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# The Calderón problem for Hilbert couples

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**Abstract.** We prove that intermediate Banach spaces $\mathcal{A}$ and $\mathcal{B}$ with respect to arbitrary Hilbert couples $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$ are exact interpolation if and only if they are exact $K$-monotonic, i.e. the condition $f^0 \in \mathcal{A}$ and the inequality $K(t, g^0; \overline{\mathcal{K}}) \leq K(t, f^0; \overline{\mathcal{H}})$, $t>0$, imply $g^0 \in \mathcal{B}$ and $\|g^0\|_{\mathcal{B}} \leq \|f^0\|_{\mathcal{A}}$ ($K$ is Peetre's $K$-functional). It is well known that this property is implied by the following: for each $\varrho>1$ there exists an operator $T: \overline{\mathcal{H}} \to \overline{\mathcal{K}}$ such that $Tf^0 = g^0$, and $K(t, Tf; \overline{\mathcal{K}}) \leq \varrho K(t, f; \overline{\mathcal{H}})$, $f \in \mathcal{H}_0 + \mathcal{H}_1$, $t>0$. Verifying the latter property, it suffices to consider the "diagonal case" where $\overline{\mathcal{H}} = \overline{\mathcal{K}}$ is finite-dimensional, in which case we construct the relevant operators by a method which allows us to explicitly calculate them. In the strongest form of the theorem, it is shown that the statement remains valid when substituting $\varrho=1$. The result leads to a short proof of Donoghue's theorem on interpolation functions, as well as Löwner's theorem on monotone matrix functions.

## 1. Preliminaries

Before we formulate our basic problem let us fix some notation and review some notions from the theory of interpolation spaces (cf. [3], [4] or [5] for comprehensive accounts of that theory). We shall denote by the letters $\mathcal{A}$, $\mathcal{B}$, etc., Banach spaces over the real or complex field, whereas $\mathcal{H}$, $\mathcal{K}$, etc., denote Hilbert spaces.

Following G. Sparr [27] we denote by $\mathcal{L}(\mathcal{A};\mathcal{B})$ the Banach space of bounded linear maps $T: \mathcal{A} \to \mathcal{B}$ provided with the operator norm

$$\|T\|_{\mathcal{L}(\mathcal{A};\mathcal{B})} = \sup_{f \in \mathcal{A} \setminus \{0\}} \frac{\|Tf\|_{\mathcal{B}}}{\|f\|_{\mathcal{A}}}.$$

Similarly for Banach couples $\bar{\mathcal{A}} = (\mathcal{A}_0, \mathcal{A}_1)$ and $\bar{\mathcal{B}} = (\mathcal{B}_0, \mathcal{B}_1)$ we define $\mathcal{L}(\bar{\mathcal{A}}; \bar{\mathcal{B}})$ as the set of linear operators $T: \mathcal{A}_0 + \mathcal{A}_1 \to \mathcal{B}_0 + \mathcal{B}_1$ such that the restriction of $T$ to $\mathcal{A}_i$ belongs to $\mathcal{L}(\mathcal{A}_i; \mathcal{B}_i)$, $i=0,1$. It is well known that $\mathcal{L}(\bar{\mathcal{A}}; \bar{\mathcal{B}})$ is a Banach space under the norm

$$\|T\|_{\mathcal{L}(\bar{\mathcal{A}};\bar{\mathcal{B}})} = \max\{\|T\|_{\mathcal{L}(\mathcal{A}_0;\mathcal{B}_0)}, \|T\|_{\mathcal{L}(\mathcal{A}_1;\mathcal{B}_1)}\}.$$

For a given $c$, let $\mathcal{L}_c(\mathcal{A};\mathcal{B})$ and $\mathcal{L}_c(\bar{\mathcal{A}};\bar{\mathcal{B}})$ denote the families of balls of radius $c$,

$$\begin{aligned} T\in\mathcal{L}_c(\mathcal{A};\mathcal{B}) &\quad \text{if and only if} \quad T\in\mathcal{L}(\mathcal{A};\mathcal{B}) \text{ and } \|T\|_{\mathcal{L}(\mathcal{A};\mathcal{B})}\le c,\\ T\in\mathcal{L}_c(\bar{\mathcal{A}};\bar{\mathcal{B}}) &\quad \text{if and only if} \quad T\in\mathcal{L}(\bar{\mathcal{A}};\bar{\mathcal{B}}) \text{ and } \|T\|_{\mathcal{L}(\bar{\mathcal{A}};\bar{\mathcal{B}})}\le c. \end{aligned}$$

In this notation, intermediate spaces $\mathcal{A}$ and $\mathcal{B}$ are *interpolation* with respect to $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ if and only if there exists $c$ with the property that

$$(c\text{-Int}) \qquad \mathcal{L}_1(\bar{\mathcal{A}};\bar{\mathcal{B}})\subset\mathcal{L}_c(\mathcal{A};\mathcal{B}),$$

(where necessarily $c\ge 1$). In the special case when $c=1$,

$$(\text{ExInt}) \qquad \mathcal{L}_1(\bar{\mathcal{A}};\bar{\mathcal{B}})\subset\mathcal{L}_1(\mathcal{A};\mathcal{B}),$$

we speak about *exact interpolation.* Of particular interest is the *diagonal case*, $\mathcal{A}=\mathcal{B}$ and $\bar{\mathcal{A}}=\bar{\mathcal{B}}$, in which we simply say that $\mathcal{A}$ is *exact interpolation with respect to* $\bar{\mathcal{A}}$.

In the present study the problem of characterizing *all* exact interpolation spaces with respect to arbitrary (possibly different) Hilbert couples is considered. Our results sharpens the theorems of Sedaev ([24], Theorem 4) and Sparr ([27], Theorem 5.1) and implies the theorems of Donoghue [9] and Löwner [17].

## 2. Main results

It is well known that many exact interpolation spaces can be described by Peetre's $K$-functional,

$$K(t,f)=K(t,f;X_0,X_1)=\inf_{f=f_0+f_1}(\|f_0\|_0+t\|f_1\|_1), \quad f\in X_0+X_1,\ t>0,$$

or more precisely by the quasi-order (relative to $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$) defined by

$$g\le f[K] \quad \text{if and only if} \quad K(t,g;\bar{\mathcal{B}})\le K(t,f;\bar{\mathcal{A}}),\ t>0.$$

We have the following basic lemma.

**Lemma 2.1.** *The inclusion* (ExInt) *is implied by the following property* ("exact $K$-monotonicity")

$$f\in\mathcal{A} \text{ and } g\le f[K] \quad \text{implies} \quad g\in\mathcal{B} \text{ and } \|g\|_{\mathcal{B}}\le\|f\|_{\mathcal{A}}.$$

For a proof see [27], Theorem 1.1, p. 232.

In general the property (ExInt) is not the same as exact $K$-monotonicity, cf. [4], Exercise 5.7.14, where a three-dimensional counterexample is given (see Section 9 for further remarks). However with respect to regular Hilbert couples the following weak form of equivalence between the two notions is known: (Recall that $(X_0, X_1)$ is *regular* if $X_0 \cap X_1$ is dense in $X_0$ and in $X_1$.) Let $\mathcal{A}$ and $\mathcal{B}$ be exact interpolation relative to the regular Hilbert couples $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$. Then they are "$\sqrt{2}$-monotonic" in the following sense:

$$f \in \mathcal{A} \text{ and } g \le f[K] \quad \text{implies} \quad g \in \mathcal{B} \text{ and } \|g\|_{\mathcal{B}} \le \sqrt{2}\, \|f\|_{\mathcal{A}}. \tag{2.1}$$

This fact is due to Sedaev [24] in the diagonal case, and to Sparr [27] in the general case. We have the following sharpening.

**Theorem 2.1.** *With respect to regular Hilbert couples,* (ExInt) *is equivalent to exact $K$-monotonicity.*

Couples having the property that (ExInt) coincides with exact $K$-monotonicity are (in this paper) called *Calderón couples*—after A. P. Calderón [6] who in 1966 found the non-trivial case $\bar{\mathcal{A}}=(L_1(\mu), L_\infty(\mu))$ and $\bar{\mathcal{B}}=(L_1(\nu), L_\infty(\nu))$, $\mu$ and $\nu$ being arbitrary $\sigma$-finite measures.([1]) In this terminology, Theorem 2.1 states that regular Hilbert couples constitute a case of Calderón couples.

We have also the following, slightly stronger theorem.

**Theorem 2.2.** *Let $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$ be regular Hilbert couples, and $f^0 \in \mathcal{H}_0 + \mathcal{H}_1$ and $g^0 \in \mathcal{K}_0 + \mathcal{K}_1$ be such that $g^0 \le f^0[K]$. Then there exists an operator $T \in \mathcal{L}_1(\overline{\mathcal{H}}; \overline{\mathcal{K}})$ such that $Tf^0 = g^0$.*

We note that our proof of the above results are fairly straightforward consequences of the following "key lemma".

**Lemma 2.2.** *Let $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$ be regular Hilbert couples with elements $f^0 \in \mathcal{H}_0 + \mathcal{H}_1$ and $g^0 \in \mathcal{K}_0 + \mathcal{K}_1$ fulfilling $K(t, g^0; \overline{\mathcal{K}}) < \varrho^{-1} K(t, f^0; \overline{\mathcal{H}})$, $t > 0$, for some number $\varrho > 1$. Then there exists an operator $T \in \mathcal{L}_1(\overline{\mathcal{H}}; \overline{\mathcal{K}})$ such that $Tf^0 = g^0$.*

It is not hard to show that Theorem 2.1 is equivalent to the above lemma, and that Theorem 2.2 follows as a limiting case. The main part of this study is devoted to the proof of Lemma 2.2.

Our next theorem concerns the mapping properties of the operators $T$ exhibited in Theorem 2.2. Evidently, for given $f^0$, $g^0$, $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$ such that $g^0 \le f^0[K]$, the set

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([1]) An equivalent characterization of the exact interpolation spaces with respect to $(L_1, L_\infty)$ was independently discovered by B. S. Mityagin [18] in 1965, whence some authors prefer to speak about *Calderón–Mityagin couples*. Also the terms *$K$-adequate couple*, *$K$-monotone couple* and *$C$-couple* exist in the literature.

of such $T$'s form a closed convex subset $\mathcal{C}_1=\{T\in\mathcal{L}_1(\overline{\mathcal{H}};\overline{\mathcal{K}}):Tf^0=g^0\}$ depending on those objects. In order to get into context, let us remark that the operators used by Sedaev and Sparr, leading to the constant $\sqrt{2}$ in (2.1) are *non-negative* in the following sense. Assume, with a rather trivial restriction, that $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$ be weighted $L_2$-couples associated with some measure spaces $\Omega_1$ and $\Omega_2$, and that $f^0$ and $g^0$ are non-negative functions on those spaces, such that $g^0\le f^0[K]$. By the Sedaev–Sparr construction, there then exists an operator $T\in\mathcal{L}_{\sqrt{2}}(\overline{\mathcal{H}},\overline{\mathcal{K}})$ such that $Tf^0=g^0$ and

$$f\ge 0 \text{ a.e. on } \Omega_1 \quad \text{implies} \quad Tf\ge 0 \text{ a.e. on } \Omega_2. \tag{2.2}$$

In view of this remark, our next theorem may come as a surprise.

