Ark. Mat., **41** (2003), 133–150

Totally real discs in non-pseudoconvex boundaries

Egmont Porten

Abstract. Let D be a relatively compact domain in $\mathbf{C}^2$ with smooth connected boundary ∂D. A compact set $K\subset\partial D$ is called removable if any continuous CR function defined on $\partial D\setminus K$ admits a holomorphic extension to D. If D is strictly pseudoconvex, a theorem of B. Jöricke states that any compact K contained in a smooth totally real disc $S\subset\partial D$ is removable. In the present article we show that this theorem is true without any assumption on pseudoconvexity.

1. Introduction

One of the most significant differences between complex analysis in several variables and the classical function theory concerns the theory of holomorphic hulls and removable singularities of holomorphic functions. Since the pioneering discoveries of F. Hartogs, a good deal of the research in several variables has been addressed to the study of analytic extension phenomena related to the geometry of the underlying complex manifolds only and not to special properties (like growth conditions) of the holomorphic functions defined on them.

In the last two decades, the investigation of removable singularities has been systematically extended to the boundary values of holomorphic functions. For a thorough introduction and good surveys on the subject, we refer to [5] and [27]. The notion of removability we shall mainly consider is the following: Let $D\Subset\mathbf{C}^n$ be a domain with smooth connected boundary ∂D. We call a compact subset $K\subset\partial D$ *removable* (for CR functions) if any continuous CR function f on $\partial D\setminus K$ admits an extension $\tilde{f}\in\mathcal{O}(D)\cap C(D\cup(\partial D\setminus K))$. Note that the definition of removability of a compact set depends on the underlying domain D. In dimension $n=2$, the dependence on D is essential whereas deep results of G. Lupacciolu [21] show that for $n\geq3$ the intrinsic properties of K are predominant (at least if we restrict to strictly pseudoconvex domains).

To formulate our main result, we need some terminology. We say that a manifold N is *embedded* in a manifold M if there is an injection $i: N \to M$ and a neighborhood U of $i(N)$ in M such that i is a proper embedding of N into U. A real submanifold S of a complex manifold is called *totally real* if the tangent spaces T_pS never contain complex lines. It is familiar that holomorphic functions defined on the complement extend through totally real submanifolds. In 1988, B. Jöricke [13] discovered the remarkable phenomenon that any compact subset of a totally real disc embedded in a strictly pseudoconvex boundary is removable for CR functions. Our main result shows that this holds true without any assumption on pseudoconvexity.

Theorem 1. *Let $D \subset \mathbf{C}^2$ be a relatively compact domain with smooth connected boundary ∂D and $S \subset \partial D$ be a smoothly embedded totally real disc. Then any compact subset $K \subset S$ is removable for CR functions, i.e. any continuous CR function f on $\partial D \setminus K$ admits a holomorphic extension $F \in \mathcal{O}(D) \cap C(D \cup (\partial D \setminus K))$.*

Some remarks about the background of Theorem 1 are in order.

Remarks. (1) In the strictly pseudoconvex case, F. Forstnerič and E. L. Stout [9] gave the remarkable generalization that the result remains true if we allow S to have finitely many isolated complex points of hyperbolic type (see Section 3 for the definition). A little later, J. Duval [6] published an elegant alternative proof, which works also for certain weakly convex domains.

(2) Also in the non-pseudoconvex case, Theorem 1 should extend to surfaces S containing isolated hyperbolic complex points. F. Forstnerič made the interesting remark that it may also be possible to admit complex points $p \in S$, at which the local hull of S points out of D. For example, one may consider elliptic points contained in the strictly pseudoconcave part of ∂D.

(3) Theorem 1 reflects special properties of complex dimension 2. Let us briefly give some indications about formal generalizations for $n \geq 3$. If one replaces S with a totally real ball of maximal real dimension n, removability becomes a corollary of more general results: For $n=3$, it is a consequence of a theorem of Jöricke [14] on real submanifolds of codimension two in the boundary. For dimension $n \geq 4$, it follows from a result of Lupacciolu and Stout [22] about the removability of metrically thin singularities. Hence it seems more attractive to look at singularities contained in a generic ball of real dimension $2n-2$. But then a recent counterexample of Jöricke and N. Shcherbina [16] exhibits a non-removable singularity contained in a generic four-ball embedded in the unit sphere $S^5 \subset \mathbf{C}^3$.

In order to focus on the essential hypotheses, it seems appropriate to formulate the main result in more local terms. So we consider a hypersurface $M \subset \mathbf{C}^2$. An open subset $W \subset \mathbf{C}^2$ is a *one-sided neighborhood* of M if for each $p \in M$ there is a small ball B around p such that at least one component of $B \setminus M$ is contained in W. We call M *globally minimal* if any pair of points in M can be joined by a CR curve $\gamma \subset M$, i.e. a piecewise differentiable curve γ whose (one-sided) derivatives are contained in the complex tangent bundle $T^c M$.

The hypersurface version of our result is the following.

Theorem 2. *Let M be a smooth globally minimal hypersurface of $\mathbf{C}^2$ and $S \subset M$ be a smoothly embedded totally real disc. Then for any compact subset $K \subset S$ there is a one-sided neighborhood W of M such that any continuous CR function f on $M \setminus K$ admits an extension $F \in \mathcal{O}(W) \cap C(W \cup (M \setminus K))$.*

Finally we mention an application of our methods to the theory of singularities of L^p-solutions of differential operators. For a general introduction to the topic, we refer to the articles [11] and [15], for further results concerning CR manifolds to [2], [18] and [24]. A closed subset $A \subset M \subset \mathbf{C}^2$ is called $(L^p_{\mathrm{loc}}, \bar\partial_b)$-removable if any function $f \in L^p_{\mathrm{loc}}(M)$ satisfying the tangential Cauchy–Riemann equations on $M \setminus A$ (in distributional sense) is CR on all of M. General structure theorems of R. Harvey and J. Polking [11], who considered the problem for general linear partial differential operators, yield in our case that any closed subset satisfying $\mathcal{H}_{3-p'}(A) < \infty$ is $(L^p_{\mathrm{loc}}, \bar\partial_b)$-removable (for $1 < p < \infty$ and $1/p + 1/p' = 1$) and that $\mathcal{H}_2(A) = 0$ implies $(L^\infty_{\mathrm{loc}}, \bar\partial_b)$-removability.

The next theorem shows that for CR functions much stronger phenomena are true. In particular, we obtain information on L^1 removability, which cannot be obtained by the methods of [11].

