Ark. Mat., **42** (2004), 325–334

Entire curves avoiding given sets in $\mathbf{C}^n$

Nikolai Nikolov and Peter Pflug

Let $F\subset\mathbf{C}^n$ be a proper closed subset of $\mathbf{C}^n$ and $A\subset\mathbf{C}^n\setminus F$ be at most countable, $n\geq 2$. The aim of this note is to discuss conditions for F and A, under which there exists a holomorphic immersion (or a proper holomorphic embedding) $\varphi\colon\mathbf{C}\to\mathbf{C}^n$ with $A\subset\varphi(\mathbf{C})\subset\mathbf{C}^n\setminus F$. Our main tool for constructing such mappings is Arakelian's approximation theorem (cf. [3] and [10]).

The first result is a generalization of the main part of Theorem 1 in [7]. More precisely, we prove the following result.

Proposition 1. *Let F be a proper convex closed set in $\mathbf{C}^n$, $n\geq 2$. Then the following statements are equivalent:*

(i) *either F is a complex hyperplane or it does not contain any complex hyperplane;*

(ii) *for any integer $k\geq 1$ and any two sets $\{\alpha_1,\dots,\alpha_k\}\subset\mathbf{C}$ and $\{a_1,\dots,a_k\}\subset\mathbf{C}^n\setminus F$, there exists a proper holomorphic embedding $\varphi\colon\mathbf{C}\to\mathbf{C}^n$ such that $\varphi(\alpha_j)=a_j$, $1\leq j\leq k$, and $\varphi(\mathbf{C})\subset\mathbf{C}^n\setminus F$.*

(iii) *the same as* (ii) *but for $k=2$.*

The equivalence of (i) and (iii) follows from the proof of Theorem 1 in [7]. For the convenience of the reader we repeat here the main idea of the proof of (iii) $\Rightarrow$ (i). Observe that condition (iii) implies that the Lempert function of the domain $D:=\mathbf{C}^n\setminus F$ is identically zero, i.e.

$$\tilde{k}_D(z,w):=\inf\{\alpha\geq 0:\text{there is } f\in\mathcal{O}(\Delta,D) \text{ with } f(0)=z \text{ and } f(\alpha)=w\}=0,$$

$z,w\in D$, where Δ denotes the open unit disc in $\mathbf{C}$. In the case when condition (i) is not satisfied we may assume (after a biholomorphic mapping) that $F=A\times\mathbf{C}^{n-1}$, where the closed convex set A, properly contained in $\mathbf{C}$, contains at least two points. Applying standard properties of $\tilde{k}$, we have $\tilde{k}_D(z,w)=\tilde{k}_{\mathbf{C}\setminus A}(z',w')$, where $(z,w)=((z',z''),(w',w''))\in D$. Since $\tilde{k}_{\mathbf{C}\setminus A}$ is not identically zero we end up with a contradiction.

Hence, we only have to prove the implication (i) $\Rightarrow$ (ii).

Proof. For simplicity of notation we shall consider only the case $n=2$.

If F is a complex line, we may assume that $F=\{z\in\mathbf{C}^2:z_2=0\}$. Considering an automorphism of the form $(z_1,z_2)\mapsto(z_1e^{\gamma z_1z_2},z_2e^{-\gamma z_1z_2})$ for a suitable constant γ, we may also assume that the second coordinates of the given points are pairwise different. Then there exist two one-variable polynomials P and Q such that the mapping $t\mapsto(t+P(e^{Q(t)}),e^{Q(t)})$ has the required property.

Assume now that F does not contain any complex line. The idea below comes from that of Theorem 8.5 in [9].