**Theorem 2.3.** *There exists regular Hilbert couples $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$, and elements $f^0$ and $g^0$ with $g^0\le f^0[K]$, such that the set $\mathcal{C}_1$ contains no element $T$ satisfying* (2.2).

It is fitting to compare our results with the work of W. F. Donoghue [9] by which a complete description of the exact interpolation *Hilbert* spaces with respect to Hilbert couples is known. Indeed, Donoghue's theorem is advantageous over our Theorem 2.1 in the respect that it yields an explicit representation formula for all possible norms in such spaces. In a later section, we shall show that Donoghue's theorem can be incorporated as a part of our theory. Our version of the theorem is as follows.

**Theorem 2.4.** (Equivalent to Donoghue [9].) *Let $\mathcal{H}_*$ be an intermediate Hilbert space with respect to a regular Hilbert couple $\overline{\mathcal{H}}$. Then $\mathcal{H}_*$ is exact interpolation if and only if there exists a positive Radon measure $\varrho$ on the compactified half-line $[0,\infty]$ such that*

$$\|f\|_*=\left(\int_{[0,\infty]}\left(1+\frac{1}{t}\right)K_2(t,f)^2\,d\varrho(t)\right)^{1/2},\quad f\in\mathcal{H}_*,$$

*where, by definition* (*cf.* [24]),

$$K_2(t,f)=K_2(t,f;\overline{\mathcal{H}})=\inf_{f=f_0+f_1}(\|f_0\|_0^2+t\|f_1\|_1^2)^{1/2}.$$

*Remark* 2.1. In Theorem 2.4, the function $k: t\mapsto(1+t^{-1})K_2(t,f)^2$ is defined by continuity at $t=0$ and $t=\infty$, i.e. $k(0)=\|f\|_1^2$ and $k(\infty)=\|f\|_0^2$, where we have used the convention: $\|f\|_i=\infty$ if $f\notin\mathcal{H}_i$, $i=0,1$.

We shall not prove Theorem 2.4 in its full generality here, but settle by proving it in the finite-dimensional case (the interesting case). We shall see in a later section that the theorem leads to a short proof of Löwner's theorem on monotone matrix functions, and that Theorem 2.4 is a 'local' version of that theorem. Concerning the infinite-dimensional case of Theorem 2.4, we prefer to get back to it in a later paper, but the interested reader may confer the author's Ph.D. thesis ([1], Chapter V), where details are given, and where it is also explained how the theorem relates to Donoghue's original version of the theorem.

*Remark* 2.2. Throughout this paper, we have been careful to avoid non-regular couples, although this restriction is strictly speaking not necessary. With minor modifications, our theorems and proofs extend to the non-regular case and to interpolation of quadratic semi-norms, cf. the remarks of Sparr [27], top of p. 235, and Donoghue [9], p. 264.

## 3. The functional $K_2$

With respect to $(X_0, X_1)$ we have the $K_2$-functional

$$K_2(t,f) = K_2(t,f;X_0,X_1) = \inf_{f=f_0+f_1} (\|f_0\|_0^2 + t\|f_1\|_1^2)^{1/2}.$$

Correspondingly, relative to $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ the quasi-order

$$g \le f[K_2] \quad \text{if and only if} \quad K_2(t,g;\bar{\mathcal{B}}) \le K_2(t,f;\bar{\mathcal{A}}),\ t>0,$$

is defined. The following theorem is crucial for what follows.

**Lemma 3.1.** *With respect to arbitrary Banach couples $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$, we have*

$$g \le f[K] \quad \textit{if and only if} \quad g \le f[K_2].$$

*Proof.* This lemma is formulated in Sparr [27], Lemma 3.2, p. 236 for weighted $L_p$-couples. However, a short consideration of the proof shows that it holds equally well with arbitrary Banach couples. □

It is immediate from the definitions that $K(t,\cdot\,)$ and $K_2(t,\cdot\,)$ enjoy the property of being *exact interpolation functors* for all $t$, viz.

$$Tf \le \|T\|_{\mathcal{L}(\bar{\mathcal{A}};\bar{\mathcal{B}})} f[K] \quad \text{and} \quad Tf \le \|T\|_{\mathcal{L}(\bar{\mathcal{A}};\bar{\mathcal{B}})} f[K_2], \tag{3.1}$$

for any $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$, $T$and $f$. An advantage of using $K_2$ and not $K$ is that the former can be conveniently calculated in the regular Hilbert case in a way which we describe below.

Given a regular Hilbert couple $(\mathcal{H}_0, \mathcal{H}_1)$ the squared norm $\|f\|_1^2$ is an (in general unbounded, but densely defined) quadratic form on $\mathcal{H}_0$, which we represent in the form

$$\|f\|_1^2 = (Af, f)_0,$$

where $A$ is a positive, injective, densely defined linear operator in $\mathcal{H}_0$ henceforth referred to as the *associated operator* of $\overline{\mathcal{H}}$. (Note that the domain of the positive square-root $A^{1/2}$ is $\mathcal{H}_0 \cap \mathcal{H}_1$. As general sources to the spectral theory of self-adjoint operators we refer to [22] and [23].)

Let us now fix $f \in \mathcal{H}_0 \cap \mathcal{H}_1$ and consider the optimal decomposition(2)

$$f = f_0(t) + f_1(t) \quad \text{and} \quad K_2(t, f)^2 = \|f_0(t)\|_0^2 + t\|f_1(t)\|_1^2.$$

Plainly $f_i(t) \in \mathcal{H}_0 \cap \mathcal{H}_1 = \text{domain}(A^{1/2})$, $i=0,1$, and moreover for all $\tilde{f}$ in this domain

$$\frac{d}{d\varepsilon}[(f_0(t)+\varepsilon\tilde{f}, f_0(t)+\varepsilon\tilde{f})_0 + t(A(f_1(t)-\varepsilon\tilde{f}), f_1(t)-\varepsilon\tilde{f})_0]\Big|_{\varepsilon=0} = 0,$$

so that

$$2\,\mathrm{Re}(f_0(t) - tAf_1(t), \tilde{f})_0 = 0, \quad \tilde{f} \in \mathcal{H}_0 \cap \mathcal{H}_1.$$

By regularity, $f_0(t) = tAf_1(t)$ and $f = f_0(t) + f_1(t) = (1+tA)f_1(t)$, which yields

$$f_0(t) = \frac{tA}{1+tA}(f) \quad \text{and} \quad f_1(t) = \frac{1}{1+tA}(f)$$

and

$$\begin{aligned} K_2(t,f)^2 &= \|f_0(t)\|_0^2 + t\|f_1(t)\|_1^2 = \left(\frac{(tA)^2}{(1+tA)^2}(f), f\right)_0 + t\left(\frac{A}{(1+tA)^2}(f), f\right)_0 \\ &= (h_t(A)f, f)_0, \quad \text{where } h_t(\lambda) = \frac{t\lambda}{1+t\lambda}. \end{aligned} \tag{3.2}$$

An important consequence of (3.2) is that $K_2(t,f)$ is a *Hilbert space norm* on $\mathcal{H}_0 + \mathcal{H}_1$ for every fixed $t > 0$. We shall denote by $\mathcal{H}_0 + t\mathcal{H}_1$ the space normed by $K_2(t,f)$; in particular $\mathcal{H}_0 + \mathcal{H}_1$ is considered as normed by $K_2(1,f)$.

The following characterization of the unit ball $\mathcal{L}_1(\overline{\mathcal{H}}; \overline{\mathcal{K}})$ with respect to regular Hilbert couples $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$ is immediate from (3.1),

$$T \in \mathcal{L}_1(\overline{\mathcal{H}}; \overline{\mathcal{K}}) \quad \text{if and only if} \quad Tf \le f[K_2],\ f \in \mathcal{H}_0 + \mathcal{H}_1,$$

which is to say that

$$\mathcal{L}_1(\overline{\mathcal{H}}; \overline{\mathcal{K}}) = \bigcap_{t>0} \mathcal{L}_1(\mathcal{H}_0 + t\mathcal{H}_1; \mathcal{K}_0 + t\mathcal{K}_1). \tag{3.3}$$

It is shown below that (3.3) implies a weak*-type compactness property of the set $\mathcal{L}_1(\overline{\mathcal{H}})$. (In the diagonal case we prefer to write $\mathcal{L}(\overline{\mathcal{H}})$ instead of $\mathcal{L}(\overline{\mathcal{H}}; \overline{\mathcal{H}})$, etc.)

(2) It is a simple exercise to verify that an optimal decomposition exists and is unique. Assume that $t=1$ and use the convexity and closedness of the subset $\{(f_0, f_1) \in \mathcal{H}_0 \times \mathcal{H}_1 : f_0 + f_1 = f\}$ of the cartesian product space $\mathcal{H}_0 \times \mathcal{H}_1$ to obtain an element of minimal norm.

**Lemma 3.2.** *The subset $\mathcal{L}_1(\overline{\mathcal{H}})\subset\mathcal{L}_1(\mathcal{H}_0+\mathcal{H}_1)$ is compact relative to the weak operator topology*(3) *on $\mathcal{L}_1(\mathcal{H}_0+\mathcal{H}_1)$.*

*Proof.* We use the well-known fact that the weak operator topology coincides on the set $\mathcal{L}_1(\mathcal{H}_0+\mathcal{H}_1)$ with the weak* topology, which is compact by Alaoglu's theorem; cf. [19], Section 4.2. Given $t>0$ it is easy to see that the subset $\mathcal{L}_1(\mathcal{H}_0+t\mathcal{H}_1)\cap\mathcal{L}_1(\mathcal{H}_0+\mathcal{H}_1)\subset\mathcal{L}_1(\mathcal{H}_0+\mathcal{H}_1)$ is weak operator closed, and hence compact relative to that topology. Thus so is $\mathcal{L}_1(\overline{\mathcal{H}})$, being an intersection of compact sets (3.3). □

## 4. Further preparations

In this section we simplify and reduce our problems; it is shown below that they reduce to the diagonal case (i.e. $\overline{\mathcal{H}}=\overline{\mathcal{K}}$) of Lemma 2.2.

