Ark. Mat., **41** (2003), 307–339

Wiener regularity for large solutions of nonlinear equations

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1. Introduction

This paper concerns *large solutions* to nonlinear elliptic equations in arbitrary bounded domains $\Omega\subset\mathbf{R}^n$, $n\geq 3$. These are solutions $u\in C^2_{\mathrm{loc}}(\Omega)$ to the nonlinear problem

$$\begin{cases} \Delta u-|u|^{q-1}u=0 & \text{in } \Omega, \\ u(x)\to+\infty, & \text{when } x\to\partial\Omega. \end{cases} \tag{1.1}$$

For the parameter q we always assume in this paper that

$$q>1. \tag{1.2}$$

Equation (1.1) is the model equation for a broad class of semilinear elliptic problems admitting comparison principle. Apart from the importance for partial differential equations, interest in large solutions in general domains comes from two different sources: the theory of spatial branching processes and conformal differential geometry. Of the two basic questions concerning problem (1.1) in arbitrary domains Ω—namely, existence and uniqueness—our main result completely resolves the first. Theorem 1.1 states that the solubility of (1.1) is equivalent to a Wiener-type test with respect to a certain capacity. As to the second question, it is well known that uniqueness for (1.1) fails in general domains [39], [14], [29]. Note that the strong maximum principle for elliptic equations implies that u from (1.1) satisfies

$$u>0, \quad \Delta u-u^q=0 \text{ in } \Omega. \tag{1.3}$$

Hence without loss of generality we need to consider only positive solutions of (1.1).

After the ground-breaking papers by Perkins [67], Dynkin [19], and Le Gall [45], solutions of (1.1) and (1.3) attracted a lot of attention from probabilists. Currently this is a very active area of research on the interface between the theory of random processes, nonlinear partial differential equations, and analysis. We refer to the ICM reports by Perkins [68] and Le Gall [48] for a survey of the progress in the field and bibliography, see also [46]. Recent monographs [21], [20], [24], and [49], are dedicated to different aspects of the theory. At the moment the probabilistic methods are limited to the case

$$1<q\le 2 \tag{1.4}$$

in (1.1) and (1.3); see [23] on this issue. Our paper was inspired by a result of Dhersin and Le Gall [17]. They proved that the existence of a solution to problem (1.1) for

$$q=2 \tag{1.5}$$

is equivalent to a Wiener-type criterion for $\Omega^c=\mathbf{R}^n\setminus\Omega$. This result is one of the milestones of the theory, see [48] and [49]. The crucial idea of Dhersin and Le Gall was to combine classical potential theory with sharp bounds on the hitting probability for the super-Brownian motion associated with positive solutions of

$$\Delta u-u^2=0. \tag{1.6}$$

Further probabilistic treatment of related problems for equation (1.6) and its parabolic counterpart can be found in [18], [16] and [15]. An open problem in this area was to extend the result from (1.5) to the full range (1.2); see, for example, [49]. Relying entirely upon analytic ideas, the present paper proves the Wiener test for solubility of (1.1) for all $q>1$. This approach also finds applications in conformal geometry; see Remark 1.2(i).

Large solutions (1.1) were initially studied by Loewner and Nirenberg [52], as well as in the earlier papers of Keller [37] and Osserman [66]. Loewner and Nirenberg considered the case

$$q=\frac{n+2}{n-2} \tag{1.7}$$

that arises in conformal differential geometry. They proved that in smooth domains Ω problem (1.1) has a unique solution. Later the questions of existence, uniqueness, and the rate of the boundary blow-up were investigated by many authors. The bibliography for the subject is very extensive [76]. For example, Brezis and Véron [11] proved that singletons are regular boundary points for (1.1) if and only if

$$1<q<\frac{n}{n-2}. \tag{1.8}$$

Aviles, Bandle, Essén, Finn, Marcus, McOwen, Véron, and others investigated the questions for domains bounded by nonsmooth hypersurfaces or manifolds of lower dimensions, as well as for more general semilinear equations. In particular, Marcus and Véron [54] found sharp asymptotics for solutions of (1.1) near conical and cuspoidal boundary points. Kondratiev and Nikishkin [39] discovered the nonuniqueness for (1.1); see also [14] and [29]. We refer to the survey [65] and the monograph [76], for further description and references. Additionally, papers [54], [28], [58], and [77] contain very recent results. However, up to this point, the analytic approach has not given necessary and sufficient conditions for existence in (1.1).

The capacity appropriate to problem (1.1) is defined as follows. Fix $x_0 \in \mathbf{R}^n$, $n \geq 3$. Let $K \subset \mathbf{R}^n$ be a compact subset of the ball $B\left(x_0, \frac{3}{2}\right)$. For $1<p<\infty$ define

$$(1.9) \qquad \mathcal{C}_p(K) = \inf\left\{ \int_{B(x_0,2)} |D^2\varphi|^p : \varphi \in C_0^\infty(B(x_0,2)),\ \varphi|_K \geq 1 \right\}.$$

Following the axiomatic potential theory approach [10], [12] and [35], we extend $\mathcal{C}_p$ to an outer capacity on the collection of sets E such that $E \Subset B\left(x_0, \frac{3}{2}\right)$. Capacities defined with different x_0 are equivalent; see Section 2 for further explanations.

The capacity $\mathcal{C}_p$ is essentially the Bessel capacity associated with the Sobolev space $W^{2,p}(\mathbf{R}^n)$. Such capacities have been carefully investigated in the theory of nonlinear potentials. The theory originates in early works by Maz′ya and Serrin in the 1960s, and was later developed during the 1970s and 1980s in papers by Adams, Fuglede, Havin, Hedberg, Maz′ya, Meyers, and many others. We will use it extensively. The main references will be the monographs [5], [61], and [80], wherein the reader can also find a rich bibliography as well as ample historical notes. Now we state the main result of this paper.

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$, be a bounded domain, and let $q>1$. The following statements are equivalent*:

(i) *Problem* (1.1) *has a solution $u \in C^2_{\mathrm{loc}}(\Omega)$.*

(ii) *The set $\Omega^c = \mathbf{R}^n \setminus \Omega$ is not thin, that is,*

$$(1.10) \qquad \int_0^1 \frac{\mathcal{C}_{q'}(\Omega^c \cap B(x,r))}{r^{n-2}} \frac{dr}{r} = +\infty \quad \text{for all } x \in \Omega^c,$$

where

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

In Remark 1.2(iii) we sketch how (1.10) and well-known properties of the capacity imply the solubility of (1.1) for specific classes of domains Ω.

For $q=2$, Theorem 1.1 was proved in [17] using probabilistic methods. More precisely, Dhersin and Le Gall proved a stronger theorem stating that the existence of a solution blowing up at a point $x_0 \in \partial\Omega$ is equivalent to the Wiener criterion being satisfied at x_0. It is very likely that the proof from [17] can be generalised to the case (1.4) using ideas from [22] and [23]. All our estimates in the proof of Theorem 1.1 are local. Thus, in a manner similar to [17], we in fact establish the stronger statement that the existence of u solving (1.3) and blowing up at a boundary point is equivalent to the Wiener test (1.10) at that point.

Capacity (1.9) has the following property:

$$\mathcal{C}_p(\{x\}) > 0 \quad \text{for } p > \tfrac{1}{2}n; \tag{1.11}$$

see, for example, [61], Chapter 7, and [5], Chapter 2. Property (1.11) implies that the integral (1.10) diverges for any domain Ω provided that q satisfies (1.8). In Section 2 we will explain that for such q, problem (1.1) admits a solution in any Ω. Therefore, to exclude this trivial case *we make the standing assumption that in all proofs in this paper*

$$q \geq \frac{n}{n-2}.$$

If $1<q<n/(n-2)$, uniqueness also holds for (1.1), provided that Ω satisfies

$$\partial\Omega = \partial((\bar{\Omega})^c);$$

see [77].

Conditions similar to (1.10) are called *Wiener criteria.* Wiener proved in his fundamental papers [79] and [78] that a condition of this type containing the classical electrostatic capacity is necessary and sufficient for solvability of the Dirichlet problem for harmonic functions. Later, Wiener tests for the solvability of the Dirichlet problem for more general *linear* second-order (degenerately) elliptic and parabolic equations were established in [51], [27], [9], [13], [25], and [26]. Recently the first complete results were obtained for linear elliptic equations of higher order [63], for an overview see [64]. The seminal papers [59] and [30] launched research on Wiener regularity of the Dirichlet problem for *quasilinear* equations of the second order by proving the sufficiency of a Wiener-type criterion. A recent paper [38] completed the investigation of the basic question by proving the necessity; see also an earlier contribution [50]. Monographs [70], [34] and [53] give a comprehensive exposition of these results. Trudinger and Wang presented in [73] an alternative, more general, and more concise approach to quasilinear equations of the second order. In [43], the Wiener criterion was proved for Hessian equations. Hessian equations [74], [72], and [71] are *fully nonlinear* (i.e. nonlinear in the second derivatives) elliptic equations. We refer to the surveys [3] and [62] for further description of this area and for the bibliography. In connection with the present paper, we mention the following result. Consider the standard (finite data) Dirichlet problem

$$\begin{cases} \Delta u - |u|^{q-1}u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

with arbitrary $q>1$. Adams and Heard [4] and [2] proved that it is solvable for all $f \in C(\partial\Omega)$ if and only if the classical Wiener test from [79] and [78] holds for Ω.