First, we shall prove by induction that for any $j\leq k$ there is an automorphism Φ_j such that the set $\operatorname{co}(\Phi_j(F))$ does not contain any complex line and it does not have a common point with the set

$$\operatorname{co}(G_j)\cup\{\Phi_j(a_{j+1}),\dots,\Phi_j(a_k)\},$$

where $G_j:=\{\Phi_j(a_1),\dots,\Phi_j(a_j)\}$ ($\operatorname{co}(M)$ denotes the convex hull of a closed set M in $\mathbf{C}^n$). Doing the induction step, we may assume that $\Phi_j=\mathrm{Id}$. Then, since F is convex and does not contain any complex line, after an affine change of coordinates one has that (cf. [2] and [7])

$$\begin{aligned} F\subset H&:=\{z\in\mathbf{C}^2:\operatorname{Re}z_1\leq-1 \text{ and } \operatorname{Re}z_2\leq-1\},\\ \operatorname{co}(G_j)&\subset\{z\in\mathbf{C}^2:\operatorname{Re}z_1\geq0\},\\ a_{j+1}&\in\{z\in\mathbf{C}^2:\operatorname{Re}z_2\geq0\}.\end{aligned}$$

In addition, we may assume that the set $A:=\{a_1,\dots,a_k\}$ of the given points and the strip $\{z\in\mathbf{C}^2:-1<\operatorname{Re}z_2<0\}$ do not have a common point. By Arakelian's theorem (cf. [3]), for $\varepsilon:=\min\{1,\operatorname{dist}(F,A)\}$ we may find an entire function f such that

$$\begin{aligned} |f(t)-a_{j+1,1}|&<\tfrac12\varepsilon, \quad \text{if } \operatorname{Re}t\leq-1,\\ |f(t)|&<\tfrac12\varepsilon, \quad \text{if } \operatorname{Re}t\geq0\end{aligned}$$

and, in addition, $f(a_{j+1,2})=0$ (here, $a_{j+1,k}$ denotes the kth coordinate of the point a_{j+1}). Then it is easy to see that the automorphism

$$\Phi_{j+1}(z_1,z_2):=(z_1+f(z_2),z_2)$$

has the required properties.

So, let F be a convex set, which does not contain any complex line and $F\cap\operatorname{co}(A)=\emptyset$. Then we may assume that (cf. [2] and [7]) $F\subset H$, $A\subset\{z\in\mathbf{C}^2:\operatorname{Re}z_1\geq1$ and $\operatorname{Re}z_2\geq0\}$, and, in addition, that $\operatorname{Re}\alpha_j\geq1$, $1\leq j\leq k$.

Note that there exists an entire function g such that $|g(t)|\leq 1$ if $\operatorname{Re} t\leq -1$ and $g(a_{j,2})=\alpha_j-a_{j,1}$ (cf. [3] and [11]; this can be proved also directly, applying a standard interpolation process and Arakelian's theorem many times). Then, applying the automorphism $(z_1,z_2)\mapsto(z_1+g(z_2),z_2)$, we may assume that $a_{j,1}=\alpha_j$ and $F\subset\{z\in\mathbf{C}^2:\operatorname{Re} z_1\leq 0 \text{ and } \operatorname{Re} z_2\leq -1\}$. Finally, we find, as above, an entire function h such that $|h(t)|<1$ on the set $\operatorname{Re} t\leq 0$ and $h(\alpha_j)=a_{j,2}$. Hence, the mapping $t\mapsto(t,h(t))$ has the required properties (in the new coordinates). $\square$

The end of the proof shows that we may also prescribe values of finitely many derivatives of φ at the points of the given planar set.

Open problem. Is it true for an F as in (i) of Proposition 1, that for any discrete set of points in $\mathbf{C}^n\setminus F$ there exists a proper holomorphic embedding of $\mathbf{C}$ into $\mathbf{C}^n$ avoiding F and passing through any of these points?

It is known that for any discrete set of points in $\mathbf{C}^n$ there exists a proper holomorphic embedding of $\mathbf{C}$ into $\mathbf{C}^n$ passing through any of the points of this set (Proposition 2 in [5]; cf. also Theorem 1 in [11] for $n\geq 3$). We have not been able to modify the proofs of [5] and [11] to get a positive answer to the above question in the general case. Nevertheless, the following result gives a positive answer to the open problem in the case when F is a complex hyperplane.

Proposition 2. *If F is a union of at most $n-1$ $\mathbf{C}$-linearly independent complex hyperplanes in $\mathbf{C}^n$, then for any discrete set of points in $\mathbf{C}^n\setminus F$ there exists a proper holomorphic embedding of $\mathbf{C}$ into $\mathbf{C}^n$ avoiding F and passing through any of these points.*

The proof of Proposition 2 will be a modification of the one in the case when F is the empty set (see Proposition 2 in [5]).

The key point is the following lemma.