**Lemma 4.1.** (1) *Lemma 2.2 is a consequence of its diagonal case.*

(2) *Theorem 2.2 is a consequence of its diagonal case.*

(3) *Theorem 2.2 is a consequence of Lemma 2.2.*

(4) *Theorem 2.1 is a consequence of Lemma 2.2.*

*Proof.* (1) Given $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$, we form the *direct sum* $\bar{\mathcal{S}}=(\mathcal{H}_0\oplus\mathcal{K}_0,\mathcal{H}_1\oplus\mathcal{K}_1)$. The splitting $\mathcal{S}_0+\mathcal{S}_1=(\mathcal{H}_0+\mathcal{H}_1)\oplus(\mathcal{K}_0+\mathcal{K}_1)$ easily yields the following expression for the $K_2$-functional with respect to a generic element $f\oplus g\in\mathcal{S}_0+\mathcal{S}_1$,

$$K_2(t,f\oplus g;\bar{\mathcal{S}})^2=K_2(t,f;\overline{\mathcal{H}})^2+K_2(t,g;\overline{\mathcal{K}})^2.$$

By Lemma 3.1, the assumptions of Lemma 2.2 translate to

$$K_2(t,0\oplus g^0;\bar{\mathcal{S}})<\frac{1}{\varrho}K_2(t,f^0\oplus 0;\bar{\mathcal{S}}),\quad t>0.$$

Applying the diagonal case of Lemma 2.2, it yields an operator $S\in\mathcal{L}(\bar{\mathcal{S}})$ such that $S(f^0\oplus 0)=0\oplus g^0$ and $\|S\|_{\mathcal{L}(\bar{\mathcal{S}})}\le 1$. Let $P$ denote the orthogonal projection

$$P\colon\mathcal{S}_0+\mathcal{S}_1\longrightarrow\mathcal{K}_0+\mathcal{K}_1.$$

Evidently $\|P\|_{\mathcal{L}(\bar{\mathcal{S}};\overline{\mathcal{K}})}=1$. Hence putting

$$\begin{aligned}T\colon\mathcal{H}_0+\mathcal{H}_1&\longrightarrow\mathcal{K}_0+\mathcal{K}_1\\ f&\longmapsto PS(f\oplus 0)\end{aligned}$$

(3) A net $T_i$ converges to $T$ in the weak operator topology on $\mathcal{L}(\mathcal{H})$ if and only if $(T_if,g)_{\mathcal{H}}$ converges to $(Tf,g)_{\mathcal{H}}$ for all $f,g\in\mathcal{H}$.

yields $Tf^0=g^0$ and $\|T\|_{\mathcal{L}(\overline{\mathcal{H}};\overline{\mathcal{K}})}\le 1$, as desired.

(2) This is very similar to (1), the simple modifications are left to the reader.

(3) By (1) and (2) it suffices to consider the diagonal case $\overline{\mathcal{H}}=\overline{\mathcal{K}}$. Let $\varrho>1$ be given together with any elements $g^0, f^0\in\mathcal{H}_0+\mathcal{H}_1$ such that $g^0\le f^0[K]$. The hypothesis that Lemma 2.2 holds true in the diagonal case then yields the existence for each $n\in\mathbf{N}$ of an operator $T_n\in\mathcal{L}_1(\overline{\mathcal{H}})$ such that $T_nf^0=ng^0/(n+1)$. By compactness (Lemma 3.2) the $T_n$'s cluster at a point $T\in\mathcal{L}_1(\overline{\mathcal{H}})$, and it remains to check that $Tf^0=g^0$. To this end, it suffices to note that

$$(Tf^0,h)_{\mathcal{H}_0+\mathcal{H}_1}=\lim_{k\to\infty}(T_{n_k}f^0,h)_{\mathcal{H}_0+\mathcal{H}_1}=(g^0,h)_{\mathcal{H}_0+\mathcal{H}_1},\qquad h\in\mathcal{H}_0+\mathcal{H}_1.$$

(4) Recall (Lemma 2.1) that exact $K$-monotonicity implies (ExInt). Under the hypothesis that Lemma 2.2 holds true, we shall prove the reverse implication. Given exact interpolation spaces $\mathcal{A}$ and $\mathcal{B}$ with respect to $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$ together with elements $g^0$ and $f^0$ such that

$$g^0\le f^0[K]\quad\text{and}\quad f^0\in\mathcal{A},$$

there then exists for each $\varrho>1$ an operator $T\in\mathcal{L}_1(\overline{\mathcal{H}};\overline{\mathcal{K}})$ such that $Tf^0=\varrho^{-1}g^0$. Hence $T\in\mathcal{L}_1(\mathcal{A};\mathcal{B})$ by (ExInt) whence

$$\|g^0\|_{\mathcal{B}}=\|\varrho Tf^0\|_{\mathcal{B}}\le\varrho\|f^0\|_{\mathcal{A}}.$$

Since $\varrho>1$ is arbitrary, it yields that $\mathcal{A}$ and $\mathcal{B}$ are exact $K$-monotonic. $\square$

## 5. Solution of the problem

This section is devoted to the diagonal case of Lemma 2.2. Our proof is divided into two parts: (i) reduction to a finite-dimensional case and (ii) explicit solution of the problem in that case.

### 5.1. Reduction to the finite-dimensional case

To fix the problem, let $\overline{\mathcal{H}}$ be given together with a number $\varrho>1$ and vectors $g^0, f^0\in\mathcal{H}_0+\mathcal{H}_1$ such that

$$K_2(t,g^0;\overline{\mathcal{H}})^2<\frac{1}{\varrho}K_2(t,f^0;\overline{\mathcal{H}})^2,\quad t>0. \tag{5.1}$$

We want to prove that there exists $T$ with the following properties

$$T\in\mathcal{L}_1(\overline{\mathcal{H}})\quad\text{and}\quad Tf^0=g^0. \tag{5.2}$$

Let $A$ be the operator associated with $\overline{\mathcal{H}}$, $E$ be the spectral measure of $A$, and let a sequence of orthogonal projections in $\mathcal{H}_0$ be defined by

$$P_n = E_{\sigma(A)\cap[1/n,n]}, \quad n \in \mathbf{N}.$$

**Lemma 5.1.** *To verify* (5.2) *we can besides* (5.1) *without loss of generality assume that* $f^0$ *and* $g^0$ *belong to the set* $P_n(\mathcal{H}_0)$ *for some* $n\in\mathbf{N}$.

*Proof.* Plainly, $\|P_n\|_{\mathcal{L}(\overline{\mathcal{H}})}=1$ for all $n\in\mathbf{N}$. Moreover, as $n\to\infty$ the projections $P_n$ converge in the strong operator topology on $\mathcal{L}(\mathcal{H}_0+\mathcal{H}_1)$ to the identity. Accordingly,

$$K_2(t, P_n g^0)^2 \le K_2(t, g^0)^2 < \frac{1}{\varrho} K_2(t, f^0)^2, \quad t > 0.$$

Moreover, by the estimate $K_2(t, P_n g^0)^2 \le C_n \min\{1, t\}$ and because the sequence of functions $K_2(t, P_m f^0)^2$ increase monotonically, converging uniformly on compact subsets of $\mathbf{R}_+$ to $K_2(t, f^0)^2$, it follows that, for each number $\varrho_0$ such that $1<\varrho_0<\varrho$, we can choose $m=m(\varrho_0, n)\ge n$ such that

$$K_2(t, P_n g^0)^2 < \frac{1}{\varrho_0} K_2(t, P_m f^0)^2, \quad t > 0. \tag{5.3}$$

Thus under the hypothesis that the implication (5.3) $\Rightarrow$ (5.2) holds true with respect to the vectors $P_m f^0, P_n g^0 \in P_m(\mathcal{H}_0)$, it yields an element $T_{nm}\in\mathcal{L}_1(\overline{\mathcal{H}})$ such that $T_{nm}P_m f^0 = P_n g^0$. By Lemma 3.2 the $T_{nm}$'s cluster at a point $T\in\mathcal{L}_1(\overline{\mathcal{H}})$, and one checks without difficulty that $Tf^0=g^0$, whence (5.2) holds. □

Define a subcouple $\overline{\mathcal{K}}\subset\overline{\mathcal{H}}$ by letting $\mathcal{K}_0=\mathcal{K}_1=P_n(\mathcal{H}_0)=P_n(\mathcal{H}_1)$, where the norm in $\mathcal{K}_i$ is defined as the restriction of the norm of $\mathcal{H}_i$, $i=0,1$. By Lemma 5.1 we can assume that $f^0, g^0 \in \mathcal{K}_0\cap\mathcal{K}_1$. Since

$$K_2(t, f; \overline{\mathcal{K}}) = K_2(t, f; \overline{\mathcal{H}}), \quad f \in \mathcal{K}_0 + \mathcal{K}_1,$$

we can (replacing $\overline{\mathcal{H}}$ by $\overline{\mathcal{K}}$ if necessary) assume that the norms of $\mathcal{H}_0$ and $\mathcal{H}_1$ are equivalent. Our next task is to approximate the problem by a finite-dimensional one.

**Lemma 5.2.** *Given* $f^0, g^0 \in P_n(\mathcal{H}_0)$ *and a number* $\varepsilon>0$, *there exists a finite-dimensional Hilbert subcouple* $\overline{\mathcal{V}}\subset\overline{\mathcal{H}}$ *such that* $f^0, g^0\in\mathcal{V}_0+\mathcal{V}_1$ *and*

$$(1-\varepsilon)K_2(t, f; \overline{\mathcal{H}})^2 \le K_2(t, f; \overline{\mathcal{V}})^2 \le (1+\varepsilon)K_2(t, f; \overline{\mathcal{H}})^2, \quad t>0, \ f\in\mathcal{V}_0+\mathcal{V}_1. \tag{5.4}$$

*Moreover,* $\overline{\mathcal{V}}$ *can be chosen so that the associated operator* $A_{\overline{\mathcal{V}}}$ *is multiplicity free.*

*Proof.* By the foregoing remarks, we can assume that the norms of $\mathcal{H}_0$ and $\mathcal{H}_1$ be equivalent. Then the associated operator $A$ is bounded and bounded below. Take $\bar{\varepsilon}>0$ and let $\{\lambda_i\}_{i=1}^n$ be a finite subset of $\sigma(A)$ such that $\sigma(A)\subset \bigcup_{i=1}^n\left(\lambda_i-\frac{1}{2}\bar{\varepsilon}, \lambda_i+\frac{1}{2}\bar{\varepsilon}\right)$. Let $\{E_i\}_{i=1}^n$ be a covering of $\sigma(A)$ consisting of disjoint intervals of length at most $\bar{\varepsilon}$ such that $\lambda_i\in E_i$. Define a Borel function $w\colon\sigma(A)\to\sigma(A)$ by $w(\lambda)=\lambda_i$ on $E_i\cap\sigma(A)$; then $\|w(A)-A\|_{\mathcal{L}(\mathcal{H}_0)}\le\bar{\varepsilon}$. The Lipschitz constants of the restrictions of the functions $h_t$ (cf. (3.2)) to $\sigma(A)$ are bounded above by $C_0\min\{1,t\}$, where $C_0$ is independent of $t$, whence

$$\|h_t(w(A))-h_t(A)\|_{\mathcal{L}(\mathcal{H}_0)}\le C_0\bar{\varepsilon}\min\{1,t\}.$$

Thus by Schwarz' inequality and the assumption on $\overline{\mathcal{H}}$,

$$\begin{aligned}|((h_t(w(A))-h_t(A))f,f)_0| &\le C_0\bar{\varepsilon}\min\{1,t\}\|f\|_0^2\\ &\le C_1\bar{\varepsilon}\min\{1,t\}\max\{\|f\|_0^2,\|f\|_1^2\},\quad f\in\mathcal{H}_0+\mathcal{H}_1.\end{aligned}\tag{5.5}$$

