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The highest smoothness of the Green function implies the highest density of a set

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Abstract. We investigate local properties of the Green function of the complement of a compact set $E\subset[0,1]$ with respect to the extended complex plane. We demonstrate that if the Green function satisfies the $\frac{1}{2}$-Hölder condition locally at the origin, then the density of E at 0, in terms of logarithmic capacity, is the same as that of the whole interval $[0,1]$.

1. Definitions and main results

Let $E\subset[0,1]$ be a compact set with positive (logarithmic) capacity $\operatorname{cap}(E)>0$. We consider E as a set in the complex plane $\mathbf{C}$ and use notions of potential theory in the plane (see [4] and [5]).

Let $\Omega:=\overline{\mathbf{C}}\setminus E$, where $\overline{\mathbf{C}}:=\{\infty\}\cup\mathbf{C}$ is the extended complex plane. Denote by $g_\Omega(z)=g_\Omega(z,\infty)$, $z\in\Omega$, the Green function of Ω with pole at ∞. In what follows we assume that 0 is a regular point of E, i.e., $g_\Omega(z)$ extends continuously to 0 and $g_\Omega(0)=0$.

The monotonicity of the Green function yields

$$g_\Omega(z)\geq g_{\overline{\mathbf{C}}\setminus[0,1]}(z),\quad z\in\mathbf{C}\setminus[0,1],$$

that is, if E has the "highest density" at 0, then g_Ω has the "highest smoothness" at the origin. In particular

$$g_\Omega(-r)\geq g_{\overline{\mathbf{C}}\setminus[0,1]}(-r)>\tfrac{1}{2}\sqrt{r},\quad 0<r<1.$$

In this regard, we would like to explore properties of E whose Green function has the "highest smoothness" at 0, that is, of E conforming to the following condition

$$g_\Omega(z)\leq c|z|^{1/2},\quad c=\text{const}>0,\ z\in\mathbf{C},$$

which is known to be the same as

$$\limsup_{r\to 0} \frac{g_\Omega(-r)}{\sqrt{r}} < \infty \tag{1.1}$$

(cf. [5, Corollary III.1.10]). Various sufficient conditions for (1.1) in terms of metric properties of E are stated in [6], where the reader can also find further references.

There are compact sets $E\subset[0,1]$ of linear Lebesgue measure 0 with property (1.1) (see e.g. [6, Corollary 5.2]), hence (1.1) may hold, though the set E is not dense at 0 in terms of linear measure. On the contrary, our first result states that if E satisfies (1.1) then its density in a small neighborhood of 0, measured in terms of logarithmic capacity, is arbitrary close to the density of $[0,1]$ in that neighborhood.

Theorem 1. *The condition* (1.1) *implies*

$$\lim_{r\to 0} \frac{\operatorname{cap}(E\cap[0,r])}{r} = \frac{1}{4}. \tag{1.2}$$

Recall that $\operatorname{cap}([0,r])=\frac{1}{4}r$ for any $r>0$.

The converse of Theorem 1 is slightly weaker.

Theorem 2. *If E satisfies* (1.2)*, then*

$$\lim_{r\to 0} \frac{g_\Omega(-r)}{r^{1/2-\varepsilon}} = 0, \quad 0<\varepsilon<\frac{1}{2}. \tag{1.3}$$

The connection between properties (1.1), (1.2) and (1.3) is quite delicate. For example, even a slight alteration of (1.1) can lead to the violation of (1.2). As an illustration of this phenomenon we formulate

Theorem 3. *There exists a regular compact set $E\subset[0,1]$ such that* (1.3) *holds and*

$$\liminf_{r\to 0} \frac{\operatorname{cap}(E\cap[0,r])}{r} = 0. \tag{1.4}$$

The next two sections contain preliminaries for the proofs. Then come the proofs of Theorems 1–3.

2. Notation

We shall use c, c_1, $c_2, \dots$, and $d_1, d_2, \dots$, to denote positive constants. These constants may be either absolute or they may depend on E depending on the context. We may use the same symbol for different constants if this does not lead to confusion.

By $|F|$ we denote the linear Lebesgue measure of a measurable subset $F \subset \mathbf{R}$ of the real line $\mathbf{R}$.

The set $\mathbf{D} := \{z : |z| < 1\}$ is the unit disk, $\mathbf{T} = \partial\mathbf{D}$ is the unit circle and for $z_1, z_2 \in \mathbf{C}$, $z_1 \neq z_2$, let

$$[z_1, z_2] := \{tz_2 + (1-t)z_1 : 0 \leq t \leq 1\}$$

be the interval between these points.

For the notions of logarithmic potential theory see e.g. [4] or [5]. In what follows μ_E denotes the equilibrium measure of E. We shall frequently use the relation

$$g_\Omega(z) = \log \frac{1}{\operatorname{cap}(E)} - \int_E \log \frac{1}{|z-t|} \, d\mu_E(t), \quad z \in \Omega. \tag{2.1}$$

We define a generalized curve to be a union of finitely many locally rectifiable Jordan curves. A Borel measurable function $\varrho \geq 0$ on $\mathbf{C}$ is called a metric if

$$0 < A(\varrho) := \iint_{\mathbf{C}} \varrho(z)^2 \, dm_z < \infty,$$

where dm_z stands for the 2-dimensional Lebesgue measure on $\mathbf{C}$. For a family $\Gamma = \{\gamma\}$ of some generalized curves let

$$L_\varrho(\Gamma) := \inf_{\gamma \in \Gamma} \int_\gamma \varrho(z) \, |dz|$$

(if the latter integral does not exist for some $\gamma \in \Gamma$, then we define the integral to be infinity). The quantity

$$m(\Gamma) := \inf_\varrho \frac{A(\varrho)}{L_\varrho(\Gamma)^2}, \tag{2.2}$$

where the infimum is taken with respect to all metrics ϱ, is called the module of the family Γ. We use the properties of $m(\Gamma)$ such as conformal invariance and comparison principle discussed in [2]. When applying results of [2] recall also that the module of a ring domain is 2π times the module of the family of curves separating its boundary components.