Lemma 3. *Let K be a polynomially convex compact set in $\mathbf{C}^n$, A a set of finitely many points in K, and H a union of at most $n-1$ $\mathbf{C}$-linearly independent complex hyperplanes in $\mathbf{C}^n$. For every $p,q\in\mathbf{C}^n\setminus(K\cup H)$ and every $\varepsilon>0$, there exists an automorphism φ of $\mathbf{C}^n$ such that $\varphi(z)=z$, $z\in H\cap A$, $\varphi(p)=q$, and $|\varphi(z)-z|\leq\varepsilon$, $z\in K$.*

In view of Lemma 3, Proposition 2 follows by repeating step by step the proof of Proposition 2 in [5]. Starting with an embedding α_0 whose graph avoids H, the desired embedding α is constructed as the limit of a sequence of embeddings α_j with $\alpha_j=\varphi_j\circ\alpha_{j-1}$, $j\geq 1$, where the φ_j are automorphisms chosen by Lemma 3. Note that the graph of α avoids H by the Hurwitz theorem.

Proof of Lemma 3. After a linear change of coordinates, we may assume that $H\subset\{z\in\mathbf{C}^n:z_1\dots z_n=0\}$ and that all the coordinates of the points in $B:=A\cup\{q\}\setminus H$ are non-zero. Applying an overshear of the form

$$w_1=z_1\exp(f(z_2,\dots,z_n)),\quad w_2=z_2,\dots,w_n=z_n,$$

where

$$f(z_2,\dots,z_n):=z_2\dots z_n\left(\varepsilon+\sum_{j=2}^{n}\varepsilon^j z_j\right)$$

and ε is small enough, provides pairwise different products of the first $n-1$ coordinates of the points in B. Repeating this argument, we may assume the same for every $n-1$ coordinates.

Now, we need the following variation of Theorem 2.1 in [6].

Lemma 4. *Let H be the union of at most $n-1$ $\mathbf{C}$-linearly independent complex hyperplanes in $\mathbf{C}^n$, D an open set in $\mathbf{C}^n$, and $K\subset D$ a compact set. Let $\Phi_t:D\to\mathbf{C}^n$, $t\in[0,1]$, be a C^2-smooth isotopy of biholomorphic maps which fix $D\cap H$ pointwise such that $\Phi_t(D\cap H)=\Phi_t(D)\cap H$. Suppose that Φ_0 is the identity map and the set $\Phi_t(K)$ is polynomially convex for every $t\in[0,1]$.*

Then Φ_1 can be approximated, uniformly on K, by automorphisms of $\mathbf{C}^n$, which fix H pointwise.

For a moment, we may assume that Lemma 4 is true. Let $\gamma:[0,1]\to\mathbf{C}^n\setminus(K\cup H)$ be a C^2-smooth path, $\gamma(0)=p$, $\gamma(1)=q$. Then we apply Lemma 4 to the following situation: Take $\Phi_t(z)$ to be z near K and to be $z+\gamma(t)-p$ near p, and choose a sufficiently small neighborhood D of the polynomially convex set $K\cup\{p\}$. For a sufficiently small $\varepsilon>0$, denote by ψ the corresponding automorphism and set $\tilde r:=\psi(r)$ for $r\in B$. Let f_1 be the Lagrange interpolation polynomial with

$$f_1(\tilde r_2\dots\tilde r_n)=\frac{1}{\tilde r_2\dots\tilde r_n}\log\frac{r_1}{\tilde r_1}$$

for every $r\in B$. Note that the overshear

$$\psi_1(z):=(z_1\exp(z_2\dots z_n f_1(z_2\dots z_n)),z_2,\dots,z_n)$$

sends $\tilde r$ to the point $(r_1,\tilde r_2,\dots,\tilde r_n)$. It is left to define $\psi_2,\dots,\psi_n$ in a similar way and to consider the composition $\psi_n\circ\dots\circ\psi_1\circ\psi$. This completes the proof of Lemma 3. □

Proof of Lemma 4. Note that under the assumptions of Lemma 4, there exists a neighborhood $U\subset D$ of K such that $U_t:=\Phi_t(U)$ is Runge for each $t\in[0,1]$

