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Approximation of infinite matrices by matricial Haar polynomials

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Abstract. The main goal of this paper is to extend the approximation theorem of continuous functions by Haar polynomials (see Theorem A) to infinite matrices (see Theorem C). The extension to the matricial framework will be based on the one hand on the remark that periodic functions which belong to $L^\infty(\mathbf{T})$ may be one-to-one identified with Toeplitz matrices from $B(l_2)$ (see Theorem 0) and on the other hand on some notions given in the paper. We mention for instance: *ms*—a unital commutative subalgebra of l^∞, $C(l_2)$ the matricial analogue of the space of all continuous periodic functions $C(\mathbf{T})$, the matricial Haar polynomials, etc.

In Section 1 we present some results concerning the space *ms*—a concept important for this generalization, the proof of the main theorem being given in the second section.

0. Introduction

0.1. The classical form of Haar's theorem

Let $\mathbf{T}$ be the one-dimensional torus identified with the interval $[0,2\pi)$. Now we consider the Haar $L^2(\mathbf{T})$-normalized functions h_k given by $h_0(t)=1$ for $t\in\mathbf{T}$ and, for $n=2^k+m$, $k\geq 0$, and $m\in\{0,\dots,2^k-1\}$, by

$$h_n(t)=\begin{cases} 2^{k/2}, & t\in\Delta_{2m}^{(k+1)},\\ -2^{k/2}, & t\in\Delta_{2m+1}^{(k+1)},\\ 0, & t\in\mathbf{T}\setminus\Delta_m^{(k)},\end{cases}\tag{1}$$

where

$$\Delta_m^{(k)}=\left[\frac{m}{2^k}\cdot 2\pi,\frac{m+1}{2^k}\cdot 2\pi\right).$$

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We can now state the following well-known theorem of approximation of continuous functions on $\mathbf{T}$ (i.e. periodic continuous functions on $[0, 2\pi]$) by means of polynomials with respect to Haar functions (extended by periodicity on $\mathbf{R}$) due to Haar:

Theorem A. *If f is a continuous function on $\mathbf{T}$ (i.e. if $f\in C(\mathbf{T})$) and if $\varepsilon>0$ then there exists a Haar polynomial of degree $n=n(\varepsilon)\in\mathbf{N}$,*

$$S_n(f)=\sum_{k=0}^{n-1}\alpha_k h_k,\quad \alpha_k\in\mathbf{C},$$

such that

$$\|f-S_n(f)\|_{L^\infty(\mathbf{T})}<\varepsilon.$$

0.2. Translation of the statement in the matricial framework

Definition 1. Let $A=(a_{ij})_{i,j\geq 1}$ be an infinite matrix. If there is a sequence of complex numbers $(a_k)_{k=-\infty}^{+\infty}$, such that $a_{ij}=a_{j-i}$ for all $i,j\in\mathbf{N}$, then A is called *a Toeplitz matrix.*

For simplicity we can write a Toeplitz matrix as $A=(a_k)_{k=-\infty}^{+\infty}$, and the class of all Toeplitz matrices will be denoted by $\mathcal{T}$.

We write $A\in B(l_2)$ if the infinite matrix A represents a bounded linear operator $T_A\colon l_2\to l_2$, that is, if $T_A(e_i)=\sum_{k=1}^{\infty} a_{ki}e_k$ for $i=1,2,\dots$, where $\{e_i\}_{i=1}^n$ constitute the standard basis in l_2. The space $B(l_2)$ is a Banach space with respect to the usual operator norm $\|A\|_{B(l_2)}=\sup_{\|x\|_{l_2}\leq 1}\|T_A x\|_{l_2}$.

The following well-known result (see [Zh], Chapter 9.1) as well as the subsequent remark constitute the starting point of whole theory presented here.

Theorem 0. *A Toeplitz matrix $A=(a_k)_{k=-\infty}^{+\infty}$ belongs to $B(l_2)$ if and only if there exists a unique function $f_A\in L^\infty(\mathbf{T})$ whose Fourier coefficients $\hat{f}_A(n)=(1/2\pi)\int_0^{2\pi} f_A(t)e^{-int}\,dt$ are equal to a_n, for all $n\in\mathbf{Z}$. Moreover*

$$\|A\|_{B(l_2)}=\|f_A\|_{L^\infty(\mathbf{T})}.$$

Remark. In order to develop the theory we find in the previous result two different "geometric" directions to be followed.

Model 1: *Diagonal matrix.* For an infinite matrix $A=(a_{ij})$, and an integer k, we denote by A_k the matrix whose entries $a'_{i,j}$ are given by

$$a'_{i,j}=\begin{cases} a_{i,j}, & \text{if } j-i=k,\\ 0, & \text{otherwise.}\end{cases}\tag{2}$$

Then A_k will be called *the* k^{th}*-diagonal matrix associated to* A.

In the preceding theorem we remark that there is a one-to-one correspondence between A_k and $\hat{f}_A(k)$ for $A \in B(l_2)$ and $f_A \in L^\infty(\mathbf{T})$.

Consequently, we may imagine $(A_k)_{k\in\mathbf{Z}}$, as the "matricial Fourier coefficients" associated to the matrix A.

Model 2: *Corner matrix.* In the sequel we use another notation, more appropriate for our aims, for the entries of the matrix A. Namely we put

$$a_k^l = \begin{cases} a_{l,l+k}, & k \geq 0, \ l = 1, 2, \dots, \\ a_{l-k,l}, & k < 0, \ l = 1, 2, \dots, \end{cases} \tag{3}$$

and write A sometimes as $A = (a_k^l)_{l \geq 1, k \in \mathbf{Z}}$.

Let $A^{(l)} = (b_k^m)_{k \in \mathbf{Z}, m \geq 1}$, where $l \in \mathbf{N} \setminus \{0\}$, be the matrix given by

$$b_k^m = \begin{cases} a_k^l, & \text{if } m = l, \\ 0, & \text{if } m \neq l. \end{cases} \tag{4}$$

We call the matrix $A^{(l)}$, *the* l^{th}*-corner matrix associated to* A.

Now, if for any corner matrix $A^{(l)} = (b_k^m)_{k \in Z, m \geq 1}$ we associate a distribution on $\mathbf{T}$, denoted by f_l such that $b_k^l = \hat{f}_l(k)$, we get, in case $A \in \mathcal{T} \cap B(l_2)$, that $f_l = f_A \in L^\infty(\mathbf{T})$ for all $l \in \mathbf{N} \setminus \{0\}$.

Using the models. (a) The model 1. In this case we recall that A_k plays the role of the "k^{th} Fourier coefficient of the matrix A".