3. Preliminaries and auxiliary conformal mappings

In this section we carry out some auxiliary constructions for the direct proof of Theorem 1 and the proof of Theorem 2. We are interested in the behavior of $g_\Omega(z)$ for small $|z|$, which depends on the geometry of E in a neighborhood of 0. In order to avoid complications outside 0, we simplify E by constructing a regular compact set $\widetilde{E}$ such that $E\subset\widetilde{E}\subset[0,1]$ and

$$\liminf_{r\to 0}\frac{\operatorname{cap}(\widetilde{E}\cap[0,r])}{r}=\liminf_{r\to 0}\frac{\operatorname{cap}(E\cap[0,r])}{r}.$$

This construction is done as follows. We use the well-known inequality

$$\left(\log\frac{1}{\operatorname{cap}(E'\cup E'')}\right)^{-1}\le\left(\log\frac{1}{\operatorname{cap}(E')}\right)^{-1}+\left(\log\frac{1}{\operatorname{cap}(E'')}\right)^{-1}, \tag{3.1}$$

which holds for any compact sets $E'\subset[0,1]$ and $E''\subset[0,1]$ (see [4, p. 130]). Let $1=r_1>r_2>\dots$, $\lim_{n\to\infty}r_n=0$, be a sequence of positive numbers such that

$$\liminf_{r\to 0}\frac{\operatorname{cap}(E\cap[0,r])}{r}=\lim_{n\to\infty}\frac{\operatorname{cap}(E\cap[0,r_n])}{r_n}.$$

Without loss of generality we may assume that the inequality

$$\left(\log\frac{4}{r_{n+1}}\right)^{-1}\le\varepsilon_n^3,\quad \varepsilon_n:=\left(\log\frac{1}{\operatorname{cap}(E\cap[0,r_n])}\right)^{-1}$$

holds for all $n\in\mathbf{N}:=\{1,2,\dots\}$. Next, for any $n\in\mathbf{N}$ we construct a compact set $\widetilde{E}_n$ which consists of a finite number of intervals and satisfies the following conditions: $r_n\in\widetilde{E}_n$,

$$[r_{n+1},r_n]\supset\widetilde{E}_n\supset E\cap[r_{n+1},r_n], \tag{3.2}$$

$$\operatorname{cap}(\widetilde{E}_n)\le\operatorname{cap}(E\cap[r_{n+1},r_n])^{1/(1+\varepsilon_n^2)}. \tag{3.3}$$

Let

$$\widetilde{E}:=\{0\}\cup\left(\bigcup_{n=1}^{\infty}\widetilde{E}_n\right).$$

The monotonicity of the capacity and (3.1) yield

$$\begin{aligned}\left(\log\frac{1}{\operatorname{cap}(\widetilde{E}\cap[0,r_n])}\right)^{-1}&\le\left(\log\frac{1}{\operatorname{cap}(\widetilde{E}_n)}\right)^{-1}+\left(\log\frac{4}{r_{n+1}}\right)^{-1}\\&\le(1+\varepsilon_n^2)\varepsilon_n+\varepsilon_n^3=(1+2\varepsilon_n^2)\varepsilon_n.\end{aligned}$$

We thus get

$$\operatorname{cap}(\widetilde{E}\cap[0,r_n])\le\operatorname{cap}(E\cap[0,r_n])^{1+2\varepsilon_n^2}.$$

Since

$$\lim_{n\to\infty} \operatorname{cap}(E\cap[0,r_n])^{\varepsilon_n^2}=1,$$

we obtain

$$\liminf_{r\to 0} \frac{\operatorname{cap}(E\cap[0,r])}{r} = \lim_{n\to\infty} \frac{\operatorname{cap}(E\cap[0,r_n])}{r_n} = \lim_{n\to\infty} \frac{\operatorname{cap}(\widetilde{E}\cap[0,r_n])}{r_n}. \tag{3.4}$$

Since

$$g_{\overline{\mathbf{C}}\setminus\widetilde{E}}(-r) \le g_\Omega(-r), \quad r>0,$$

as a consequence of all these we can see that in the proof of Theorem 1 we may assume

$$E=\widetilde{E}=\{0\}\cup\left(\bigcup_{j=1}^{\infty}[a_j,b_j]\right), \tag{3.5}$$

where $1=b_1>a_1>b_2>a_2>\dots$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0. \tag{3.6}$$

Let either E consist of a finite number of intervals or conform to the conditions (3.5)–(3.6). In both cases we write

$$E=\bigcup_{j=1}^{N}[a_j,b_j], \quad 1=b_1>a_1>b_2>\dots>a_N=0,$$

where N is either finite or ∞.

Denote by $\mathbf{H}:=\{z:\operatorname{Im} z>0\}$ the upper half-plane and consider the function

$$f(z):=\exp\left(\int_E \log(z-\zeta)\,d\mu_E(\zeta)-\log\operatorname{cap}(E)\right), \quad z\in\mathbf{H}. \tag{3.7}$$

It is analytic in $\mathbf{H}$ and has the following obvious properties (cf. (2.1)):

$$|f(z)|=e^{g_\Omega(z)}>1, \qquad z\in\mathbf{H},$$

$$\operatorname{Im} f(z)=e^{g_\Omega(z)}\sin\int_E \arg(z-\zeta)\,d\mu_E(\zeta)>0, \quad z\in\mathbf{H}.$$

Moreover, f can be extended from $\mathbf{H}$ continuously to $\overline{\mathbf{H}}$ such that

$$|f(z)|=1, \qquad z\in E,$$

$$f(x)=e^{g_\Omega(x)}>1, \qquad x\in\mathbf{R},\ x>1,$$

$$f(x)=-e^{g_\Omega(x)}<-1, \quad x\in\mathbf{R},\ x<-1.$$

Next, for any $1\le n\le N-1$ and $b_{n+1}\le x_1<x_2\le a_n$, we have

$$\arg\frac{f(x_2)}{f(x_1)}=\arg\exp\int_E\log\frac{x_2-\zeta}{x_1-\zeta}\,d\mu_E(\zeta)=0,$$

that is,

$$\arg f(x_1)=\arg f(x_2),\quad b_{n+1}\le x_1<x_2\le a_n.$$

Our next objective is to prove that f is univalent in $\mathbf{H}$. We shall use the following result. Let $\sqrt{z^2-1}$, $z\in\overline{\mathbf{C}}\setminus[-1,1]$, be the analytic function defined in a neighborhood of infinity as

$$\sqrt{z^2-1}=z\left(1-\frac{1}{2z^2}+\dots\right).$$

Lemma 1. *For any* $-1\le x\le 1$ *and* $z\in\mathbf{H}$,

$$u_x(z):=\operatorname{Re}\frac{\sqrt{z^2-1}}{z-x}\ge 0. \tag{3.8}$$

Proof. First, we consider the particular case when $x=0$. For $z\in\mathbf{H}$, let $h(z):=\sqrt{z^2-1}/z$. Then $h(z)^2=1-1/z^2$. Thus, the image of the upper half-plane under h^2 is disjoint from $(-\infty,1]$. Thus the image of $\mathbf{H}$ under h lies entirely in the left half-plane or entirely in the right half-plane. On taking account of the determination of the square root for large z, it follows that the real part of h is always positive.