Now let $c>0$ be such that $A\ge c$. Using that $h_t(c)\ge\frac{1}{2}\min\{1,ct\}$, we get

$$(h_t(A)f,f)_0\ge h_t(c)\|f\|_0^2\ge C_2\min\{1,t\}\max\{\|f\|_0^2,\|f\|_1^2\},\quad f\in\mathcal{H}_0+\mathcal{H}_1.$$

This and (5.5) yields

$$|(h_t(w(A))f,f)_0-(h_t(A)f,f)_0|\le C_3\bar{\varepsilon}(h_t(A)f,f)_0,\quad f\in\mathcal{H}_0+\mathcal{H}_1.\tag{5.6}$$

Choose unit vectors $e_i$ and $f_i$ supported by the spectral sets $E_i$ such that $f^0$ and $g^0$ belong to the space $\mathcal{W}$ spanned by $\{e_i,f_i\}_{i=1}^n$. Put $\mathcal{W}_0=\mathcal{W}_1=\mathcal{W}$ (as sets) and define the norms by

$$\|f\|_{\mathcal{W}_0}^2=\|f\|_{\mathcal{H}_0}^2 \text{ and } \|f\|_{\mathcal{W}_1}^2=(w(A)f,f)_{\mathcal{H}_0},\quad f\in\mathcal{W}.$$

The operator associated with $\overline{\mathcal{W}}$ is then the compression $A_{\overline{\mathcal{W}}}$ of $w(A)$ to $\mathcal{W}_0$, i.e.

$$\|f\|_{\mathcal{W}_1}^2=(A_{\overline{\mathcal{W}}}f,f)_{\mathcal{W}_0}=(w(A)f,f)_{\mathcal{H}_0},\quad f\in\mathcal{W}.\tag{5.7}$$

Let $\varepsilon=2C_3\bar{\varepsilon}$ and observe that (5.6) and (5.7) yield

$$|K_2(t,f;\overline{\mathcal{W}})^2-K_2(t,f;\overline{\mathcal{H}})^2|\le\tfrac{1}{2}\varepsilon K_2(t,f;\overline{\mathcal{H}})^2,\quad f\in\mathcal{W}.\tag{5.8}$$

In general the eigenvalues of the operator $A_{\overline{\mathcal{W}}}$ have multiplicity 2. To remedy this, perturb $A_{\overline{\mathcal{W}}}$ slightly to a positive matrix $A_{\overline{\mathcal{V}}}$, all of whose eigenvalues have multiplicity 1, such that $\|A_{\overline{\mathcal{W}}}-A_{\overline{\mathcal{V}}}\|_{\mathcal{L}(\mathcal{H}_0)}<\varepsilon/2C_3$. Let $\overline{\mathcal{V}}$ be the couple associated with $A_{\overline{\mathcal{V}}}$, i.e.

$$\|f\|_{\mathcal{V}_0}^2=\|f\|_{\mathcal{W}_0}^2 \text{ and } \|f\|_{\mathcal{V}_1}^2=(A_{\overline{\mathcal{V}}}f,f)_{\mathcal{V}_0},\quad f\in\mathcal{W}.$$

By a calculation analogous to the one leading to (5.8), one gets without difficulty

$$|K_2(t,f;\overline{\mathcal{W}})^2-K_2(t,f;\overline{\mathcal{V}})^2|\le\tfrac{1}{2}\varepsilon K_2(t,f;\overline{\mathcal{H}})^2,\quad f\in\mathcal{W}.$$

Together with (5.8), this yields (5.4). □

The following lemma finishes the reduction to the finite-dimensional case.

**Lemma 5.3.** *Verifying* (5.2) *one can besides* (5.1) *without loss of generality assume that $\overline{\mathcal{H}}$ is finite-dimensional and that the associated operator is multiplicity free.*

*Proof.* Let $\overline{\mathcal{H}}$, $\varrho$, $g^0$ and $f^0$ fulfill our basic assumption (5.1). By Lemma 5.1 we can assume that $g^0, f^0 \in P_n(\mathcal{H}_0)$ for large enough $n$. In this case, for a given $\varepsilon>0$ (to be fixed later) Lemma 5.2 provides us with a couple $\overline{\mathcal{V}}$ of the desired form such that

$$K_2(t,g^0;\overline{\mathcal{V}})^2 \le (1+\varepsilon)K_2(t,g^0;\overline{\mathcal{H}})^2$$
$$< \frac{1}{\varrho}K_2(t,f^0;\overline{\mathcal{V}})^2+\varepsilon(K_2(t,f^0;\overline{\mathcal{H}})^2+K_2(t,g^0;\overline{\mathcal{H}})^2).$$

Choosing $\varepsilon>0$ sufficiently small in this inequality we can arrange that

$$K_2(t,g^0;\overline{\mathcal{V}})^2 < \frac{1}{\varrho_0}K_2(t,f^0;\overline{\mathcal{V}})^2, \quad t>0, \tag{5.9}$$

where $\varrho_0$ is any number in the interval $1<\varrho_0<\varrho$. By using (5.9) instead of our basic assumption (5.1) and the hypothesis that the conclusion (5.2) holds true with respect to the couple $\overline{\mathcal{V}}$, we infer that for each $\varrho_1$ in the interval $1<\varrho_1<\varrho_0$ there exists an operator $T\in\mathcal{L}_{1/\varrho_1}(\overline{\mathcal{V}})$ such that $Tf^0=g^0$. Denote the canonical inclusion and projections by

$$I\colon \mathcal{V}_0+\mathcal{V}_1 \longrightarrow \mathcal{H}_0+\mathcal{H}_1 \quad \text{and} \quad P\colon \mathcal{H}_0+\mathcal{H}_1 \longrightarrow \mathcal{V}_0+\mathcal{V}_1,$$

then by (5.4), $\|I\|^2_{\mathcal{L}(\overline{\mathcal{V}};\overline{\mathcal{H}})} \le (1-\varepsilon)^{-1}$, and $\|P\|^2_{\mathcal{L}(\overline{\mathcal{H}};\overline{\mathcal{V}})} \le 1+\varepsilon$. Let $\varepsilon>0$ be sufficiently small, so that

$$\frac{1}{\varrho_1}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \le 1,$$

and put $S=ITP$. Then $Sf^0=g^0$ and $S\in\mathcal{L}_1(\overline{\mathcal{H}})$. Thus (5.2) is satisfied with respect to $\overline{\mathcal{H}}$ and the operator $S$. $\square$

### 5.2. The finite-dimensional case

Let $\overline{\mathcal{H}}$ be of the type described in Lemma 5.3. Henceforth we shall assume complex scalars (the real case is postponed to the end of the proof). Let $A\in\mathcal{L}(\mathcal{H}_0)$ be the operator associated with $\overline{\mathcal{H}}$ and $\lambda=\{\lambda_i\}_{i=1}^n$ its distinct eigenvalues, ordered in increasing order. Denote by $\{\zeta_i\}_{i=1}^n$ the corresponding orthonormal basis of $\mathcal{H}_0$ consisting of eigenvectors of $A$. Then for a generic vector $f=\sum_{i=1}^n f_i\zeta_i\in\mathcal{H}_0$,

$$\|f\|_0^2 = \sum_{i=1}^n |f_i|^2 \quad \text{and} \quad \|f\|_1^2 = \sum_{i=1}^n \lambda_i |f_i|^2. \tag{5.10}$$

Working in the coordinate system $\zeta_i$ the couple $\overline{\mathcal{H}}$ becomes identified with the weighed $n$-dimensional $l_2$-couple $(l_2^n, l_2^n(\lambda))$ defined by (5.10) for $f=\{f_i\}_{i=1}^n \in l_2^n$. Put (see (3.2))

$$K_{2,\lambda}(t,f)^2 = K_2(t,f;l_2^n,l_2^n(\lambda))^2 = \sum_{i=1}^n |f_i|^2 \frac{t\lambda_i}{1+t\lambda_i}. \tag{5.11}$$

Let $\varrho>1$ be given such that

$$\varrho\lambda_i < \lambda_{i+1}, \quad i=1,\dots,n-1. \tag{5.12}$$

The problem then becomes the following: given $f^0, g^0 \in l_2^n$ such that

$$K_{2,\lambda}(t^{-1},g^0)^2 < \frac{1}{\varrho} K_{2,\lambda}(t^{-1},f^0)^2, \quad t \geq 0, \tag{5.13}$$

we must verify the existence of a matrix $T=T_{f^0,g^0} \in M_n(\mathbf{C}) := \mathcal{L}(l_2^n)$ such that

$$Tf^0 = g^0 \text{ and } K_{2,\lambda}(t,Tf) \leq K_{2,\lambda}(t,f), \quad t>0, \ f \in l_2^n. \tag{5.14}$$

To simplify the problem, let us first suppose that our problem is soluble with respect to the elements $|f^0| = \{|f_i^0|\}_{i=1}^n$, $|g^0| = \{|g_i^0|\}_{i=1}^n$, i.e. there exists $T_0 \in M_n(\mathbf{C})$ such that

$$T_0|f^0| = |g^0| \text{ and } K_{2,\lambda}(t,T_0f) \leq K_{2,\lambda}(t,f), \quad t>0, \ f \in l_2^n.$$

Choosing for each $k$ numbers $\theta_k, \varphi_k \in \mathbf{R}$ such that $f_k^0 = e^{i\theta_k}|f_k^0|$ and $g_k^0 = e^{i\varphi_k}|g_k^0|$, we infer that (5.14) holds with respect to the matrix

$$T = \operatorname{diag}(e^{i\varphi_k}) T_0 \operatorname{diag}(e^{-i\theta_k}),$$

whence there is no loss of generality in assuming that the entries $f_i^0$ and $g_i^0$ be non-negative. Moreover, replacing $g^0$ and $f^0$ by small perturbations if necessary, we can besides (5.13), assume

$$f_i^0 > 0 \quad \text{and} \quad g_i^0 > 0. \tag{5.15}$$

Put $\beta_i = \lambda_i$ and $\alpha_i = \varrho\beta_i$; then (5.12) becomes

$$0 < \beta_1 < \alpha_1 < \dots < \beta_n < \alpha_n. \tag{5.16}$$

It is a simple matter to check that

$$K_{2,\beta}(t,f)^2 \leq K_{2,\alpha}(t,f)^2 \leq \varrho K_{2,\beta}(t,f)^2, \quad t>0, \ f \in l_2^n, \tag{5.17}$$

whence (5.13) yields

$$K_{2,\alpha}(t^{-1},g^0) < K_{2,\beta}(t^{-1},f^0), \quad t\geq 0. \tag{5.18}$$

Moreover, (5.17) yields that, verifying (5.14), it suffices to verify the existence of a matrix $T{=}T_{\varrho,f^0,g^0}\in M_n(\mathbf{C})$ such that

$$Tf^0 = g^0 \text{ and } K_{2,\alpha}(t^{-1},Tf) \leq K_{2,\beta}(t^{-1},f), \quad t>0,\ f\in l_2^n. \tag{5.19}$$

Starting the construction of $T\in M_n(\mathbf{C})$ fulfilling (5.19), we put

$$L_\beta(t) = \prod_{i=1}^n (t+\beta_i), \quad L_\alpha(t) = \prod_{i=1}^n (t+\alpha_i) \quad \text{and} \quad L(t) = L_\beta(t)L_\alpha(t),$$

and note that (5.16) yields

$$L'(-\beta_i) > 0 \quad \text{and} \quad L'(-\alpha_i) < 0. \tag{5.20}$$

We now define an important polynomial $P\in\mathcal{P}_{2n-1}(\mathbf{R})$ by

$$\frac{P(t)}{L(t)} = K_{2,\beta}(t^{-1},f^0)^2 - K_{2,\alpha}(t^{-1},g^0)^2 = \sum_{i=1}^n (f_i^0)^2 \frac{\beta_i}{t+\beta_i} - \sum_{i=1}^n (g_i^0)^2 \frac{\alpha_i}{t+\alpha_i}. \tag{5.21}$$

By (5.18) we have $P(t)>0$ for $t\geq 0$. Moreover, $P$ is uniquely determined by the $2n$ conditions

$$P(-\beta_i) = (f_i^0)^2 \beta_i L'(-\beta_i) \quad \text{and} \quad P(-\alpha_i) = -(g_i^0)^2 \alpha_i L'(-\alpha_i). \tag{5.22}$$

We note that (5.15), (5.20) and (5.22) yields

$$P(-\beta_i) > 0 \quad \text{and} \quad P(-\alpha_i) > 0. \tag{5.23}$$

We claim that it suffices to prove (5.19) in the case when

(5.24) $\quad P$ has exact degree $2n-1$ and all zeros of $P$ have multiplicity 1.