(Lemma 2.2 in [6]). We shall follow the proofs of Theorem 1.1 in [6] and Theorem 2.5 in [13]. Consider the vector field $X_t := (d/dt)\Phi_t \circ \Phi_t^{-1}$ defined on U_t. For a sufficiently large positive integer N and $0 \le j \le N-1$ set

$$X_{j,t} := \begin{cases} 0, & t \notin [j/N, (j+1)/N], \\ X_{j/N}, & t \in [j/N, (j+1)/N]. \end{cases}$$

Note that $X_{j/N}$ vanishes on $U_{j/N} \cap H$. It is easy to see that it can be approximated by holomorphic vector fields on $\mathbf{C}^n$ which vanish on H, since $U_{j/N}$ is Runge (here and below, the approximations are locally uniformly). On the other hand, these vector fields can be approximated by Lie combinations of complete vector fields vanishing on H (Proposition 5.13 in [13]). Thus we may assume that $X_{j/N}$ is a Lie combination of complete vector fields vanishing on H. Note that the local flow of $\sum_{j=0}^{N-1} X_{j,t}$ at time 1 is $h_{N-1} \circ \dots \circ h_0$, where h_j is the local flow of $X_{j/N}$ at time $1/N$. If $N \to \infty$, then this composition converges to the time one map Φ_1 of the flow of X_t. To finish the proof of Lemma 4, it is enough to note that every h_j can be approximated by finite compositions of automorphisms of $\mathbf{C}^n$ which fix H (cf. the proof of Theorem 2.5 in [13]). $\square$

In this way Proposition 2 is completely proved.

Remark. It is an open question whether every holomorphic vector field in $\mathbf{C}^n$, which vanishes on the set $L := \{z \in \mathbf{C}^n : z_1 \dots z_n = 0\}$, can be locally uniformly approximated by Lie combinations of complete vector fields vanishing on L [13]. If this would be so, then the above proof shows that Proposition 2 is also true for every union of $\mathbf{C}$-linearly independent complex hyperplanes in $\mathbf{C}^n$, $n \ge 3$. To see this, choose, for example, the starting embedding

$$\alpha_0(\eta) := \left(\exp(-\eta^2), \exp\left(-\eta\sqrt{2}\,\right), \exp(\eta), \dots, \exp(\eta)\right).$$

It remains an unsolved problem (for us) if there exists a proper holomorphic embedding of $\mathbf{C}$ into $\mathbf{C}^2$ whose graph avoids both coordinate axes.

We are also able to answer the open problem, posed after Proposition 1, in the bounded case.

Proposition 5. *If K is a polynomially convex compact set in $\mathbf{C}^n$, then for any discrete set C of points in $\mathbf{C}^n \setminus K$ there exists a proper holomorphic embedding H of $\mathbf{C}$ into $\mathbf{C}^n$ avoiding K and passing through any of these points. In addition, for a given point $c \in C$ and $X \in \mathbf{C}^n \setminus \{0\}$ we can choose H such that $H'(H^{-1}(c)) = X$. In particular, the Lempert function and the Kobayashi pseudometric of $\mathbf{C}^n \setminus K$ vanish.*

Proof. The proof is a modification of the proof of Proposition 2 in [5].

We may assume that $X=(1,0,\dots,0)$ and that K does not intersect the first coordinate axis. Note that there exists a smooth non-negative plurisubharmonic exhaustion function φ on $\mathbf{C}^n$ that is strongly plurisubharmonic on $\mathbf{C}^n\setminus K$ and vanishes precisely on K (cf. [1]). For any $\varepsilon>0$, put

$$G_\varepsilon:=\{z\in\mathbf{C}^n:\phi(z)<\varepsilon\}\quad\text{and}\quad K_\varepsilon:=\{z\in\mathbf{C}^n:\phi(z)\le\varepsilon\}.$$

In particular, K_ε is polynomially convex. By Sard's theorem we may choose a strictly decreasing sequence $(\varepsilon_j)_{j\ge0}$, bounded from below by a positive constant, such that the boundary of $G_j:=G_{\varepsilon_j}$ is smooth for any j and $K_0:=K_{\varepsilon_0}$ does not intersect the first coordinate axis. In particular, $K_j:=K_{\varepsilon_j}$ has finitely many connected components.