In the general case, $-1\le x\le 1$, a continuity argument allows us to restrict our proof of (3.8) to the case $x^2\ne 1$. Consider a linear fractional transformation of $\mathbf{H}$ onto itself given by

$$\zeta=\frac{z-x}{-xz+1}\quad\text{and}\quad z=\frac{\zeta+x}{x\zeta+1}.$$

A straightforward calculation shows that

$$u_x(z)=\frac{u_0(\zeta)}{\sqrt{1-x^2}},$$

from which (3.8) follows. □

Using the reflection principle we can extend f to a function analytic in $\overline{\mathbf{C}}\setminus[0,1]$ by the formula

$$f(z):=\overline{f(\bar z)},\quad z\in\mathbf{C}\setminus\overline{\mathbf{H}},$$

and consider the function

$$h(w):=\frac{1}{f(J(w))},\quad w\in\mathbf{D},$$

where J is a linear transformation of the Joukowski mapping, namely

$$J(w):=\frac{1}{2}\left(\frac{1}{2}\left(w+\frac{1}{w}\right)+1\right),$$

which maps the unit disk $\mathbf{D}$ onto $\overline{\mathbf{C}}\setminus[0,1]$. Note that the inverse mapping is defined as follows

$$w=J^{-1}(z)=(2z-1)-\sqrt{(2z-1)^2-1}\,,\quad z\in\overline{\mathbf{C}}\setminus[0,1].$$

Therefore, for $z\in\mathbf{H}$ and $w=J^{-1}(z)\in\mathbf{D}$, we obtain

$$\begin{aligned}\frac{wh'(w)}{h(w)}&=w(\log h(w))'\\&=-w\left(\int_E \log(J(w)-\zeta)\,d\mu_E(\zeta)\right)'\\&=-wJ'(w)\int_E\frac{d\mu_E(\zeta)}{z-\zeta}\\&=-\frac{1}{4}\left(w-\frac{1}{w}\right)\int_E\frac{d\mu_E(\zeta)}{z-\zeta}\\&=\frac{1}{2}\int_E\frac{\sqrt{(2z-1)^2-1}}{z-\zeta}\,d\mu_E(\zeta)\\&=\int_E\frac{\sqrt{(2z-1)^2-1}}{(2z-1)-(2\zeta-1)}\,d\mu_E(\zeta).\end{aligned}$$

According to (3.8) for w under consideration we obtain

$$\operatorname{Re}\frac{wh'(w)}{h(w)}\geq 0.$$

Because of the symmetry and the maximum principle for harmonic functions we have

$$\operatorname{Re}\frac{wh'(w)}{h(w)}>0,\quad w\in\mathbf{D}.$$

It means that h is a conformal mapping of $\mathbf{D}$ onto a starlike domain (cf. [3, p. 42]).

Hence, f is univalent and maps $\overline{\mathbf{C}}\setminus[0,1]$ onto a (with respect to ∞) starlike domain $\overline{\mathbf{C}}\setminus K$ (see Figure 1) with the following properties: $\overline{\mathbf{C}}\setminus K$ is symmetric with respect to the real line $\mathbf{R}$ and coincides with the exterior of the unit disk with $2N-2$ slits, i.e.,

$$\overline{\mathbf{H}}\cap K=(\overline{\mathbf{H}}\cap\overline{\mathbf{D}})\cup\left(\bigcup_{j=1}^{N-1}[e^{i\theta_j},r_je^{i\theta_j}]\right),$$

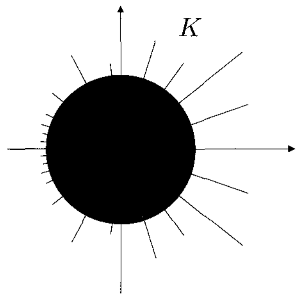

Figure 1. The set K.

where $r_j>1$, $0<\theta_1<\theta_2<...<\pi$, and in the case $N=\infty$,

$$\lim_{j\to\infty}\theta_j=\pi \quad \text{and} \quad \lim_{j\to\infty} r_j=1.$$

Note that

$$f([b_{j+1},a_j]):=\bigcup_{b_{j+1}\le x\le a_j}\left(\lim_{\substack{z\in\mathbf{H}\\ z\to x}} f(z)\right)=[e^{i\theta_j},r_je^{i\theta_j}]$$

and any point of $[e^{i\theta_j},r_je^{i\theta_j})$ has exactly two preimages. Besides,

$$f(E):=\bigcup_{x\in E}\left(\lim_{\substack{z\in\mathbf{H}\\ z\to x}} f(z)\right)=\mathbf{T}\cap\overline{\mathbf{H}}.$$

There is a close connection between the capacities of the compact sets K and E, namely

$$\operatorname{cap}(E)=\frac{1}{4\operatorname{cap}(K)}. \tag{3.9}$$

Indeed, let $w\in\mathbf{C}\setminus\overline{\mathbf{D}}$, $z=J(1/w)$ and $\xi=f(z)$. Now

$$\operatorname{cap}(E)=\lim_{z\to\infty}\left|\frac{z}{\xi}\right| \quad \text{and} \quad \operatorname{cap}(K)=\lim_{w\to\infty}\left|\frac{\xi}{w}\right|,$$

where the first relation follows from the definition (3.7) of f, and the second one follows from the fact that $w\mapsto\xi$ is the canonical conformal map $w\mapsto\Phi(w)$ of $\overline{\mathbf{C}}\setminus\overline{\mathbf{D}}$ onto $\overline{\mathbf{C}}\setminus K$ (note that $\log|\Phi^{-1}(w)|$ is the Green function of $\overline{\mathbf{C}}\setminus K$ with pole at infinity and apply (2.1)). Thus,

$$\operatorname{cap}(E)\operatorname{cap}(K)=\lim_{w\to\infty}\left|\frac{z}{w}\right|=\frac{1}{4}.$$

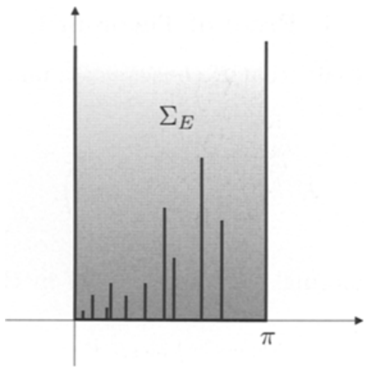

Figure 2. The domain Σ_E.