Theorem 3. *Let M be a smooth globally minimal hypersurface of $\mathbf{C}^2$ and $S \subset M$ be a smoothly embedded totally real disc. Then any compact subset $K \subset S$ is $(L^p_{\mathrm{loc}}, \bar\partial_b)$-removable for $p \geq 1$.*

For strictly pseudoconvex boundaries this result was established in [2]. In the general setting, very easy examples, where M may be chosen as a real hyperplane of $\mathbf{C}^2$ for instance, show that $(L^1_{\mathrm{loc}}, \bar\partial_b)$-removability cannot be true without any assumptions on the CR orbits.

This paper is organized as follows: In Section 2 we sketch the lines of our proof in the familiar strictly pseudoconvex case and discuss the additional difficulties arising in the general proof. In Section 3 we collect some preliminary material concerning CR orbits and semi-local analytic extension including the reduction of Theorem 1 to Theorem 2. In Section 4 we prove Theorems 2 and 3 modulo a deformation lemma about holomorphic discs, which is postponed to Section 5.

Acknowledgements. It is a pleasure to thank my academical teacher B. Jöricke for the introduction to the subject. Next I would like to thank J. Merker, who rearoused my interest in the topic by a clever question, and N. Eisen, F. Forstnerič and W. Klingenberg for fruitful discussions. I am greatful to J. Globevnik for his kind support in the proof of Lemma 6.

2. The strictly pseudoconvex case

In this section we sketch a new proof of Theorem 1 for strictly pseudoconvex domains. Afterwards we will discuss the specific difficulties arising for arbitrary hypersurfaces. We hope that the detours of the general proof of Theorem 2 are less confusing after the examination of the simplified setting.

Proof of Theorem 1 *for D strictly pseudoconvex.* Given a CR function f on $\partial D\backslash K$, we have to construct a holomorphic extension to D. By the Hartogs–Bochner theorem it is enough to find a holomorphic extension F on a one-sided neighborhood of ∂D (which will of course be contained in D by strict pseudoconvexity). The construction of F shall be performed in three steps.

Step 1. *Semi-local extension near $\partial D\backslash K$.* As D is strictly pseudoconvex, a classical local result of H. Lewy [19] allows us to extend f holomorphically to a small one-sided neighborhood of each point in $\partial D\backslash K$. These extensions glue together and yield an extension on a one-sided neighborhood $V\subset D$ of $\partial D\backslash K$. After deforming $\partial D\backslash K$ slightly into V, we may assume that f is holomorphic near $\partial D\backslash K$.

Step 2. *Construction of nice holomorphic discs.* In order to construct an analytic extension of f to a one-sided neighborhood W attached to a neighborhood of K in ∂D, we shall employ Bishop discs. In [13], a convenient family is constructed explicitly. With regard to the non-pseudoconvex case, we shall instead apply the powerful existence theorem of E. Bedford and W. Klingenberg [3] that every generic two-sphere contained in a strictly pseudoconvex boundary can be filled by a Levi-flat three-ball. More precisely, we will embed K in a two-sphere $\Sigma\subset\partial D$ and remove K by using analytic discs glued to Σ and to slightly translated copies of Σ.

Before proceeding, we have to recall the main ingredients of the Bedford–Klingenberg theorem. It is formulated for spheres Σ with finitely many isolated complex points. As observed by Bishop [4], Σ may be written near a complex point as a graph of the form

$$w=z\bar{z}+\gamma(z^2+\bar{z}^2)+o(|z|^2),$$

where $\gamma \geq 0$ is an invariant. The complex point is called hyperbolic, parabolic, or elliptic if $\gamma < \frac{1}{2}$, $\gamma = \frac{1}{2}$, or $\gamma > \frac{1}{2}$, respectively. Generically complex points are either hyperbolic or elliptic. Then the Bedford–Klingenberg theorem (in a refined version due to Kruzhilin [17]) implies that a sphere Σ of class C^6 contained in a strictly pseudoconvex boundary, bounds a unique Levi-flat three-ball B. Furthermore B is foliated by a one-parameter family of analytic discs Δ_t attached to Σ. An analytic disc attached to Σ is a holomorphic mapping $\Delta: \mathbf{D} \to \mathbf{C}^2$ which extends continuously to $\mathbf{T}$ and fulfills $\Delta(\mathbf{T}) \subset \Sigma$ ($\mathbf{D}$ denoting the open unit disc and $\mathbf{T}$ its boundary). We shall often write Δ and $\partial\Delta$ also for the unparametrized sets $\Delta(\bar{\mathbf{D}})$ and $\Delta(\mathbf{T})$, respectively. The discs Δ_t are differentiable up to the boundary and transverse to ∂D except at finitely many points where they touch Σ tangentially at hyperbolic complex points.

In the strictly pseudoconvex case, the construction of $\Sigma \subset \partial D$ is straightforward. Let $S' \Subset S$ be a disc with smooth boundary containing K. We construct Σ by choosing a nearby almost parallel copy $S'' \subset \partial D \setminus S$, and gluing S' and S'' along the boundaries. If necessary, we put Σ in general position by a slight deformation leaving S' unchanged. As the normal bundle of Σ in ∂D is trivial, the choice of a thin tubular neighborhood V of $\Sigma = \Sigma_0$ in ∂D will give us a foliation of V by generic spheres Σ_t, $-\varepsilon < t < \varepsilon$.

Next the Bedford–Klingenberg theorem associates to every Σ_t, a Levi-flat three-ball $B_t \subset D$ with $\partial B_t = \Sigma_t$. Each B_t is fibered by a one-parameter family of holomorphic discs $\Delta_{t,s}$. By transversality, we see that the $\Delta_{t,s}$ induce a foliation on a domain $W \subset D$ containing K in its boundary. We may choose W as a one-sided neighborhood attached to some neighborhood of K in ∂D and suppose that it is divided into two parts by the hypersurface $H = W \cap B_0$.

Step 3. *Extension to* W. As already observed, it remains to extend f to W. Applying the continuity principle along the three-balls B_t, $t \neq 0$, we get a holomorphic extension of f to $W \setminus H$. Since S is totally real, none of the discs $\Delta_{0,s} \subset B_0$ has boundary in K. Indeed, we may apply the classical Poincaré–Bendixson theory to the foliation $\mathcal{F}$ induced on Σ by the boundaries $\partial\Delta_{0,s}$. The singular points of $\mathcal{F}$ are precisely centers at elliptic complex points and saddle points at hyperbolic complex points of Σ. If a boundary $\partial\Delta_{0,s}$ does not meet the complex points, then $\Sigma \setminus \Delta_{0,s}$ is a union of two discs, each containing at least one elliptic complex point of Σ. Being totally real, S cannot contain such a boundary.