Claim. The inclusion $K_j\subset\psi_j(K_{j-1})$ holds for any automorphism ψ_j of $\mathbf{C}^n$ which is close enough to the identity map on K_{j-1}.

Let now $C=(\alpha_l)_{l\ge1}$ with $\alpha_1=c$. Set $H_0(\zeta)=(\zeta,0,\dots,0)$ and $\varrho_0=0$. In view of the claim and the proof of Proposition 2 in [5], for any $j\ge1$ we may find, by induction, numbers $\varrho_j\ge\varrho_{j-1}+1$, $\zeta_j\in\mathbf{C}$, and an automorphism ψ_j such that for $H_j=\psi_j\circ H_{j-1}$ one has:

(a) $H_j'(\zeta_1)=X$ and $H_j(\zeta_l)=\alpha_l$, $1\le l\le j$;

(b) $|H_j(\zeta)|>|\alpha_j|-1$ if $|\zeta|\ge\varrho_j$ and $K_j\subset\{z\in\mathbf{C}^n:|z|\le|\alpha_j|-\frac12\}$;

(c) $|H_j(\zeta)-H_{j-1}(\zeta)|\le\delta_j\le2^{-j}$ if $|\zeta|\le\varrho_j$;

(d) $H_j(\mathbf{C})\cap K_j=\emptyset$.

It is easy to check that the limit map $H:=\lim_{j\to\infty}H_j$ exists and that it has the required properties except properness. The last one can be provided by the choice of δ_j. Note that the only modifications that have to be made in the proof of Proposition 2 in [5] are the choice of the ψ_j with the additional property that $\psi_j'(\zeta_1)$ is the identity matrix and the replacing of the set

$$F:=\{z\in\mathbf{C}^n:|z|\le|\alpha_j|-\tfrac12\}\cup H_{j-1}\{\zeta\in\mathbf{C}:|\zeta|\le\varrho\}$$

by the set

$$F:=K_j\cup H_{j-1}\{\zeta\in\mathbf{C}:|\zeta|\le\varrho\}$$

if

$$K_j\not\subset\{z\in\mathbf{C}^n:|z|\le|\alpha_j|-\tfrac12\}.$$

Proof of the claim. Since K_j has finitely many connected components $K_{j,1},\dots,K_{j,m}$, we have that $\operatorname{dist}(K_j,\partial K_{j-1})>0$. Then we find $r>0$ with $\operatorname{dist}(K_j,\partial K_{j-1})>r$

and some ball B_l with radius r belonging to $K_{j,l}$, $1 \le l \le m$. It follows that $K_j \subset \psi_j(K_{j-1})$ if

$$\max\{|\psi_j(z)-z| : z \in K_{j-1}\} \le r.$$

Indeed, suppose the contrary, i.e., $\psi_j(a) \in K_j$ for some $a \notin K_{j-1}$. We may assume that $\psi_j(a) \in K_{j,1}$. Denote by b_1 the image of the center of B_1 under ψ_j. Then there exists a path γ in $K_{j,1}$ joining $\psi_j(a)$ and b_1. Note that $\psi_j^{-1}(\gamma) \cap \partial K_{j-1} \neq \emptyset$. If $c \in \psi_j^{-1}(\gamma) \cap \partial K_{j-1}$, then $\psi(c) \in K_j$. Hence $r \ge |\psi_j(c)-c| \ge \operatorname{dist}(K_j, \partial K_{j-1})$; a contradiction. $\square$

Note that if F is a proper subset in $\mathbf{C}^2$ such that for any point in $\mathbf{C}^2 \setminus F$ there exists a non-constant entire curve $\gamma: \mathbf{C} \to \mathbf{C}^2 \setminus F$ which passes through this point, then the interior of F is pseudoconvex, since $\mathbf{C}^2 \setminus \overline{\gamma(\mathbf{C})}$ is pseudoconvex [12]. Moreover, if F is compact and for any point $a \in \mathbf{C}^2 \setminus F$ there exists a proper holomorphic mapping $\varphi: \mathbf{C} \to \mathbf{C}^2$ with $a \in \varphi(\mathbf{C}) \subset \mathbf{C}^2 \setminus F$, then F is rational convex [4]. The same does not hold in higher dimensions. For example, if F and G are two proper closed subsets of $\mathbf{C}^k$ and $\mathbf{C}^l$, respectively, then for any point in $\mathbf{C}^{k+l} \setminus (F \times G)$ there exists a proper holomorphic embedding of $\mathbf{C}$ into $\mathbf{C}^{k+l}$ avoiding $F \times G$ and passing through this point.