For $r>1$ we have

$$\operatorname{cap}(\overline{\mathbf{D}}\cup[1,r]) = \frac{(1+r)^2}{4r}$$

(see [4, Section 5.2, p. 135]), therefore, (3.9), the equality $\operatorname{cap}(r\overline{\mathbf{D}})=r$ and the monotonicity of the capacity imply

$$\frac{1}{4\sup\limits_{1\le j\le N-1} r_j} \le \operatorname{cap}(E) \le \frac{\sup\limits_{1\le j\le N-1} r_j}{\Bigl(1+\sup\limits_{1\le j\le N-1} r_j\Bigr)^2}. \tag{3.10}$$

In addition to the conformal map f we introduce the conformal map

$$F(z) := \pi + i\log f(z), \quad z\in\mathbf{H},$$

which maps $\mathbf{H}$ onto Σ_E bounded by the set (see Figure 2)

$$\partial\Sigma_E = \{\zeta = iv : v\ge 0\}\cup\{\zeta=\pi+iv : v\ge 0\}\cup[0,\pi]\cup\left(\bigcup_{j=1}^{N-1}[u_j, u_j+iv_j]\right),$$

where

$$u_j = \pi-\theta_j, \quad v_j = \log r_j,$$

and in the case $N=\infty$,

$$\lim_{j\to\infty} u_j = \lim_{j\to\infty} v_j = 0.$$

If we extend F continuously to the real line, we obtain the following boundary correspondence:

$$F(0)=0, \quad F([b_{j+1}, a_j]) = [u_j, u_j+iv_j] \quad\text{and}\quad g_\Omega(z) = \operatorname{Im} F(z), \; z\in\overline{\mathbf{H}}\setminus E, \tag{3.11}$$

where the last equality is a consequence of (2.1).

4. Proof of Theorem 1

We assume (cf. (3.5)–(3.6)) that E consists of countably many closed intervals accumulating at the origin, i.e.,

$$E=\{0\}\cup\left(\bigcup_{j=1}^{\infty}[a_j,b_j]\right).$$

Let $F:\mathbf{H}\to\Sigma_E$ be the conformal mapping defined in the previous section. The regularity of $0\in E$ implies

$$\lim_{j\to\infty}u_j=\lim_{j\to\infty}v_j=0. \tag{4.1}$$

We divide the proof of Theorem 1 into several steps.

Step I. First we prove that much more can be established than (4.1) if the Green function satisfies the $\frac{1}{2}$-Hölder condition: *if* (1.1) *is true then*

$$\lim_{j\to\infty}\frac{v_j}{u_j}=0. \tag{4.2}$$

We carry out the proof by contradiction. Assume that (4.2) is false, i.e., there exists a constant $0<c<1$ and a (monotone) sequence of natural numbers $\{j_k\}_{k=1}^{\infty}$ such that

$$v_{j_k}\ge cu_{j_k},\quad 2u_{j_{k+1}}<u_{j_k}\quad\text{and}\quad u_{j_1}\le 1.$$

Let the domain $\Sigma^*\supset\Sigma_E$ be bounded by the set

$$\partial\Sigma^*=\{\zeta=iv:v\ge 0\}\cup\{\zeta=\pi+iv:v\ge 0\}\cup[0,\pi]\cup\left(\bigcup_{k=1}^{\infty}[u_{j_k},u_{j_k}(1+ic)]\right).$$

Consider the auxiliary conformal mapping $F^*:\mathbf{H}\to\Sigma^*$ normalized by the conditions

$$F^*(\infty)=\infty,\quad F^*(0)=0\quad\text{and}\quad F^*(1)=\pi.$$

Comparing modules of the quadrilaterals

$$\Sigma_E(0,\pi,\infty,F(-r))\quad\text{and}\quad\Sigma^*(0,\pi,\infty,F(-r)),$$

for $r>0$ we have

$$|F^*(-r)|\le|F(-r)|=g_\Omega(-r)\le c_1\sqrt{r}\,. \tag{4.3}$$

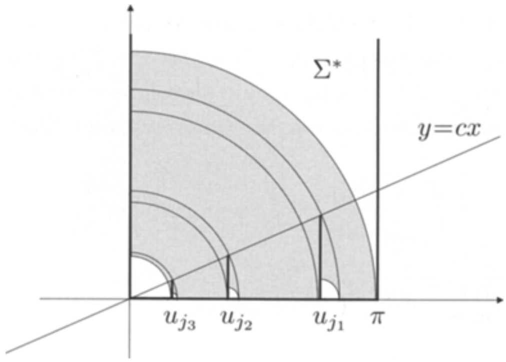

Figure 3.

Let $\Gamma(t,\Sigma^*)$, $t>0$, be the family of all cross-cuts of Σ^* which join in Σ^* its boundary intervals $[0,it]$ and $\{\zeta=\pi+iv:v\geq 0\}$. Let

$$\Gamma_{1,k}:=\Gamma(u_{j_k},\Sigma^*),\ \Gamma'_{1,k}:=(F^*)^{-1}(\Gamma_{1,k})=\{\gamma:F^*(\gamma)\in\Gamma_{1,k}\},\ r^*_{j_k}:=-(F^*)^{-1}(iu_{j_k}).$$

For k large enough we have $r^*_{j_k}\leq 1$. Therefore, for such k by (4.3) and [2, Chapter II, (1.2) and (2.10)] the module of the family $\Gamma'_{1,k}$ satisfies the inequalities

$$m(\Gamma'_{1,k})\geq\pi\left(\log\frac{16(1+r^*_{j_k})}{r^*_{j_k}}\right)^{-1}\geq\frac{\pi}{2}\left(\log\frac{4c_1\sqrt{2}}{u_{j_k}}\right)^{-1}. \tag{4.4}$$

Our next objective is to estimate (for large k) the quantity $m(\Gamma_{1,k})$ from above.