Capacity (1.9) has been used in previous works on potential theory for semilinear equations. Baras and Pierre [8] used it to characterise removable singularities for solutions of (1.3). In [6] and [36] this capacity was used to investigate a different class of semilinear equations. We also mention the continuing series of papers by Marcus and Véron [55], [56], [58], and [57] on Riesz–Herglotz-type effects for equation (1.3) and its parabolic counterpart, questions that are also under current active study from the probabilistic point of view by Dynkin, Kuznetsov, Le Gall, and others [22], [23], [41], [40], and [47].

The crucial fact about solutions of (1.3) that will be used constantly in this paper is the elliptic *comparison principle*. As a consequence of this principle, local regularity estimates hold for solutions of (1.3). In particular, if $u \in L^q_{\mathrm{loc}}(\Omega)$ is a distributional solution of (1.3) in Ω, then, in fact, $u \in C^\infty_{\mathrm{loc}}(\Omega)$ and u is the classical solution. Another consequence of the comparison principle is the existence of a *maximal solution* $U_\Omega \in C^\infty_{\mathrm{loc}}(\Omega)$ of (1.3) such that the inequality

$$U_\Omega \geq u$$

holds for any u solving (1.3); see Section 2. To illustrate the main phenomenon behind Theorem 1.1 we now formulate our crucial estimate in the model form. Let $K \subset B(0,1)$ be a compact set in $\mathbf{R}^n$ with $n \geq 3$, let $\Omega = \mathbf{R}^n \setminus K$, and let $q > n/(n-2)$. Then

$$U_\Omega(x) \asymp \frac{\mathcal{C}_{q'}(K)}{|x|^{n-2}} \quad \text{for } |x| \geq 2. \tag{1.12}$$

Theorems 3.1 and 3.2 provide the sharper versions of estimate (1.12) that will actually be used in the proof of Theorem 1.1. For $q=2$ estimate (1.12) has a probabilistic interpretation, see [17].

Remark 1.2. (i) In [44] we apply the techniques from the present paper to the singular Yamabe problem in the case of *negative* scalar curvature. The problem which arose in the work of Loewner and Nirenberg [52] and of Schoen and Yau [69], consists of finding a complete metric on an open set Ω of, say, the unit sphere in $\mathbf{R}^n$, $n\geq 3$, that is conformally equivalent to the standard metric g_0 and has constant scalar curvature. Analytically, one seeks a solution of (1.3) with q satisfying (1.7) such that the metric $u^{4/(n-2)}g_0$ is complete in Ω. The latter replaces the condition of pointwise blow-up. A further description and references can be found in the survey [65]. In [44] we prove that a Wiener test similar to (1.10) characterises the open sets admitting such complete conformal metrics with negative scalar curvature.

(ii) In the present paper we consider only the model problem (1.1). It is interesting to extend Theorem 1.1 to the more general nonlinearities considered by Dynkin and Kuznetsov [23] and [42], and to more general linear elliptic operators. Generalisations to some nonlinear equations admitting the comparison principle are straightforward. Another open question is to adapt the techniques from the present paper to the problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u(x) \to +\infty, & \text{when } x \to \partial\Omega, \end{cases}$$

in plane domains $\Omega\subset\mathbf{R}^2$ for exponential nonlinearities f, see [75] and [76]. Capacities suitable for the exponential nonlinearities were recently introduced and investigated by Grillot and Véron [31].

(iii) We illustrate how Theorem 1.1 implies the solubility of (1.1) for domains subject to some transparent geometric conditions:

(1) The connection (4.6) between capacity and Lebesgue measure implies at once that (1.1) is solvable whenever there exist constants $C>0$ and α satisfying

$$\alpha \leq 1 + \frac{2}{(n-2)q-n}$$

such that for any $x\in\partial\Omega$ we have

$$|\Omega^c \cap B(x,r)| \geq C|B(x,r)|^\alpha \quad \text{for all } 0<r<1.$$

Using (4.6) it is also possible to derive an analogous result for $q=n/(n-2)$ in a logarithmic scale.

(2) We set

$$d(q) = \frac{(n-2)q-n}{q-1}.$$

Exploiting the well-known relationship between capacity and Hausdorff measure (e.g. [5], Chapter 5) we deduce that (1.1) is solvable in any domain Ω such that

$\partial\Omega=\Gamma_1\cup\Gamma_2\cup\ldots$, where Γ_j for $j=1,2,\ldots$, is an immersed submanifold of class C^1, say, with

$$\dim\Gamma_j > d(q).$$

Sets with finite Hausdorff $d(q)$-measure have capacity zero. Consequently, from (1.10) we also recover the result of Loewner and Nirenberg [52] stating that (1.1) is not solvable if

$$H_{d(q)}(\partial\Omega) < +\infty.$$

(3) The well-known formulae for the capacity of cylinders [1], and [61], Chapter 7, allow us to analyse the solubility of (1.1) for all values of q and n in the case when Ω^c is the Lebesgue cusp. For $q=2$ this was done in [17] and the general case can be treated in the same way.

Notation. If $E\subset\mathbf{R}^n$, then $E^c=\mathbf{R}^n\setminus E$ is the complement of E in $\mathbf{R}^n$, $|E|$ is the Lebesgue measure of E, and χ_E is the indicator (characteristic) function of E. For $x\in\mathbf{R}^n$ and $r>0$ we denote by $B(x,r)$ the open Euclidean ball of radius r centered at x. For $j\in\mathbf{Z}$ we put $r_j=1/2^j$. By B_j we denote the dyadic ball, $B_j=B(0,r_j)$. We denote the Green's function and the Poisson's kernel for the Laplacian in $B(0,R)$ by G_R and P_R, respectively. By C, $\widetilde{C}$, $C_1,\ldots$, we denote positive constants depending only on the dimension n and the parameter $q>1$ from (1.1) and (1.3). The value of C, $\widetilde{C}$, $C_1,\ldots$, may vary even within the same line. We write

$$A\lesssim B \quad (A\gtrsim B)$$

if

$$A\le CB \quad (A\ge CB)$$

for some C. We write

$$A\asymp B$$

if $A\lesssim B\lesssim A$.

Organisation of the paper. In Section 2 we recall some known results about solutions of (1.3) and the capacity (1.9), and prove preliminary estimates. In Section 3 we establish estimates of type (1.12) for the maximal solution of (1.1) or (1.3) near the boundary. In Section 4 we conclude by proving the main Theorem 1.1, relying on the estimates from Section 3.

Acknowledgments. I wish to thank members of the ETH analysis group: Michael Struwe, Tom Ilmanen, and Reiner Schätzle for their support and interest. Part of the work was done during my visit to the Newton Institute, University of Cambridge. I am grateful to Neil Trudinger and John Toland for their invitation. I wish to thank Nina Ivochkina, Vladimir Maz′ya, and Laurent Véron for very interesting discussions. I also thank the referee for remarks and corrections.

2. Preliminaries on equation and capacity

In this section we set about proving some preliminary estimates and state some well-known facts about the solutions of

$$u \geq 0, \quad \Delta u - u^q = 0, \tag{2.1}$$

with $q>1$, and the capacity (1.9). The proofs that we omit can be found in [52], [11], [76], [5], [61], and [80].

Solutions to (2.1) exhibit the following dilation invariance: for all $a>0$ and $r>0$,

$$u \text{ solves (2.1) in } B(0,r) \quad \Longrightarrow \quad a^{2/(q-1)} u(a\,\cdot) \text{ solves (2.1) in } B(0,r/a). \tag{2.2}$$

Let u solve (2.1) in a domain Ω. Then

$$u(x_0) \lesssim \frac{1}{\operatorname{dist}(x_0,\partial\Omega)^{2/(q-1)}} \qquad \text{for all } x_0 \in \Omega. \tag{2.3}$$

This estimate, first discovered by Keller [37] and Osserman [66], follows from the comparison principle. It allows us to define large solutions in the following way.