To prove this, we note that polynomials fulfilling (5.24) constitute an open, dense subset $G$ of the cone $C$ of polynomials $P\in\mathcal{P}_{2n-1}(\mathbf{R})$ fulfilling $P(-\beta_i)>0$, $P(-\alpha_i)>0$ and $P(t)>0$ for $t\geq 0$. Since the formulas (5.22) for the coefficients $f^0$ and $g^0$ define continuous (positive) functions of $P\in C$, we can, replacing $f^0$ and $g^0$ by small perturbations if necessary, assure that (5.24) holds. The set $G$ shall henceforth be referred to as the set of *generic polynomials.*

Let $P\in G$, i.e. $P$ fulfills (5.23) and (5.24) and $P(t)>0$ for $t\geq 0$. We split the zeros of $P$ according to

$$P^{-1}(\{0\}) = \{-r_i\}_{i=1}^{2m-1} \cup \{-c_i\}_{i=1}^{n-m} \cup \{-\bar{c}_i\}_{i=1}^{n-m},$$

where the $r_i$'s are positive, and the $c_i$ are in the open upper half plane. We have the following lemma.

**Lemma 5.4.** *The following inequalities hold:*

$$L'(-\beta_i)P(-\beta_i)>0 \quad \text{and} \quad L'(-\alpha_i)P(-\alpha_i)<0, \tag{5.25}$$

*and there is a splitting* $\{r_i\}_{i=1}^{2m-1}=\{\delta_i\}_{i=1}^{m}\cup\{\gamma_i\}_{i=1}^{m-1}$, *where*

$$L(-\delta_j)P'(-\delta_j)>0 \quad \text{and} \quad L(-\gamma_k)P'(-\gamma_k)<0. \tag{5.26}$$

*Proof.* The inequalities (5.25) follow immediately from (5.20) and (5.23), so it remains only to prove (5.26). Let $-h$ denote the leftmost real zero of the polynomial $LP$. We claim that $-h$ is a zero of $P$. In order to prove this, we assume the contrary, i.e. $h=\alpha_n$. Since the degree of $P$ is odd, and $P(t)>0$ for $t>0$, this polynomial must be negative for large negative values of $t$, which implies $P(-\alpha_n)<0$, contradicting (5.23) and proving our claim. It follows that $L(-h)P'(-h)=(LP)'(-h)>0$, which justifies the notation $h=\delta_m$. Putting $P_*(t)=P(t)/(t+\delta_m)$ and noting that $t+\delta_m>0$ for $t\in\{-\alpha_i\}_{i=1}^{n}\cup\{-\beta_i\}_{i=1}^{n}$, (5.23) yields

$$P_*(-\beta_i)>0 \quad \text{and} \quad P_*(-\alpha_i)>0.$$

Thus $P_*$ has either 0 or an even number of zeros between each pair of zeros of $L$. Let $\{-r_j^*\}_{j=1}^{2m-2}$ denote the real zeros of $P_*$. Since the degree of $LP_*$ is even, and the polynomial $(LP_*)'$ has alternating signs in the set $\{-\alpha_i\}_{i=1}^{n}\cup\{-\beta_i\}_{i=1}^{n}\cup\{-r_i^*\}_{i=1}^{2m-2}$, this yields that we may split the zeros of $P_*$ as $\{-\delta_i\}_{i=1}^{m-1}\cup\{-\gamma_i\}_{i=1}^{m-1}$, where

$$L(-\delta_i)P_*'(-\delta_i)>0 \quad \text{and} \quad L(-\gamma_i)P_*'(-\gamma_i)<0. \tag{5.27}$$

Since $P'(-r_j^*)=(\delta_m-r_j^*)P_*'(-r_j^*)$ and $\delta_m>r_j^*$, we get $\operatorname{sign}P'(-r_j^*)=\operatorname{sign}P_*'(-r_j^*)$ for all $j$, and (5.26) follows from (5.27). $\square$

Recall that $\{-c_i\}_{i=1}^{n-m}$ denotes those zeros of $P$ which are in the open upper half-plane, and put

$$L_\delta(t)=\prod_{i=1}^{m}(t+\delta_i), \quad L_\gamma(t)=\prod_{i=1}^{m-1}(t+\gamma_i) \quad \text{and} \quad L_c(t)=\prod_{i=1}^{n-m}(t+c_i).$$

(If $m=1$, we define $L_\gamma=1$, and if $n=m$, we define $L_c=1$.) Then $P$ belongs to the $n$-dimensional space $V\subset\mathcal{P}_{2n-1}(\mathbf{C})$ defined by

$$V=\{L_cL_\delta q : q\in\mathcal{P}_{n-1}(\mathbf{C})\}. \tag{5.28}$$

Indeed, we have that $P=L_c L_\delta(aL_\gamma \bar{L}_c)$, where $a$ is the leading coefficient of $P$. For $Q\in V$, we now put

$$\frac{|Q(t)|^2}{L(t)P(t)}=\sum_{i=1}^{n}|f_i|^2\frac{\beta_i}{t+\beta_i}-\sum_{i=1}^{n}|g_i|^2\frac{\alpha_i}{t+\alpha_i}-\sum_{i=1}^{m-1}|h_i|^2\frac{\gamma_i}{t+\gamma_i}, \tag{5.29}$$

where

$$\begin{aligned} f_i&=\frac{Q(-\beta_i)}{(\beta_i L'(-\beta_i)P(-\beta_i))^{1/2}},\\ g_i&=\frac{Q(-\alpha_i)}{(-\alpha_i L'(-\alpha_i)P(-\alpha_i))^{1/2}},\\ h_i&=\frac{Q(-\gamma_i)}{(-\gamma_i L(-\gamma_i)P'(-\gamma_i))^{1/2}}. \end{aligned} \tag{5.30}$$

The identities (5.30) define linear maps

$$\begin{aligned} l_2^n &\longrightarrow V \quad\text{and}\quad V\longrightarrow l_2^n\oplus l_2^{m-1}\\ f&\longmapsto Q \quad\text{and}\quad Q\longmapsto g\oplus h. \end{aligned}$$

Their composition is thus a matrix $\widetilde{T}\in M_{n\times(n+m-1)}(\mathbf{C})$. Putting $Q=P$ in (5.29), we see that $\widetilde{T}f^0=g^0\oplus 0$. Let $T\in M_n(\mathbf{C})$ be defined by $T=E\widetilde{T}$, where $E$ is the orthogonal projection onto the first $n$ coordinates of $l_2^n\oplus l_2^{m-1}$. Then $Tf^0=g^0$ and by (5.29),

$$\begin{aligned} K_{2,\beta}(t^{-1},f)^2-K_{2,\alpha}(t^{-1},Tf)^2&=\sum_{i=1}^{n}|f_i|^2\frac{\beta_i}{t+\beta_i}-\sum_{i=1}^{n}|g_i|^2\frac{\alpha_i}{t+\alpha_i}\\ &\geq\sum_{i=1}^{n}|f_i|^2\frac{\beta_i}{t+\beta_i}-\sum_{i=1}^{n}|g_i|^2\frac{\alpha_i}{t+\alpha_i}-\sum_{i=1}^{m-1}|h_i|^2\frac{\gamma_i}{t+\gamma_i}\\ &=\frac{|Q(t)|^2}{L(t)P(t)}\geq 0,\quad t>0,\ f\in l_2^n. \end{aligned}$$

Thus $T$ satisfies (5.19), and Lemma 2.2 is proved in the case of complex scalars. In the case of real scalars, the proof must be modified by showing that any matrix $T$ satisfying (5.19) can be replaced by a real matrix fulfilling the same condition. Hereby the argument is as follows: since the complex matrix $T\in M_n(\mathbf{C})$ satisfies (5.19), so does the complex conjugate $\overline{T}$, and therefore, so does any convex combination of $T$ and $\overline{T}$. In particular, (5.19) is satisfied by the real matrix $\operatorname{Re} T=\frac{1}{2}(T+\overline{T})$. This finishes the proof of Lemma 2.2. □

## 6. Formula for a solution matrix and the proof of Theorem 2.3

In this section, we start with an increasing sequence $\lambda=\{\lambda_i\}_{i=1}^n\subset\mathbf{R}_+$ and two non-negative vectors $f^0, g^0\in l_2^n(=l_2^n(\mathbf{C}))$ fulfilling $f_i^0\geq 0$, $g_i^0\geq 0$ and

$$K_{2,\lambda}(t,g^0)\leq K_{2,\lambda}(t,f^0),\quad t>0.$$

We shall explain how the material from the previous section can be used to explicitly calculate good approximations of a real matrix $T=T_{f^0,g^0}\in M_n(\mathbf{R}):=\mathcal{L}(l_2^n(\mathbf{R}))$ fulfilling

$$Tf^0=g^0 \text{ and } K_{2,\lambda}(t,Tf)\leq K_{2,\lambda}(t,f),\quad t>0,\ f\in l_2^n. \tag{6.1}$$

For $\varrho>1$ fulfilling (5.12) we perturb $f^0$ and $g^0$ slightly to vectors $f^\varrho$ and $g^\varrho$ such that the following conditions are satisfied:

(i) $f_i^\varrho>0$ and $g_i^\varrho>0$ for all $i$;

(ii) $f^\varrho\to f^0$ and $g^\varrho\to g^0$, as $\varrho\searrow 1$;

(iii) $K_{2,\lambda}(t^{-1},g^\varrho)<\varrho^{-1}K_{2,\lambda}(t^{-1},f^\varrho)$, $t\geq 0$;

(iv) the polynomial $P(t)=L(t)(K_{2,\beta}(t^{-1},f^\varrho)^2-K_{2,\alpha}(t^{-1},g^\varrho)^2)$ satisfies (5.24) (i.e. $P$ belongs to the set $G$ of generic polynomials).