The next proposition is in the spirit of the above remark and it generalizes Proposition 1 in [8].

Proposition 6. *If F and G are two sets in $\mathbf{C}^k$ and $\mathbf{C}^l$, respectively, then for any countable set C of points in $\mathbf{C}^{k+l}$ with $\operatorname{dist}(C, F \times G) > 0$, there exists a holomorphic immersion of $\mathbf{C}$ into $\mathbf{C}^{k+l}$ avoiding $F \times G$ and passing through any point of C.*

Proof. The idea for the proof comes from the proof of Theorem 2 in [11].

For any point c in $\mathbf{C}^{k+l}$ denote by c' and c'' its projections onto $\mathbf{C}^k$ and $\mathbf{C}^l$, respectively. Set $\varepsilon := \operatorname{dist}(C, F \times G) > 0$, $C' := \{c \in C : \operatorname{dist}(c', \mathbf{C}^k \setminus F) \ge \varepsilon\}$ and $C'' := C \setminus C'$. We may assume that both sets are infinite and enumerate them, i.e. $C' = (a_j)_{j \ge 0}$ and $C'' = (b_j)_{j \ge 0}$. Denote by $\mathbf{D}_n(c, r)$ the polydisc in $\mathbf{C}^n$ with center at c and radius r. Note that $\mathbf{D}_k(a_j', \varepsilon) \subset \mathbf{C}^k \setminus F$ and $\mathbf{D}_l(b_j'', \varepsilon) \subset \mathbf{C}^l \setminus G$ for any $j \ge 0$. Define

$$A_j := \{z \in \mathbf{C} : \operatorname{Re} z \le -3 \text{ and } |\operatorname{Im} z - 7j| \le 3\}, \qquad j \ge 1,$$

$$A_0 := \{z \in \mathbf{C} : \operatorname{Re} z \ge -1\} \setminus \bigcup_{j=1}^{\infty} \{z \in \mathbf{C} : \operatorname{Re} z > 5 \text{ and } |\operatorname{Im} z - 7j| < 1\}.$$

Choose a number $t \in (0,1)$ such that

$$t \exp\left(\sqrt[3]{x} - \sqrt[4]{x+2}\right) \ge 4e(1-t)\sqrt[3]{(x+2)^4}, \quad x \ge 0.$$

For $1\le m\le k$, combining the extensions of Arakelian's theorem in [3] and [11] gives an entire function f_m such that

$$\begin{gathered}\bigl|f_m(z)-a_{0,m}-\tfrac{1}{2}\varepsilon t\exp\bigl(-\sqrt[4]{z+2}\,\bigr)\bigr|<\tfrac{1}{2}\varepsilon(1-t)\exp\bigl(-\sqrt[3]{|z|}\,\bigr),\quad z\in A_0,\\ \bigl|f_m(z)-a_{j,m}-\tfrac{1}{2}\varepsilon t\exp\bigl(-\sqrt[4]{-z-2}\,\bigr)\bigr|<\tfrac{1}{2}\varepsilon(1-t)\exp\bigl(-\sqrt[3]{|z|}\,\bigr),\quad z\in A_j,\ j\ge 1,\end{gathered}$$

$$f_m(2)=a_{0,m},\quad f_m(-2)=b_{0,m},\quad f_m(-7+i7j)=a_{j,m},\quad f_m(7+i7j)=b_{j,m}$$

for $j\ge1$ ($\sqrt[4]{z}$ is the branch with $\sqrt[4]{1}=1$ and $c_{j,m}$ denotes the mth coordinate of the point c_j). Note that $|f_m(z)-a_{j,m}|<\varepsilon$ if $z\in A_j$. For $k+1\le m\le k+l$ we choose analogously an entire function f_m such that