Hence every $\Delta_{0,s}$ passes through the region where f is holomorphic. Contracting W if necessary, we may suppose that each leaf of the Levi-flat manifold $H = W \cap B_0$ does so. But then Lemma 4.5 of [15] (see Lemma 6 in Section 4) implies that f extends holomorphically to W. $\square$

Let us now discuss the *modifications required in the general case.*

In Step 1 the local result of Lewy is of course no longer applicable. Instead standard results about CR orbits and propagation of analytic extension readily fill in the gap.

As ∂D is no longer strictly pseudoconvex, not every two-sphere in ∂D must be fillable by a Levi-flat three-ball [7]. In order to carry over Step 2, we employ an additional argument: The key observation is that the squared distance function δ_S^2 is strictly plurisubharmonic in a neighborhood of S. Modifying the level sets of δ_S^2, we shall construct a family of strictly pseudoconvex domains G_t (all diffeomorphic to the four-ball) such that the intersections $\Sigma_t = \partial D \cap \partial G_t$ give the desired two-spheres.

Step 3 seems to offer the most serious resistance to generalization. Indeed, the discs $\Delta_{t,s}$ will surely not sweep out an open set. For example the presence of large Levi-flat parts in ∂D may imply that all $\Delta_{t,s}$ are contained in ∂D! In addition changes of sign of the Levi-form of ∂D may imply that discs $\Delta_{t,s}$ flip over to the other side of ∂D. Hence we cannot hope for the nice global geometry of the pseudoconvex case.

Fortunately the final argument can be localized in the following manner: We shall replace K by the smaller compact set K' of points where one-sided holomorphic extension of f fails. If $K' \neq \emptyset$, then we shall derive a contradiction by repeating the argument of Step 3 locally near a well-chosen point $p \in K'$.

3. Semi-local extension near hypersurfaces

In this section, we recall some known material on semi-local extension of CR functions. For the reader's comfort, we sketch proofs where the special cases we need are much easier than the original results in the literature.

Let $M \subset \mathbf{C}^2$ be a smooth hypersurface. Two points $p, q \in M$ are contained in the same CR orbit $\mathcal{O}(p, M)$ of M if they are joined by a piecewise smooth CR curve $\gamma \subset M$, i.e. a chain of smooth curves tangent to $T^c M$. By a fundamental observation of H. Sussmann [28], [14], a CR orbit is either an open subset of M or an injectively immersed Riemann surface, and the union of all CR orbits of codimension one is relatively closed in M. If M has only one CR orbit, it is called globally minimal. We recall from [14] the fact which we use in the reduction of Theorem 1 to Theorem 2.

Lemma 1. *Let $D \Subset \mathbf{C}^2$ be a domain with smooth connected boundary ∂D. Then ∂D is globally minimal.*

Proof. If not, then the union A of the lower-dimensional orbits would be a closed union of Riemann surfaces. A standard argument yields a contradiction to the maximum principle. □

Lemma 1 easily implies the *reduction of Theorem* 1 *to Theorem* 2. Theorem 2 gives an analytic extension of f to a one-sided neighborhood W of ∂D. We may choose W connected and derive Theorem 1 by an application of the Hartogs–Bochner theorem.

Next we look at the interplay between the orbits and the singularity sets.

Lemma 2. *Suppose that M and K are as in Theorem* 2. *Then $M\backslash K$ is globally minimal.*

Proof. Being contained in a totally real surface, K cannot contain CR orbits of M. To reach a contradiction, we assume the existence of $p\in M\backslash K$ such that $\mathcal{O}(p, M\backslash K)$ is a Riemann surface. Obviously it suffices to show that

$$\mathcal{O}(p, M\backslash K) = \mathcal{O}(p, M)\backslash K$$

in order to derive a contradiction to the global minimality of M. We explain how to reconstruct $\mathcal{O}(p, M)$ from $\mathcal{O}(p, M\backslash K)$ by transverse gluing along K.

Since S is totally real, the intersection $l_z = T_zS\cap T^cM$ is a real line for each $z\in S$. Integration of the line field l_z gives a foliation by curves on S. Because S is a disc, Poincaré–Bendixson theory tells us that no trajectory can be contained in K. Hence for any $q\in\mathcal{O}(p, M)\cap K$, there is a trajectory γ connecting q with some point $p'\in\mathcal{O}(p, M)\cap(S\backslash K)$.

Near γ we choose a non-vanishing CR vector field X tangent to S. This means, in particular, that X is collinear with l_z along S. Let D be a small disc in $\mathcal{O}(p, M\backslash K)$ around p'. There is $t_0\in\mathbf{R}$ such that $q=\Phi_{X,t_0}(p')$, where Φ_{X,t_0} denotes the time-t_0-map defined by integrating X. As Φ_{X,t_0} is a local diffeomorphism, $\Phi_{X,t_0}(D)$ is again a smooth disc. Since S is totally real, we derive that D and $\Phi_{X,t_0}(D)$ are transverse to S. More precisely, we see that $\Phi_{X,t_0}(D)\cap S$ is contained in the integral curve through q. Now $\Phi_{X,t_0}(D)\backslash S$ is tangent to T^cM, since $\Phi_{X,t_0}(D)\backslash S$ is an open subset of $\mathcal{O}(p, M\backslash K)$. By continuity, $\Phi_{X,t_0}(D)$ is tangent to T^cM everywhere. Hence $\Phi_{X,t_0}(D)$ is a Riemann surface.

As $\Phi_{X,t_0}(D)$ is a local integral manifold of T^cM, the germ of $\Phi_{X,t_0}(D)$ in q does not depend on the choices made during the construction. For the same reason we obtain globally an immersed Riemann surface $\mathcal{O}$ by applying the above procedure to every point of $\mathcal{O}(p, M)\cap K$. By construction $\mathcal{O}$ is the union $\mathcal{O}(p, M\backslash K)\cup\mathcal{O}(p, M)\cap K$. The very definition of CR orbits yields $\mathcal{O}=\mathcal{O}(p, M)$, a contradiction. □

4. Non-pseudoconvex hypersurfaces

In this section we prove Theorem 2 using an auxiliary argument on deformation of analytic discs, which shall be treated in Section 5.

Proof of Theorem 2. As far as possible, we follow the subdivision of the proof sketched in Section 2.

