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Boundaries and random walks on finitely generated infinite groups

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Abstract. We prove that almost every path of a random walk on a finitely generated non-amenable group converges in the compactification of the group introduced by W. J. Floyd. In fact, we consider the more general setting of ergodic cocycles of some semigroup of one-Lipschitz maps of a complete metric space with a boundary constructed following Gromov. We obtain in addition that when the Floyd boundary of a finitely generated group is non-trivial, then it is in fact maximal in the sense that it can be identified with the Poisson boundary of the group with reasonable measures. The proof relies on works of Kaimanovich together with visibility properties of Floyd boundaries. Furthermore, we discuss mean proximality of $\partial\Gamma$ and a conjecture of McMullen. Lastly, related statements about the convergence of certain sequences of points, for example quasigeodesic rays or orbits of one-Lipschitz maps, are obtained.

1. Introduction

In several situations concerning an infinite group it has been useful to consider an auxiliary space which in some sense is a boundary. For a finitely generated group Γ one could start with the Cayley graph $K(\Gamma, S)$ with respect to some finite set of generators S. The *end-compactification* of the group is the graph itself union the space of ends of this graph and was introduced by H. Freudenthal, see [S]. There are however certain groups with only one end, but for which one would like to have a non-trivial boundary. For example this is the case for fundamental groups of compact negatively curved manifolds. A finer compactification $\bar{\Gamma}=\Gamma\cup\partial\Gamma$ was first used by W. J. Floyd [F1] and it is obtained by rescaling the length one edges in a certain way so that the graph gets finite diameter, then taking the completion of the graph as a metric space. Indeed, such a boundary $\partial\Gamma$ of a fundamental group Γ of a compact surface of genus at least two is the circle. Starting with [F1] this compactification has been used in Kleinian group theory. In this context, we wish to draw the reader's attention to a conjecture stated in a recent paper by C. McMullen [Mc] concerning the existence of a boundary map from the Floyd boundary of a fundamental group into the boundary of hyperbolic three-space.

In this note we prove the convergence of quasigeodesics and paths of certain random walks in a geodesic space to points in a boundary which is constructed following M. Gromov [G] extending Floyd. In particular, when Γ is a finitely generated non-amenable group with a measure μ whose support generates the group, we obtain that the Floyd boundary $\partial\Gamma$ is a μ-boundary in the sense of H. Furstenberg, see [Fu2] and [Ka2]. Using a different approach, by demonstrating certain visibility properties and then relying on work of V. Kaimanovich we obtain that this μ-boundary is in fact either trivial or maximal; in the latter case it is the *Poisson boundary.* In general, it may happen that the boundary is trivial: the Floyd boundary of the product of two finitely generated infinite groups is one point ([F1]).

It was previously known that the Poisson boundary (for reasonable measures) of a group with infinitely many ends can be identified with the space of ends with the hitting measure. See W. Woess [W], D. I. Cartwright and P. M. Soardi [CS], and Kaimanovich [Ka2]. Furthermore, Kaimanovich obtained an identification of the Poisson boundary for hyperbolic groups, see [Ka1] and [Ka2]. See also the work of A. Ancona [A].

Note also the somewhat related results of Floyd [F2] and C. W. Stark [St], which extend the result in the original paper [F1] and concern the comparison of $\partial\Gamma$ with the Furstenberg boundary for rank one symmetric spaces.

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2. Preliminaries on metric spaces

The following material is standard; we borrow some notation from [BHK].

Let (Y,d) be a metric space. The *length* of a continuous curve $\alpha\colon[a,b]\to Y$ is defined to be

$$L(\alpha)=\sup\sum_{i=1}^{k}d(\alpha(t_{i-1}),\alpha(t_i)),$$

where the supremum is taken over all finite partitions $a=t_0<t_1<\ldots<t_k=b$. When this supremum is finite, α is said to be *rectifiable.* For such α we can define the *arc length* $s\colon[a,b]\to[0,\infty)$ by

$$s(t)=L(\alpha|_{[a,t]}),$$

which is a function of bounded variation.