It is well known that for each $f \in L^\infty(\mathbf{T})$ whose Fourier coefficients are a_n, $n \in \mathbf{Z}$, we have

$$f \in C(\mathbf{T}) \quad \text{if and only if} \quad \lim_{n \to \infty} \|\sigma_n(f) - f\|_{L^\infty(\mathbf{T})} = 0,$$

where

$$\sigma_n(f)(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) a_k e^{ikt}.$$

Let us recall the following definition (see [BPP]).

Definition 2. Let $A \in B(l_2)$ and

$$\sigma_n(A) = \sum_{k=-n}^{n} A_k \left(1 - \frac{|k|}{n+1}\right), \quad n = 1, 2, \dots,$$

for $n \in \mathbf{N} \setminus \{0\}$, the *matricial Fejér sum of order* n *associated to* A.

Then we call a matrix A *continuous* and we write $A\in C(l_2)$ if the following relation holds:

$$\lim_{n\to\infty} \|\sigma_n(A)-A\|_{B(l_2)}=0.$$

Obviously $C(l_2)$ endowed with the operator norm becomes a Banach space.

Remark 3. The space $C(l_2)$ does not depend on the specific choice of an approximate unit, for instance the Cesàro means from above.

Proof. Indeed, by Theorem 4.2, [BPP], it follows that $A\in C(l_2)$ if and only if $f_A(t)\stackrel{\text{def}}{=}\sum_{k=-\infty}^{\infty} A_k e^{ikt}$ is a $B(l_2)$-valued continuous function. So, reasoning as in [K], Theorem 2.11, we get that the convolution between an approximate unit and the matrix A (that is, the Schur product between the Toeplitz matrix associated with the given approximate unit and the matrix A) converges to A in the $B(l_2)$-norm. □

Theorem 0 allows us to write the formula

$$[\mathcal{T}\cap B(l_2)]^*=L^\infty(\mathbf{T}),$$

where by $[H]^*$ we denote the image of the space H of matrices by the correspondence $A\mapsto f_A$.

Remark 4. For brevity in what follows we write equations like the previous one in the following manner:

$$\mathcal{T}\cap B(l_2)=L^\infty(\mathbf{T}),$$
$$\mathcal{T}\cap C(l_2)=C(\mathbf{T}).$$

(b) Model 2. We can identify the matrix $A=(A^{(l)})_{l\in\mathbf{N}^*}$ with its sequence of associated distributions $\mathbf{f}\stackrel{\text{def}}{=}(f_l)_{l\in\mathbf{N}^*}$, writing this fact as

$$A=A_{\mathbf{f}}.$$

By Theorem 0 we have

$$fg\in L^\infty(\mathbf{T}) \quad \text{if and only if} \quad A_{\mathbf{fg}}\in\mathcal{T}\cap B(l_2), \quad \text{where } \mathbf{fg}=(fg,fg,fg,...).$$

The matrix $A=(a_{ij})$ is said to be of n-band type if $a_{ij}=0$ for $|i-j|>n$.

Having these notions in mind, we introduce a commutative product of infinite matrices.

Definition 5. Let $A=A_{\mathbf{f}}$ and $B=A_{\mathbf{g}}$ be two infinite matrices of finite band type. We introduce now the commutative product $\square$ given by

$$A\square B \stackrel{\text{def}}{=} A_{\mathbf{fg}}.$$

Remark 6. (1) We mention that in the previous definition we took $A=A_{\mathbf{f}}$ and $B=A_{\mathbf{g}}$ to be infinite matrices of finite band type since f and g being trigonometric polynomials, we may consider the product fg.

(2) This product can be defined also for all matrices $A, B\in B(l_2)$, but $A\square B$ does not belong in general to $B(l_2)$ as the reader may easily see.

(3) Of course, if $A_{\mathbf{f}}, A_{\mathbf{g}}\in \mathcal{T}\cap B(l_2)$ then it follows that $A_{\mathbf{f}}\square A_{\mathbf{g}}=A_{\mathbf{fg}}\in\mathcal{T}\cap B(l_2)$.

We conclude the presentation of this model taking into account an important particular case:

Let $\alpha=(\alpha^1,\alpha^2,...)$ be a sequence of complex numbers and $B=A_{\mathbf{f}}\in B(l_2)$, where $\mathbf{f}=(f_1,f_2,...)$. Taking α as a sequence of constant functions on $\mathbf{T}$, we get, by Definition 5,

$$A_\alpha\square B=A_{\alpha\mathbf{f}},$$

where $\alpha\mathbf{f}=(\alpha^1f_1,\alpha^2f_2,...)$.

For brevity we denote $A_\alpha\square B$ by $\alpha\odot B$.

In what follows it will be important to know more about the sequences α satisfying the condition $B\in B(l_2)\Rightarrow \alpha\odot B\in B(l_2)$.

Actually, the entire next section will be devoted to this, but for the moment, to understand its implications, we will rewrite the operation $\odot$ in a different form.

In order to do this let us recall some classical concepts.

Definition 7. Let A and B be infinite matrices. Then

$$C=A*B$$

is called the Schur product of the matrices $A=(a_{ij})$ and $B=(b_{ij})$ if the entries of $C=(c_{ij})$ satisfy the relation $c_{ij}=a_{ij}b_{ij}$.

Definition 8. An infinite matrix A is called a Schur multiplier if $A*B\in B(l_2)$ for all $B\in B(l_2)$.

The space $M(l_2)$ of all Schur multipliers, endowed with the norm

$$\|A\|_{M(l_2)}=\sup_{\|B\|_{B(l_2)}\leq 1}\|A*B\|_{B(l_2)}$$

becomes a Banach space.

We associate to any sequence $\alpha=(\alpha^1,\alpha^2,...)$, the matrix $[\alpha]$ whose entries $[\alpha]_k^l$ are equal to α^l, for $l\geq 1$ and $k\in\mathbf{Z}$.

Then it is clear that

$$\alpha\odot B=[\alpha]*B. \tag{5}$$

Definition 9. Define ms to be the space of all sequences α such that

$$\alpha\odot B\in B(l_2) \quad \text{for all } B\in B(l_2),$$

or equivalently,

$$[\alpha]\in M(l_2).$$

On ms we consider the norm $\|\alpha\|_{ms}\stackrel{\text{def}}{=}\|[\alpha]\|_{M(l_2)}$. Then ms is a unital commutative Banach algebra with respect to the usual multiplication of sequences.

Remark 10. Any constant complex sequence $\alpha=(\alpha,\alpha,...)$ belongs to ms.