Step 1. *Semi-local extension near* $M\backslash K$. Let f be a continuous CR function on $M\backslash K$. By Lemma 2, $M\backslash K$ is globally minimal. Hence $M\backslash K$ contains a minimal point p of M, i.e. there is no local holomorphic curve contained in M which passes through p (a hypersurface without minimal points is Levi-flat). By Trépreau's theorem [29], f extends analytically to a one-sided neighborhood attached to some neighborhood of p in $M\backslash K$. But one-sided analytic extensions propagates along CR curves, according to another result of Trépreau [30]. By a standard gluing argument, we obtain a holomorphic extension of f to a one-sided neighborhood U of $M\backslash K$. After deforming $M\backslash K$ slightly into U, we may henceforth assume that f is holomorphic near $M\backslash K$.

Step 2. *Embedding K in a pseudoconvex boundary.* The following lemma contains the construction of good spheres in M.

Lemma 3. *In the situation of Theorem* 2, *let $S'\Subset S$ be a relative neighborhood of K in S and V be a neighborhood of S in $\mathbf{C}^2$. Then there is a smoothly embedded hypersurface $N\subset V$ with the following properties*:

(1) *N is strictly pseudoconvex and diffeomorphic to the three-sphere*;
(2) *N intersects M transversally in a two-sphere $\Sigma=M\cap N$*;
(3) $S'\subset\Sigma$.

Proof. After enlarging S', we may assume that S' is a disc with smooth boundary. We shall use the following well-known fact: Let $\mathcal{S}$ be a totally real submanifold of a complex manifold $\mathcal{M}$ equipped with a smooth riemannian metric μ. Then the distance function $\delta^2_{\mathcal{S},\mu}$ is strictly pseudoconvex in a neighborhood of $\mathcal{S}$. (In [1] the reader finds a proof for the case $\mathcal{M}=\mathbf{C}^2$ equipped with the euclidean metric, which generalizes easily.)

We denote by $\mathbf{D}$ the open unit disc in $\mathbf{C}$. By the tubular neighborhood theorem, there is a smooth diffeomorphism $G:\mathbf{D}\times\mathbf{D}\to\mathbf{C}^2$ taking $\mathbf{D}\times\{0\}$ to some neighborhood of $\bar{S}'$ in S. One easily arranges $G(\mathbf{D}\times\{(x,y)\in\mathbf{D}:y=0\})\subset M$. For μ we take the pushforward of the euclidean distance on $\mathbf{D}\times\mathbf{D}$ with respect to G. For some $\delta\in(0,1)$ we have $\bar{S}'\subset G(\{\zeta\in\mathbf{D}:|\zeta|<1-\delta\}\times\{0\})$. Choose a function $\eta\in C^2([0,1])$

with $\eta|_{[0,1-\delta]}\equiv 0$ and $\eta'(t)>0$ for $t>1-\delta$. Let π_1 and π_2 denote the projections to the first and second coordinate of $(w_1, w_2)\in\mathbf{C}^2$, respectively. Then for a sufficiently small $\varepsilon_1>0$, the function

$$\varrho(z)=|\pi_2\circ G^{-1}(z)|^2+\varepsilon_1\eta(|\pi_1\circ G^{-1}(z)|)^2$$

is strictly plurisubharmonic in some neighborhood V_1 of $G(\{\zeta\in\mathbf{D}:|\zeta|\le 1-\delta\}\times\{0\})$. For small $t>0$, the level sets $\{z:\varrho(z)=t\}$ are diffeomorphic to three-spheres intersecting M transversally in two-spheres.

We take V_2 with $\overline{S}'\subset V_2\Subset V_1$. For a very small $\varepsilon_2>0$, the translated function

$$\varrho(z)=|\pi_2\circ G^{-1}(z)+(\varepsilon_2, 0)|^2+\varepsilon_1\eta(|\pi_1\circ G^{-1}(z)|)^2$$

is still strictly plurisubharmonic on V_2, and its level sets for small positive t are transverse to M. By construction we have $S'\subset\{z:\varrho(z)=\varepsilon_2^2\}$ and get therefore the desired strictly pseudoconvex three-sphere by setting $N=\{z:\varrho(z)=\varepsilon_2^2\}$. $\square$

In order to apply the Bedford–Klingenberg theorem, we put Σ into general position without changing it near S'. Of course, this may be achieved by deforming N conveniently and taking the intersection with M. As explained in Section 2, Σ bounds a Levi-flat three-ball B, which is foliated by a family Δ_s of analytic discs.

Step 3. *Localization of the argument.* Now we have to reorganize the logic of the proof in Section 2 in order to localize the concluding argument.

Let K' be the set of all points $p\in K$ for which there is no one-sided neighborhood U_p attached to some neighborhood V_p of p in $M\setminus K$ to which f extends holomorphically (in other words, there is no function $f_p\in\mathcal{O}(U_p)\cap C(U_p\cup(V_p\setminus K))$ coinciding with f on $V_p\setminus K$). By definition, K' is a compact subset of K. Clearly, it is enough to show $K'=\emptyset$.

Assume that $K'\neq\emptyset$ to derive a contradiction. We distinguish two cases.

First, we consider the case when K' is contained in a finite union of boundaries $\partial\Delta_{s_1}, \dots, \partial\Delta_{s_k}$. As K' cannot contain the whole boundary of a disc Δ_s, K' is contained in a finite union A of proper subarcs of $\partial\Delta_{s_1}, \dots, \partial\Delta_{s_k}$. According to Theorem 4 of [25], A is removable in the sense of one-sided analytic extension, and we get a contradiction to the definition of K'. (If $M=\partial D$, a combination of the theorem of G. Stolzenberg [26] on polynomial convexity of arcs and a removability theorem of C. Laurent-Thiébaut [20] can also be applied.)

It remains to examine the case when K' is not contained in a finite union of boundaries $\partial\Delta_s$. Then there is a disc $\Delta_{s'}$ whose boundary $\partial\Delta_{s'}$ intersects K' non-trivially and lies in the totally real part of Σ. Indeed, the construction of

Bedford–Klingenberg shows that only the boundaries of finitely many of the discs Δ_s pass through (hyperbolic) complex points of Σ.

As $\partial\Delta_{s'}\not\subset K'$, there is a point $\tilde{p}\in\partial\Delta_{s'}\cap K'$ such that there are locally no points of K' on the right-hand side of $\tilde{p}$ in $\partial\Delta_{s'}$. We may fix, for a moment, a holomorphic parametrization $\Delta_{s'}:\mathbf{D}\to\mathbf{C}^2$ mapping $1\in\mathbf{T}$ to $\tilde{p}$. If $(\partial/\partial\zeta)\Delta_{s'}(1)\notin T^c_{\tilde{p}}M$, then $\Delta_{s'}$ touches M in $\tilde{p}$ tranversally, and we can immediately pass to Step 4.