For $m\in\mathbf{N}$ let

$$S_m:=\left\{\zeta\in\Sigma^*:|\zeta-u_{j_m}|\leq u_{j_m}\left(\sqrt{1+c^2}-1\right)\text{ and }0\leq\arg(\zeta-u_{j_m})\leq\tfrac{1}{2}\pi\right\}.$$

Consider the metric (see Figure 3, where the shaded area is the support of $\varrho_{1,k}$ for $k=3$)

$$\varrho_{1,k}(\zeta):=\begin{cases}|\zeta|^{-1}, & \text{if }\zeta\in\Sigma^*\setminus\left(\bigcup_{m=1}^{k}S_m\right)\text{ and }u_{j_k}\leq|\zeta|\leq\pi,\\ 0, & \text{elsewhere.}\end{cases}$$

We claim that for any $\gamma\in\Gamma_{1,k}$,

$$\int_\gamma\varrho_{1,k}(\zeta)\,|d\zeta|\geq\log\frac{\pi}{u_{j_k}}. \tag{4.5}$$

Indeed, let

$$R(a,b):=\{\zeta:a<|\zeta|<b\},\quad 0<a<b<\infty.$$

For $\gamma\in\Gamma_{1,k}$ denote by $\gamma(a,b)\subset\gamma\cap R(a,b)$, $u_{j_k}\le a<b\le\pi$, a subarc of γ joining in $R(a,b)$ its boundary circular components. Note that in Figure 3, each $\gamma\in\Gamma_{1,k}$ must make at least one crossing of each ring within the shaded area (even after removing the quarter-disks). Taking into account the obvious inequalities

$$\int_{\gamma(a,b)}\frac{|d\zeta|}{|\zeta|}\ge\left|\int_{\gamma(a,b)}\frac{d\zeta}{\zeta}\right|\ge\log\frac{b}{a},$$

with $u_{j_0}:=\pi$ we have (cf. Figure 3)

$$\begin{aligned}\int_\gamma \varrho_{1,k}(\zeta)\,|d\zeta| &\ge \sum_{m=1}^{k}\left(\int_{\gamma(u_{j_m},u_{j_m}\sqrt{1+c^2})}\frac{|d\zeta|}{|\zeta|}+\int_{\gamma(u_{j_m}\sqrt{1+c^2},u_{j_{m-1}})}\frac{|d\zeta|}{|\zeta|}\right)\\ &\ge \sum_{m=1}^{k}\left(\log\sqrt{1+c^2}+\log\frac{u_{j_{m-1}}}{\sqrt{1+c^2}\,u_{j_m}}\right)=\log\frac{\pi}{u_{j_k}},\end{aligned}$$

which proves (4.5).

Recalling the definition of the module of a family of curves (2.2), we have

$$\begin{aligned}m(\Gamma_{1,k}) &\le \left(\inf_{\gamma\in\Gamma_{1,k}}\int_\gamma \varrho_{1,k}(\zeta)\,|d\zeta|\right)^{-2}\iint_{\mathbf{C}}\varrho_{1,k}(\zeta)^2\,dm_\zeta\\ &\le\left(\log\frac{\pi}{u_{j_k}}\right)^{-2}\left(\frac{\pi}{2}\log\frac{\pi}{u_{j_k}}-A_k\right),\end{aligned}\tag{4.6}$$

where

$$A_k:=\sum_{m=1}^{k}\iint_{S_m}\frac{dm_\zeta}{|\zeta|^2}\ge\sum_{m=1}^{k}\frac{\pi\left(\sqrt{1+c^2}-1\right)^2}{4(1+c^2)}=c_2k.\tag{4.7}$$

Comparing (4.4) and (4.6), we obtain $A_k\le c_3$, which contradicts (4.7). This contradiction proves (4.2).

Step II. We can rewrite (4.2) as follows: for

$$w_k:=\sup_{j\ge k}\frac{v_j}{u_j},\quad k\in\mathbf{N},\tag{4.8}$$

we have

$$\lim_{k\to\infty}w_k=0\quad\text{and}\quad v_j\le w_ku_k,\ j\ge k.\tag{4.9}$$

Let

$$1=r_1>r_2>\dots,\qquad \lim_{n\to\infty} r_n=0,$$

be the sequence of real numbers from the definition of the set $\widetilde{E}$ in (3.2)–(3.3). Recall that we assume that $E=\widetilde{E}$. By splitting some of the intervals $[a_j,b_j]$ into two subintervals we may assume without loss of generality that $r_n=b_{j_n}$ with some j_n's. For $n\in\mathbf{N}$ let

$$E_n:=E\cap[0,r_n]=\{0\}\cup\biggl(\bigcup_{j=j_n}^{\infty}[a_j,b_j]\biggr).$$

Denote by $F_n\colon\mathbf{H}\to\Sigma_{E_n}$ the corresponding conformal mapping from Section 3, where

$$\partial\Sigma_{E_n}=\{\zeta=i\eta:\eta\ge 0\}\cup\{\zeta=\pi+i\eta:\eta\ge 0\}\cup[0,\pi]\cup\biggl(\bigcup_{j=j_n}^{\infty}[\xi_j,\xi_j+i\eta_j]\biggr)$$

and

$$\pi>\xi_1>\xi_2>\dots,\qquad \lim_{j\to\infty}\xi_j=0,\qquad \eta_j>0.$$

Note that ξ_j and η_j depend also on n. The mapping F_n adheres to the following boundary correspondence:

$$F_n(0)=0,\quad F_n(r_n)=\pi,\quad F_n(\infty)=\infty\quad\text{and}\quad F_n(b_{j+1})=F_n(a_j)=\xi_j \text{ for } j\ge j_n.$$

Step III. Our next aim is to prove that *if* g_Ω *satisfies* (1.1), *then*

$$\lim_{n\to\infty}\sup_{j\ge j_n}\eta_j=0. \tag{4.10}$$

To this end for any fixed n and $j\ge j_n$ consider the set

$$\gamma_\eta:=\Sigma_{E_n}\cap[i\eta,\xi_j+i\eta],\qquad 0<\eta\le\eta_j,$$

consisting of a finite number of open intervals. Denote by $\Gamma_{2,j}$ the family of all these sets, i.e.,

$$\Gamma_{2,j}:=\{\gamma_\eta\}_{0<\eta\le\eta_j}.$$

By the comparison principle for the module of $\Gamma_{2,j}$ we have

$$m(\Gamma_{2,j})\ge m\bigl(\{[i\eta,\xi_j+i\eta]\}_{0<\eta\le\eta_j}\bigr)=\frac{\eta_j}{\xi_j}\ge\frac{\eta_j}{\pi}. \tag{4.11}$$

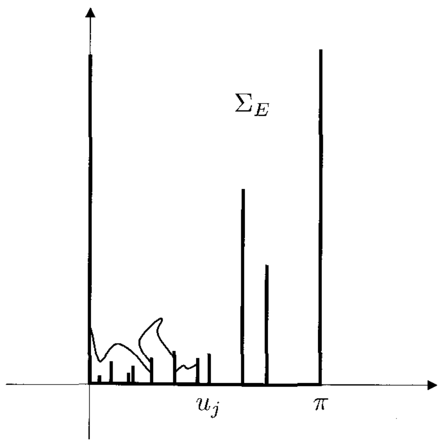

Figure 4. A typical element of $\Gamma'_{2,j}$.