First, let Ω be a bounded domain with, say, $\partial\Omega \in C^2$. Then, as was discovered by Loewner and Nirenberg [52], there exists a unique solution to the problem

$$\begin{cases} \Delta u - u^q = 0 & \text{in } \Omega, \\ u(x) \to +\infty, & \text{when } x \to \partial\Omega. \end{cases} \tag{2.4}$$

Moreover, for $x_0 \in \partial\Omega$ and $r>0$, let u be a solution of (2.1) in Ω such that

$$u(x) \to +\infty, \quad \text{when } x \to (\partial\Omega) \cap B(x_0,r).$$

Then

$$u(x) \operatorname{dist}(x,\partial\Omega)^{2/(q-1)} \to \left(\frac{2(q+1)}{(q-1)^2}\right)^{1/(q-1)}, \quad \text{when } x \to (\partial\Omega) \cap B(x_0,r). \tag{2.5}$$

Now, let Ω be an arbitrary domain, not even necessarily bounded. Take a sequence of bounded smooth domains $\{\Omega_j\}_{j=1}^{\infty}$ such that

$$\Omega_1 \subset \ldots \subset \Omega_j \subset \Omega_{j+1} \subset \ldots, \quad \bigcup_{j=1}^{\infty} \Omega_j = \Omega.$$

Let u_j be the unique solution to (2.4) in Ω_j. In combination, the comparison principle, regularity, and (2.3) imply that the sequence $\{u_j\}_{j=1}^{\infty}$ decreases to a function $U=U_\Omega$, $U_\Omega\in C^\infty_{\mathrm{loc}}(\Omega)$, and

$$u_j \to U_\Omega \text{ in } C^\infty_{\mathrm{loc}}(\Omega), \quad \text{when } j\to\infty. \tag{2.6}$$

Moreover, U_Ω is the *maximal solution* to (2.1) in Ω. It means that the inequality

$$u\le U_\Omega \quad \text{in } \Omega \tag{2.7}$$

holds for any classical solution u to (2.1) in Ω. From (2.7) it follows at once that

$$U_{\Omega_1}\le U_{\Omega_2} \text{ in } \Omega_2, \quad \text{when } \Omega_1\supset\Omega_2. \tag{2.8}$$

Let $K_1, \dots, K_m$ be compact sets, let

$$K=K_1\cup\dots\cup K_m,$$

and let U, $U_1, \dots, U_m$ be the maximal solutions of (2.1) in K^c, $K_1^c, \dots, K_m^c$, respectively. Then on the one hand

$$U_1+\dots+U_m$$

is a supersolution of (2.1) in K^c, but on the other hand Hölder inequality ensures that

$$\frac{1}{m^{1/q}}(U_1+\dots+U_m)$$

is a subsolution of (2.1) in K^c. Hence, smooth approximation of K and application of the comparison principle imply that

$$\frac{1}{m^{1/q}}\sum_{i=1}^m U_i\le U\le\sum_{i=1}^m U_i \quad \text{in } K^c. \tag{2.9}$$

Suppose now that

$$1<q<\frac{n}{n-2}.$$

Then simple calculations show that the function

$$\sigma(x)=\left(\frac{2(n-q(n-2))}{(q-1)^2}\right)^{1/(q-1)}\frac{1}{|x|^{2/(q-1)}}$$

solves (2.1) in $\mathbf{R}^n\setminus\{0\}$. Let Ω be an arbitrary domain. Take any $x_0\in\partial\Omega$. From (2.7) we conclude that

$$U_\Omega(x)\ge\sigma(x-x_0),$$

and therefore that U_Ω solves (2.4). From now on we always assume that

$$q \ge \frac{n}{n-2}. \tag{2.10}$$

Our next goal in this section is to discuss the properties of the capacity $\mathcal{C}_p$, $1<p<\infty$, defined in (1.9). Later when dealing with (1.10) we will take $p=q'$. Thus according to (2.10) we can restrict consideration to the case

$$1<p\le \tfrac{1}{2}n.$$

Fix $x_0\in\mathbf{R}^n$. Take a compact set $K\subset B\big(x_0,\frac{3}{2}\big)$. Then

$$\begin{aligned} \mathcal{C}_p(K) \asymp \inf\bigg\{ \int_{\mathbf{R}^n} (|D^2\varphi|^p+|D\varphi|^p+|\varphi|^p) : \varphi\in C_0^\infty(\mathbf{R}^n), \\ \text{and } \varphi|_O=1 \text{ for some open set } O\supset K\bigg\}. \end{aligned} \tag{2.11}$$

Following the standard scheme of axiomatic potential theory, we define $\mathcal{C}_p(E)$ as the corresponding outer capacity for any set $E\Subset B\big(x_0,\frac{3}{2}\big)$. The function $\mathcal{C}_p$ defined in this way is a capacity in the sense of Choquet. The equivalence (2.11) implies that, by fixing a different point $\tilde{x}_0$, $K\subset B\big(\tilde{x}_0,\frac{3}{2}\big)$, we obtain an equivalent capacity $\widetilde{\mathcal{C}}_p$, i.e.

$$\widetilde{\mathcal{C}}_p(K)\asymp \mathcal{C}_p(K)$$

for compact subsets of $B\big(x_0,\frac{3}{2}\big)\cap B\big(\tilde{x}_0,\frac{3}{2}\big)$. Hence conditions of type (1.10) do not depend on the choice of x_0.

In this paper, we will need the following (partially known) result concerning the behaviour of the capacity with respect to the dilation scaling.

Lemma 2.1. *Let E be a Borel set with $\overline{E}\subset B\big(0,\frac{3}{2}\big)$, and let $1<p<\frac{1}{2}n$. Then*

$$\mathcal{C}_p(tE)\asymp t^{n-2p}\mathcal{C}_p(E) \qquad \textit{for } 0<t<\tfrac{2}{3}, \tag{2.12}$$

$$\frac{1}{\mathcal{C}_{n/2}(tE)}\asymp \frac{1}{\mathcal{C}_{n/2}(E)}+\frac{1}{\mathcal{C}_{n/2}(B(0,t))} \quad \textit{for } 0<t<\tfrac{2}{3}. \tag{2.13}$$

Proof. (1) The proof of (2.12) is a straightforward application of (2.11) and is well known. The proof of (2.13) is not available in the literature except for the particular case $n=4$ [17], when the linear theory can be applied. In what follows we prove (2.13).

(2) For a Radon measure $\mu\geq 0$ we define the function $F_\mu\colon(0,+\infty)\to\mathbf{R}^1$ by writing

$$F_\mu(\tau)=\int_{\mathbf{R}^n}\int_0^\tau \mu(B(x,r))^{2/(n-2)}\,\frac{dr}{r}\,d\mu(x),\qquad \tau>0.$$

After approximation, we can assume that in (2.13),

$$E=K,$$

where K is a compact set that is the disjoint union of finitely many closed domains with smooth boundaries. For such K a basic result in nonlinear potential theory, combined with the Wolff inequality [5], Chapters 2 and 4, implies that

(2.14)
$$\frac{1}{\mathcal{C}_{n/2}(K)^{2/(n-2)}}\asymp\inf\bigl\{F_\mu(1):\operatorname{supp}\mu\subset K,\ \|\mu\|=1 \text{ and } \mu \text{ is absolutely continuous}\bigr\}.$$

We claim that

$$\frac{dF_\mu}{d\tau}(\tau)\asymp\frac{1}{\tau}\quad\text{for } \tau\geq 1, \tag{2.15}$$

for any absolutely continuous Radon measure $\mu\geq 0$ such that $\operatorname{supp}\mu\subset K$ and $\|\mu\|=1$. Indeed, applying the dominated convergence theorem we discover that

$$\frac{dF_\mu}{d\tau}(\tau)=\frac{1}{\tau}\int_{\mathbf{R}^n}\mu(B(x,\tau))^{2/(n-2)}\,d\mu(x).$$

Now the condition $\|\mu\|=1$ implies at once that

$$\frac{dF_\mu}{d\tau}(\tau)\leq\frac{1}{\tau}.$$

To establish the lower bound for F'_μ we cover the set K by the fixed number of balls $B\bigl(a_j,\frac12\bigr)$, $j=1,\dots,N(n)$. Clearly there exists a number i, $1\leq i\leq N(n)$, such that

$$\mu\bigl(B\bigl(a_i,\tfrac12\bigr)\bigr)\geq\frac{1}{N(n)}.$$

For $\tau\geq 1$ we infer that

$$\begin{aligned}\frac{dF_\mu}{d\tau}(\tau)&\geq\frac{1}{\tau}\int_{B(a_i,1/2)}\mu(B(x,1))^{2/(n-2)}\,d\mu(x)\\&\geq\frac{1}{\tau}\int_{B(a_i,1/2)}\mu\bigl(B\bigl(a_i,\tfrac12\bigr)\bigr)^{2/(n-2)}\,d\mu(x)\gtrsim\frac{1}{\tau}.\end{aligned}$$

This finishes the proof of (2.15).

(3) We claim that

$$\frac{1}{\mathcal{C}_{n/2}(tK)^{2/(n-2)}} \lesssim \frac{1}{\mathcal{C}_{n/2}(K)^{2/(n-2)}} + \log\frac{1}{t} \quad \text{for } 0<t<1. \tag{2.16}$$

In fact, utilising (2.14), we can find an absolutely continuous Radon measure $\mu\geq 0$ such that

$$\|\mu\|=1, \quad \operatorname{supp}\mu\subset K, \quad \text{and} \quad \frac{1}{\mathcal{C}_{n/2}(K)^{2/(n-2)}} \asymp F_\mu(1).$$

For $t>0$ we define $T_t\colon \mathbf{R}^n\to\mathbf{R}^n$ by $x\mapsto tx$, and consider the corresponding push-forward of μ,

$$(T_{t*}\mu)(E)=\mu(T_t^{-1}E)=\mu\left(\frac{E}{t}\right),$$

for $E\subset\mathbf{R}^n$ with $\operatorname{supp}(T_{t*}\mu)\subset tK$. Then we deduce from (2.14) that

$$\frac{1}{\mathcal{C}_{n/2}(tK)^{2/(n-2)}} \lesssim F_{T_{t*}\mu}(1) = \int_{\mathbf{R}^n}\int_0^1 \mu\left(B\left(\frac{x}{t},\frac{r}{t}\right)\right)^{2/(n-2)}\frac{dr}{r}\,d\mu\left(\frac{x}{t}\right) \lesssim F_\mu\left(\frac{1}{t}\right).$$

Hence, invoking (2.15), we discover that

$$\frac{1}{\mathcal{C}_{n/2}(tK)^{2/(n-2)}} \lesssim F_\mu(1) + \int_1^{1/t} F'_\mu(\tau)\,d\tau \lesssim \frac{1}{\mathcal{C}_{n/2}(K)^{2/(n-2)}} + \log\frac{1}{t}.$$

This establishes (2.16).