The case when $(\partial/\partial\zeta)\Delta_{s'}(1)\in T^c_{\tilde{p}}M$ causes additional technical difficulties. The strategy consists of deforming Σ slightly so that a deformed disc passing through K' gets locally transverse to M. In preparation, we first have to replace $\tilde{p}$ by a nearby point p enjoying more convenient properties with respect to the geometry of K'. The idea is to choose p together with a hypersurface $L\subset M$ through p such that K' lies locally in the closure of one side of L. We shall frequently denote by $\Delta_{s(p)}$ the disc whose boundary passes through the point p.

Lemma 4. *There are a point p as close to $\tilde{p}$ as we please, a neighborhood $U=U_p$ of p in M, and a smooth embedded disc $L\subset U$ with the following properties*:

(a) *L is transverse in p to the boundary of the disc $\Delta_{s(p)}$ passing through p*;

(b) *L divides U into two components U^+ and U^-, and $K'\cap U\subset U^+\cup L$.*

Proof. The boundaries of the discs Δ_s induce a smooth foliation $\mathcal{G}$ of S near $\tilde{p}$. Let v be a smooth non-vanishing vector field on a neighborhood of $\tilde{p}$ in M which is tangent to $\mathcal{G}$ along S. Let $F\subset M$ be a small smooth disc passing through $\tilde{p}$ which is transverse to v. Near $\tilde{p}$ we get smooth coordinates (w,t), $w\in F$, by integrating v up to the time t with initial values $w\in F$. Of course we may assume that the coordinates (w,t) range over $\{w\in\mathbf{R}^2:|w|<\varepsilon\}\times(-\varepsilon,\varepsilon)$ and that $\tilde{p}$ corresponds to the origin.

As $K'\cap\Delta_{s(\tilde{p})}$ lies to the right of $\tilde{p}$, for any $t'<0$ of small modulus, the point $(0,t')$ does not lie in K'. As K' is closed, the same holds for the set $\{(w,t'):|w|<\delta\}$, if $\delta>0$ is sufficiently small. Following [23], we consider for $\tau>0$ the family of ellipsoids

$$E_\tau=\left\{(w,t):|t-t'|^2+\frac{|w|^2}{\tau}<\delta\right\}.$$

Evidently $\overline{E}_\tau\cap K'=\emptyset$ for sufficiently small τ. Hence we get the desired disc L as a hemisphere of ∂E_{τ_0}, where τ_0 is minimal with the property $\overline{E}_\tau\cap K'\neq\emptyset$. $\square$

Choosing p as in Lemma 4 sufficiently close to $\tilde{p}$, we may assume that the boundary of the disc $\Delta_{s(p)}$ through p is contained in the totally real part of Σ. Of course, $\Delta_{s(p)}$ may touch M tangentially also in p. But now the following deformation lemma is available.

Lemma 5. *Let $q \in \partial\Delta_{s(p)}, q \neq p$, and U_q be a neighborhood of q. Then there are smooth manifolds M^d, N^d and $\Sigma^d = N^d \cap M$ with the following properties:*

(1) *The manifolds M^d and N^d are as C^2-close as we please to the original manifolds M and N, respectively, and coincide with them outside of U_q. Furthermore N^d is again strictly pseudoconvex.*

(2) *The manifolds M^d and N^d intersect transversely in a two-sphere Σ^d. The construction of Bedford–Klingenberg applies to Σ^d and furnishes a filling by holomorphic discs Δ_s^d.*

(3) *The disc $\Delta_{s(p)}^d$ passing through p is as C^2-close to $\Delta_{s(p)}$ as we please and touches M transversally in p.*

Similar results were proved by Trépreau [30], Tumanov [31], and others, for small discs attached to CR manifolds (of positive CR dimension). In addition, the result seems completely natural in view of what was proved by Forstnerič [8], Globevnik [10], and others, for large discs attached to totally real manifolds. As there does not seem to be a proof in the literature, we shall supply the argument in the next section.

Step 4. *Holomorphic extension to a one-sided neighborhood.* We choose deformed objects as in Lemma 5. For the sake of simplicity, we denote Σ^d, N^d and Δ^d again by Σ, N and Δ. We may include $N = N_0$ into a one-parameter family N_t of strictly pseudoconvex boundaries forming a foliation near N_0. For each t, we obtain a filling of $\Sigma_t = N_t \cap M$ by analytic discs $\Delta_{t,s}$.

Consider the local situation near p: As $\Delta_{0,s(p)}$ (the analytic disc provided by Lemma 5) is C^2-close to the original disc (before deformation), its boundary $\partial\Delta_{0,s(p)}$ is still transverse in p to the disc L of Lemma 4 (we have left L unchanged during the deformation). Therefore the intersection $\partial\Delta_{0,s(p)} \cap K'$ still lies locally on the right-hand side of p in $\partial\Delta_{0,s(p)}$.

We may now choose a holomorphic parametrization $\Delta_{0,s(p)}: \mathbf{D} \to \mathbf{C}^2$ such that $\Delta_{0,s(p)}(1) = p$ and extend it to the nearby discs $\Delta_{t,s}$ with C^2 dependence on the parameters s and t. Since $\Delta_{0,s(p)}$ touches M transversally in p, the set

$$W = \bigcup_{\substack{|t|<\varepsilon \\ |s-s(p)|<\varepsilon}} \Delta_{t,s}(\omega_\varepsilon), \quad \text{where } \omega_\varepsilon = \{\zeta \in \mathbf{D} : |1-\zeta| < \varepsilon\},$$

is a one-sided neighborhood of p, if $\varepsilon > 0$ is sufficiently small. Furthermore W is foliated by the analytic curves $\Delta_{t,s}(\omega_\varepsilon)$.

The set $H = \bigcup_{|s-s(p)|<\varepsilon} \Delta_{0,s}(\omega_\varepsilon)$ is a Levi-flat hypersurface of W. Applying the continuity principle along the discs attached to the spheres Σ^t, $t \neq 0$, we extend f

holomorphically to $W\setminus H$. It remains to extend f holomorphically to W, where we are allowed to contract W near p. This is a consequence of Lemma 4.5 of [15]. For the reader's convenience, we state the required special case and sketch the proof.

Lemma 6. *Let $\Omega\subset\mathbf{C}^2$ be a domain and $H\subset\Omega$ be an embedded hypersurface of class C^2. Then for every closed set $A\subset H$ which does not contain a CR orbit of H we have $\mathcal{O}(\Omega\setminus A)=\mathcal{O}(\Omega)$.*

Proof. Fix $g\in\mathcal{O}(\Omega\setminus A)$ and let A' be the set of all $z\in A$ such that g does not extend holomorphically to a neighborhood of z. To obtain a contradiction we assume that $A'\neq\emptyset$.