Next we estimate from above the module of the family $\Gamma'_{2,j}:=F\circ F_n^{-1}(\Gamma_{2,j})$ (see Figure 4). Without loss of generality we may assume $w_j<\frac{1}{2}$, and consider the metric

$$\varrho_{2,j}(w):=\begin{cases} |w-u_j|^{-1}, & \text{if } w\in\Sigma_E \text{ and } w_ju_j\le|w-u_j|\le u_j\sqrt{1+w_j^2}\,, \\ 0, & \text{elsewhere,}\end{cases}$$

where w_j are the numbers from (4.8)–(4.9). We claim that for any $\gamma\in\Gamma'_{2,j}$,

$$\int_\gamma \varrho_{2,j}(w)\,|dw|\ge\frac{1}{3}\log\frac{1}{w_j}. \tag{4.12}$$

To demonstrate the validity of (4.12), for $w_ju_j\le r<R\le u_j$ we define (see Figure 5)

$$B(r,R):=\Sigma_E\cap\Bigg(\{w=u+iv: u_j-R\le u\le u_j-r \text{ and } 0\le v\le w_ju_j\}$$
$$\cup\left\{w=u_j+\varrho e^{i\theta}: r^2+w_j^2u_j^2\le\varrho^2\le R^2+w_j^2u_j^2,\ 0\le\theta\le\pi-\arcsin\frac{w_ju_j}{\varrho}\right\}\Bigg).$$

For $\gamma\in\Gamma'_{2,j}$ and $R\le 2r$ we have

$$\int_{\gamma\cap B(r,R)}\varrho_{2,j}(w)\,|dw|\ge\frac{R-r}{\sqrt{w_j^2u_j^2+R^2}}\ge\frac{R-r}{\sqrt{5}\,r}>\frac{1}{3}\log\frac{R}{r},$$

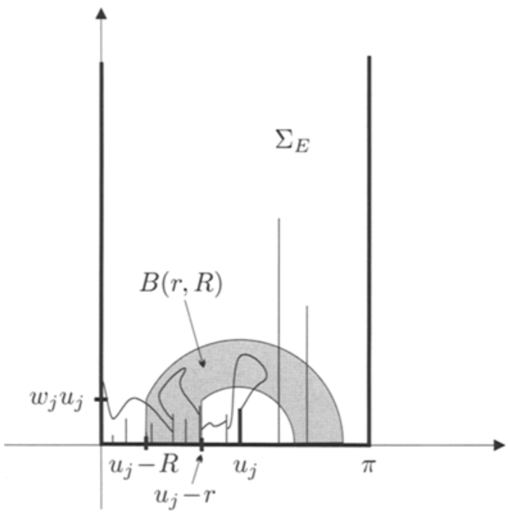

Figure 5. The domain $B(r,R)$.

from which (4.12) immediately follows.

Recalling the definition of module (2.2), we obtain

$$m(\Gamma'_{2,j}) \le 9\left(\log\frac{1}{w_j}\right)^{-2}\iint_{\mathbf{C}}\varrho_{2,j}(w)^2\,dm_w = 9\pi\left(\log\frac{1}{w_j}\right)^{-2}\log\frac{\sqrt{1+w_j^2}}{w_j}. \tag{4.13}$$

Comparing (4.11) and (4.13) we get for large n

$$\sup_{j\ge j_n}\eta_j \le 10\pi^2\left(\log\frac{1}{w_{j_n}}\right)^{-1}.$$

Therefore, (4.10) follows directly from (4.9).

Step IV. According to (3.10) we have

$$\frac{\operatorname{cap}(E\cap[0,r_n])}{r_n} \ge \frac{1}{4\sup_{j\ge j_n} e^{\eta_j}}.$$

Hence, by (4.10),

$$\lim_{n\to\infty}\frac{\operatorname{cap}(E\cap[0,r_n])}{r_n} = \frac{1}{4}.$$

This and (3.4) yield (1.2), and that concludes the proof of Theorem 1. □

5. Proof of Theorem 2

Let

$$h_E(r) := \sup_{0<t\le r}\left(\frac{1}{4}-\frac{\operatorname{cap}(E_t)}{t}\right), \quad 0<r\le 1,$$

where $E_t := E\cap[0,t]$.

Note that $h_E(r)$ is a nonnegative, monotonically increasing (with respect to r) function which satisfies

$$\lim_{r\to 0} h_E(r) = 0. \tag{5.1}$$

Below we derive estimates for g_Ω in terms of h_E in the case of a "simple" E and then extend this estimate to the arbitrary E under consideration. It is important to emphasize that it is the uniformity of these estimates with respect to the function h_E which makes it possible to deduce the result for arbitrary E.

Let E consist of a finite number (greater than one) of intervals, i.e.,

$$E = \bigcup_{j=1}^{N}[a_j,b_j], \quad 0=a_N<b_N<\ldots<a_1<b_1=1.$$

Let $a_k<r=r_k<b_k$, $k=1,\ldots,N-1$, be an arbitrary but fixed number. Denote by $F_r\colon \mathbf{H}\to\Sigma_{E_r}$ the appropriate conformal mapping from Section 3, where

$$\partial\Sigma_{E_r} = \{\zeta=i\eta:\eta\ge 0\}\cup\{\zeta=\pi+i\eta:\eta\ge 0\}\cup[0,\pi]\cup\left(\bigcup_{j=k}^{N-1}[\xi_j,\xi_j+i\eta_j]\right).$$

Note that ξ_j and η_j depend on r. Taking r sufficiently close to a_k we can ensure $\frac{1}{2}\pi<\xi_k<\pi$. Let

$$\eta_k^* := \max_{k\le j\le N-1}\eta_j.$$

According to (3.10) we have

$$\frac{1}{4}-h_E(r)\le\frac{\operatorname{cap}(E_r)}{r}\le\frac{e^{\eta_k^*}}{(1+e^{\eta_k^*})^2},$$

which implies

$$\eta_k^*\le\log\bigl(A(r)+\sqrt{A(r)^2-1}\bigr), \quad A(r):=\frac{1+4h_E(r)}{1-4h_E(r)}. \tag{5.2}$$