A *geodesic* is a curve β for which

$$d(\beta(t),\beta(t'))=L(\beta|_{[t,t']})=|t-t'|$$

for any t and t'. A metric space is called *geodesic* if any two points can be joined by a geodesic segment.

Given a continuous, (strictly) positive function f on Y, we define the *f-length* of a rectifiable curve α to be

$$L_f(\alpha)=\int_\alpha f\,ds=\int_a^b f(\alpha(t))\,ds(t).$$

If $f\equiv 1$, then $L_f=L$.

Assume from now on that (Y,d) is a geodesic space. A new distance d_f is defined by

$$d_f(x,y)=\inf L_f(\alpha),$$

where the infimum is taken over all rectifiable curves α with $\alpha(a)=x$ and $\alpha(b)=y$. It is straightforward to verify that (Y,d_f) indeed is a metric space and that the two metrics induce the same topology. Note that for a geodesic β, we have the simple bound

$$L_f(\beta)\le L(\beta)\max_{x\in\beta} f(x).$$

3. Definition of f-boundaries

This section defines certain boundaries of a complete metric space. The construction here is a somewhat more restrictive version of the one given by Gromov [G], Section 7.2.K "A conformal view on the boundary", which extends Floyd [F1], which in turn is "based on an idea of Thurston's and inspired by a construction of Sullivan's".

Let (Y,d) be a complete metric space which is geodesic and let f be a continuous, (strictly) positive function on Y. We will assume that for some point y in Y, f is bounded by a monotone real function F in the way

(F1) $$f(x)\le F(d(y,x))\quad \text{for every } x.$$

Furthermore we require that this F is summable:

(F2) $$\sum_{r=1}^{\infty} F(r)<\infty,$$

and that for every $c>0$ there is a number N such that

(F3) $$F(cr)\le NF(r)\quad \text{for all } r\ge 0.$$

Let the *f-boundary* of Y be the space $\partial_f Y:=\overline{Y}_f-Y$, where $\overline{Y}_f$ denotes the metric space completion of (Y,d_f).

Floyd's boundary

This is essentially the construction introduced in [F1]. Let Γ be a group generated by a finite set of elements S. Associated to S there is a left-invariant metric (*word metric*) d on Γ and a one-complex (*Cayley graph*) $K(\Gamma, S)$. The vertices of this graph consist of the elements of Γ, and two vertices are connected by an (unoriented) edge if they differ by an element of S on the left. When the edges are assigned to have length one, the distance d on Γ is simply the geodesic distance in the graph.

Let F be a monic, summable function $F\colon \mathbf{N}\to\mathbf{R}$, such that given $k\in\mathbf{N}$ there exists $L, M, N>0$ so that $MF(r)\leq F(kr)\leq NF(r)$ and $L^{-1}f(r)\leq f(s)\leq Lf(r)$ for all natural numbers r and $r-k\leq s\leq r+k$. (It is common to consider $F(r-1):= r^{-2}$.) We insist for convenience that F is monotonically decreasing and let $f(x)= F(d(x,e))$. Since F is summable the graph now has finite diameter. The *group completion in the sense of Floyd* $\bar{\Gamma}=\Gamma\cup\partial\Gamma$ is (just as above) the completion of the Cayley graph with the new distance d_f as a metric space (i.e. the equivalence classes of Cauchy sequences). The group Γ acts on $\bar{\Gamma}$ by homeomorphisms. If $\partial\Gamma$ consists of only zero, one, or two points, we say that the boundary is trivial.

Examples

It is easy to see that there is a surjection from $\partial\Gamma$ to the space of ends. Therefore, groups with infinitely many ends have a non-trivial Floyd boundary. When Γ is a word hyperbolic group then, under some conditions, the f-boundary of Γ coincides with the standard hyperbolic boundary, see [G], or [CDP] for an exposition with more details. The conjecture stated in [Mc] predicts that the Floyd boundary of a finitely generated fundamental group of a hyperbolic three-manifold is at least as large as the limit set. For geometrically finite Kleinian groups (in every dimension) this was already proven in [F1] and [T1].