(4) We claim that

$$\frac{1}{\mathcal{C}_{n/2}(tK)^{2/(n-2)}} \gtrsim \frac{1}{\mathcal{C}_{n/2}(K)^{2/(n-2)}} + \log\frac{1}{t} \quad \text{for } 0<t<1. \tag{2.17}$$

In fact, we can find a Radon measure in (2.14) such that

$$\frac{1}{\mathcal{C}_{n/2}(tK)^{2/(n-2)}} \asymp F_\mu(1).$$

Consider the push-forward $T_{t^{-1}*}\mu$, $T_{t*}(T_{t^{-1}*}\mu)=\mu$. Arguing as in the previous step, we deduce that

$$\begin{aligned}\frac{1}{\mathcal{C}_{n/2}(tK)^{2/(n-2)}} &\gtrsim F_{T_{t*}(T_{t^{-1}*}\mu)}(1) = F_{T_{t^{-1}*}\mu}\left(\frac{1}{t}\right)\\ &= F_{T_{t^{-1}*}\mu}(1) + \int_1^{1/t} F'_{T_{t^{-1}*}\mu}(\tau)\,d\tau \gtrsim \frac{1}{\mathcal{C}_{n/2}(K)^{2/(n-2)}} + \log\frac{1}{t}.\end{aligned}$$

(5) For the capacity of a ball we have the estimates [5], Chapter 5:

$$\mathcal{C}_p(B(0,r)) \asymp r^{n-2p}, \qquad 0<r<\tfrac{9}{10}, \tag{2.18}$$

$$\mathcal{C}_{n/2}(B(0,r)) \asymp \left(\log\frac{1}{r}\right)^{(2-n)/2}, \quad 0<r<\tfrac{9}{10}. \tag{2.19}$$

Combining (2.19) with (2.16) and (2.17), we arrive at (2.13). □

Next we derive a preliminary a priori estimate for solutions of (2.1).

Lemma 2.2. *Let $K\subset B(0,\frac{3}{2})$ be a compact set, and let u be a solution to (2.1) in K^c. Then there exists a function $\varphi\in C_0^\infty(B(0,2))$ satisfying $0\le\varphi\le 1$ in $B(0,2)$ and $\varphi=1$ in an open neighbourhood of K such that*

$$\int_{B(0,2)} |D^2\varphi|^{q'} \lesssim \mathcal{C}_{q'}(K), \tag{2.20}$$

and such that $\eta=(1-\varphi)^{2q'}$ satisfies

$$\int_{\mathbf{R}^n} u(|D\eta|+|\Delta\eta|) \lesssim \mathcal{C}_{q'}(K). \tag{2.21}$$

Proof. (1) The open set K^c can be approximated from the interior by domains with smooth boundaries. Consequently, by standard continuity properties of capacity, we can assume in the proof that K is a disjoint union of a finite number of closed domains with smooth boundaries. Take any $\varepsilon>0$. Appealing to (2.3), we choose $R>4$ such that

$$u\le\varepsilon \quad \text{on } \partial B(0,R).$$

Set $B=B(0,R)$. Let v solve the problem

$$\begin{cases} \Delta v-v^q=0 & \text{in } B\setminus K,\\ v(x)\to+\infty, & \text{when } x\to K,\\ v=0 & \text{on } \partial B.\end{cases}$$

Then

$$\Delta(v+\varepsilon)-(v+\varepsilon)^q\le 0 \quad \text{in } B\setminus K.$$

Hence by (2.5) and the comparison principle

$$u\le v+\varepsilon \quad \text{in } B\setminus K. \tag{2.22}$$

(2) We set $\widehat{B}=B(0,2)$ and note that $\widehat{B}\Subset B$. We claim that there exists a function $\varphi\in C_0^\infty(\widehat{B})$ with $0\le\varphi\le 1$ in $\widehat{B}$ and $\varphi=1$ in an open neighbourhood of K such that (2.20) holds. To prove this, we first recall a well-known result in nonlinear potential theory [33] (see also [5], Chapter 2, and [61], Chapter 9) that states that there exists a function $\widetilde{\varphi}\in C_0^\infty(\widehat{B})$ such that

$$\widetilde{\varphi}|_K\ge 1,\quad \int_{\widehat{B}}|D^2\widetilde{\varphi}|^{q'}\asymp \mathcal{C}_{q'}(K),\quad\text{and}\quad \|\widetilde{\varphi}\|_{L^\infty(\widehat{B})}\lesssim 1.$$

Next, take a function $H\in C^\infty(\mathbf{R}^1)$ such that

$$H(t)=0 \text{ for } t<\tfrac{1}{3},\quad\text{and}\quad H(t)=1 \text{ for } t>\tfrac{1}{2}.$$

Now we take φ to be the smooth truncation of $\widetilde{\varphi}$, $\varphi=H(\widetilde{\varphi})$. Then

$$\int_{\widehat{B}}|D^2\varphi|^{q'}\lesssim\int_{\widehat{B}}|H''(\widetilde{\varphi})|^{q'}|D\widetilde{\varphi}|^{2q'}+\int_{\widehat{B}}|H'(\widetilde{\varphi})|^{q'}|D^2\widetilde{\varphi}|^{q'}.$$

To obtain (2.20), we just apply the Gagliardo–Nirenberg interpolation inequality [61], Chapter 9, to the first term: if $1<r<\infty$, then

$$\|Df\|_{L^{2r}(\widehat{B})}\lesssim\|D^2f\|_{L^r(\widehat{B})}^{1/2}\|f\|_{L^\infty(\widehat{B})}^{1/2}\quad\text{for any } f\in C_0^\infty(\widehat{B}). \tag{2.23}$$

We remark that arguments of this type first appeared in [60] and [7] (see also [61], Chapter 9, and [5], Chapter 3).

(3) Let $\psi=1-\varphi$. We claim that

$$\int_B v^q\psi^m\le C(m,n,q)\,\mathcal{C}_{q'}(K)\quad\text{for } m\ge 2q'. \tag{2.24}$$

In fact, by Green's formula

$$\int_B v^q\psi^m=\int_B(\Delta v)\psi^m=\int_B v\Delta(\psi^m)+\int_{\partial B}\left(\psi^m\frac{\partial v}{\partial\nu}-v\frac{\partial\psi^m}{\partial\nu}\right),$$

where ν is the outer normal on ∂B. Since $\psi|_{\{x:|x|\ge 2\}}=1$ we conclude that

$$\frac{\partial\psi^m}{\partial\nu}=0\quad\text{on } \partial B.$$

By the comparison principle, $v|_{B\setminus K}>0$. Hence

$$\int_{\partial B}\psi^m\frac{\partial v}{\partial\nu}\le 0.$$

Using the Hölder inequality, we compute

$$
\begin{aligned}
\int_B v^q\psi^m &\le \int_B v\Delta(\psi^m)\\
&\le \int_B v|\Delta(\psi^m)|\\
&\le m\int_B v\psi^{m-1}|\Delta\psi| + m(m-1)\int_B v\psi^{m-2}|D\psi|^2\\
&\le m\biggl(\int_B v^q\psi^m\biggr)^{1/q}\biggl(\int_{\widehat{B}} \psi^X|\Delta\varphi|^{q'}\biggr)^{1/q'}\\
&\quad + m(m-1)\biggl(\int_B v^q\psi^m\biggr)^{1/q}\biggl(\int_{\widehat{B}} \psi^Y|D\varphi|^{2q'}\biggr)^{1/q'},
\end{aligned}
\tag{2.25}
$$

where

$$X = m-q', \quad \text{and} \quad Y = m-2q'.$$

We can assume that the left-hand side in (2.24) is positive. From (2.25) it then follows that

$$\int_B v^q\psi^m \le m^{2q'}\int_{\widehat{B}} (|\Delta\varphi|^{q'} + |D\varphi|^{2q'}).$$

Applying the inequality (2.23), we obtain

$$\int_B v^q\psi^m \le C(m,n,q)\int_{\widehat{B}} |D^2\varphi|^{q'},$$

and (2.24) follows from (2.20).