Since A' does not contain a CR orbit of H, there is a point $z'\in A'$ and a differentiable curve $\gamma:[0,1]\to H$ which is tangent to T^cH such that $\gamma([0,1))\cap A'=\emptyset$ and $\gamma(1)=z'$. Deforming H slightly along $\gamma([0,1))$ it is possible to produce a hypersurface $H^d\supset A'$ such that z' is a minimal point of H^d. Then the theorem of Trépreau [29] implies that g extends holomorphically to a neighborhood of z', in contradiction to the definition of A'. □

The CR orbits of H are the intersections $\Delta_{0,s}(\omega)\cap W$. As we chose p such that there are no points of K' to the left of p on $\partial\Delta_{0,s(p)}$, all discs passing through W will meet the domain U where f is already holomorphic. Hence Lemma 6 concludes the proof of Theorem 2. □

Proof of Theorem 3. As the passage from holomorphic hulls to statements on L^p removability is treated in detail in the literature, we shall only briefly indicate how to complete the arguments of the previous proof. The reader may find careful expositions of the required methods in [15] and [24].

Let $f\in L^1_{\mathrm{loc}}(M)$ be CR on $M\setminus K$. The first step of the proof of Theorem 2 translates without pain to locally integrable functions. After a slight deformation of $M\setminus K$, we may assume f holomorphic near $M\setminus K$.

In the sequel all arguments carry over with the only (obvious) modification that we have to consider the set K' of points where f is not locally CR. To derive a contradiction to the assumption $K'\neq\emptyset$, we construct again a holomorphic extension F of $f|_{M\setminus K'}$ on a one-sided neighborhood of some well chosen point $q\in K'$. So it remains to verify local Hardy space estimates for F in order to recover the restriction of f to a small neighborhood of q as a weak L^1-limit of F. Since L^1-limits of holomorphic functions are CR, this finishes the proof. The local Hardy space estimates can be derived from Carleson's embedding theorem in the same way as in [2] and [15]. □

5. Deformation of analytic discs

In this section we prove Lemma 5. Let us first simplify the notation by eliminating those elements which were only relevant in the context of Section 4. We are given an analytic disc Δ ($\Delta_{t(p)}$ in Section 4) whose boundary is contained in a totally real surface $R\subset\mathbf{C}^2$. On $\partial\Delta$ we consider two distinguished points $p\neq q$. After reparametrization we may suppose that $p=\Delta(1)$ and $q=\Delta(-1)$. Recall that we use Δ to denote both the mapping itself as well as its image $\Delta(\overline{\mathbf{D}})$. Likewise $\partial\Delta$ denotes $\Delta(\mathbf{T})$.

The interplay between the geometry of R and Δ is characterized by the Maslov index of Δ with respect to R. In our context the following version of the normal Maslov index is very convenient (see [12] for a careful introduction and relations to different formulations of the Maslov index): Choose a trivialization $N\Delta\cong\Delta\times\mathbf{R}^2=\Delta\times\mathbf{C}$ of the oriented normal bundle of Δ. Then the restricted tangent bundle $TR|_{\partial\Delta}$ defines a one-dimensional subbundle L of $N\Delta|_{\partial\Delta}\cong S^1\times\mathbf{C}$. A continuous unitary section of L over $\partial\Delta\setminus\{p\}$ corresponds to a function $g: S^1\setminus\{1\}\to S^1$. It is immediate that its square g^2 extends continuously to S^1. The winding number of g^2 is called the normal Maslov index $\mu(\Delta, R)$ of Δ.

From the construction of Bedford–Klingenberg, it follows that each disc of the constructed family whose boundary is contained in the totally real part of Σ has normal Maslov index zero. Hence it is enough to prove the following.

Lemma 7. *Let Δ be an embedded analytic disc attached to a smooth totally real surface R. Assume that the normal Maslov index of Δ is zero. Then for every neighborhood U of $q=\Delta(-1)$, there is a deformation R^d of R arbitrarily close to R in the C^2-sense and coinciding with R outside of U, and an analytic disc Δ^d attached to R^d, with $\Delta^d(1)=p$, such that $(d/d\zeta)\Delta^d(1)$ is not a complex multiple of $(d/d\zeta)\Delta(1)$.*

Before giving the proof we first derive Lemma 5. We choose $R\subset\Sigma$ as a thin annulus containing $\partial\Delta_{s(p)}$. The preceding lemma gives a C^2-small deformation R^d of R such that the corresponding disc $\Delta^d_{s(p)}$ touches R^d transversally in p. The support of the deformation being contained in a sufficiently small neighborhood of q, we obtain a deformed sphere Σ^d by an obvious gluing. Finally we easily find adapted deformations M^d and N^d satisfying all the requirements of Lemma 5.

Proof. The rough idea is as follows: The union of the discs attached to R which are close to Δ forms a Levi-flat hypersurface H, which contains some open part of R in its boundary. Near q we construct a one-parameter family R_t by deforming R in the direction transverse to H. As a consequence, the direction of the deformed discs in p shall also turn out of H.

To describe the local arguments, we shall closely follow the paper [8] of F. Forstnerič. Observe first that Forstnerič introduces a slightly different notion of index, which equals one in our situation. Lemma 10 in [8] gives us $\mathbf{C}^2$-valued vector fields $X(\theta)$ and $Y(\theta)$ ($\theta\in\mathbf{R}/2\pi\mathbf{Z}$ denoting the arc length on $\mathbf{T}$) and a 2×2 matrix-valued function B with the following properties:

(1) The functions B and B^{-1} have coefficients in $\mathcal{O}(\mathbf{D})\cap C^{k,1/2}(\overline{\mathbf{D}})$, where we shall assume that k is sufficiently large in what follows.

(2) The vector fields $X(\theta)$ and $Y(\theta)$ form a basis of $T_{\Delta(\theta)}R$ for all θ. Furthermore we have $X(\theta)=(d/d\theta)\Delta(\theta)$ for all θ (this can be read off from the proof of Lemma 10 in [8]).

(3) The identities $X(\theta)=B(\theta)X_0(\theta)$ and $Y(\theta)=B(\theta)Y_0(\theta)$ hold, where $X_0(\theta)=(ie^{i\theta},0)$, $Y_0(\theta)=(\alpha(\theta),1)$ and $\alpha\in\mathcal{O}(\mathbf{D})\cap C^{k,1/2}(\overline{\mathbf{D}})$.