In order to get an extension of Haar's theorem we had to find the appropriate analogues in the matrix context. They are summarized below.

	The function case	The matrix case
1	norm $\|\cdot\|_{L^\infty}(\mathbf{T})$	norm $\|\cdot\|_{B(l_2)}$
2	space $C(\mathbf{T})$	space $C(l_2)$
3	multiplication of a function by a scalar	multiplication $\odot$

The correspondence given by (3) becomes more transparent if we remark that for $\alpha\in\mathbf{C}$ and for $f\in L^\infty(\mathbf{T})$, denoting by $\tilde{\alpha}$ the sequence $(\alpha,\alpha,...)$, and by $\mathbf{f}$ the constant sequence $(f,f,...)$, we get that $\tilde{\alpha}\odot A_{\mathbf{f}}=[\alpha]*A_{\mathbf{f}}=A_{\alpha\mathbf{f}}$.

Denoting by H_k the Toeplitz matrix associated, like in Theorem 0, to the Haar function h_k, for $k=0,1,...$, and by $S_n(\mathbf{f},\alpha_k)$ the sequence $(S_n(f,\alpha_k),\, S_n(f,\alpha_k),...)$, where $S_n(f,\alpha_k)=\sum_{k=0}^{n-1}\alpha_k h_k$, for $f\in C(\mathbf{T})$, $\alpha_k\in\mathbf{C}$, and $k\in\{0,n-1\}$, we get the following translation of Theorem A in the Toeplitz matrices setting.

Theorem B. *Let $A=A_{\mathbf{f}}\in C(l_2)$ be a Toeplitz matrix and let $\varepsilon>0$. Then there is a matricial polynomial given by*

$$A_{S_n(\mathbf{f},\alpha_k)}=\sum_{k=0}^{n-1}\alpha_k H_k=\sum_{k=0}^{n-1}\tilde{\alpha}_k\odot H_k$$

such that

$$\|A-A_{S_n(\mathbf{f},\alpha_k)}\|_{B(l_2)}<\varepsilon,$$

where $\widetilde{\alpha}_k=(\alpha_k,\alpha_k,\dots)$.

Now it is natural to ask ourselves about the existence of a class of matrices larger than $\mathcal{T}\cap C(l_2)$ such that Theorem B still holds.

The aim of our paper is to give an answer to this question. More precisely we prove the following result.

Theorem C. *Let* $A=(a_k^l)_{l\geq 1,\,k\in\mathbf{Z}}$ *be a matrix which belongs to* $C(l_2)$ *such that all sequences* $\mathbf{a}_k\stackrel{\text{def}}{=}(a_k^l)_{l\geq 1}$, $k\in\mathbf{Z}$, *belong to ms.*

Then, for any $\varepsilon>0$ *there are an* $n=n_\varepsilon\in\mathbf{N}\setminus\{0\}$ *and sequences* $\alpha_k\in ms$, $k\in\{0,\dots,n-1\}$ *such that*

$$\left\|A-\sum_{k=0}^{n-1}\alpha_k\odot H_k\right\|_{B(l_2)}<\varepsilon.$$

It is also worthwhile to mention the following open problem.

Open problem. *Does Theorem C still hold if the matrix A satisfies only condition* $A\in C(l_2)$? *If not, what is the best version of Theorem C?*

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1. About the space *ms*

As we remarked in the previous section (see also the statement of Theorem C) the space *ms* plays an important role for our theory and, consequently, it is desirable to know more facts about it.

In this context, we saw in Remark 10 that any constant sequence belongs to *ms*. Our primary goal here is to prove that this algebra is far richer than that; this richness will quantify the level of extension of the theorem of Haar in the matrix case, since in the function case, corresponding to Toeplitz matrices, (see Theorem B) the algebra *ms* is reduced to exactly the constant sequences.

Here is an outlook for this section:

We give some sufficient conditions for a sequence to belong to *ms*, following two complementary ways:

The first one is based on defining a particular algebra *pms* and showing that *pms* is intimately connected with *ms*. (See Proposition 12.)

As a consequence we derive properties for *ms* displaying some necessary and sufficient conditions for a sequence to belong to *pms* (see Theorem 13); the second approach (Theorem 15) is concerned with the structure of *ms* rather than that of *pms*.

For an infinite matrix $A=(a_{ij})_{i\geq 1, j\geq 1}$, let us define its upper triangular projection

$$P_T(A)=\begin{cases} a_{i,j}, & \text{if } i\leq j, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 11. A sequence $b=(b_n)_{n\geq 1}$ belongs to *pms* if and only if

$$B \stackrel{\text{def}}{=} \{b\} = P_T([b]) \in M(l_2). \tag{6}$$

Then *pms* endowed with the norm $\|b\|=\|\{b\}\|_{M(l_2)}$ becomes a Banach algebra with respect to the usual product of sequences.

Proposition 12. *Let $b=(b_n)_{n\geq 1}$ be a sequence of complex numbers. Then*
(1) *$b\in pms \Rightarrow b\in ms$ (so $pms\subset ms$);*
(2) *If we write*

$$(b_1,b_2,\dots,b_n,\dots)=(b_1,0,b_3,\dots,b_{2n-1},\dots)+(0,b_2,0,b_4,\dots,b_{2n},\dots),$$

or equivalently $b=b^{10}+b^{20}$, and denoting with $b^1=(b_1,b_3,\dots,b_{2n-1},\dots)$ and $b^2=(b_2,b_4,\dots,b_{2n},\dots)$, we have $b^i\in pms \Leftrightarrow b^{i0}\in ms$ for $i\in\{1,2\}$ and so if $b^i\in pms$, $i\in\{1,2\}$, then $b\in ms$.

Proof. (1) The statement is obvious.

(2) We first show that $[b^{10}]\in M(l_2)$ implies that $\{b^1\}\in M(l_2)$ and similarly for $[b^{20}]$.

For, let $(\beta_{ij})\in B(l_2)$ and remark that

$$A \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \beta_{11} & 0 & \beta_{12} & 0 & \beta_{13} & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \beta_{21} & 0 & \beta_{22} & 0 & \beta_{23} & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \beta_{31} & 0 & \beta_{32} & 0 & \beta_{33} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in B(l_2).$$

Put $A*[b^{10}]:=X$ and, since $[b^{10}]\in M(l_2)$, then $X\in B(l_2)$.