It was suggested by Gromov that a Floyd type boundary of a Hadamard space (CAT(0)-space) consists of the set of Tits components of the usual ray boundary.

Let Y be a bounded convex domain equipped with Hilbert's metric and consider its natural extrinsic boundary $\partial_e Y:=\overline{Y}\setminus Y\subset\mathbf{R}^N$. Some calculations of G. Noskov and the author indicate that a Floyd type boundary is $\partial_e Y$ modulo collapsing faces to points and identifying adjacent faces.

4. Ergodic cocycles and μ-boundaries

Let (Y,d), y, f, and $\partial Y := \partial_f Y$ be as in Section 3.

Let (X,ν) be a measure space with $\nu(X)=1$ and let $L: X \to X$ be an ergodic measure-preserving transformation. Let $w: X \to S$ be a measurable map into a semigroup S of selfmaps of Y which does not increase d-distances. Assume that the integrability condition

$$\int_X d(y, w(x)y)\, d\nu(x) < \infty$$

holds. Define the associated *ergodic cocycle* (or *random product*)

$$u(n,x) = w(x)w(Lx) \ldots w(L^{n-1}x)$$

and

$$A := \lim_{n\to\infty} \frac{1}{n} \int_X d(e, u(n,x))\, d\mu(x).$$

A basic observation is that $a(n,x) := d(y, u(n,x)y)$ is a subadditive cocycle. The following purely subadditive ergodic statement was proved in [KM].

Lemma. *For each $\varepsilon>0$, let E_ε be the set of x for which there exist an integer $K=K(x)$ and infinitely many n such that*

$$(\dagger) \qquad a(n,x) - a(n-k, L^k x) \geq (A-\varepsilon)k$$

for all k, $K \leq k \leq n$. Then $\nu\big(\bigcap_{\varepsilon>0} E_\varepsilon\big)=1$.

Using this lemma we can prove the following result.

Theorem. *Assume that $A>0$. Then for almost every x the trajectory $u(n,x)y$ converges to a point $\xi=\xi(x)\in\partial Y$.*

Proof. Fix an $x \in E_\varepsilon$ for some $\varepsilon < A$. For each m denote by y_m the point $u(m,x)y$. Consider an n_i and k, $K \leq k \leq n_i$, such that $(\dagger)$ in the lemma holds, and let β be a geodesic joining y_{n_i} and y_k. Let J be the smallest integer larger than $d(y_{n_i}, y_k)$ and j_0 be the smallest integer larger than $1/\min\{A-\varepsilon, \frac{1}{2}\}$. In the case $J=1$ we have that

$$\begin{aligned} d_f(y_{n_i}, y_k) &= \inf_\alpha L_f(\alpha) \leq \max_t f(\beta(t)) d(y_{n_i}, y_k) \\ &\leq F((A-\varepsilon)k-1) \leq NF(k) \leq N \sum_{r=k-j_0}^{\infty} F(r), \end{aligned}$$

where N corresponds to $c=A-\varepsilon-1/K$ in (F3). In the case $J>1$, note that $\frac{1}{2}\le d(y_{n_i},y_k)/J\le 1$ and let

$$t_j=\frac{j}{J}\,d(y_{n_i},y_k).$$

Using the monotonicity of F, (F1), the triangle inequality, the inequality (†), and (F3) we have

$$\begin{aligned}
d_f(y_{n_i},y_k) &\le L_f(\beta)\\
&\le \sum_{j=1}^{J}\max_{t_{j-1}\le t\le t_j} f(\beta(t))d(\beta(t_{j-1}),\beta(t_j))\\
&\le \sum_{j=1}^{J} F(d(y,\beta(t_j))-1)\\
&\le \sum_{j=1}^{J} F\biggl(a(n_i,x)-\biggl(d(y_{n_i},y_k)-\frac{jd(y_{n_i},y_k)}{J}\biggr)-1\biggr)\\
&\le \sum_{j=1}^{J} F\biggl(a(n_i,x)-a(n_i-k,L^kx)+\frac{j}{2}-1\biggr)\\
&\le \sum_{j=1}^{J} F\biggl((A-\varepsilon)k+\frac{j}{2}-1\biggr)\\
&\le N\sum_{j=1}^{J} F(j+k-j_0)\\
&\le N\sum_{r=k-j_0}^{\infty} F(r).
\end{aligned}$$