We may suppose p to be the origin of $\mathbf{C}^2$. Let Z be a smooth extension of the vector field iY to a neighborhood $U\subset R$ of q. Fix a smooth bump function $\chi\geq0$ on R, with small support in U, which equals one near q. For small $t\geq0$, we define R_t as the hypersurface which coincides with the image of the mapping $z\in R\mapsto z+tZ(z)$ near q, and with R elsewhere. Then Theorem 1 in [8] says that there is, up to reparametrization by automorphisms of the unit disc, a unique analytic disc Δ_t attached to R_t with $0\in\partial\Delta_t$. We choose holomorphic parametrizations $\Delta_t\colon\mathbf{D}\to\mathbf{C}^2$, with $\Delta_t(1)=0$, which depend C^3 on t.

As observed in [8], $\Delta_t|_\mathbf{T}$ may be parametrized (as any differentiable mapping from $\mathbf{T}$ to $\mathbf{C}^2$ which is close to $\Delta|_\mathbf{T}$) by the expression

$$(1)\qquad \Delta_t|_\mathbf{T}=\Delta|_\mathbf{T}+u_{1,t}X+u_{2,t}Y+i(f_{1,t}+i\mathcal{T}_0f_{1,t})X+i(f_{2,t}+i\mathcal{T}_0f_{2,t})Y,$$

where u_1, u_2, f_1 and f_2 are real-valued functions on the circle and $\mathcal{T}_zf$ denotes the harmonic conjugate of f vanishing at z. Such an expression is obtained by applying the implicit function theorem to the mapping $(u_1,u_2,f_1,f_2)\mapsto\Delta|_\mathbf{T}+u_1X+u_2Y+i(f_1+i\mathcal{T}_0f_1)X+i(f_2+i\mathcal{T}_0f_2)Y$ and solving for (f_1,f_2). Decompose

$$\frac{d}{d\theta}\Delta_t(\theta)\bigg|_{\theta=0}=(a(t)+ib(t))X(0)+(c(t)+id(t))Y(0)$$

at $\theta=0$. As R_t coincides with R near the origin and $X(0)$ and $Y(0)$ span T_0R, we obtain $b(t)\equiv0$ and $d(t)\equiv0$. By property (2) we have $a(0)=1$ and $c(0)=0$. In order to show that the direction of Δ_t at the origin can be different from the direction of $\Delta_0=\Delta$ it is clearly enough to prove the following claim.

Claim. *The coefficient $c(t)$ is not constant zero on any interval $[0,t_0], 0<t_0$.*

Proof of the claim. To derive a contradiction, assume that $c_t \equiv 0$ for $0 \le t \le t_0$.

First we check that $u_{2,t}(\theta)$ is constant in θ for all t. Indeed, multiplying (1) with B^{-1} we see that

$$u_{1,t}X_0 + u_{2,t}Y_0 + i(f_{1,t} + i\mathcal{T}_0 f_{1,t})X_0 + i(f_{2,t} + i\mathcal{T}_0 f_{2,t})Y_0 = B^{-1}(\Delta_t|_{\mathbf{T}} - \Delta|_{\mathbf{T}})$$

extends to $\mathbf{D}$ as a holomorphic vector-valued function. Its second component reads

$$u_{2,t} + i(f_{2,t} + i\mathcal{T}_0 f_{2,t}).$$

and hence $u_{2,t}$ must be constant.

As $\Delta_t(1)=0$, we deduce

$$u_{2,t} + i(f_{2,t} + i\mathcal{T}_0 f_{2,t}) = i(f_{2,t} + i\mathcal{T}_0 f_{2,t}).$$

By a well-known formula,

$$c(t) = -\frac{d}{d\theta}\mathcal{T}_0 f_{2,t}\bigg|_{\theta=0} = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f_{2,t}(\theta)}{|e^{i\theta}-1|^2}\,d\theta.$$

For small t, we shall establish a lower bound

$$\left|\int_{-\pi}^{\pi} \frac{f_{2,t}(\theta)}{|e^{i\theta}-1|^2}\,d\theta\right| \ge ct, \quad c>0, \tag{2}$$

which obviously implies the desired contradiction to the assumption that $c(t)$ is constant zero near the origin.

To this end, we fix some small $\delta>0$, and consider for each $\varepsilon\in(0,\delta)$ the following open cover of $\mathbf{T}=\{\zeta:|\zeta|=1\}$:

$$\begin{aligned} \alpha(\varepsilon) &= \{\zeta : |\zeta|=1 \text{ and } |\zeta-1|<\varepsilon\}, \\ \beta(\varepsilon) &= \left\{\zeta : |\zeta|=1,\ |\zeta-1|>\tfrac{1}{2}\varepsilon \text{ and } |\zeta+1|>\tfrac{1}{2}\delta\right\}, \\ \gamma &= \{\zeta : |\zeta|=1 \text{ and } |\zeta+1|<\delta\}. \end{aligned}$$

We assume that R is only deformed near $\left\{\zeta:|\zeta|=1 \text{ and } |\zeta+1|<\frac{1}{4}\delta\right\}$. We shall show three elementary estimates: First there are uniform constants $c_1, c_2>0$ such that for small $t\ge 0$ we have

$$\left|\int_{\alpha(\varepsilon)} \frac{f_{2,t}(\theta)}{|e^{i\theta}-1|^2}\,d\theta\right| \le c_1\varepsilon t, \tag{3}$$

and

$$\int_{\gamma} \frac{f_{2,t}(\theta)}{|e^{i\theta}-1|^2}\, d\theta \geq c_2 t. \tag{4}$$

Finally we find for each ε a constant $c(\varepsilon)>0$ with

$$\left| \int_{\beta(\varepsilon)} \frac{f_{2,t}(\theta)}{|e^{i\theta}-1|^2}\, d\theta \right| \leq c(\varepsilon) t^2. \tag{5}$$

Taking $\varepsilon>0$ with $c_1\varepsilon<c_2$, we get the lower bound (2).

To prove (5), we recall that the integral measures the increment of the smoothly varying curves $\Delta_t(\mathbf{T})$ in the direction of iY for varying t. Near $\beta(\varepsilon)$, the surfaces R_t and R coincide. Furthermore, the vector fields X and Y span TR along $\Delta(\beta(\varepsilon))$. So the increment in the direction of iY can at most grow quadratically, whence (5). A similar argument shows that the bump as a deformation in the direction of iY contributes linearly to the integral in (4).