(It is of course possible that $f^0$ and $g^0$ already satisfy (i), (iii) and (iv), in which case we simply choose $f^\varrho=f^0$ and $g^\varrho=g^0$.)

We shall first consider the problem of constructing for each $\varrho$, $f^\varrho$ and $g^\varrho$ as above, a matrix $T_\varrho=T_{\varrho,f^\varrho,g^\varrho}$ such that

$$T_\varrho f^\varrho=g^\varrho \text{ and } K_{2,\alpha}(t,T_\varrho f)\leq K_{2,\beta}(t,f),\quad t>0,\ f\in l_2^n, \tag{6.2}$$

where it is understood that $\beta_i=\lambda_i$ and $\alpha_i=\varrho\lambda_i$ for all $i$. As $\varrho$, $f^\varrho$ and $g^\varrho$ approach 1, $f^0$ and $g^0$, respectively, it is clear that any cluster point of the corresponding set $T_\varrho$ of matrices fulfilling (6.2) must be a matrix fulfilling (6.1).

**Theorem 6.1.** *The matrix* $T_\varrho=T_{\varrho,f^\varrho,g^\varrho}=(t_{ik})_{i,k=1}^n$, *where*

$$t_{ik}=\operatorname{Re}\left(\frac{1}{\alpha_i-\beta_k}\,\frac{f_k^\varrho\beta_kL_\delta(-\alpha_i)L_c(-\alpha_i)L_\alpha(-\beta_k)}{g_i^\varrho\alpha_iL_\delta(-\beta_k)L_c(-\beta_k)L'_\alpha(-\alpha_i)}\right) \tag{6.3}$$

*satisfies* (6.2). *Moreover, each accumulation point* $T$ *of the* $T_\varrho$*'s, as* $\varrho\searrow 1$, *satisfies* (6.1).

*Remark* 6.1. We emphasize that each of the quantities $\alpha_i$, $\delta_i$, $\gamma_i$ and $c_i$ appearing in the formula (6.3) for $T_\varrho$ depend in an essential way on the parameter $\varrho$, whereas the vectors $f^\varrho$ and $g^\varrho$ depend on $\varrho$ only in the pathological cases when $f^0$ and $g^0$ fail to satisfy (i), (iii) and (iv).

*Proof of Theorem* 6.1. We have the following polynomials

$$L_\beta(t)=\prod_{i=1}^{n}(t+\beta_i),\quad L_\alpha(t)=\prod_{i=1}^{n}(t+\alpha_i),\qquad L(t)=L_\beta(t)L_\alpha(t),$$
$$L_\delta(t)=\prod_{i=1}^{m}(t+\delta_i),\quad L_\gamma(t)=\prod_{i=1}^{m-1}(t+\gamma_i),\quad L_c(t)=\prod_{i=1}^{n-m}(t+c_i),$$
$$P(t)=\sum_{i=1}^{n}(\beta_i(f_i^0)^2-\alpha_i(g_i^0)^2)L_\delta(t)L_\gamma(t)|L_c(t)|^2.$$

(To verify the expression for the leading coefficient of $P$ in the last formula, multiply (5.21) by $t$ and let $t\to\infty$.) We introduce the basis of the space $V$ (cf. (5.28))

$$Q_k(t)=\frac{L_\delta(t)L_c(t)L_\beta(t)}{t+\beta_k}\,\frac{(\beta_k L'(-\beta_k)P(-\beta_k))^{1/2}}{L_\delta(-\beta_k)L_c(-\beta_k)L'_\beta(-\beta_k)},\quad k=1,\dots,n.$$

Then

$$\frac{Q_k(-\beta_i)}{(\beta_i L'(-\beta_i)P(-\beta_i))^{1/2}}=\delta_{ik}\quad\text{(the Kronecker delta)}.$$

Denoting the canonical basis in $l_2^n$ by $\{\zeta_i\}_{i=1}^n$ and using (5.30), we get

(6.4)
$$\begin{aligned}t_{ik}=(T_\varrho\zeta_k)_i&=\frac{Q_k(-\alpha_i)}{(-\alpha_i L'(-\alpha_i)P(-\alpha_i))^{1/2}}\\&=\frac{1}{\beta_k-\alpha_i}\,\frac{L_\delta(-\alpha_i)L_c(-\alpha_i)L_\beta(-\alpha_i)}{L_\delta(-\beta_k)L_c(-\beta_k)L'_\beta(-\beta_k)}\left(\frac{\beta_k L'(-\beta_k)P(-\beta_k)}{-\alpha_i L'(-\alpha_i)P(-\alpha_i)}\right)^{1/2},\quad 1\le i,k\le n.\end{aligned}$$

Inserting the expressions (5.22) for $P(-\beta_k)$ and $P(-\alpha_i)$ into (6.4) yields

$$t_{ik}=\frac{1}{\alpha_i-\beta_k}\,\frac{f_k^\varrho\beta_k L_\delta(-\alpha_i)L_c(-\alpha_i)L_\alpha(-\beta_k)}{g_i^\varrho\alpha_i L_\delta(-\beta_k)L_c(-\beta_k)L'_\alpha(-\alpha_i)}.$$

This matrix solves (6.2) in the case of complex scalars. In order to get the solution in the form (6.3), we recall from the concluding remarks of the proof of Lemma 2.2, that (6.2) remains valid if we replace $T_\varrho$ by its real part. □

*Example* 6.1. Let $n=5$ and

$$\lambda=(1,2,4,5,6),\quad f^0=\left(1,\frac{7}{\sqrt{24}},1,\sqrt{\frac{19}{3}},1\right)^t,\quad g^0=\left(\sqrt{\frac{19}{15}},1,\sqrt{\frac{61}{12}},1,\sqrt{\frac{121}{40}}\right)^t.$$

One finds that

$$K_{2,\lambda}(t^{-1},f^0)^2-K_{2,\lambda}(t^{-1},g^0)^2=\frac{t(t-3)^2}{L_\lambda(t)}\geq 0,\quad t>0. \tag{6.5}$$

Let us use Theorem 6.1 to compute a matrix $T$ such that $Tf^0=g^0$ and $\|T\|_{\mathcal{L}(\overline{\mathcal{H}})}\leq 1$. We used the following data

$$\varrho=1.001,\quad f^\varrho=\varrho f^0+(\varrho-1)u,\quad g^\varrho=\varrho^{-1}g^0+(\varrho-1)u,\quad u=(1,1,1,1,1)^t$$

and some computer algebra to obtain the matrix

$$\begin{pmatrix} 0.8888 & 0.2685 & 0 & -0.0585 & 0\\ 0 & 0.6997 & 0 & 0 & 0\\ 0 & 0.2989 & 0.4437 & 0.5497 & 0.0002\\ 0 & 0 & 0 & 0.3973 & 0\\ 0 & -0.1462 & 0 & 0.5456 & 0.5749 \end{pmatrix}.$$

We also obtained

$$\begin{aligned} g^0-T_\varrho f^0&=(0.0002,0.0002,0.0005,0.0001,0.0003)^t,\\ \|T\|_{\mathcal{L}(l_2^n)}&=0.9999=\|T\|_{\mathcal{L}(l_2^n(\lambda))}. \end{aligned}$$

Note that this matrix has two negative entries. We shall now prove that this is no coincidence.

*Proof of Theorem* 2.3. We shall prove the following. Let $n$, $\lambda$, $f^0$ and $g^0$ be as in Example 6.1, then

$$Tf^0\neq g^0\quad\text{for all non-negative operators } T\in\mathcal{L}_1(\overline{\mathcal{H}}). \tag{6.6}$$

To prove this, first note that (6.5) is satisfied, and that $\|f^0\|_i=\|g^0\|_i$, $i=0,1$. We shall invoke the following theorem, which is a special case of Sparr [28], Lemma 5.4.

**Theorem 6.2.** *Suppose that $f^0$ and $g^0$ are non-negative vectors and $\|g^0\|_i=\|f^0\|_i$ for $i=0,1$. Suppose further that there exists a non-negative matrix $T\in\mathcal{L}_1(\overline{\mathcal{H}})$ such that $Tf^0=g^0$. Then*

$$L(t,g^0)\leq L(t,f^0),\quad t>0, \tag{6.7}$$

*where*(4)

$$L(t,f)=\sum_{i=1}^{n} f_i^2 \min\{1,t\lambda_i\}.$$

Since the conditions of Theorem 6.2 are satisfied with the above choice of $n$, $\lambda$, $f^0$ and $g^0$, it suffices to show that (6.7) is violated for $t=\frac{1}{4}$ in this case. But, by calculation,

$$L(\tfrac{1}{4},f^0)-L(\tfrac{1}{4},g^0)=-\tfrac{77}{240}<0.$$

This shows (6.6), and the proof is finished. □

*Example* 6.2. Let $L_2$ be the Lebesgue space of square integrable functions on $\mathbf{R}^d$, $m$ a positive integer and $H^m$ the Sobolev space on $\mathbf{R}^d$, i.e. the set of tempered distributions $g$ on $\mathbf{R}^d$ for which all derivatives $D^\alpha g$ of order $|\alpha|\le m$ belong to $L_2$. It is well known that $H^m$ is a Hilbert space under the norm (cf. [16], p. 5)

$$\|g\|_{H^m}=\left(\int_{\mathbf{R}^d}(1+|y|^2)^m|\hat{g}(y)|^2\,dy\right)^{1/2}, \tag{6.8}$$

where $\hat{g}$ is the Fourier transform

$$\hat{g}(y)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbf{R}^d} f(x)e^{-ixy}\,dx, \quad y\in\mathbf{R}^d,$$

$xy=x_1y_1+\ldots+x_dy_d$ and $|y|^2=yy$. By (3.2), (6.8) and Plancherel's theorem we have

$$K_2(t,g;L_2,H^m)^2=\int_{\mathbf{R}^d}\frac{t(1+|y|^2)^m}{1+t(1+|y|^2)^m}|\hat{g}(y)|^2\,dy, \quad g\in L_2.$$

Thus Theorem 2.2 yields that the inequality ($g^0,f^0\in L_2$)

$$\int_{\mathbf{R}^d}\frac{t(1+|y|^2)^m}{1+t(1+|y|^2)^m}|\hat{g}^0(y)|^2\,dy\le\int_{\mathbf{R}^d}\frac{t(1+|y|^2)^m}{1+t(1+|y|^2)^m}|\hat{f}^0(y)|^2\,dy, \quad t>0,$$

is sufficient to guarantee the existence of a linear operator $T$ such that $Tf^0=g^0$ and

$$\int_{\mathbf{R}^d}\frac{t(1+|y|^2)^m}{1+t(1+|y|^2)^m}|\widehat{Tf}(y)|^2\,dy\le\int_{\mathbf{R}^d}\frac{t(1+|y|^2)^m}{1+t(1+|y|^2)^m}|\hat{f}(y)|^2\,dy, \quad f\in L_2.$$

In particular, $\|g^0\|_{\mathcal{A}}\le\|f^0\|_{\mathcal{A}}$ whenever $\mathcal{A}$ is an exact interpolation space with respect to the couple $(L_2,H^m)$.