Since N and j_0 depend only on $A-\varepsilon$, and because of (F2), it follows that $u(n_i,x)y$ is a Cauchy sequence and moreover that the whole sequence $u(m,x)y$ hence converges to a point in ∂Y. □

We also have the following consequence.

Corollary. *Let Γ be a finitely generated group and μ be a probability measure with finite first moment such that the support of μ generates Γ as a semigroup. Assume that the Poisson boundary of (Γ,μ) is non-trivial (which is the case if Γ is non-amenable). Then almost every sample path of the random walk (Γ,μ) converges to a (random) point of $\partial\Gamma$ (which is defined as in the example in Section 3). The space $\partial\Gamma$ with the resulting limit measure is a μ-boundary.*

Proof. We will apply the theorem to (Y, d) being the group Γ equipped with a left-invariant word-distance. Furthermore, we let $S=\Gamma$, which acts on $Y=\Gamma$ by translations preserving the word-metric. Let (X, ν) be the infinite product of (Γ, μ) and L be the shift. It is a known fact that L is an ergodic measure preserving transformation, and the finiteness of the first moment simply translates into the integrability assumed above.

It is known that when the Poisson boundary is non-trivial, the word length of the trajectory grows linearly (so $A>0$) for almost every trajectory, see [Gu] or Theorem 5.5 in [Ka2]. Hence by the theorem, almost every sample path converges to a point in $\partial\Gamma$.

The resulting measure space, $\partial\Gamma$ with the hitting measure, is a μ-boundary, see [Fu1] or, e.g., [Ka2], p. 660. □

Note that it does not seem to be clear whether the μ-boundary obtained is non-trivial or not. In the next sections however, by combining some observations about the Floyd boundary with works of Kaimanovich, we obtain that whenever the Floyd boundary is non-trivial, then it is in fact maximal.

5. Visibility of Floyd's boundary

Let Γ be a finitely generated infinite group with a boundary $\partial\Gamma$ of Floyd type, see the example in Section 3. We start with a lemma.

Lemma. *Let z and w be two points in Γ and let $[z, w]$ be a geodesic segment connecting z and w. Then*

$$d'(z, w) \le 4rF(r) + 2\sum_{j=r}^{\infty} F(j),$$

where $r = d(e, [z, w])$.

Proof. Let a denote the distance to z from a point m on $[z, w]$ closest to e. Let x_j, $j=0, \dots, a$, be the points (vertices) of the geodesic segment $[m, z] \subset [w, z]$. Because of the minimality of r and the triangle inequality we have the estimates

$$d(e, x_j) \ge r \quad \text{and} \quad d(e, x_j) \ge j - r.$$

For the usual reasons, we hence get

$$\begin{aligned} d'(m, z) &\le \sum_{j=0}^{a} F(\min\{d(e, x_j), d(e, x_{j+1})\}) \\ &\le \sum_{j=0}^{2r-1} F(r) + \sum_{j=2r}^{a} F(j-r) \le 2rF(r) + \sum_{j=r}^{\infty} F(j). \end{aligned}$$

(If $a<2r-1$, then we would not decompose the sum, and so the second term would not be present.) By the same consideration with w instead of z, the lemma is proved. □

Note that with only minor modifications, this proof works for any f-boundary of a geodesic space Y.

Two boundary points γ and ξ are said to be *connected by a geodesic line* if there is a d-geodesic α such that $\alpha(n)\to\gamma$ and $\alpha(-n)\to\xi$, as $n\to\infty$.