(4) Define $\eta = \psi^{2q'}$. We claim that

$$\int_B v(|\Delta\eta| + |D\eta|) \lesssim \mathcal{C}_{q'}(K). \tag{2.26}$$

In fact, for $s = 2q'$ we have by the same calculations as in (2.25),

$$
\begin{aligned}
\int_B v|\Delta\eta| &\le s\biggl(\int_B v^q\psi^{(s-1)q}\biggr)^{1/q}\biggl(\int_{\widehat{B}} |\Delta\varphi|^{q'}\biggr)^{1/q'}\\
&\quad + s(s-1)\biggl(\int_{B_N} v^q\psi^{(s-2)q}\biggr)^{1/q}\biggl(\int_{\widehat{B}} |D\varphi|^{2q'}\biggr)^{1/q'},
\end{aligned}
\tag{2.27}
$$

$$\int_B v|D\eta| \le s\biggl(\int_B v^q\psi^{(s-1)q}\biggr)^{1/q}\biggl(\int_{\widehat{B}} |D\varphi|^{q'}\biggr)^{1/q'}. \tag{2.28}$$

For $s=2q'$ we have

$$(s-2)q=2q', \quad \text{and} \quad (s-1)q=2q'+q.$$

Thus we can use (2.24) to estimate the integrals containing v^q in (2.27) and (2.28). Applying the interpolation inequality (2.23) to the last term in (2.27), we conclude on the basis of (2.20) that

$$\int_B v|\Delta\eta| \lesssim \mathcal{C}_{q'}(K)^{1/q}\left(\int_{\widehat{B}} |D^2\varphi|^{q'}\right)^{1/q'} \lesssim \mathcal{C}_{q'}(K).$$

Similarly, applying the Poincaré inequality to the last integral in (2.28) gives

$$\int_B v|D\eta| \lesssim \mathcal{C}_{q'}(K).$$

We conclude that (2.26) indeed holds.

(5) From (2.22) and (2.26) we obtain

$$\int_{\mathbf{R}^n} u(|D\eta|+|\Delta\eta|)=\int_{\widehat{B}} u(|D\eta|+|\Delta\eta|) \lesssim \mathcal{C}_{q'}(K)+\varepsilon\int_{\widehat{B}}(|D\eta|+|\Delta\eta|).$$

To establish (2.21) we let $\varepsilon\to 0$ both in (2.22) and in the last inequality. $\square$

Finally, for later use we record the following elementary inequality, (see for example [5] or [61]). Let $J\in\mathbf{Z}$, and let the function $\phi\colon(0,r_J)\to\mathbf{R}^1$ be either nondecreasing or nonincreasing. Then for any $\varkappa\in\mathbf{R}$,

$$\sum_{j=J+1}^{\infty} \phi(r_j)r_j^{\varkappa} \lesssim \int_0^{r_J} \phi(r)r^{\varkappa}\,\frac{dr}{r} \lesssim \sum_{j=J}^{\infty} \phi(r_j)r_j^{\varkappa}. \tag{2.29}$$

3. Capacitary estimates

Let $K\subset\mathbf{R}^n$ be a compact set. In this section we establish estimates from above and below of type (1.12) for solutions of

$$u\ge 0, \quad \Delta u-u^q=0 \text{ in } K^c. \tag{3.1}$$

The lower bound will be used in Section 4 to prove the sufficiency of (1.10) for the solubility of (1.1), whereas the estimate from above will be used in the proof of the necessity. The following theorem provides a lower bound.

Theorem 3.1. *Let $K \subset \{x : \varrho < |x| < 1\}$ be a compact set, where $0 < \varrho < 1$, and let $U \in C^2_{\mathrm{loc}}(K^c)$ be the maximal solution of* (3.1). *Then*

$$U(0) \gtrsim \sum_{j=0}^{\infty} \frac{\mathcal{C}_{q'}(K \cap B_j)}{r_j^{n-2}}. \tag{3.2}$$

We emphasize that the implicit constant in inequality (3.2) is independent of ϱ.

Proof. (1) First utilising (2.6) we approximate K and will assume further that K in (3.2) is the closure of a finite number of domains with smooth boundaries, $K \subset B(0,1) \setminus \overline{B}(0,\varrho)$.

The Bessel kernel $\mathcal{J}_2 \in C^\infty_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\})$ is defined via the formula

$$(1-\Delta)^{-1} f = \mathcal{J}_2 * f \quad \text{for all } f \in \mathcal{S}.$$

It satisfies the estimates (see, for instance, [5], Chapter 1)

$$\begin{aligned} \mathcal{J}_2(x) &\asymp \frac{1}{|x|^{n-2}} && \text{for } x \in B(0,1), \\ \mathcal{J}_2(x) &\asymp \frac{1}{e^{|x|}|x|^{(n-1)/2}} && \text{for } x \in B(0,1)^c. \end{aligned} \tag{3.3}$$

For $j \in \mathbf{Z}$ define

$$S_j = \{x : r_j \le |x| \le r_{j-1}\}.$$

Fix a positive integer J such that $2^{-J} < \varrho$. Consider the sets $K \cap S_j$, $j = 1, \dots, J$. A basic theorem in nonlinear potential theory (see [5], Chapter 2) states that for any j there exists a nonnegative Radon measure μ_j such that

$$\operatorname{supp} \mu_j \subset K \cap S_j$$

and

$$\mathcal{C}_{q'}(K \cap S_j) \asymp \mu_j(K \cap S_j) \asymp \int_{\mathbf{R}^n} (\mathcal{J}_2 * \mu_j)^q.$$

Consequently, after a suitable regularisation of μ_j and an additional smooth approximation of K, we obtain J functions $g_j \in C_0^\infty(\mathbf{R}^n)$, $g_j \ge 0$, $j = 1, \dots, J$, such that

$$\operatorname{supp} g_j \subset K \cap S_j, \tag{3.4}$$

$$\mathcal{C}_{q'}(K \cap S_j) \asymp \int_{\mathbf{R}^n} g_j \asymp \int_{\mathbf{R}^n} (\mathcal{J}_2 * g_j)^q. \tag{3.5}$$

(2) We set $B=B(0,2)$. Take $\varepsilon>0$ and let the functions $g_1,\dots,g_J$ be as in the previous step. Consider the Dirichlet problem

$$\begin{cases} \Delta u=-\varepsilon \sum_{j=1}^{J} g_j+u^q & \text{in } B, \\ u=0 & \text{on } \partial B. \end{cases}$$

The problem has a unique solution $u\in C^2_{\mathrm{loc}}(B)\cap C(\overline{B})$ such that $u\geq 0$ [52], [54], and [76]. Of course, uniqueness here is a straightforward consequence of the comparison principle. Our goal will be to show that there exists $\varepsilon=\varepsilon(n,q)>0$ such that

$$u(0)\gtrsim \sum_{j=0}^{J} \frac{\mathcal{C}_{q'}(K\cap B_j)}{r_j^{n-2}}. \tag{3.6}$$

To prove this we first note that for $x\in B$,

$$u(x)\leq -\varepsilon \int_B G_2(x,y) \sum_{j=1}^{J} g_j(y)\,dy.$$

Thus

$$\begin{aligned} u(0)&=-\varepsilon \int_B G_2(0,x) \sum_{j=1}^{J} g_j(x)\,dx+\int_B G_2(0,x)u^q(x)\,dx \\ &\geq \varepsilon \int_B |G_2(0,x)| \sum_{j=1}^{J} g_j(x)\,dx \\ &\quad -\varepsilon^q \int_B |G_2(0,x)| \biggl(\int_B |G_2(x,y)| \sum_{j=1}^{J} g_j(y)\,dy\biggr)^q dx \\ &=\varepsilon I-\varepsilon^q II. \end{aligned} \tag{3.7}$$

To obtain (3.6) we will estimate I from below and II from above.

(3) By a simple estimate for the Green's function and an application of (3.4) and (3.5),

$$I\gtrsim \int_B \frac{1}{|x|^{n-2}} \sum_{j=1}^{J} g_j(x)\,dx\gtrsim \sum_{j=1}^{J} \frac{\mathcal{C}_{q'}(K\cap S_j)}{r_j^{n-2}}.$$

We rewrite this estimate in terms of the balls B_j rather than the shells S_j. From the inequality

$$\mathcal{C}_{q'}(K\cap B_{j-1})\leq \mathcal{C}_{q'}(K\cap S_j)+\mathcal{C}_{q'}(K\cap B_j)$$

we deduce that

$$
\begin{aligned}
I &\gtrsim \sum_{j=1}^{J} \frac{\mathcal{C}_{q'}(K\cap S_j)}{r_{j-1}^{n-2}} \gtrsim \sum_{j=1}^{J}\left(\frac{\mathcal{C}_{q'}(K\cap B_{j-1})}{r_{j-1}^{n-2}} - \frac{\mathcal{C}_{q'}(K\cap B_j)}{(2r_j)^{n-2}}\right) \\
&\gtrsim \left(1-\frac{1}{2^{n-2}}\right)\sum_{j=0}^{J}\frac{\mathcal{C}_{q'}(K\cap B_j)}{r_j^{n-2}} \gtrsim \sum_{j=0}^{J}\frac{\mathcal{C}_{q'}(K\cap B_j)}{r_j^{n-2}}.
\end{aligned} \tag{3.8}
$$

(4) To obtain an upper estimate for II in (3.7) we first apply (3.3) and write

$$
II \lesssim \int_B |x|^{2-n}\left(\mathcal{J}_2 * \sum_{j=1}^{J} g_j\right)^q(x)\,dx \lesssim \sum_{j=1}^{J}\frac{1}{r_j^{n-2}}\int_{S_j}\left(\mathcal{J}_2 * \sum_{k=1}^{J} g_k\right)^q(x)\,dx. \tag{3.9}
$$