But

$$X' := \begin{pmatrix} b_1\beta_{11} & 0 & 0 & 0 & \dots \\ b_1\beta_{21} & b_3\beta_{22} & 0 & 0 & \dots \\ b_1\beta_{31} & b_3\beta_{32} & b_5\beta_{33} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in B(l_2),$$

since $\|X'\|_{B(l_2)} \le \|X\|_{B(l_2)}$.

Now observe that $(\beta_{ij})*[b^1]^t = X'$.

For the converse just observe that $\{b^1\} \in M(l_2)$ implies that $P_T([b^{01}]) \in M(l_2)$ and apply (1). □

We pass now to the study of the algebra *pms*.

Let us introduce a new method to estimate the norm on the space $B(l_2)$.

We associate to every sequence $x=(x_j)_{j\ge1}$ from $l_2(\mathbf{N})$ the function $h(t)=\sum_{j=1}^{\infty} x_j e^{2\pi i j t} \in H_0^2([0,1])$, where $H_0^2([0,1])$ consists of all functions $h\colon [0,1]\to\mathbf{C}$ from the Hardy space H^2 such that $\int_0^1 h(t)\,dt=0$.

If $A=(a_{kj})\in B(l_2)$, let us denote by $\mathcal{L}_k(t)=\sum_{j=1}^{\infty} a_{kj}e^{2\pi i j t}\in H^2([0,1])$.

It follows that

$$(7) \qquad \|A\|_{B(l_2)} = \sup_{\|h\|_2\le1} \left(\sum_{k=1}^{\infty} \left| \int_0^1 \mathcal{L}_k(t) h(s-t)\,dt \right|^2 \right)^{1/2} < \infty \quad \text{(for every } s\text{)}.$$

Theorem 13. *Let $b=(b_n)_{n\ge1}$ be a sequence of complex numbers.*

(1) *If $(i_n)_{n\ge1}$ is a strictly increasing sequence of natural numbers with $i_1=0$, define $z_{i_n}=\max_{i_n<k\le i_{n+1}}|b_k|$. Then there exists a constant $R>0$ such that*

$$\|\{b\}\|_{M(l_2)} = \|B\|_{M(l_2)} \le R \inf_{(i_n)_{n\ge1}} \left(\|(z_{i_n})_{n\ge1}\|_2 + \|(z_{i_n}\log(i_{n+1}-i_n))_n\|_\infty \right).$$

(2) *If $b\in pms$ then*

$$\sup_{n\ge1;p\ge1} \frac{(\log n)^2}{n} \sum_{k=p}^{n+p} |b_k|^2 < \infty.$$

(3) *If $(|b_k|)_{k\ge1}$ is a decreasing sequence then $b\in pms$ if and only if $|b_k| = \mathcal{O}(1/\log k)$.*

Proof. (1) Let $A\in B(l_2)$ and $x\in l_2(\mathbf{N})$. It is easy to see, by (7), that

$$\|(B*A)x\|_2^2 = \sum_{k=1}^{\infty} |b_k|^2 \left| \int_0^1 \mathcal{L}_k(t)(h-S_{k-1}(h))(-t)\,dt \right|^2,$$

where $S_k(h)$ is the Fourier partial sum of order k (i.e. if D_k is the Dirichlet kernel, then $S_k(h)(t)=(h*D_k)(t)$ is the convolution of h and D_k).

Therefore, we have

$$\begin{aligned}\|(B*A)x\|_2^2 &\leq 2\sum_{k=1}^{\infty}|b_k|^2\left[\left|\int_0^1 \mathcal{L}_k(t)S_{k-1}(h)(-t)\,dt\right|^2+\left|\int_0^1 \mathcal{L}_k(t)h(-t)\,dt\right|^2\right]\\ &\leq 2\sum_{k=1}^{\infty}|b_k|^2\left|\int_0^1 \mathcal{L}_k(t)S_{k-1}(h)(-t)\,dt\right|^2+2\|b\|_\infty^2\sum_{k=1}^{\infty}\left|\int_0^1 \mathcal{L}_k(t)h(-t)\,dt\right|^2\\ &\leq 2\sum_{k=1}^{\infty}|b_k|^2\left|\int_0^1 \mathcal{L}_k(t)S_{k-1}(h)(-t)\,dt\right|^2+2\|b\|_\infty^2\|A\|_{B(l_2)}^2\|h\|_2^2.\end{aligned}$$

Let $(i_n)_{n\geq 1}$ be a strictly increasing sequence of natural numbers such that $i_1=0$.

Then we have

$$\|b\|_\infty^2 \leq \|(z_{i_n})_{n\geq 1}\|_2^2 \tag{8}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}|b_k|^2\left|\int_0^1 \mathcal{L}_k(t)S_{k-1}(h)(-t)\,dt\right|^2 &\\ &=\sum_{n=1}^{\infty}\sum_{k=i_n+1}^{i_{n+1}}|b_k|^2\left|\int_0^1 \mathcal{L}_k(t)S_{k-1}(h)(-t)\,dt\right|^2\\ &\leq 2\left(\sum_{n=1}^{\infty}\sum_{k=i_n+1}^{i_{n+1}}|b_k|^2\left|\int_0^1 \mathcal{L}_k(t)S_{i_n}(h)(-t)\,dt\right|^2\right)\\ &\quad+2\left(\sum_{n=1}^{\infty}\sum_{k=i_n+1}^{i_{n+1}}|b_k|^2\left|\int_0^1 \mathcal{L}_k(t)(S_{k-1}-S_{i_n})(h)(-t)\,dt\right|^2\right)\\ &\leq 2\sum_{n=1}^{\infty}z_{i_n}^2\left(\sum_{k=i_n+1}^{i_{n+1}}\left|\int_0^1 \mathcal{L}_k(t)S_{i_n}(h)(-t)\,dt\right|^2\right)\\ &\quad+2\sum_{n=1}^{\infty}z_{i_n}^2\left(\sum_{k=i_n+1}^{i_{n+1}}\left|\int_0^1 \mathcal{L}_k(t)(S_{k-1}-S_{i_n})(h)(-t)\,dt\right|^2\right).\end{aligned}$$

Using the formula (7), we get

$$\sum_{k=i_n+1}^{i_{n+1}}\left|\int_0^1 \mathcal{L}_k(t)S_{i_n}(h)(-t)\,dt\right|^2 \leq \|A\|_{B(l_2)}^2\|h\|_2^2.$$

