L^{p,q_2}}^{q_1}|Q|^{-q_1/p}\\
&\approx 2^{q_1}\sup_{Q\in\mathcal{R}}\left\|S^2\left[\sum_{j=1}^{M}\frac{P_Q\phi_{N_j}}{\|\varphi_{N_j}\|_{L^{p,q_2}}^{1/2}}\right]\right\|_{L^{p,q_2}}^{q_1}|Q|^{-q_1/p}\\
&\leq B_{p,q_1,q_2}^{q_1/q_2}(M+\varepsilon)^{q_1/q_2}
\end{aligned}$$

by Lemma 4.2. Letting $M\to\infty$, we obtain a contradiction. $\square$

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