If we set $g_{J+2}=g_{J+1}=g_0=g_{-1}=0$, then (3.9) can be continued:

$$
\begin{aligned}
II &\lesssim \sum_{j=1}^{J}\frac{1}{r_j^{n-2}}\int_{S_j}\left(\left(\sum_{k=-1}^{j-2}+\sum_{k=j-1}^{j+1}+\sum_{k=j+2}^{J+2}\right)(\mathcal{J}_2*g_k)(x)\right)^q dx \\
&\lesssim \sum_{j=1}^{J}\frac{1}{r_j^{n-2}}\int_{S_j}\left(\sum_{k=j-1}^{j+1}(\mathcal{J}_2*g_k)(x)\right)^q dx \\
&\quad + \sum_{j=1}^{J} r_j^2\left(\sum_{k=-1}^{j-2}\|\mathcal{J}_2*g_k\|_{L^\infty(S_j)}\right)^q + \sum_{j=1}^{J} r_j^2\left(\sum_{k=j+2}^{J+2}\|\mathcal{J}_2*g_k\|_{L^\infty(S_j)}\right)^q \\
&= X+Y+Z.
\end{aligned} \tag{3.10}
$$

In this calculation we have used the simple inequality

$$
\int_{\mathcal{O}}(f_1+\ldots+f_N)^q \le |\mathcal{O}|\,\|(f_1+\ldots+f_N)^q\|_{L^\infty(\mathcal{O})} \le |\mathcal{O}|(\|f_1\|_{L^\infty(\mathcal{O})}+\ldots+\|f_N\|_{L^\infty(\mathcal{O})})^q
$$

valid for measurable functions $f_j \ge 0$, $j=1,\ldots,N$. Thus in order to estimate II in (3.7) we must estimate X, Y, and Z in (3.10).

(5) In light of (3.5) it is clear at once that

$$
X \lesssim \sum_{j=1}^{J}\frac{1}{r_j^{n-2}}\sum_{k=j-1}^{j+1}\int_{\mathbf{R}^n}(\mathcal{J}_2*g_k)^q \lesssim \sum_{j=1}^{J}\frac{\mathcal{C}_{q'}(K\cap S_j)}{r_j^{n-2}} \lesssim \sum_{j=0}^{J}\frac{\mathcal{C}_{q'}(K\cap B_j)}{r_j^{n-2}}. \tag{3.11}
$$

(6) Next, we deduce from (3.3) that

$$
\mathcal{J}_2(x-y) \lesssim \frac{1}{r_k^{n-2}} \quad \text{for all } x\in S_j,\ y\in S_k,\ k\le j-2.
$$

Recalling (3.4) and (3.5), we derive the estimate

$$
\begin{aligned}
Y &\lesssim \sum_{j=1}^{J} r_j^2 \biggl(\sum_{k=-1}^{j-2} \frac{1}{r_k^{n-2}} \int_{S_k} g_k \biggr)^q \lesssim \sum_{j=1}^{J} r_j^2 \biggl(\sum_{k=-1}^{j} \frac{\mathcal{C}_{q'}(K\cap S_k)}{r_k^{n-2}} \biggr)^q \\
&\lesssim \sum_{j=0}^{J} r_j^2 \biggl(\sum_{k=0}^{j} \frac{\mathcal{C}_{q'}(K\cap B_k)}{r_k^{n-2}} \biggr)^q .
\end{aligned} \tag{3.12}
$$

(7) We assert next that

$$
Z \lesssim \sum_{j=0}^{J} r_j^2 \biggl(\frac{\mathcal{C}_{q'}(K\cap B_j)}{r_j^{n-2}} \biggr)^q . \tag{3.13}
$$

To prove this we conclude with the help of (3.3) that

$$
\mathcal{J}_2(x-y) \lesssim \frac{1}{r_j^{n-2}} \quad \text{for all } x\in S_j,\ y\in S_k,\ k\geq j+2.
$$

Consequently,

$$
Z \lesssim \sum_{j=1}^{J} r_j^2 \biggl(\sum_{k=j+2}^{J+2} \frac{1}{r_j^{n-2}} \int_{S_k} g_k \biggr)^q \lesssim \sum_{j=1}^{J} \frac{1}{r_j^{(n-2)q-2}} \biggl(\sum_{k=j+2}^{J} \mathcal{C}_{q'}(K\cap B_k) \biggr)^q .
$$

To estimate the last sum we introduce the function $\Psi\colon (0,1)\to\mathbf{R}^1$ by writing

$$
\Psi(r) = \mathcal{C}_{q'}(K\cap B(0,r)), \quad 0<r<1.
$$

Bringing (2.29) into play, we continue the estimate for Z,

$$
Z \lesssim \int_0^1 \frac{1}{r^{(n-2)q-1}} \biggl(\int_0^r \Psi(t)\,\frac{dt}{t} \biggr)^q dr.
$$

Note that

$$
(n-2)q-1>1 \quad \text{when } n\geq 3 \text{ and } q\geq \frac{n}{n-2}.
$$

Hence we can apply the Hardy inequality [32], Chapter 9, to discover that

$$
\int_0^1 \frac{1}{r^{(n-2)q-1}} \biggl(\int_0^r \Psi(t)\,\frac{dt}{t} \biggr)^q dr \lesssim \int_0^1 \Psi(r)^q \, \frac{1}{r^{(n-2)q-1}}\, dr.
$$

Applying (2.29) to the last integral we confirm (3.13).

(8) We claim that

$$Y+Z \lesssim \sum_{j=0}^{J} \frac{\mathcal{C}_{q'}(K\cap B_j)}{r_j^{n-2}}. \tag{3.14}$$

Indeed, we combine (3.12) and (3.13) and find that

$$Y+Z \lesssim \sum_{j=0}^{J} r_j^2 \biggl(\sum_{k=0}^{j} \frac{\mathcal{C}_{q'}(K\cap B_k)}{r_k^{n-2}} \biggr)^q. \tag{3.15}$$

Next we introduce the function $\Phi\colon (0,1)\to\mathbf{R}^1$ by writing

$$\Phi(r) = \int_r^1 \frac{\mathcal{C}_{q'}(K\cap B(0,t))}{t^{n-2}} \,\frac{dt}{t}, \quad 0<r<1.$$

The function Φ is nonincreasing. Consequently, we can employ (2.29) to rewrite (3.15) as

$$Y+Z \lesssim \int_0^1 r\Phi(r)^q \,dr. \tag{3.16}$$

We estimate the integral in (3.16). The function Φ is absolutely continuous. Therefore, integrating by parts and noting that

$$\lim_{r\to 1-0} \Phi(r) = 0,$$

we compute

$$\int_0^1 r\Phi(r)^q \,dr = -\frac{q}{2} \int_0^1 r^2\Phi(r)^{q-1}\Phi'(r)\,dr.$$

From (2.18) and (2.19) we deduce that

$$\Phi(r) \lesssim \int_r^1 \frac{t^{n-2q'}}{t^{n-2}}\,\frac{dt}{t} \lesssim \frac{1}{r^{2/(q-1)}} \quad \text{when } 0<r<1,$$

whence

$$\int_0^1 r\Phi(r)^q \,dr \lesssim \int_0^1 r^2 \biggl(\frac{1}{r^{2/(q-1)}} \biggr)^{q-1} (-\Phi'(r))\,dr \lesssim \lim_{r\to 0+0} \Phi(r).$$

According to (2.29)

$$\lim_{r\to 0+0} \Phi(r) \lesssim \sum_{j=0}^{J} \frac{\mathcal{C}_{q'}(K\cap B_j)}{r_j^{n-2}}.$$

In view of (3.16) claim (3.14) is established.

(9) Combining (3.11), (3.14), and (3.10), we now estimate II in (3.7),

$$II \lesssim \sum_{j=0}^{J} \frac{\mathcal{C}_{q'}(K \cap B_j)}{r_j^{n-2}}.$$

In conjunction with (3.8) we obtain from this estimate and (3.7),

$$u(0) \geq (\varepsilon C_1(n,q) - \varepsilon^q C_2(n,q)) \sum_{j=0}^{J} \frac{\mathcal{C}_{q'}(K \cap B_j)}{r_j^{n-2}}.$$

Choosing $\varepsilon = \varepsilon(n,q) > 0$ small enough, we obtain (3.6).

(10) From (2.7) we deduce that

$$U \geq u \quad \text{on } \partial(B \setminus K).$$

Hence (3.4) and (2.7) allow us to apply the comparison principle to U and u in $B \setminus K$. We conclude that

$$U(0) \geq u(0).$$

Now (3.2) follows directly from (3.6) because the terms with $j \geq J$ in (3.2) vanish. □

Next we derive an upper bound for solutions of (3.1).

Theorem 3.2. *Let $K \subset \{x : \varrho < |x| < 1\}$ be a compact set, where $0 < \varrho < 1$, and let $U \in C^2_{\mathrm{loc}}(K^c)$ be the maximal solution of (3.1). Then*

$$U(0) \lesssim \sum_{j=0}^{\infty} \frac{\mathcal{C}_{q'}(K \cap B_j)}{r_j^{n-2}} \qquad \text{for } q > \frac{n}{n-2}, \tag{3.17}$$

$$U(0) \lesssim \sum_{j=0}^{\infty} \frac{\mathcal{C}_{n/2}((K \cap B_j)/r_j)}{r_j^{n-2}} \qquad \text{for } q = \frac{n}{n-2}. \tag{3.18}$$

First we prove the following lemma.