Proposition. *Every two points in $\partial\Gamma$ can be connected by a geodesic line.*

Proof. It is known and easy to show (see [F1]) that every point $\gamma\in\partial\Gamma$ can be represented as an endpoint of a geodesic ray from e. Consider the geodesic segments $[\gamma(n),\xi(n)]$ for any two distinct boundary points. It follows from the lemma that the distance r_n from these curve segments to e must be bounded (due to the summability of F and since $\gamma(\infty)\neq\xi(\infty)$). Thanks to the local finiteness of Γ (the Cayley graph) we may now extract a desired geodesic line using Cantor's diagonal argument. □

6. Poisson boundaries

The results in the previous section lead to (in notation and definitions as in [Ka2]) the following theorem.

Theorem. *If $\partial\Gamma$ contains at least three points, then the compactification $\bar{\Gamma}$ of Floyd type satisfies Kaimanovich's conditions* (CP), (CS), *and* (CG).

Proof. It is known that the (isometric) action of Γ by left translation on itself extends to an action on $\bar{\Gamma}$ by homeomorphisms, see for example [K2]. Since $\bar{\Gamma}$ is a compact metric space it is separable. Any two sequences of bounded distance from each other clearly cannot converge to different boundary points; this is (CP).

As shown in the previous section, any two boundary points can be joined by d-geodesics. Therefore the sets (*strips*) $S(\gamma,\xi)$ which equal the union of all joining geodesic lines are non-empty and constitute an equivariant family of Borel maps. Furthermore, for any three distinct boundary points ξ_i, $i=1,2,3$, we may find small neighborhoods U_i in $\bar{\Gamma}$ of each of the three points so that

$$S(U_1,U_2)\cap U_3=\emptyset.$$

To see this, take for U_i small disjoint ε-neighborhoods in the metric d'. Now assume that we can find geodesic lines γ_k in $S(U_1,U_2)$, parametrized so that $\gamma_k(0)$ is a point closest to e and indices $n_k\to\infty$ (because every geodesic line in S intersects a ball around e) such that $\gamma_k(n_k) \to \xi$, where ξ is some point in $U_3 \cap \partial\Gamma$. But then, since the d'-length of every d-geodesic ray is uniformly bounded, it follows that $d'(\gamma_k(n_k), \gamma_k(\infty)) \to 0$, and this cannot happen because the neighborhoods were disjoint. This proves (CS).

The condition (CG): d is a left-invariant metric on Γ and the corresponding gauge is temperate (just meaning in this context that Γ is finitely generated). Finally, as already discussed, for any two boundary points every joining geodesic line intersects the same d-finite radius ball around e (a consequence of the lemma in Section 5). □

We can now invoke the nice arguments of Kaimanovich in [Ka2], Sections 2, 3, 4, and 6, which in particular involve entropy considerations of certain conditional random walks (the strip approximation criterion) as well as a version of the martingale convergence theorem ([Fu2]). Corresponding to [Ka2, Theorem 6.6] we hence have the following result.

Corollary. *Let Γ be a finitely generated group and assume that a Floyd type boundary $\partial\Gamma$ contains more than three points. Let μ be a probability measure on Γ with finite entropy and finite first logarithmic moment, and whose support generates a subgroup which is non-elementary with respect to $\bar{\Gamma}$, that is, it does not fix a finite subset of $\partial\Gamma$. Then the compactification $\bar{\Gamma}$ is μ-maximal.*

The Floyd boundary of a finitely generated amenable group consists of zero, one, or two points since otherwise the group contains non-commutative free subgroups ([K2]). There are however solvable groups for which the Poisson boundary is non-trivial for any (non-degenerate) probability measure with finite entropy (due to Kaimanovich, Vershik and Erschler).

7. Mean proximality

Let us record some further consequences of the previous two sections. For the definitions and a basic account of Furstenberg's boundary theory, we refer to the original source [Fu2] as well as [M, Chapter VI].