Since $h_E(1)<\frac{1}{4}$, inequality (5.2) yields

$$\eta_k^* \le \log\big(A(1)+\sqrt{A(1)^2-1}\,\big) =: d_1 = d_1(E). \tag{5.3}$$

The set

$$\gamma_v := \Sigma_E \cap [iv, u_k+iv], \quad 0<v\le v_k,$$

consists of a finite number of open intervals. Let $\Gamma_{3,k}$ denote the family of all these sets, i.e.,

$$\Gamma_{3,k} := \{\gamma_v\}_{0<v\le v_k}.$$

By the comparison principle for the module of $\Gamma_{3,k}$ we have

$$m(\Gamma_{3,k}) \ge m(\{[iv, u_k+iv]\}_{0<v\le v_k}) = \frac{v_k}{u_k}. \tag{5.4}$$

Let $\Gamma'_{3,k} := F_r \circ F^{-1}(\Gamma_{3,k})$. Consider the metric

$$\varrho_{3,k}(\zeta) := \begin{cases} 1, & \text{if } \zeta\in\Sigma_{E_r} \text{ and } 0<\operatorname{Im}\zeta<\eta_k^*+1, \\ 0, & \text{elsewhere.} \end{cases}$$

Since for any $\gamma\in\Gamma'_{3,k}$,

$$\int_\gamma \varrho_{3,k}(\zeta)\,|d\zeta| \ge 1,$$

we have

$$m(\Gamma'_{3,k}) \le \iint_{\mathbf{C}} \varrho_{3,k}(\zeta)^2\,dm_\zeta = \pi(\eta_k^*+1). \tag{5.5}$$

Combining (5.3)–(5.5) we obtain

$$\frac{v_k}{u_k} \le \pi(d_1+1) =: d_2 = d_2(E). \tag{5.6}$$

For small η_k^*, (5.6) can be improved in the following manner. If $\eta_k^*\le\frac{1}{4}\pi$, then

$$\frac{v_k}{u_k} \le 27\pi\left(\log\frac{\pi}{2\eta_k^*}\right)^{-1}. \tag{5.7}$$

Indeed, setting

$$\varrho_{3,k}(\zeta) := \begin{cases} |\zeta-\xi_k|^{-1}, & \text{if } \zeta\in\Sigma_{E_r} \text{ and } \eta_k^*\le|\zeta-\xi_k|\le\sqrt{\xi_k^2+(\eta_k^*)^2}\,, \\ 0, & \text{elsewhere,} \end{cases}$$

and repeating word for word the proof of (4.13), we obtain

$$m(\Gamma'_{3,k}) \leq 9\pi\left(\log\frac{\xi_k}{\eta_k^*}\right)^{-2}\log\frac{\sqrt{\xi_k^2+(\eta_k^*)^2}}{\eta_k^*} < 27\pi\left(\log\frac{\pi}{2\eta_k^*}\right)^{-1}.$$

The above inequality and (5.4) yield (5.7).

Let $\Gamma_{4,x}$, $0<x<1$, $x\in E$, be the family of all cross-cuts of $\mathbf{H}$ which join $(-\infty,0]$ with $[x,1]$. Then

$$m(\Gamma_{4,x}) \leq \frac{1}{\pi}\log\frac{16}{x} \tag{5.8}$$

(see [2, Chapter II, (1.2) and (2.10)]). For the module of the family $\Gamma'_{4,x}:=F(\Gamma_{4,x})$ we have

$$m(\Gamma'_{4,x}) \geq \frac{2}{\pi}\log\frac{\pi}{\sqrt{1+d_2^2}\,F(x)}. \tag{5.9}$$

Indeed, in the nontrivial case when $\sqrt{1+d_2^2}\,F(x)<\pi$, we compare $\Gamma'_{4,x}$ with the family Γ of all circular cross-cuts of the domain

$$\left\{w=\varrho e^{i\theta}:\sqrt{1+d_2^2}\,F(x)<\varrho<\pi \text{ and } 0<\theta<\tfrac{1}{2}\pi\right\}$$

to obtain

$$m(\Gamma'_{4,x}) \geq m(\Gamma) = \frac{2}{\pi}\log\frac{\pi}{\sqrt{1+d_2^2}\,F(x)}.$$

Comparison of (5.8) and (5.9) gives

$$x \leq d_3F(x)^2, \quad d_3=d_3(E):=\frac{16}{\pi^2}(1+d_2^2). \tag{5.10}$$

Let $x_0\in E$ be any point satisfying $0<x_0<1$ and

$$h_E(x_0) \leq \frac{1}{4}\left(\frac{e^{\pi/8}-e^{-\pi/8}}{e^{\pi/8}+e^{-\pi/8}}\right)^2.$$

Such an x_0 exists because of (5.1). For any k such that $a_k<x_0$ we can choose $r=r_k<x_0$ in the above discussion. Hence, by (5.2), $\eta_k^*\leq\frac{1}{4}\pi$. Taking into account (5.2), (5.7) and (5.10), for such k we have

$$\frac{v_k}{u_k} \leq 27\pi\left(\log\frac{\pi}{2\eta_k^*}\right)^{-1} \leq 27\pi\left(\log\frac{\pi}{2B_k}\right)^{-1}, \tag{5.11}$$

where

$$B_k := \log\left(A_k + \sqrt{A_k^2 - 1}\right), \quad A_k := A(2d_3 u_k^2)$$

and where we have used that

$$r = r_k \leq d_3 F(r_k)^2 \leq 2d_3 u_k^2,$$

since for r sufficiently close to a_k we obtain $F(r_k) \leq 2F(a_k) = 2u_k$.