Lemma 3.3. *Let $K \subset B_1$ be a compact set, and let $u \in C^2_{\mathrm{loc}}(K^c)$ satisfy (3.1). Then*

$$u(x) \lesssim \mathcal{C}_{q'}(K) \quad \text{when } |x| \geq 3. \tag{3.19}$$

Proof. Take $\varepsilon > 0$. Using (2.3), we choose $N \in \mathbf{Z}$ with $N < -2$ such that

$$u \leq \varepsilon \quad \text{on } \partial B_N.$$

We then set $B=B_N$ and $R=r_N$. We next fix a function η as in Lemma 2.2 with $\eta|_{\{x:|x|\geq 2\}}=1$, and consider x satisfying $|x|\geq 3$. Using the properties of the Green's function and the Poisson's kernel, we obtain

$$\begin{aligned}
u(x) &= \int_B G_R(x,y)\Delta(u\eta)(y)\,dy+\int_{\partial B} P_R(x,\xi)(u\eta)(\xi)\,d\sigma(\xi)\\
&\leq \int_B G_R(x,y)(\eta\Delta u+u\Delta\eta+2DuD\eta)(y)\,dy+\varepsilon\int_{\partial B} P_R(x,\xi)\,d\sigma(\xi)\\
&\leq \int_B G_R(x,y)(u\Delta\eta+2DuD\eta)(y)\,dy+\varepsilon \qquad\qquad (3.20)\\
&= -\int_B G_R(x,y)(u\Delta\eta)(y)\,dy-2\int_B D_yG_R(x,y)D\eta(y)u(y)\,dy+\varepsilon\\
&\lesssim \varepsilon+\int_{\mathbf{R}^n} u(|\Delta\eta|+|D\eta|).
\end{aligned}$$

To obtain (3.19) we apply the estimate (2.21) from Lemma 2.2 to the second term in (3.20) and let $\varepsilon\to 0$. $\square$

With the aid of the scaled estimate (3.19), we now prove Theorem 3.2.

Proof of Theorem 3.2. (1) For $j\in\mathbf{Z}$ we define the shells

$$S_j=\{x:r_j\leq|x|\leq r_{j-1}\}.$$

Fix $j\geq 1$. Cover S_j by $N=N(n)$ number of closed balls $\overline{B}(a_k,\varrho_j)$, $k=1,\dots,N(n)$, where $a_k\in S_j$ and $\varrho_j=\frac{1}{100}r_j$. For $k=1,\dots,N$, let $V_k=V_{kj}$ be the maximal solution to (3.1) with K replaced by $K\cap\overline{B}(a_k,\varrho_j)$. Let U_j be the maximal solution to (3.1) with K replaced by $K\cap S_j$.

(2) Fix $q\geq n/(n-2)$. Utilising (2.2), we scale estimate (3.19) in Lemma 3.3 to discover that

$$V_k(0)\lesssim\frac{C_{q'}((K\cap B_{j-1})/r_{j-1})}{r_{j-1}^{2/(q-1)}}.$$

Consequently by (2.9),

$$U_j(0)\leq\sum_{k=1}^N V_k(0)\lesssim\frac{C_{q'}((K\cap B_{j-1})/r_{j-1})}{r_{j-1}^{2/(q-1)}}.$$

Choose an integer J such that $2^{-J}<\varrho$. We employ (2.9) and the previous estimate on $U_j(0)$ to see that

$$U(0)\leq\sum_{j=1}^J U_j(0)\lesssim\sum_{j=0}^J\frac{C_{q'}((K\cap B_j)/r_j)}{r_j^{2/(q-1)}}.$$

The estimate (3.18) is thereby proved. Moreover, (3.17) also follows immediately by scaling (2.12). $\square$

4. Proof of the Wiener criterion

In this section we prove our main result, namely Theorem 1.1. We use the capacity estimates from Section 3.

Proof of Theorem 1.1. (ii) $\Rightarrow$ (i) Assume that (1.10) holds. Let U be the maximal solution to (1.3) in Ω, $U\in C^2_{\mathrm{loc}}(\Omega)$. Fix $x_0\in\partial\Omega$. To prove the solubility of (1.1) we will demonstrate that

$$\lim_{\substack{x\in\Omega\\ x\to x_0}} U(x)=+\infty. \tag{4.1}$$

In proving this, we may assume that $x_0=0$. Take any $M>0$. By (1.10) there exists $\varrho>0$ such that

$$\int_{3\varrho}^{1}\frac{C_{q'}(\Omega^c\cap B(0,r))}{r^{n-2}}\,\frac{dr}{r}\geq M.$$

Let U_ϱ be the maximal solution to (1.3) in $\Omega\cup B(0,3\varrho)$. By monotonicity (2.8), we have

$$U\geq U_\varrho \quad \text{in } \Omega.$$

To estimate U_ϱ from below fix any $z_0\in B(0,\varrho)$. Then apply both estimate (3.2) from Theorem 3.1 and (2.29),

$$U_\varrho(z_0)\gtrsim\int_{2\varrho-|z_0|}^{1}\frac{C_{q'}(\Omega^c\cap B(z_0,r))}{r^{n-2}}\,\frac{dr}{r}\gtrsim\int_{3\varrho}^{1}\frac{C_{q'}(\Omega^c\cap B(0,r))}{r^{n-2}}\,\frac{dr}{r}\gtrsim M.$$

Consequently

$$U\gtrsim M \quad \text{in } B(0,\varrho)\cap\Omega,$$

and (4.1) follows.

(i) $\Rightarrow$ (ii) (1) Assume that (1.1) has a solution u. Take any $x\in\partial\Omega$. We need to prove that (1.10) holds. We again assume to economise on notation that $x=0$. We set

$$K=\Omega^c,$$

and define the dyadic shells

$$S_j=\{x:r_{j+1}<x<r_j\},$$
$$\widetilde{S}_j=\{x:r_{j+2}\leq x\leq r_{j-1}\}.$$

Let U_1, U_2, U_3, and U_4 be the maximal solutions to (1.3) for the exteriors of $K\cap \overline{B}_{j+2}$, $K\cap\widetilde{S}_j$, $(K\cap\overline{B}_1)\setminus B_{j-1}$, and $K\setminus B_1$, respectively. Of course, these functions also depend on j. From (2.9) and (2.7) we learn that

$$u\leq U_1+U_2+U_3+U_4 \quad \text{in } S_j.$$

Consequently,

$$\inf_{S_j} u\leq \|U_1\|_{L^\infty(S_j)}+\|U_3\|_{L^\infty(S_j)}+\|U_4\|_{L^\infty(S_j)}+\inf_{S_j} U_2. \tag{4.2}$$

The definition of u ensures that

$$\inf_{S_j} u\to+\infty, \quad \text{when } j\to\infty. \tag{4.3}$$

The crux of the proof lies in obtaining an upper capacity estimate for u in S_j. Estimates (3.17) and (3.18) from Theorem 3.2 provide the necessary upper bounds for the first three terms in (4.2). We now proceed to estimate the infimum of U_2 in S_j from above.

(2) Denote $E=K\cap\widetilde{S}_1$. Let w be the maximal solution to (1.3) in E^c. In other words, denote U_2 for $j=1$ by w. Fix functions φ and η as in Lemma 2.2, where K is replaced by the compact set E. We claim that

$$\int_{B(0,2)} w\eta\lesssim C_{q'}(E). \tag{4.4}$$

To prove this we set $B=B(0,4)$, and take $\theta\in C_0^\infty(B)$ such that $\theta\geq 0$ and $\theta|_{B(0,3)}=1$. For $x\in B$ we use an argument similar to the one that proved (3.20) to show that

$$(w\eta\theta)(x)\leq -\int_B G_4(x,y)(w\Delta(\eta\theta))(y)\,dy-2\int_B D_y(G_4(x,y))D(\eta\theta)(y)w(y)\,dy.$$

Hence by Fubini's theorem

$$\int_{B(0,2)} w\eta\lesssim \max_{y\in\overline{B}}\biggl(\int_B(|G_4(x,y)|+|D_yG_4(x,y)|)\,dx\biggr)\int_B w(|D(\eta\theta)|+|\Delta(\eta\theta)|). \tag{4.5}$$

The definitions of η and θ ensure that

$$\operatorname{supp}(D^\alpha\eta)\cap\operatorname{supp}(D^\beta\theta)=\emptyset \quad \text{for } |\alpha|,|\beta|\geq 1.$$

As a result,

$$|\Delta(\eta\theta)| = |(\Delta\eta)\theta + \eta\Delta\theta| \lesssim |\Delta\eta| + \chi_{B\setminus B(0,3)},$$
$$|D(\eta\theta)| = |(D\eta)\theta + \eta D\theta| \lesssim |\nabla\eta| + \chi_{B\setminus B(0,3)}.$$

Returning to (4.5) we compute

$$\int_{B(0,2)} w\eta \lesssim \|w\|_{L^\infty(B\setminus B(0,3))} + \int_{B(0,2)} w(|D\eta| + |\Delta\eta|).$$

We apply estimate (3.19) from Lemma 3.3 to the first term, and estimate (2.21) from Lemma 2.2 to the second term. This concludes the proof of (4.4).