Theorem. *Let Γ be a finitely generated group and assume that $\partial\Gamma$ is a non-trivial Floyd boundary. Then $\partial\Gamma$ is a mean proximal Γ-space.*

Proof. In [K2] it is shown that $\partial\Gamma$ is a boundary of Γ in the sense of [Fu2]. Therefore, in particular, Γ does not leave a finite subset of $\partial\Gamma$ invariant. In view of this, the theorem in Section 6, and the elegant arguments of Kaimanovich in [Ka2, Section 2.4], we have that $\partial\Gamma$ is mean proximal. □

If the Floyd boundary is non-trivial for a finitely generated fundamental group of a hyperbolic three-manifold, then by knowing that it can be identified with the Poisson boundary as we now do (Section 6), we hence get a unique (up to null sets) Γ-equivariant *measurable* map onto the limit set. (This is part of the theory of Poisson boundaries, see [Ka2]; for the uniqueness we refer in addition to [M, Chapter VI, Corollary 2.10].) When does this map agree a.e. with a continuous map? This question, together with the issue of the non-triviality of $\partial\Gamma$, is in a sense now the content of the conjecture stated in [Mc].

Added in proof. The non-triviality of $\partial\Gamma$ is in fact not an issue: Let Γ be a finitely generated fundamental group of a hyperbolic three-manifold. It is known from Scott's core theorem and Thurston's uniformization theorem that the group Γ can be made to act as a geometrically finite Kleinian group, see [E], pp. 266–267. Hence we know from [F1] that the boundary $\partial\Gamma$ is infinite (provided that Γ is non-elementary of course) and we have.

Theorem. *Let N be a hyperbolic three-manifold with finitely generated fundamental group and denote by Λ its limit set on the boundary of the hyperbolic three-space. Then there exists a unique (up to null sets) measurable, $\pi_1(N)$-equivariant map*

$$F:\partial\pi_1(N)\to\Lambda.$$

It is perhaps interesting to compare this result with [T2], which also considers measurable boundary maps. In view of contractivity properties, quite generally, a map such as F either agrees with a continuous map or is very discontinuous (the F-image of every neighborhood of a point is dense in Λ). For references to other papers discussing the conjecture, see [Mc].

8. Convergence of certain sequences

Let (Y,d), y, f, and $\partial Y:=\partial_f Y$ be as in the previous section.

Proposition. *Let y_n be a sequence of points in Y for which there exists two positive constants A and C such that $d(y_n,y_{n+1})<C$ and $d(y_n,y)>An$ for all large n. Then y_n converges to a point in ∂Y.*

Proof. Let β be a geodesic joining y_n and y_{n+1}. For every large n we have

$$\begin{aligned} d_f(y_n,y_{n+1}) &= \inf_\alpha L_f(\alpha)\le L_f(\beta)\\ &\le \max_t f(\beta(t))\,d(y_n,y_{n+1})\le F(An-C)C\le CNF(n). \end{aligned}$$

Hence the sequence of points in question is a d'-Cauchy sequence, because F is summable, and N and C are independent of n. As $y_n \to \infty$ in Y, the sequence therefore converges to a point in the boundary of Y. □

As a corollary of the proposition we have that any quasigeodesic ray converges. If $F(r-1)=r^{-(2+\varepsilon)}$, for some positive ε, then a similar argument also shows that any regular sequence in the sense of [Ka2] converges.

Using the argument in the proof of Proposition 5.1 in [K1], the lemma in Section 5 and the inequality

$$(z|w)_y \le d(y,[z,w])$$

one can prove the following proposition.

Proposition. *Let $\phi: Y \to Y$ be a one-Lipschitz map of a complete, geodesic metric space Y and let $\partial_f Y$ be an f-boundary. If $d(\phi^n(y), y) \to \infty$, then $\phi^n(y)$ converges to a point in $\partial_f Y$, as $n \to \infty$.*

Let us emphasize that here, as well as in some other statements in this paper, no local compactness is assumed.

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