(4) The functional $L$ is denoted by $\mathcal{K}_{(2,2)}$ in Sparr's paper [28].

## 7. The proof of Theorem 2.4

We settle here by proving the finite-dimensional part of Theorem 2.4; to prove it in full generality would bring us too far from the main course of our investigations (cf. however [1]).

To fix the notation. Given a positive definite matrix $A\in M_n(\mathbf{C}):=\mathcal{L}(l_2^n)$ let a Hilbert couple $\overline{\mathcal{H}}=(\mathcal{H}_0,\mathcal{H}_1)$ be defined by $\mathcal{H}_0=l_2^n$ and $\|f\|_1^2=(Af,f)_0$, $f\in l_2^n$. It will be convenient to use the following alternative notation for the operator norms with respect to $\mathcal{H}_0$ and $\mathcal{H}_1$ ($T\in M_n(\mathbf{C})$),

$$\|T\|^2=\|T\|^2_{\mathcal{L}(\mathcal{H}_0)}=\sup_{(f,f)_0\le 1}(T^*Tf,f)_0,$$
$$\|T\|_A^2=\|T\|^2_{\mathcal{L}(\mathcal{H}_1)}=\sup_{(Af,f)_0\le 1}(T^*ATf,f)_0.$$

A positive function $h$ defined on $\sigma(A)$ is said to be *interpolation with respect to* $A$ and to *belong to the set* $C_A$ if and only if

$$\|T\|_{h(A)}\le\max\{\|T\|,\|T\|_A\},\quad T\in M_n(\mathbf{C}),\tag{7.1}$$

where, naturally, $\|T\|^2_{h(A)}=\sup_{(h(A)f,f)_0\le 1}(T^*h(A)Tf,f)_0$.

The following result is our principal result in this section; it is equivalent to Donoghue's theorems [8], Theorem III, and [9], Theorem 1, and we shall see later that it is equivalent to Theorem 2.4.

**Theorem 7.1.** *Let $A\in M_n(\mathbf{C})$ be a positive definite matrix. For a positive function $h$ defined on $\sigma(A)$ to belong to the class $C_A$, it is necessary and sufficient that $h$ be representable in the form*

$$h(\lambda)=\int_{[0,\infty]}\frac{(1+t)\lambda}{1+t\lambda}\,d\varrho(t),\quad \lambda\in\sigma(A),\tag{7.2}$$

*for some positive Radon measure $\varrho$ on $[0,\infty]$. (The measure is not unique.)*

*Proof.* Let $A\in M_n(\mathbf{C})$ be positive definite and put $\sigma(A)=\{\lambda_i\}_{i=1}^n\subset\mathbf{R}_+$. Working in the coordinate system formed by the eigenvectors of $A$ yields a canonical isomorphism identifying $\overline{\mathcal{H}}$ with the couple (see 5.10)

$$\overline{\mathcal{H}}=(l_2^n,l_2^n(\lambda)),$$

and it becomes evident that, for functions $h$ defined on $\sigma(A)$, the membership of $h$ in the cone $C_A$ is equivalent to that the space $l_2^n(h(\lambda))$ be exact interpolation with respect to the couple $(l_2^n,l_2^n(\lambda))$, i.e.

$$\|T\|_{\mathcal{L}(l_2^n(h(\lambda)))}\le\max\{\|T\|_{\mathcal{L}(l_2^n)},\|T\|_{\mathcal{L}(l_2^n(\lambda))}\},\quad T\in M_n(\mathbf{C}).\tag{7.3}$$

Our problem is thus to show that for a function to satisfy (7.3), it is necessary and sufficient that it be representable in the form (7.2) for some positive Radon measure $\varrho$ on $[0,\infty]$.

*Sufficiency.* Let $h$ be of the form (7.2). Then by (5.11),

$$\begin{aligned}(h(A)f,f)_0 &= \sum_{i=1}^n |f_i|^2 \left(\int_{[0,\infty]} \frac{(1+t)\lambda_i}{1+t\lambda_i}\,d\varrho(t)\right)\\ &= \int_{[0,\infty]} \left(1+\frac{1}{t}\right)\left(\sum_{i=1}^n |f_i|^2 \frac{t\lambda_i}{1+t\lambda_i}\right) d\varrho(t) \qquad (7.4)\\ &= \int_{[0,\infty]} \left(1+\frac{1}{t}\right) K_{2,\lambda}(t,f)^2\,d\varrho(t).\end{aligned}$$

The latter expression is the square of an exact interpolation norm with respect to $(l_2^n, l_2^n(\lambda))$ (use (3.1) and integrate with respect to $d\varrho(t)$). Thus (7.3) holds.

*Necessity.* Let $h$ satisfy (7.3). Denote by $C$ the unital $C^*$-algebra of continuous complex functions on $[0,\infty]$ with the sup-norm $\|u\|_\infty = \sup_{t\geq 0}|u(t)|$. Put

$$e_i(t) = \frac{(1+t)\lambda_i}{1+t\lambda_i}, \quad i=1,\dots,n.$$

Then

$$K_{2,\lambda}(t,f)^2 = \frac{1}{1+t^{-1}} \sum_{i=1}^n |f_i|^2 e_i(t). \tag{7.5}$$

Let $V\subset C$ be the linear span of the $e_i$'s, and let a linear functional on $V$ be defined by

$$\begin{aligned}\phi\colon V &\longrightarrow \mathbf{C}\\ \sum_{i=1}^n a_i e_i &\longmapsto \sum_{i=1}^n a_i h(\lambda_i).\end{aligned}$$

We have the following lemma.

**Lemma 7.1.** *Let $h\in C_A$. Then $\phi$ is a* positive functional *on $V$, viz. the conditions $u\in V$ and $u(t)\geq 0$, $t>0$, implies $\phi(u)\geq 0$.*

*Proof.* Let $u(t)=\sum_{i=1}^n a_i e_i(t)\geq 0$, $t>0$, and put $a_i=|f_i|^2-|g_i|^2$ with some $f,g\in l_2^n$. Then by (7.5),

$$\left(1+\frac{1}{t}\right)K_{2,\lambda}(t,f)^2 = \sum_{i=1}^n |f_i|^2 e_i(t) \geq \sum_{i=1}^n |g_i|^2 e_i(t) = \left(1+\frac{1}{t}\right)K_{2,\lambda}(t,g)^2. \tag{7.6}$$

Dividing (7.6) by the positive number $1+t^{-1}$ yields $K_{2,\lambda}(t,f)\geq K_{2,\lambda}(t,g)$, $t>0$. Because $l_2^n(h(\lambda))$ is exact interpolation with respect to $(l_2^n, l_2^n(\lambda))$, we are in a position to use Theorem 2.1. It yields $\|f\|_{l_2^n(h(\lambda))}\geq\|g\|_{l_2^n(h(\lambda))}$, or

$$\phi(u)=\sum_{i=1}^{n}(|f_i|^2-|g_i|^2)h(\lambda_i)=\|f\|^2_{l_2^n(h(\lambda))}-\|g\|^2_{l_2^n(h(\lambda))}\geq 0. \quad \square$$

Replacing $\lambda_i$ by $c\lambda_i$ with a suitable constant $c>0$, we can without loss of generality assume that $1\in\sigma(A)$, i.e. that the unit $1\in C$ belongs to $V$. By the positivity of $\phi$ (Lemma 7.1)

$$\|\phi\|=\sup_{\substack{u\in V\\ \|u\|_\infty\leq 1}}|\phi(u)|=\phi(1).$$

Let $\Phi: C\to\mathbf{C}$ be a Hahn–Banach extension of $\phi$; then

$$\|\Phi\|=\|\phi\|=\phi(1)=\Phi(1).$$

Hence $\Phi$ is a positive functional on $C$ (cf. [19], Corollary 3.3.4), and the Riesz representation theorem yields a positive Radon measure $\varrho$ on $[0,\infty]$ such that

$$\Phi(u)=\int_{[0,\infty]}u(t)\,d\varrho(t), \quad u\in C.$$

In particular,

$$h(\lambda_i)=\phi(e_{\lambda_i})=\Phi(e_{\lambda_i})=\int_{[0,\infty]}\frac{(1+t)\lambda_i}{1+t\lambda_i}\,d\varrho(t), \quad i=1,\dots,n.$$

Thus $h$ has the required representation (7.2) and the proof of Theorem 7.1 is finished. $\square$

To finish the proof of Theorem 2.4, it now suffices to combine Theorem 7.1 with the equation (7.4).

## 8. The theorems of Löwner and Foiaş–Lions

A real function $h$ on $\mathbf{R}_+$ is said to be *monotone of order* $n$ and to *belong to the class* $P_n$ if and only if for any positive definite matrices $A, B\in M_n(\mathbf{C})$

$$A\leq B \quad \text{implies} \quad h(A)\leq h(B).$$

A function is *matrix monotone* if and only if it is monotone of all finite orders.

This definition along with the principal theorem (stated below in an equivalent form) is due to K. Löwner [17].

**Theorem 8.1.** *For a positive function $h$ to be matrix monotone, it is necessary and sufficient that $h$ be representable in the form*

$$h(\lambda)=\int_{[0,\infty]}\frac{(1+t)\lambda}{1+t\lambda}\,d\varrho(t),\quad \lambda>0, \tag{8.1}$$

*for some positive Radon measure $\varrho$ on the compactified half-line $[0,\infty]$.*

Simplifications and new proofs of this theorem have been provided by several authors, cf. [2], [14], [20], [28], [13], cf. also Donoghue's book [10] which contains a fairly comprehensive exposition of the theory up until 1974.

In our approach, we shall obtain a new proof of Löwner's theorem, based on the theory of interpolation functions developed in the previous section (notably Theorem 7.1), and on a characterization of the positive matrix monotone functions on $\mathbf{R}_+$ due to F. Hansen [12]. In detail, Hansen's theorem states that a continuous function $h$ defined on $\mathbf{R}_+$ is matrix monotone if and only if for any $n\in\mathbf{N}$ and any positive definite matrix $A\in M_n(\mathbf{C})$, we have the following inequality

$$T^*h(A)T\le h(T^*AT),\quad T\in M_n(\mathbf{C}),\ T^*T\le 1. \tag{8.2}$$

(It is well known that matrix monotone functions are continuous, i.e. (8.2) is valid in general, but we shall not use this fact.)

By definition, we shall say that a positive function $h$ defined of $\mathbf{R}_+$ is *interpolation in the sense of Foiaş–Lions* if and only if for every $n\in\mathbf{N}$, and every positive definite matrix $A\in M_n(\mathbf{C})$,

$$\|T\|_{h(A)}\le \max\{\|T\|,\|T\|_A\},\quad T\in M_n(\mathbf{C}). \tag{8.3}$$

By homogeneity of the norms it is clear that (8.3) is equivalent to the following: for every positive definite matrix $A\in M_n(\mathbf{C})$, we have the following implication

$$T^*T\le 1 \quad\text{and}\quad T^*AT\le A \text{ implies } T^*h(A)T\le h(A),\quad T\in M_n(\mathbf{C}), \tag{8.4}$$

which is to hold for every positive definite matrix $A\in M_n(\mathbf{C})$. This latter form (8.4) will turn out to be convenient for the applications we have in mind.