(3) We recall the well-known connection between capacity and Lebesgue measure, see [5], Chapter 5, or [61], Chapter 7. There exists a constant $C>0$ such that the following holds: if $t\geq 0$ and $1<p\leq\frac{1}{2}n$ and if the function h is defined by

$$h(t) = h(t,p) = \begin{cases} t^{n/(n-2p)}, & \text{when } 1<p<\frac{1}{2}n, \\ \exp(-Ct^{-2/(n-2)}), & \text{when } p=\frac{1}{2}n, \end{cases}$$

then for any $F\subset\mathbf{R}^n$ with $\overline{F}\subset B\big(x_0, \frac{3}{2}\big)$ we have

$$|F| \lesssim h(\mathcal{C}_p(F), p). \tag{4.6}$$

Now we claim that for any $\varepsilon>0$,

$$\mathcal{C}_{q'}(E) \leq \varepsilon \quad\Longrightarrow\quad \big|\big\{x\in S_1 : \eta(x) \leq \tfrac{1}{100}\big\}\big| \lesssim h(\varepsilon, q'). \tag{4.7}$$

Indeed, denote the set in the right-hand side of (4.7) by E_1. Then

$$E_1 = \left\{ x\in S_1 : \frac{\varphi(x)}{1-100^{-s}} \geq 1 \right\},$$

where $s=2q'$. By definition (1.9) with $x_0=0$, we find, in view of (2.20), that

$$\mathcal{C}_{q'}(E_1) \leq \int_{\mathbf{R}^n} \left(\frac{1}{1-100^{-s}}\right)^{q'} |D^2\varphi|^{q'} \lesssim \mathcal{C}_{q'}(E).$$

We then infer (4.7) from (4.6).

(4) We assert that there exists small enough $\varepsilon>0$, $\varepsilon=\varepsilon(n,q)$, such that

$$\mathcal{C}_{q'}(E)\le\varepsilon \quad\Longrightarrow\quad \inf_{S_1} w\lesssim\mathcal{C}_{q'}(E). \tag{4.8}$$

To prove this, we choose $\varepsilon>0$, $\varepsilon=\varepsilon(n,q)$, in (4.7) so small that

$$\left|\left\{x\in S_1:\eta(x)\le\tfrac{1}{100}\right\}\right|\le\tfrac{1}{100}|S_1|.$$

Then for the set

$$E_2=\left\{x\in S_1:\eta(x)>\tfrac{1}{100}\right\}$$

we have

$$|E_2|\ge\tfrac{99}{100}|S_1|.$$

Hence by (4.4),

$$\mathcal{C}_{q'}(E)\gtrsim\int_{B(0,2)} w\eta\gtrsim\int_{S_1} w\eta\gtrsim\frac{1}{|E_2|}\int_{E_2} w\gtrsim\inf_{S_1} w,$$

and (4.8) is proved.

(5) Let $q>n/(n-2)$. We prove (1.10) in this case.

In fact, take any large $j\in\mathbf{N}$, say, $j\ge10$. On the basis of the definitions we can state that

$$(K\cap\widetilde{S}_j)/r_{j-1}\subset\widetilde{S}_1.$$

Thus we can scale estimate (4.8) using (2.2) and (2.12), and determine that there exists $\varepsilon>0$, $\varepsilon=\varepsilon(n,q)$, such that

$$\frac{\mathcal{C}_{q'}(K\cap B_{j-2})}{r_{j-2}^{n-2q'}}\le\varepsilon \quad\Longrightarrow\quad \inf_{S_j} U_2\lesssim\frac{\mathcal{C}_{q'}(K\cap B_{j-2})}{r_{j-2}^{n-2}}. \tag{4.9}$$

First assume that there exists $J\in\mathbf{N}$, $J\ge100$, such that

$$\frac{\mathcal{C}_{q'}(K\cap B_{j-2})}{r_{j-2}^{n-2q'}}\le\varepsilon \quad\text{for all } j\ge J.$$

Then for $j\ge J$ we estimate the first and second terms in (4.2) by invoking (3.17). The third term in (4.2) is estimated by (2.3). We estimate the last term in (4.2) by appealing to (4.9). In summary

$$\inf_{S_j} u\lesssim\frac{\mathcal{C}_{q'}(K\cap B_j)}{r_j^{n-2}}+\sum_{k=J}^{j}\frac{\mathcal{C}_{q'}(K\cap B_k)}{r_k^{n-2}}+1+\frac{\mathcal{C}_{q'}(K\cap B_{j-2})}{r_{j-2}^{n-2}}\lesssim\sum_{k=0}^{j}\frac{\mathcal{C}_{q'}(K\cap B_k)}{r_k^{n-2}}+1.$$

Because of (4.3), this estimate and (2.29) yield (1.10).

Next assume, alternatively, that

$$\frac{\mathcal{C}_{q'}(K\cap B_j)}{r_j^{n-2q'}}>\varepsilon$$

for infinitely many j. Then for any such j

$$\frac{\mathcal{C}_{q'}(K\cap B_j)}{r_j^{n-2}}\geq\frac{\varepsilon}{r_j^{2/(q-1)}}\to+\infty,\quad\text{when } j\to\infty.$$

In this case (1.10) follows at once from (2.29).

(6) Let $q=n/(n-2)$. We prove (1.10) in this case.

First by scaling (2.13) we find a constant $\widetilde{C}>0$ such that for all $j\geq1$,

$$\frac{1}{\mathcal{C}_{n/2}(K\cap B_{j-1})}\leq\widetilde{C}\left(\frac{1}{\mathcal{C}_{n/2}((K\cap B_{j-1})/r_{j-1})}+\frac{1}{\mathcal{C}_{n/2}(B_{j-1})}\right).$$

We can assume that

$$\mathcal{C}_{n/2}(K\cap B_j)\leq\frac{1}{2\widetilde{C}}\mathcal{C}_{n/2}(B_j)\quad\text{for all } j \text{ large.}$$

Otherwise, we could infer from (2.19) that, for infinitely many j,

$$\frac{\mathcal{C}_{n/2}(K\cap B_j)}{r_j^{n-2}}\gtrsim\frac{\mathcal{C}_{n/2}(B_j)}{r_j^{n-2}}\gtrsim\left(\frac{2^j}{j^{1/2}}\right)^{n-2}\to+\infty,\quad\text{when } j\to+\infty,$$

and (1.10) would follow at once from (2.29). Thus, without loss of generality, we may assume that there exists $J\in\mathbf{N}$ such that for all $j\geq J$,

$$\mathcal{C}_{n/2}((K\cap B_{j-1})/r_{j-1})\leq\frac{\widetilde{C}\mathcal{C}_{n/2}(K\cap B_{j-1})\mathcal{C}_{n/2}(B_{j-1})}{\mathcal{C}_{n/2}(B_{j-1})-\widetilde{C}\mathcal{C}_{n/2}(K\cap\widetilde{S}_j)}\lesssim\mathcal{C}_{n/2}(K\cap B_{j-1}).\tag{4.10}$$

Next fix any $j\in\mathbf{N}$, say $j\geq J+10$. From the definition, we have

$$(K\cap\widetilde{S}_j)/r_{j-1}\subset\widetilde{S}_1.$$

Hence we can scale estimate (4.8) using (2.2). Then taking (4.10) into account, we discover that there exists $\varepsilon>0$ such that

$$\mathcal{C}_{n/2}(K\cap B_{j-2})\leq\varepsilon\quad\Longrightarrow\quad\inf_{S_j}U_2\lesssim\frac{\mathcal{C}_{n/2}(K\cap B_{j-2})}{r_{j-2}^{n-2}}.$$

Since

$$\mathcal{C}_{n/2}(K\cap B_j)\to 0, \quad \text{when } j\to\infty,$$

we can take advantage of the last implication and conclude, after possibly increasing J, that

$$\inf_{S_j} U_2 \lesssim \frac{\mathcal{C}_{n/2}(K\cap B_{j-2})}{r_{j-2}^{n-2}} \quad \text{for all } j\geq J. \tag{4.11}$$

Now for $j\geq J$ we estimate the first and second terms in (4.2) using (3.18) and (4.10). The third term in (4.2) is estimated by (2.3). We obtain a bound on the last term in (4.2) from (4.11). In summary

$$\begin{aligned}\inf_{S_j} u &\lesssim \frac{\mathcal{C}_{n/2}(K\cap B_j)}{r_j^{n-2}}+\sum_{k=J}^{j}\frac{\mathcal{C}_{n/2}(K\cap B_k)}{r_k^{n-2}}+A(\Omega,J)+1+\frac{\mathcal{C}_{n/2}(K\cap B_{j-2})}{r_{j-2}^{n-2}}\\ &\lesssim \sum_{k=0}^{j}\frac{\mathcal{C}_{n/2}(K\cap B_k)}{r_k^{n-2}}+A(\Omega,J),\end{aligned}$$

where $A(\Omega,J)$ is a positive constant. This estimate and (2.29) give (1.10) due to (4.3). $\square$

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Received December 20, 2001

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