$$\begin{gathered}\bigl|f_m(z)-b_{0,m}-\tfrac{1}{2}\varepsilon t\exp\bigl(-\sqrt[4]{-z-2}\,\bigr)\bigr|<\tfrac{1}{2}\varepsilon(1-t)\exp\bigl(-\sqrt[3]{|z|}\,\bigr),\quad -z\in A_0,\\ \bigl|f_m(z)-b_{j,m}-\tfrac{1}{2}\varepsilon t\exp\bigl(-\sqrt[4]{z+2}\,\bigr)\bigr|<\tfrac{1}{2}\varepsilon(1-t)\exp\bigl(-\sqrt[3]{|z|}\,\bigr),\quad -z\in A_j,\ j\ge 1,\end{gathered}$$

$$f_m(2)=a_{0,m},\quad f_m(-2)=b_{0,m},\quad f_m(-7+i7j)=a_{j,m},\quad f_m(7+i7j)=b_{j,m}$$

for $j\ge1$. Then the mapping $(f_1,\dots,f_{k+l})$ will have the required properties if it is non-singular. To see this, note that applying the triangle inequality and the Cauchy inequality gives

$$\biggl|\frac{\varepsilon t}{8\sqrt[3]{|z+2|^4}}\exp\bigl(-\sqrt[4]{|z+2|}\,\bigr)\biggr|-|f_m'(z)|<\frac{\varepsilon}{2}(1-t)\exp\bigl(1-\sqrt[3]{|z|}\,\bigr)$$

for $1\le m\le k$ and

$$z\in E_0:=\{z\in\mathbf{C}:\operatorname{Re}z\ge0\}\setminus\bigcup_{j=1}^{\infty}\{z\in\mathbf{C}:\operatorname{Re}z>4\text{ and }|\operatorname{Im}z-7j|<2\}.$$

Then the choice of t shows that $f_m'(z)\ne0$ if $1\le m\le k$ and $z\in E_0$; a similar argument gives that $f_m'(z)\ne0$ if

$$z\in E:=\bigcup_{j=1}^{\infty}\{z\in\mathbf{C}:\operatorname{Re}z\le-4\text{ and }|\operatorname{Im}z-7j|\le2\}.$$

We analogously obtain that $f_m'(z)\ne0$ if $k+1\le m\le k+l$ and $-z\in E_0\cup E$, which implies that the mapping is non-singular. $\square$

Note that, in general, the mapping in Proposition 6 cannot be chosen to be proper. For example, let $F:=\mathbf{C}\setminus\mathbf{D}_1(0,1)$ and let $f:=(f_1,f_2)$ be a proper holomorphic map of $\mathbf{C}$ into $\mathbf{C}^2$ which avoids $F\times F$. Choose R such that $\max\{|f_1(z)|,|f_2(z)|\}$

≥ 2 for $|z|>R$. Assume that f_1 is not a polynomial. Then by Picard's theorem there is a point $a\in\mathbf{C}$, $|a|>R$, with $|f_1(a)|=1$. Thus $|f_2(a)|\geq 2$. On the other hand, using that $f(\mathbf{C})\cap(F\times F)=\emptyset$ implies that $|f_1(a)|<1$, a contradiction. In conclusion, one of the functions f_1 and f_2 is a polynomial and the other one is a constant smaller than 1.

It follows from Proposition 6 that if F and G are two closed proper subsets of $\mathbf{C}^k$ and $\mathbf{C}^l$, respectively, then the Lempert function of $\mathbf{C}^{k+l}\setminus(F\times G)$ vanishes. The next proposition implies that the same holds for the Kobayashi pseudometric.

Proposition 7. *If F and G are two proper closed sets in $\mathbf{C}^k$ and $\mathbf{C}^l$, respectively, then for any point $c\in\mathbf{C}^{k+l}\setminus(F\times G)$ and any vector $X\in\mathbf{C}^{k+l}$ there exists a holomorphic mapping of $\mathbf{C}$ into $\mathbf{C}^{k+l}\setminus(F\times G)$ with $f(0)=c$ and $f'(0)=X$.*