*Definition* 8.1. Following [9], we shall denote the cone of functions representable in the form (8.1) by the letter $P'$.

**Theorem 8.2.** *Let $h$ be a positive function defined on $\mathbf{R}^+$. The following are equivalent*:

(1) $h\in P'$;

(2) *$h$ is interpolation in the sense of Foiaş–Lions*;

(3) *$h$ is matrix monotone.*

*Remark* 8.1. The equivalence (1) $\Leftrightarrow$ (2) is due to Foiaş and Lions [11], whereas (1) $\Leftrightarrow$ (3) is of course Löwner's theorem.

*Proof of Theorem* 8.2. (1) $\Leftrightarrow$ (2) This follows from Theorem 7.1 and the following observation: for a function to belong to $P'$ it is necessary and sufficient that its restriction to every finite subset of $\mathbf{R}_+$ coincide on that set with a $P'$ function. This latter property is a consequence of the well-known fact that the cone $P'$ is compact relative to the topology of pointwise convergence on $\mathbf{R}_+$ (use Helly's theorem, cf. [8], top of p. 154).

(2) $\Rightarrow$ (3) (Cf. Donoghue [9], pp. 266–267.) Let $h$ fulfill (8.4), and let $A,B\in M_n(\mathbf{C})$ be positive definite matrices such that $A\le B$. Form the $2n\times 2n$-matrices

$$\tilde{A}=\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad T=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Evidently $\tilde{A}$ is positive definite,

$$T^*T=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\le 1 \quad \text{and} \quad T^*\tilde{A}T=\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\le \tilde{A}.$$

Hence (8.4) yields $T^*h(\tilde{A})T\le h(\tilde{A})$, or

$$\begin{pmatrix} h(A) & 0 \\ 0 & 0 \end{pmatrix}\le\begin{pmatrix} h(B) & 0 \\ 0 & h(A) \end{pmatrix}.$$

Comparing the elements in the upper-left corners now yield the desired conclusion, $h(A)\le h(B)$.

(3) $\Rightarrow$ (2) Let $h$ be positive and matrix monotone on $\mathbf{R}_+$. Let us further assume that $h$ be continuous; this extra assumption will be removed at the end of the proof. Then, given matrices $A,T\in M_n(\mathbf{C})$ such that $A$ is positive definite, $T^*T\le 1$ and $T^*AT\le A$, using (8.2) and the monotonicity of $h$,

$$T^*h(A)T\le h(T^*AT)\le h(A),$$

i.e. $h$ fulfills (8.4). Now let $h$ be an arbitrary (not necessarily continuous) matrix monotone function on $\mathbf{R}_+$. Let $\varphi$ be a smooth non-negative function on $\mathbf{R}_+$ such that $\int_0^\infty \varphi(t)\,dt/t=1$ and define a sequence $h_k$ by

$$h_k(\lambda)=\frac{1}{k}\int_0^\infty \varphi\biggl(\frac{\lambda^k}{t^k}\biggr)h(t)\,\frac{dt}{t}.$$

Since the class of matrix monotone functions is a convex cone, closed under pointwise convergence (cf. [10], p. 68), the $h_k$'s are matrix monotone for all $k\in\mathbf{N}$. Since they are evidently positive and continuous, the first part of the proof yields that they satisfy (8.4). Since the property (8.4) is finitary in nature, it is easy to see (the same proof as for monotone functions) that the set of interpolation functions in the sense of Foiaş–Lions is closed under pointwise convergence on $\mathbf{R}_+$. Hence the limiting function $h=\lim_{k\to\infty} h_k$ is also in that set, as desired. $\square$

### 8.1. A closer look at monotone and interpolation functions

This subsection comprises a finer study of the relation between interpolation functions and matrix monotone functions of finite order. (Recall that a real function $h$ is said to belong to the class $P_n$ of matrix monotone functions of order $n$ if $A,B\in M_n(\mathbf{C})$ and $0<A\le B$ yields $h(A)\le h(B)$.) In the following, it will be convenient to use the letter $P_n'$ to denote the set of positive functions in the class $P_n$, $n\in\mathbf{N}$. A further scale of function classes, which we shall denote by $C_n$, $n=1,2,,\dots,$ is obtained by the definition

$$h\in C_n \quad \text{if and only if} \quad \|T\|_{h(A)}\le\max\{\|T\|,\|T\|_A\},\ A,T\in M_n(\mathbf{C}),\ A>0.$$

It is fitting to refer to elements of $C_n$ as *interpolation functions of order $n$*.

It is easy to see that $C_n$ and $P_n'$ are convex cones, and that $C_n\supset C_{n+1}$ and $P_n'\supset P_{n+1}'$ for all $n$. Moreover, Theorem 8.2 shows that

$$\bigcap_{n=1}^{\infty} C_n=\bigcap_{n=1}^{\infty} P_n'=P'.$$

A closer look at the proof yields that $P_{2n}'\subset C_n$ and $C_{2n}\subset P_n'$ for all $n$. In this section, we shall refine that result by showing that $\{C_n\}_{n=1}^{\infty}$ is a "finer scale" than $\{P_n'\}_{n=1}^{\infty}$.

**Theorem 8.3.** *It holds that $P_{n+1}'\subset C_{2n}\subset P_n'$.*

*Proof.* The inclusion $C_{2n}\subset P_n'$ is contained in the proof of (2) $\Rightarrow$ (3) of Theorem 8.2. It remains to prove $P_{n+1}'\subset C_{2n}$. In order to accomplish this, we need to invoke the class $P$ of *Pick functions* on $\mathbf{R}_+$. This is by definition the class of functions $h$ having a representation of the form (cf. [10], Chapter II)

$$h(\lambda)=\alpha\lambda+\beta+\int_{-\infty}^{0}\left(\frac{1}{t-\lambda}-\frac{t}{t^2+1}\right)d\mu(t)$$

for some $\alpha\ge0$, $\beta\in\mathbf{R}$ and some positive Borel measure $\mu$ on $(-\infty,0)$ such that $\int(t^2+1)^{-1}\,d\mu(t)$ is finite. It is easy to see that $P'=\{h\in P:h\ge0\}$. In [8], pp. 153–154, it was noted that the class $P$ has an important *compactness property*; if $h_k(\lambda)$ is a sequence in $P$ such that for a pair of distinct points $\lambda'$, $\lambda''\in\mathbf{R}_+$ the sequences $h_k(\lambda')$ and $h_k(\lambda'')$ remain bounded, then there exists a subsequence of those functions converging uniformly on closed subintervals of $\mathbf{R}_+$ to a function $h$ in the class $P$.

For our purposes, the usefulness of the class $P$ depends on the following interpolation theorem due to Löwner, cf. [10], Theorem I, p. 128.

**Lemma 8.1.** *Let $h\in P_{n+1}$ and let $S\subset\mathbf{R}_+$ consist of $2n+1$ points. There then exists a function $\bar{h}\in P$ such that $\bar{h}=h$ on $S$.*

Assume that $h\in P'_{n+1}$ and take $A\in M_{2n}(\mathbf{C})$ positive definite. By Lemma 8.1, there exists a sequence $\bar{h}_k$ of Pick functions such that for each $k$, $\bar{h}_k=h$ on the set $\sigma(A)\cup\{1/k\}$. Thus by compactness, there exists a subsequence of the $\bar{h}_k$'s converging uniformly on compact subsets of $\mathbf{R}_+$ to a function $\bar{h}$ in the class $P$. Since $h$ is positive, moreover

$$\bar{h}(0)\geq\liminf_{k\to\infty}h\left(\frac{1}{k}\right)\geq 0.$$

Together with the simple fact that Pick functions are increasing, this yields that $\bar{h}\in P'$. Finally, since $\bar{h}=h$ on $\sigma(A)$, the implication (1) $\Rightarrow$ (2) of Theorem 8.2 yields that $h$ is exact interpolation with respect to $A$. Since $A$ was arbitrary, we obtain $h\in C_{2n}$ as desired. □

*Remark* 8.2. A closer study shows that Theorem 8.3 yields a strengthening of Sparr [28], Lemma 1, p. 267. It is not hard to prove that Sparr's lemma implies $P'_{n+2}\subset C_{2n}\subset P'_n$ but not the sharper statement of Theorem 8.3.

## 9. Two conjectures

Let $1<p<\infty$, $\lambda$ be a positive weight function on some measure space $(X,\mu)$, and $L_p$, $L_p(\lambda)$ be the corresponding $L_p$-spaces. Relative to the couple $\bar{A}=(L_p,L_p(\lambda))$ it is interesting besides $K$ to study the functional $K_p$ given by

$$K_p(t,f;\bar{A})=\inf_{f=f_0+f_1}(\|f_0\|_0^p+t\|f_1\|_1^p)^{1/p}.$$

Defining the corresponding quasi-order $g\leq f[K_p]$ in the obvious way, we have (cf. Sparr [27], Lemma 3.3)

$$g\leq f[K]\quad\text{if and only if}\quad g\leq f[K_p]. \tag{9.1}$$

It is not hard to verify the following formula for $K_p$,

$$K_p(t,f;L_p,L_p(\lambda))^p=\int_X \frac{t\lambda(x)}{(1+(t\lambda(x))^{1/(p-1)})^{p-1}}|f(x)|^p\,d\mu(x),\quad f\in L_p+L_p(\lambda),$$

which yields an analytic expression for the quasi-order (9.1) for all $1<p<\infty$, resembling in several aspects the case $p=2$. Our first conjecture is the following.

**Conjecture 9.1.** *It holds that the couples* $\bar{A}=(L_p(a_0),L_p(a_1))$ *and* $\bar{B}=(L_p(b_0),L_p(b_1))$ *are Calderón,* $1\le p\le\infty$.

A somewhat weaker conjecture arises when we restrict our study to interpolation spaces $L_p^*=L_p(h(\lambda))$ which are themselves weighted $L_p$-spaces relative to $(X,\mu)$. We have the following conjecture.

**Conjecture 9.2.** *A weighted* $L_p$*-space* $L_p^*$ *is exact interpolation with respect to* $(L_p,L_p(\lambda))$ *if and only if there exists a positive Radon measure* $\varrho$ *on* $[0,\infty]$ *such that*

$$\|f\|_*=\left(\int_{[0,\infty]}\left(1+\frac{1}{t}\right)K_p(t,f;L_p,L_p(\lambda))^p\,d\varrho(t)\right)^{1/p},\quad f\in L_p^*.$$

It is not hard to see that the arguments in the proof of Theorem 2.4 admit generalization to all $p$'s, in the sense that Conjecture 9.2 is a consequence of Conjecture 9.1 (for all $1<p<\infty$).

## References


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