On the other hand

$$\sum_{k=i_n+1}^{i_{n+1}}\left|\int_0^1 \mathcal{L}_k(t)(S_{k-1}-S_{i_n})(h)(-t)\,dt\right|^2$$
$$\sim \sum_{k=1}^{i_{n+1}-i_n}\left|\int_0^1 D_{k-1}(t)\{[\mathcal{L}_{k+i_n}*(S_{i_{n+1}}-S_{i_n})(h)](-t)e^{2\pi i_n t}\}\,dt\right|^2$$
$$\leq \sup_{1\leq k\leq i_{n+1}-i_n}\|D_{k-1}\|_{L^1(0,1)}$$
$$\times \sum_{k=1}^{i_{n+1}-i_n}\int_0^1 |D_{k-1}(s)|\,|(\mathcal{L}_{k+i_n}*(S_{i_{n+1}}-S_{i_n})(h))(-s)|^2\,ds$$
$$\leq C\log(i_{n+1}-i_n)\left(\int_0^1 \sup_{1\leq k\leq i_{n+1}-i_n}|D_{k-1}(s)|\,ds\right)$$
$$\times\left\|\sum_{k=1}^{i_{n+1}-i_n}|(\mathcal{L}_{k+i_n}*(S_{i_{n+1}}-S_{i_n})(h))(\,\cdot\,)|^2\right\|_{L^\infty}$$
$$\leq C'\|A\|^2_{B(l_2)}\|(S_{i_{n+1}}-S_{i_n})(h)\|_2^2[\log(i_{n+1}-i_n)]^2.$$

Thus, using (8), we get

$$\|(A*B)x\|_2\leq R\|A\|_{B(l_2)}\|h\|_2(\|(z_{i_n})_{n\geq 1}\|_2+\|z_{i_n}\log(i_{n+1}-i_n)\|_\infty).$$

(2) Let $B\in M(l_2)$.

Taking $A\in\mathcal{T}\cap B(l_2)$ such that $a_j^l=1/j$ for all $j\in\mathbf{Z}\setminus\{0\}$ and for all $l\in\mathbf{N}\setminus\{0\}$ and $a_0^l=0$ for all $l\in\mathbf{N}$, we obtain that $\widetilde{B}*\tilde{A}\in B(l_2)$, where

$$\widetilde{B}:=\begin{pmatrix} b_1 & b_1 & b_1 & b_1 & \dots\\ b_2 & b_2 & b_2 & b_2 & \dots\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ b_n & b_n & b_n & b_n & \dots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\quad\text{and}\quad \tilde{A}:=\begin{pmatrix} 1 & \frac12 & \frac13 & \dots\\ \frac12 & 1 & \frac12 & \dots\\ \frac13 & \frac12 & 1 & \dots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

Letting $x_p^n\overset{\text{def}}{=}(x_k)_{k\geq 1}$ with

$$x_k=\begin{cases}1, & \text{if } k\in\{p,\dots,n+p\},\\ 0, & \text{otherwise,}\end{cases}$$

where $p,n\in\mathbf{N}\setminus\{0\}$ are fixed, we get

$$(\log(n+1))^2\sum_{k=p}^{n+p}|b_k|^2\leq C\|(\widetilde{B}*\tilde{A})x_p^n\|_2^2\leq C(n+1).$$

Thus

$$\sup_{\substack{n\geq 1\\ p\geq 1}} \frac{(\log(n+1))^2}{n+1} \sum_{k=p}^{n+p} |b_k|^2 < \infty.$$

(3) Let $(|b_k|)_{k\geq 1}$ be a decreasing sequence. Then by (2) we get $|b_n| = \mathcal{O}(1/\log n)$. Conversely, defining $r=(r_n)_{n\geq 1}$ with $r_n = 1/\log(n+1)$ for all $n\geq 1$, we have

$$\|B\|_{M(l_2)} = \|\{b\}\|_{M(l_2)} \leq C\|\{r\}\|_{M(l_2)}.$$

For $i_n = 2^n$ for all $n\geq 2$ and $i_1=0$, it follows by (1) that $z_{i_n} = r_{2^n+1} \sim 1/n$.

Consequently

$$\|\{r\}\|_{M(l_2)} \leq R\left\{ \left\| \left(\frac{1}{n}\right)_{n\geq 1} \right\|_2 + \left\| \left(\frac{1}{n}\log 2^n\right)_{n\geq 1} \right\|_\infty \right\} < \infty.$$

That is, $B\in M(l_2)$. □

Observe that results like Theorem 7.1 or Theorem 8.6 in [B] cannot be applied in our situation.

Remark 14. From the previous results we deduce that

$$l_2(\mathbf{N}) \subset ms \subset l_\infty(\mathbf{N})$$

and that $\{(b_n)_{n\geq 1} \mid |b_n| = \mathcal{O}(1/\log n)\} \subset ms$, with proper inclusions.

Now changing the point of view we will obtain another set of sufficient conditions so that $b\in ms$. These results use the estimate on the absolute value of differences of terms rather than the absolute value of the terms themselves.

Theorem 15. *Let $b=(b_n)_{n\geq 1}$ be a sequence of complex numbers.*

(1) *If $\sup_{n\geq 1}(\sum_{j=1}^n |b_j - b_n|^2) < \infty$ then $b\in ms$.*

(2) *If $\|b\|_{BV(\mathbf{N})} \overset{\text{def}}{=} |b_1| + \sum_{n=1}^\infty |b_{n+1} - b_n| < \infty$ then $b\in ms$.*

Proof. (1) We will use the following result from [B].

Theorem. *A matrix M belongs to $M(l_2)$ if and only if there exists a $P\in B(l_2, l_\infty)$ and $Q\in B(l_1, l_2)$ such that*

$$M = PQ \quad \textit{and} \quad \|M\|_{M(l_2)} \leq \|P\|_{2,\infty}\|Q\|_{1,2}.$$

We recall that if $Q=(q_{jk})_{j\geq 1, k\geq 1}$ and $P=(p_{jk})_{j\geq 1, k\geq 1}$ then

$$\|Q\|_{1,2} = \sup_{k\geq 1}\left(\sum_{j\geq 1} |q_{jk}|^2\right)^{1/2} \quad \text{and} \quad \|P\|_{2,\infty} = \sup_{j\geq 1}\left(\sum_{k\geq 1} |p_{jk}|^2\right)^{1/2}.$$

Let $[b]=B_b+C_b$, where

$$B_b=\begin{pmatrix} b_1 & b_2 & b_3 & \dots \\ b_1 & b_2 & b_3 & \dots \\ b_1 & b_2 & b_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C_b=\begin{pmatrix} 0 & b_1-b_2 & b_1-b_3 & \dots \\ 0 & 0 & b_2-b_3 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly $B_b\in M(l_2)$ for any $b\in l_\infty$ and, since $\|C_b\|_{1,2}=\sup_n(\sum_{j=1}^n |b_j-b_n|^2)^{1/2}<\infty$, by Bennett's theorem it follows that $C_b\in M(l_2)$.