Proof. We may assume that $c'\in\mathbf{C}^k\setminus F$ and $\mathbf{D}_l(0,1)\subset\mathbf{C}^l\setminus G$. The statement is trivial if $X'=0$. Otherwise, we may assume $c'=0$ and the ball in $\mathbf{C}^k$ with center at the origin and radius $(e+1)\sqrt{k}$ belongs to $\mathbf{C}^k\setminus F$. After a unitary transformation of $\mathbf{C}^k$ we may also assume that $X'=(r,\dots,r)$ for some $r>0$. Note that $\mathbf{D}_k(0,e+1)\subset\mathbf{C}^k\setminus F$ and if $|e^{rz}-1|\geq e+1$, then $\operatorname{Re} z\geq 1/r$. By Arakelian's theorem, there exists an entire function f_m such that $f_m(0)=0$, $f'_m(0)=X_m$, and $|f_m(z)|<1$ if $\operatorname{Re} z\geq 1/r$, $k+1\leq m\leq k+l$. Setting $f_m(z):=e^{rz}-1$ for $1\leq m\leq k$ implies that the mapping $(f_1,\dots,f_{k+l})$ has the required properties. □

Acknowledgments. A part of this note was prepared during the stay of the first author at the University of Oldenburg (January, 2003), supported by DFG, and during the stays of both authors at the Jagiellonian University in Kraków (February, 2003) with support for the second author by DFG. We like to thank these institutions.

Proposition 2 was obtained after the first version of this note was accepted for publication. We would like to thank our colleagues Franc Forstnerič and Josip Globevnik for extremely helpful discussions about this proposition.

The present article is an extended version of the originally submitted manuscript with the same title. It contains also results from a second manuscript entitled *Entire curves avoiding hyperplanes in $\mathbf{C}^n$*, which was submitted to *Arkiv för matematik* on November 3, 2003.

References

1. Catlin, C., *Boundary Behavior of Holomorphic Functions on Weakly Pseudoconvex Domains*, Ph.D. Dissertation, Princeton University, Princeton, NJ, 1978.
2. Drinovec Drnovšek, B., Proper holomorphic discs avoiding closed convex sets, *Math. Z.* **241** (2002), 593–596.

3. Gauthier, P. M. and Pouryayevali, M. R., Approximation by meromorphic functions with Mittag-Leffler type constraints, *Canad. Math. Bull.* **44** (2001), 420–428.
4. Gunning, R. C. and Rossi, H., *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, NJ, 1965.
5. Forstnerič, F., Globevnik, J. and Rosay, J.-P., Nonstraightenable complex lines in $\mathbf{C}^2$, *Ark. Mat.* **34** (1996), 97–101.
6. Forstnerič, F. and Rosay, J.-P., Approximation of biholomorphic mappings by automorphisms of $\mathbf{C}^n$, *Invent. Math.* **112** (1993), 323–349. Erratum in *Invent. Math.* **118** (1994), 573–574.
7. Jarnicki, W. and Nikolov, N., Concave domains with trivial biholomorphic invariants, *Ann. Polon. Math.* **79** (2002), 63–66.
8. Nikolov, N., Entire curves in complements of Cartesian products in $\mathbf{C}^n$, *Univ. Iagel. Acta Math.* **40** (2002), 13–14.
9. Rosay, J.-P. and Rudin, W., Holomorphic maps from $\mathbf{C}^n$ into $\mathbf{C}^n$, *Trans. Amer. Math. Soc.* **310** (1988), 47–86.
10. Rosay, J.-P. and Rudin, W., Arakelian's approximation theorem, *Amer. Math. Monthly* **96** (1989), 432–434.
11. Rosay, J.-P. and Rudin, W., Holomorphic embeddings of $\mathbf{C}$ in $\mathbf{C}^n$, in *Several Complex Variables (Stockholm, 1987/88)*, Math. Notes **38**, pp. 563–569, Princeton Univ. Press, Princeton, NJ, 1993.
12. Ueda, T., Local structure of analytic transformations of two complex variables, I, *J. Math. Kyoto Univ.* **26** (1986), 233–261.
13. Varolin, D., The density property for complex manifolds and geometric structures, *J. Geom. Anal.* **11** (2001), 135–160.

Received March 11, 2003
in revised form December 14, 2003

Nikolai Nikolov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
BG-1113 Sofia
Bulgaria
email: nik@math.bas.bg

Peter Pflug
Carl von Ossietzky Universität Oldenburg
Fakultät V
Institut für Mathematik
Postfach 2503
DE-26111 Oldenburg
Germany
email: pflug@mathematik.uni-oldenburg.de