Only for (3) we need the assumption $c(t)\equiv 0$. This implies that $\partial\Delta_t$ is always tangent to X at the origin. Taylor's formula yields

$$f_{2,t}(\theta) = \left(\frac{d^2}{d\theta^2} f_{2,t} \bigg|_{\theta=0} \right) \theta^2 + e(t,\theta)$$

near $\theta=0$, where $e(t,\theta)$ admits a uniform estimate by $|\theta|^3$. Now θ^2 and the denominator in the integral are comparable, and we get (3) after expanding in t to the first order. □

References

1. ALEXANDER, H. and WERMER, J., *Several Complex Variables and Banach Algebras*, Springer-Verlag, New York, 1997.
2. ANDERSON, J. T. and CIMA, J. A., Removable singularities of L^p CR-functions, *Michigan Math. J.* **41** (1994), 111–119.
3. BEDFORD, E. and KLINGENBERG, W., On the envelope of holomorphy of a 2-sphere in $\mathbf{C}^2$, *J. Amer. Math. Soc.* **4** (1991), 623–646.
4. BISHOP, E., Differentiable manifolds in complex Euclidean space, *Duke Math. J.* **32** (1965), 1–22.
5. CHIRKA, E. M. and STOUT, E. L., Removable singularities in the boundary, in *Contributions to Complex Analysis and Analytic Geometry* (Skoda, H. and Trépreau, J.-M., eds.), pp. 43–104, Vieweg, Braunschweig, 1994.
6. DUVAL, J., Surfaces convexes dans un bord pseudo-convexe, in *Colloque d'Analyse Complexe et Géométrie (Marseille, 1992)*, Astérisque **217**, pp. 103–118, Soc. Math. France, Paris, 1993.

7. FORNÆSS, J. E. and MA, D., A 2-sphere in $\mathbf{C}^2$ that cannot be filled in with analytic disks, *Internat. Math. Res. Notices* **1995** (1995), 17–22.
8. FORSTNERIČ, F., Analytic disks with boundaries in a maximal real submanifold of $\mathbf{C}^2$, *Ann. Inst. Fourier (Grenoble)* **37**:1 (1987), 1–44.
9. FORSTNERIČ, F. and STOUT, E. L., A new class of polynomially convex sets, *Ark. Mat.* **29** (1991), 51–62.
10. GLOBEVNIK, J., Perturbation by analytic discs along maximally real submanifolds of $\mathbf{C}^N$, *Math. Z.* **217** (1994), 287-316.
11. HARVEY, R. and POLKING, J., Removable singularities of linear partial differential equations, *Acta Math.* **125** (1970), 39–56.
12. HOFER, H., LIZAN, V. and SIKORAV, J.-C., On genericity for holomorphic curves in four-dimensional almost-complex manifolds, *J. Geom. Anal.* **7** (1998), 149–159.
13. JÖRICKE, B., Removable singularities of CR-functions, *Ark. Mat.* **26** (1988), 117–143.
14. JÖRICKE, B., Boundaries of singularity sets, removable singularities, and CR-invariant subsets of CR-manifolds, *J. Geom. Anal.* **9** (1999), 257–300.
15. JÖRICKE, B., Removable singularities of L^p CR-functions on hypersurfaces, *J. Geom. Anal.* **9** (1999), 429–456.
16. JÖRICKE, B. and SHCHERBINA, N., A non-removable generic 4-ball in the unit sphere of $\mathbf{C}^3$, *Duke Math. J.* **102** (2000), 87–100.
17. KRUZHILIN, N. G., Two-dimensional spheres in the boundaries of strictly pseudoconvex domains in $\mathbf{C}^2$, *Izv. Akad. Nauk SSSR Ser. Mat.* **55** (1991), 1194–1237 (Russian). English transl.: *Math. USSR-Izv.* **39** (1992), 1151–1187.
18. KYTMANOV, A. M. and REA, C., Elimination of L^1 singularities on Hölder peak sets for CR functions, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **22** (1995), 211–226.
19. LEWY, H., On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables, *Ann. of Math.* **64** (1956), 514–522.
20. LAURENT-THIÉBAUT, C., Sur l'extension des fonctions CR dans une variété de Stein, *Ann. Mat. Pura Appl.* **150** (1988), 141–151.
21. LUPACCIOLU, G., Characterization of removable sets in strongly pseudoconvex boundaries, *Ark. Mat.* **32** (1994), 455–473.
22. LUPACCIOLU, G. and STOUT, E. L., Removable singularities for $\bar\partial_b$, in *Several Complex Variables (Stockholm, 1987/1988)* (Fornæss, J. E., ed.), Math. Notes **38**, pp. 507–518, Princeton Univ. Press, Princeton, N. J., 1993.
23. MERKER, J., On removable singularities in higher codimension, *Internat. Math. Res. Notices* **1** (1997), 21–56.
24. MERKER, J. and PORTEN, E., On the local meromorphic extension of CR meromorphic mappings, in *Complex Analysis and Applications (Warsaw, 1997)* (Chollet, A.-M., Chirka, E., Dwilewicz, R., Jacobowitz, H., and Siciak, J., eds.), Ann. Pol. Math. **70**, pp. 163–193, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 1998.
25. MERKER, J. and PORTEN, E., On removable singularities for integrable CR functions, *Indiana Univ. Math. J.* **48** (1999), 805–856.
26. STOLZENBERG, G., Uniform approximation on smooth curves, *Acta Math.* **115** (1966), 185–198.

27. Stout, E. L., Removable singularities for the boundary values of holomorphic functions, in *Several Complex Variables: Proceedings of the Mittag-Leffler Institute, 1987–1988* (Fornæss, J. E., ed.), Math. Notes **38**, pp. 600–629, Princeton Univ. Press, Princeton, N. J., 1993.
28. Sussmann, H. J., Orbits of families of vector fields and integrability of distributions, *Trans. Amer. Math. Soc.* **180** (1973), 171–188.
29. Trépreau, J.-M., Sur le prolongement holomorphe des fonctions CR définies sur une hypersurface reelle de classe C^2, *Invent. Math.* **83** (1986), 583–592.
30. Trépreau, J.-M., Sur la propagation des singularités dans les variétés CR, *Bull. Soc. Math. France* **118** (1990), 403–450.
31. Tumanow, A. E., Connections and propagation of analyticity for CR-functions, *Duke Math. J.* **70** (1994), 1–24.

Received October 22, 2001

Egmont Porten
Humboldt-Universität zu Berlin
Rudower Chaussee 25
DE-12489 Berlin
Deutschland
email: egmont@mathematik.hu-berlin.de