(2) If $A\in M(l_2)$, $(\mathcal{L}_k)_{k\geq 1}$, $h\in H^2([0,1])$, $x\in l^2(\mathbf{N})$ and $h_k=h*D_k$ are as before (7) and defining $f(t)=\sum_{j=1}^\infty b_j e^{2\pi ijt}$ (in the sense of distributions) we get that

$$\begin{aligned}\|([b]*A)x\|_2^2 &= \sum_{k=1}^\infty \left| \int_0^1 f(t)(\mathcal{L}_k*h_k)(-t)\,dt + \int_0^1 f(t)(\mathcal{L}_k*(h-h_k))(0)e^{-2\pi ikt}\,dt\right|^2 \\ &\leq 2\sum_{k=1}^\infty \left|\int_0^1 g_k(s)(\mathcal{L}_k*h)(-s)\,ds\right|^2 + 2\|b\|_\infty^2\|A\|_{B(l_2)}^2\|x\|_2^2,\end{aligned}$$

where $g_k(s)=\sum_{j=1}^k(\hat{f}(k)-\hat{f}(j))e^{2\pi ijs}=\sum_{j=1}^{k-1}(\hat{f}(j+1)-\hat{f}(j))D_j(s)$.

But

$$\begin{aligned}\sum_{k=1}^\infty \left|\int_0^1 g_k(s)(\mathcal{L}_k*h)(-s)\,ds\right|^2 &\leq \left(\sum_{j=1}^\infty |\hat{f}(j+1)-\hat{f}(j)|\right) \\ &\quad \times \sum_{j=1}^\infty |\hat{f}(j+1)-\hat{f}(j)| \left(\sum_{k=2}^\infty |(h_j*\mathcal{L}_k)(0)|^2\right) \\ &\leq \left(\sum_{j=1}^\infty |\hat{f}(j+1)-\hat{f}(j)|\right)^2 \|h\|_2^2\|A\|_{B(l_2)}^2,\end{aligned}$$

so that

$$\|([b]*A)x\|_2 \leq C\|A\|_{B(l_2)}\|x\|_2\left(\sum_{j=1}^\infty |b_{j+1}-b_j|+\|b\|_\infty\right).$$

That is $[b]\in M(l_2)$. □

A continuous version of this result was obtained in [AJPR]. We thank the referee for pointing out this fact.

2. Extension of Haar's theorem

As we announced, this section will be dedicated to proving the generalized Haar theorem—see Theorem C from the introduction. We will start our exposition by introducing a vector space $E(l_2)$. After that, we will define the notion of generalized scalar product for matrices which allows us to give a more useful form for $E(l_2)$ and also to see some similarities with the function case.

Remark 16 is needed to identify the constraints of the definition of $E(l_2)$ and also to remark some of the difficulties of this theory.

Finally, we define the space $C_r(l_2)$ and prove that this one admits a Schauder type basis (see Theorem 18). In this vein Theorem C will follow as a corollary.

Let us consider the vector space given by:

$$E(l_2)=\left\{A=\sum_{k=0}^{n}\alpha_k\odot H_k\in B(l_2)\,|\,\alpha_k\odot H_k\in B(l_2)\text{ for all }0\le k\le n,\ n\in\mathbf{N}\right\},$$

where $\alpha_k\in l_\infty$, and $\alpha_k\odot H_k=[\alpha_k]*H_k$ with the notation in (5).

We introduce a *generalized scalar product of matrices* $\langle A,B\rangle$ for $A=A_{\mathbf{f}}$ and $B=B_{\mathbf{g}}$, where $\mathbf{f}=(f_1,f_2,...)$ and $\mathbf{g}=(g_1,g_2,...)$, in the following way:

$$\langle A,B\rangle=(\langle f_1,g_1\rangle,\langle f_2,g_2\rangle,...). \tag{9}$$

We say that a family of matrices $(\Phi_k)_{k\in\mathbf{N}}$ is *an orthonormal system* if the following orthogonality relations hold: $\langle\Phi_k,\Phi_l\rangle=\mathbf{0}\in l_\infty$ for $k\neq l$ and $\langle\Phi_k,\Phi_k\rangle=\mathbf{1}\in l_\infty$ for all $k\in\mathbf{N}^*$.

By the orthogonality of the system $(H_k)_{k\geq1}$ we deduce that $A\in E(l_2)$ implies $A=\sum_{l=1}^{n}\langle A,H_l\rangle\odot H_l\in E(l_2)$.

Therefore

$$E(l_2)=\left\{A=\sum_{l=1}^{n}\langle A,H_l\rangle\odot H_l\in B(l_2)|\langle A,H_l\rangle\odot H_l\in B(l_2)\text{ for }l\le n,\ n\in\mathbf{N}\setminus\{0\}\right\}.$$

Remark 16. (1) There is $A\in B(l_2)$ such that $\langle A,H_1\rangle\in l_\infty$ and $\langle A,H_1\rangle\odot H_1\notin B(l_2)$.

(2) If $0<p\leq2$ and $A\in S_p$, where S_p is the Schatten class of order p, (see, for instance, [Zh] for the definition of a Schatten class) then $[\langle A,H_k\rangle]\in M(l_2)$, which in turn implies that $\langle A,H_k\rangle\odot H_k\in B(l_2)$ for any $k\in\mathbf{N}\setminus\{0\}$.

Proof. (1) Let $A=A_1$ with $a_1^{2k-1}=1$, $a_1^{2k}=0$ for $k\in\mathbf{N}\setminus\{0\}$ and $a_k^l=0$, if $k\neq1$ and $k\in\mathbf{N}\setminus\{0\}$.

Then $\langle A,H_1\rangle=(x_1,0,x_1,0,...)\in l_\infty$, where x_1 is some constant.

Hence $\langle A, H_1\rangle \odot H_1 = 2iBx_1/\pi$, where

$$B = \begin{pmatrix} 0 & 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \dots \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{3} & \dots \\ -\frac{1}{3} & 0 & -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ -\frac{1}{5} & 0 & -\frac{1}{3} & 0 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

But, clearly, $\infty = \|I - P_T(B)\|_{B(l_2)} \leq \|B\|_{B(l_2)}$, where I is the unit for the usual non-commutative multiplication of infinite matrices.