Let

$$\begin{aligned} u_0 &:= \left(\frac{x_0}{d_3}\right)^{1/2}, \\ A^*(u) &:= A(2d_3u^2), && 0 < u < u_0, \\ B^*(u) &:= \log\left(A^*(u) + \sqrt{A^*(u)^2 - 1}\right), && 0 < u < u_0, \\ C^*(u) &:= 27\pi\left(\log\frac{\pi}{2B^*(u)}\right)^{-1}, && 0 < u < u_0, \\ v_0 &:= \max\{C^*(u_0), d_2\}, \\ C(u) &:= \begin{cases} C^*(u), & \text{if } 0 < u < u_0, \\ v_0, & \text{if } u_0 \leq u \leq \pi. \end{cases} \end{aligned}$$

The function $C(u)$ is monotonically increasing and it satisfies

$$\lim_{u \to 0} C(u) = 0 \quad \text{and} \quad \frac{v_k}{u_k} \leq C(u_k), \; k = 1, \ldots, N-1.$$

For an arbitrary but fixed $0 < \varepsilon < \frac{1}{2}$ denote by u_ε any point such that $0 < u_\varepsilon < u_0$ and

$$\delta = \delta_\varepsilon := \frac{1}{\pi} \arctan C(u_\varepsilon) \leq \frac{\varepsilon}{2}.$$

Fix any $0 < r_\varepsilon < 1$ such that $F(-r_\varepsilon) \leq e^{-\pi} u_\varepsilon$. Let $\Gamma_{5,r}$, $0 < r < 1$, be the family of all cross-cuts of $\mathbf{H}$ which join $(-\infty, -r]$ with $[0, 1]$. Comparing $\Gamma_{5,r}$ with the family Γ^* of all circular cross-cuts of the domain

$$\{w = \varrho e^{i\theta} : r < \varrho < 1 \text{ and } 0 < \theta < \pi\}$$

we have

$$m(\Gamma_{5,r}) \geq m(\Gamma^*) = \frac{1}{\pi} \log \frac{1}{r}. \tag{5.12}$$

Next we estimate from above the module of the family $\Gamma'_{5,r}:=F(\Gamma_{5,r})$. Let $iR:=F(-r)$, $0<r<r_\varepsilon$. Consider the metrics

$$\varrho^*_{5,r}(w):=\begin{cases}\dfrac{2}{|w|(1-2\delta)\pi}, & \text{if } Re^{-\pi/2}\le|w|\le u_\varepsilon \text{ and } \pi\delta\le\arg w\le\dfrac{\pi}{2},\\ 0, & \text{elsewhere},\end{cases}$$

$$\varrho^{**}_{5,r}(w):=\begin{cases}\dfrac{e^{\pi/2}}{u_\varepsilon\cos\delta\pi}, & \text{if } w\in\Sigma_E \text{ and } 0<\operatorname{Im} w\le v_0\pi+u_\varepsilon\cos\delta\pi/e^{\pi/2},\\ 0, & \text{elsewhere},\end{cases}$$

$$\varrho_{5,r}(w):=\max\{\varrho^*_{5,r}(w),\varrho^{**}_{5,r}(w)\}.$$

By a straightforward calculation we obtain that for any $\gamma\in\Gamma'_{5,r}$,

$$\int_\gamma \varrho_{5,r}(w)\,|dw|\ge1.$$

Therefore,

$$\begin{aligned}m(\Gamma_{5,r})&\le\iint_{\mathbf{C}}\varrho_{5,r}(w)^2\,dm_w\le\iint_{\mathbf{C}}\varrho^*_{5,r}(w)^2\,dm_w+\iint_{\mathbf{C}}\varrho^{**}_{5,r}(w)^2\,dm_w\\ &=\frac{2}{\pi(1-2\delta)}\left(\log\frac{1}{R}+d_4\right),\end{aligned}\tag{5.13}$$

where

$$d_4=d_4(E,\varepsilon):=\log e^{\pi/2}u_\varepsilon+\frac{e^\pi}{u_\varepsilon^2\cos^2\delta\pi}\pi^2\left(v_0\pi+\frac{u_\varepsilon\cos\delta\pi}{e^{\pi/2}}\right)\frac{1-2\delta}{2}.$$

Comparing (5.12) and (5.13) we get

$$R\le e^{d_4}r^{(1-\varepsilon)/2},\tag{5.14}$$

and in view of (3.11) this proves (1.3) for E consisting of a finite number of intervals.

Let now E be an arbitrary compact set in the statement of Theorem 2, and let

$$[0,1]\supset E_1\supset E_2\supset\dots,$$

be a sequence of sets each of which consists of a finite number of intervals, $1\in E_n$, $E\subset E_n$,

$$\lim_{n\to\infty} g_{\overline{\mathbf{C}}\setminus E_n}(-r)=g_\Omega(-r),\quad r>0.\tag{5.15}$$

Note that $g_{\overline{\mathbf{C}}\setminus E_n}$ satisfies (5.14). Since

$$h_E(r)\ge h_{E_n}(r),$$

careful analysis of constants and (5.15) show that, as before, we have

$$g_\Omega(-r)\le e^{d_4}r^{(1-\varepsilon)/2},\quad 0<r<r_\varepsilon,\tag{5.16}$$

with d_4 and r_ε which depend only on ε and E. Thus, (1.3) follows from (5.16). □

6. Proof of Theorem 3

By [6, Chapter II, Theorem 2.2] (see also [1, Corollary 1]), for any compact set $E\subset[0,1]$ with positive capacity we have

$$(6.1)\qquad g_\Omega(-r)\le c_1\sqrt{r}\,\exp\left(c_2\int_r^1\frac{\theta_E(x)^2}{x^3}\,dx\right)\log\frac{2}{\operatorname{cap}(E)},\qquad 0<r<1,$$

where $\theta_E(x):=|[0,x]\setminus E|$, $0<x\le 1$, is the linear measure of $[0,x]\setminus E$.

In order to prove Theorem 3, it is enough to find two monotonic sequences of positive numbers $\{a_j\}_{j=1}^\infty$ and $\{b_j\}_{j=1}^\infty$ converging to 0 such that for the set

$$E:=\{0\}\cup\left(\bigcup_{j=1}^\infty[a_j,b_j]\right)$$

the following properties hold

$$(6.2)\qquad \lim_{r\to 0}\left(\log\frac{1}{r}\right)^{-1}\int_r^1\frac{\theta_E(x)^2}{x^3}\,dx=0,$$

$$(6.3)\qquad \lim_{j\to\infty}\frac{b_{j+1}}{a_j}=0.$$

Indeed, in this case (1.3) follows from (6.2) and (6.1). Moreover, since

$$\frac{\operatorname{cap}(E\cap[0,a_j])}{a_j}\le\frac{b_{j+1}}{4a_j},$$

(6.3) implies (1.4).

With the choice

$$b_j:=2^{-2^{j-1}},\quad a_j:=b_{j+1}\log(j+1),\qquad j\in\mathbf{N},$$

both properties (6.2) and (6.3) hold. □

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