This result is not surprising, since using Proposition 12 and Theorem 13 we obtain that

$$(x_1, 0, x_1, 0, \dots) \in ms \iff x_1 = 0.$$

(2) Let $p \leq 2$. By [Zh], $A \in B(l_2)$ belongs to S_p if and only if, for any orthonormal basis $(e_k)_{k\geq 1}$ in l_2, we have $\sum_{k=1}^{\infty} \|Ae_k\|^p < \infty$.

Thus, for $A = (a_{k,j})$, we get

$$\sum_{k=1}^{\infty}\Big(\sum_{j=1}^{\infty}|a_{kj}|^2\Big)^{p/2} < \infty \quad \text{and} \quad \sum_{j=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{kj}|^2\Big)^{p/2} < \infty.$$

Then, using the Cauchy–Schwarz inequality and the above inequalities, we get that $\|\langle A, H_k\rangle\|_p^p \leq C\|H_k\|_{B(l_2)}^p \|A\|_{S_p}^p < \infty$ for some constant $C > 0$.

By Remark 14 it follows that $[\langle A, H_k\rangle] \in M(l_2)$. The last implication is now obvious. □

Observe also that there exists $A \in B(l_2)$ such that $\langle A, H_k\rangle \odot H_k \in B(l_2)$, for all $k \in \mathbf{N}$, but for some $k_0 \in \mathbf{N}$, we get $\langle A, H_{k_0}\rangle \notin ms$. Indeed $A = A_0 = (a_n)_{n\geq 1} \in l_\infty \setminus ms$ gives an answer to the above problem for $k_0 = 0$.

Therefore, in the definition of $E(l_2)$, we prefer the weaker condition $\langle A, H_k\rangle \odot H_k \in B(l_2)$ for all k rather than $\langle A, H_k\rangle \in ms$ for all k.

On the ms-module $E(l_2)$ we consider the norm

$$\|\!|A|\!\| = \sup_{m\leq n}\Big\|\sum_{k=0}^{m}\langle A, H_k\rangle \odot H_k\Big\|_{B(l_2)} < \infty. \tag{10}$$

Since $\mathcal{T} \cap E(l_2)$ can be identified with $E_d([0,1])$, the space of all dyadic step functions, whose completion with respect to supremum norm is equal to the space of all countable piecewise continuous functions with discontinuities at dyadic points of $[0,1]$, a space denoted by $C_r([0,1])$, we call $C_r(l_2)$ the completion of $(E(l_2), \|\!|\cdot|\!\|)$.

In what follows we will give some known classes of matrices which are embedded in $C_r(l_2)$.

Examples. (1) Obviously all Toeplitz matrices associated to functions from $C_r([0,1])$ belong to $C_r(l_2)$.

(2) The Hilbert–Schmidt matrices $A=(a_j^l)_{j\in\mathbf{Z},\, l\geq 1}$, with

$$\|A\|_{\mathrm{HS}}=\Big(\sum_{j=-\infty}^{\infty}\sum_{l=1}^{\infty}|a_j^l|^2\Big)^{1/2}<\infty$$

belong to $C_r(l_2)$ and $\|A\|_{C_r(l_2)}\leq\sqrt{2}\,\|A\|_{\mathrm{HS}}$.

We write $\check{g}(t)=g(-t)$ and $P_l(\sum_{j=-\infty}^{\infty}a_je^{2\pi ijt})=\sum_{j=l}^{\infty}a_je^{2\pi ijt}$, where $l\in\mathbf{Z}$. Then by Fubini's theorem and the Cauchy–Schwarz inequality we get

$$\begin{aligned}\|P_T(S_n(A))\|_{B(l_2)}^2&=\sup_{\|g\|_{H^2([0,1])}\leq 1}\sum_{l=1}^{\infty}\bigg|\int_0^1 S_n(f_l)(P_lg)(-t)\,dt\bigg|^2\\&=\sup_{\|g\|_{H^2([0,1])}\leq 1}\sum_{l=1}^{\infty}\bigg|\int_0^1 f_lS_n(P_l\check{g})(-t)\,dt\bigg|^2\\&\leq\|A\|_{\mathrm{HS}}^2\sup_{\|g\|_{H^2([0,1])}\leq 1}\|P_l\check{g}\|_{L^2}^2\\&=\|A\|_{\mathrm{HS}}^2.\end{aligned}$$

Hence $\|A\|_{C_r(l_2)}\leq\sqrt{2}\,\|A\|_{\mathrm{HS}}$.

(3) Let A be a diagonal matrix having as non-zero entries the elements of the sequence $\alpha=(\alpha_i)_{i\geq 1}\in ms$. Then $A\in C_r(l_2)$ and $\|A\|_{C_r(l_2)}\leq\|\alpha\|_{ms}$.

The proof is straightforward using the trivial observations that ms is an algebra with respect to usual multiplication and $C_r(l_2)$ is an ms-module with $|||\alpha\odot X|||\leq\|\alpha\|_{ms}|||X|||$.

(4) If $A=(a_j^l)_{j\in\mathbf{Z},l\geq 1}$ is such that $\sum_{j=-\infty}^{\infty}\|a_j\|_{ms}<\infty$, where $a_j\overset{\mathrm{def}}{=}(a_j^l)_{l\geq 1}$, then $\|A\|_{C_r(l_2)}\leq\sum_{j=-\infty}^{\infty}\|a_j\|_{ms}$ and $A\in C_r(l_2)$.

The statement follows easily by (3).

(5) If A is the main diagonal matrix having as non-zero entries the elements a_j with $(a_j)_{j\geq 1}\in l_\infty$, then $A\in C_r(l_2)$ and $\|A\|_{B(l_2)}=\|A\|_{C_r(l_2)}$. (Note that $(a_j)_{j\geq 1}$ may not belong to ms.)

Proposition 17. *If the sequence of matrices $(A^n)_{n\geq 1}$ is a Cauchy sequence in $E(l_2)$ with respect to the norm $|||\cdot|||$, then $\langle A^n,H_k\rangle\odot H_k$ converges to some $\alpha_k\odot H_k$ in this norm. Moreover $\alpha_k\odot H_k\in B(l_2)$ and $\langle A^n,H_k\rangle\to_n\alpha_k$ in l_∞.*

Proof. Step I. We first prove that

$$\|\langle A,H_k\rangle\|_{l_\infty}\leq 2\|A\|_{B(l_2)}\quad\text{for all }k\in\mathbf{N}\text{ and }A\in B(l_2).\tag{11}$$

If $A=A_{\mathbf{f}}$, where $\mathbf{f}=(f_1,f_2,...)$, and Q_lA is the matrix with entries

$$[Q_lA]_j^k=\begin{cases} a_j^l, & k=l,\ j\in\mathbf{Z},\\ 0, & k\neq l,\end{cases}$$

by the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}\|\langle A,H_k\rangle\|_\infty=\|((\langle f_l,h_k\rangle))_{l\geq 1}\|_\infty &\leq \sup_{l\in\mathbf{N}\setminus\{0\}}\|f_l\|_{L^2}=\sup_{l\in\mathbf{N}\setminus\{0\}}\left(\sum_{j=-\infty}^{\infty}|a_j^l|^2\right)^{1/2}\\ &\leq\sqrt{2}\sup_{l\in\mathbf{N}\setminus\{0\}}\|Q_lA\|_{B(l_2)}\leq 2\|A\|_{B(l_2)}.\end{aligned}$$

Step II. Let now $(A^n)_{n\geq 1}$ be a Cauchy sequence in $E(l_2)$. Then, for a fixed $k\in\mathbf{N}$, we have that $\langle A^n,H_k\rangle\to\alpha_k$, as $n\to\infty$, in l_∞.

Indeed, using (11) and the fact that $\|A\|_{B(l_2)}\leq|\!|\!|A|\!|\!|$, the statement follows by Step I.

Step III. If $(A^n)_{n\geq 1}$ is a Cauchy sequence in $E(l_2)$, then $(\langle A^n,H_k\rangle\odot H_k)_{n\geq 1}$ is a Cauchy sequence in $E(l_2)$ for all k and hence $\langle A^n,H_k\rangle\odot H_k\to B^k\in C_r(l_2)$, as $n\to\infty$, in the norm $|\!|\!|\cdot|\!|\!|$. Thus, by (11) it follows that $\lim_{n\to\infty}\|\langle A^n,H_k\rangle-\langle B^k,H_k\rangle\|_\infty=0$, and by Step II it follows that $\alpha_k=\langle B^k,H_k\rangle$.

Step IV. If we show that $B^k=\langle B^k,H_k\rangle\odot H_k$ then Proposition 17 is proved. But by Step III we have that $\langle A^n,H_k\rangle\odot H_k\to B^k$, as $n\to\infty$, in $B(l_2)$. Then the entries of the matrices $\langle A^n,H_k\rangle\odot H_k$ converge with respect to n to the corresponding entries of the matrix B^k. By Step I, $\langle A^n,H_k\rangle\to\langle B^k,H_k\rangle$, as $n\to\infty$, in l_∞, hence it follows that $\langle B^k,H_k\rangle\odot H_k=B^k$. □

We use Proposition 17 in order to prove the existence of some kind of Schauder basis in $C_r(l_2)$ given by the sequence $(H_k)_{k\geq 0}$.

More specifically, we have the following result.

Theorem 18. *Let $A\in C_r(l_2)$. Then we have the decomposition*

$$A=\sum_{k=0}^{\infty}\langle A,H_k\rangle\odot H_k,$$

in the norm $|\!|\!|\cdot|\!|\!|$.

Proof. Let $A\in C_r(l_2)$. Then there is a Cauchy sequence $A^n\in E(l_2)$ such that $A=\lim_{n\to\infty}A^n$. By Proposition 17 we get $\lim_{n\to\infty}|\!|\!|\langle A^n,H_k\rangle\odot H_k-\alpha_k\odot H_k|\!|\!|=0$

for all $k \geq 0$. Let $\varepsilon > 0$. Then there is $n_\varepsilon \geq 0$ such that for all $n \geq n_\varepsilon$ and all $k > j$, we get

$$\left\|\left|\sum_{i=j}^{k} \langle A^n, H_i\rangle \odot H_i - \sum_{i=j}^{k} \alpha_i \odot H_i \right|\right\| \leq \limsup_{m\to\infty} \left\|\left|\sum_{i=j}^{k} \langle A^n, H_i\rangle \odot H_i - \sum_{i=j}^{k} \langle A^m, H_i\rangle \odot H_i \right|\right\|$$

$$(12) \qquad + \lim_{m\to\infty} \sum_{i=j}^{k} \||\langle A^m, H_i\rangle \odot H_i - \alpha_i \odot H_i\|| \leq \varepsilon$$

Using (12) and the orthogonality relations satisfied by the sequence $(H_k)_{k\geq 0}$ we find that there exists l_ε such that $\|| \sum_{i=j}^{k} \alpha_i \odot H_i \|| < \varepsilon$ for all $k > j > l_\varepsilon$.

Therefore $\sum_{i=0}^{\infty} \alpha_i \odot H_i = B \in C_r(l_2)$. Taking $j=0$ and $k \geq \max\{k(n), l_\varepsilon\}$, where $\sum_{i=0}^{k(n)} \langle A^n, H_i\rangle \odot H_i = A^n$, in (12), we get that $\|| A^n - \sum_{i=0}^{k(n)} \alpha_i \odot H_i \|| < \varepsilon$ for all $\varepsilon > 0$ and for all $n \geq n_\varepsilon$.

Thus $A = B = \sum_{i=0}^{\infty} \alpha_i \odot H_i$ and, using the orthogonality relations satisfied by $(H_k)_{k\geq 0}$ and the fact that the operator $A \mapsto \langle A, H_i\rangle : C_r(l_2) \mapsto l_\infty$ is continuous, we get $A = \sum_{i=0}^{\infty} \langle A, H_i\rangle \odot H_i$. □

Now we get the extension of Haar's theorem for matrices.

Corollary 19. *Let $A \in C_r(l_2)$. Then $A = \sum_{k=0}^{\infty} \langle A, H_k\rangle \odot H_k$ in the norm of $B(l_2)$.*

Of course there exists $A \in C(l_2) \setminus C_r(l_2)$. For instance, A being the diagonal matrix A_1 given by the sequence $(a_n)_{n\geq 1}$, where $a_{2n-1} = 1$ and $a_{2n} = 0$ for all $n = 1, 2, \dots$.

Proof of Theorem C. Let A be an infinite matrix as in Theorem C and let $\varepsilon > 0$. Since $A \in C(l_2)$ there is $k \in \mathbf{N}$ such that $\|\sigma_k(A) - A\|_{B(l_2)} < \frac{1}{2}\varepsilon$. Then by hypothesis and by Example (4) it follows that $\sigma_k(A) \in C_r(l_2)$. Consequently, by Theorem 18, there is a Haar polynomial $\sum_{i=0}^{n-1} \alpha_i \odot H_i$ such that $\|\sigma_k(A) - \sum_{i=0}^{n-1} \alpha_i \odot H_i\|_{B(l_2)} < \frac{1}{2}\varepsilon$. □

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