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Dynamics of polynomial automorphisms of $\mathbf{C}^k$

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Abstract. We study the dynamics of polynomial automorphisms of $\mathbf{C}^k$. To an algebraically stable automorphism we associate positive closed currents which are invariant under f, considering f as a rational map on $\mathbf{P}^k$. These currents give information on the dynamics and allow us to construct a canonical invariant measure which is shown to be mixing.

Introduction

The dynamics of polynomial automorphisms of $\mathbf{C}^2$ has been studied quite intensively in the past decade. We refer to the survey articles by Bedford and Smillie [BS3] and the second author [S] which contain a quite extensive bibliography. We recall a few basic facts.

The algebraic structure of the group $\mathrm{Aut}(\mathbf{C}^k)$ of polynomial automorphisms of $\mathbf{C}^k$ is well understood when $k=2$. Any polynomial automorphism is conjugate either to an *elementary automorphism*

$$e(z,w)=(\alpha z+P(w),\beta w+\gamma),$$

where P is a (holomorphic) polynomial, or to a finite composition of *Hénon maps* h_j defined as follows

$$h_j(z,w)=(P_j(z)-a_jw,z),$$

where P_j are polynomials of degree $d_j\geq 2$. We denote by $\mathcal{H}$ the semigroup generated by Hénon maps (see [FM]).

It is clear that only the elements of $\mathcal{H}$ are dynamically interesting. If $h\in\mathcal{H}$ is of degree d, then $h^n=h\circ...\circ h$ is of degree d^n. One can define the Green function

$$G^+(z,w)=\lim_{n\to+\infty}\frac{1}{d^n}\log^+\|h^n(z,w)\|,$$

and the associated current $T_+=dd^cG^+$, where $d^c=i(\bar\partial-\partial)/2\pi$. There are similar objects G^- and T_- associated to the inverse map h^{-1} and one can define a probability measure $\mu:=T_+\wedge T_-$. Here are some important properties of these objects.

- The function G^+ satisfies the invariance property $G^+\circ f=dG^+$. It is Hölder continuous and $\{p|G^+(p)=0\}=K^+:=\{p|(h^n(p))_{n=0}^{\infty}$ is bounded$\}$.
- The support of T_+ coincides with the boundary of K^+, which also equals the Julia set of h (i.e. the complement of the largest open set on which the family $(h^n)_{n=0}^{\infty}$ is equicontinuous).
- The current T_+ is extremal among positive closed currents in $\mathbf{C}^2$ and is—up to a multiplicative constant—the unique positive closed current supported on K^+ [FS1].
- The measure μ is invariant and has support in the compact set ∂K, where $K=\{p\in\mathbf{C}^2|(h^n(p))_{n=-\infty}^{\infty}$ is bounded$\}$.

This type of properties has interesting dynamical consequences: connectedness of ∂K^+ [BS1], density of stable manifolds in ∂K^+ [BS1], mixing of μ [BS2].

The measure μ has been studied by Bedford–Smillie–Lyubich [BS2] and [BLS]. They show in particular that μ maximise entropy and is well approximated by Dirac masses at saddle points.

Much less is known in the study of the dynamics of polynomial automorphisms of $\mathbf{C}^k$, $k\geq 3$. Indeed the algebraic structure of $\mathrm{Aut}(\mathbf{C}^k)$, $k\geq 3$, is poorly understood.

Bedford and Pambuccian [BP] have introduced the class of shift-like maps in $\mathbf{C}^k$. A shift like automorphism of type $\nu\in\{1,\dots,k-1\}$ has the form

$$f(z_1,\dots,z_k)=(z_2,\dots,z_k,P(z_{k-\nu+1})-az_1).$$

They introduced the corresponding currents T_+ and T_- and constructed the invariant measure $\mu=T_+^{\nu}\wedge T_-^{k-\nu}$.

Coman and Fornæss [CF] have studied the Green function of interesting classes of polynomial automorphisms of degree 2 in $\mathbf{C}^3$. They study in particular the rate of escape at infinity of orbits.

In this paper we consider polynomial automorphisms of $\mathbf{C}^k$ as rational maps on $\mathbf{P}^k$. The behaviour under iteration of the hyperplane at infinity plays a central role. Before describing the results we obtain, we first recall a few notions. For more details, we refer to [S].

Let $f=(f_1,\dots,f_k)$ be a polynomial map in $\mathbf{C}^k$. Let $d=\deg f:=\max_j \deg f_j\geq 2$. We denote by $\mathrm{End}(\mathbf{C}^k)$ the space of maps of generic rank k. We denote by $z=(z_1,\dots,z_k)$ the coordinates in $\mathbf{C}^k$ and $[z_1:\dots:z_k:t]$ the homogeneous coordinates in $\mathbf{P}^k$. So the hyperplane at infinity is identified with $\{[z:0]\}$.

We consider the extension $\bar{f}$ of f to $\mathbf{P}^k$ as a rational map. In homogeneous coordinates

$$\bar{f}[z:t] = [F_1(z,t): \dots : F_k(z,t) : t^d],$$

where $F_j(z,1)=f_j(z)$. The mapping $\bar{f}$ has an indeterminacy set I which is an analytic subset of codimension ≥ 2 contained in $\{[z:0]\}$. Let I_n denote the indeterminacy set of $\bar{f}^n$. When f is an automorphism we denote its indeterminacy set by I^+, and I^- denotes the indeterminacy set of f^{-1}. Similarly $d_+=\deg f$ and $d_-=\deg f^{-1}$.

We will say that f is *algebraically stable* if and only if $\bar{f}^n(\{[z:0]\}\setminus I_n)$ is not contained in I for any $n>0$. This is equivalent to the fact that $\deg f^n = (\deg f)^n$.

Elements of $\mathcal{H}$ are algebraically stable. When f is algebraically stable, one can associate to f a Green function

$$G(z) = \lim_{n\to+\infty} \frac{1}{d^n} \log^+ \|f^n(z)\|.$$

If we define $T=dd^cG$, one can show that T is a *non-zero* positive closed current. More precisely if ω denotes the standard Fubini–Study Kähler form on $\mathbf{P}^k$, then $\bar{T}=\lim(\bar{f}^n)^*\omega/d^n$ is a positive closed current on $\mathbf{P}^k$ of mass one which gives zero mass to the hyperplane $\{[z:0]\}$ (Theorem 1.8.1 [S]). So $T=\bar{T}|_{\mathbf{C}^k}$ has mass one in $\mathbf{C}^k$.

From now on we identify f and $\bar{f}$. If $f\in\operatorname{End}(\mathbf{C}^k)$ is algebraically stable we define inductively the analytic sets X_j by

$$X_1 = \overline{f(\{[z:0]\}\setminus I)}, \quad X_{j+1} = \overline{f(X_j\setminus I)}.$$

The sequence is decreasing, X_j is non-empty because f is algebraically stable. Hence it is stationary. Let X be the corresponding limit set (when $f\in\operatorname{Aut}(\mathbf{C}^k)$, we denote this set by X^+). Replacing f by an appropriate iterate, we can always assume that $X=\overline{f(\{[z:0]\}\setminus I)}$. In the automorphism case, the notation is X^+ if f is algebraically stable and X^- when f^{-1} is algebraically stable. Observe that X is always contained in the hyperplane at infinity.

For an algebraically stable endomorphism of $\mathbf{C}^k$, we define U to be the basin of attraction of X, i.e.

$$U = \left\{ z\in\mathbf{C}^k \,\middle|\, \lim_{n\to+\infty} f^n(z)\in X \right\} \quad \text{and} \quad \mathcal{K} := \mathbf{C}^k\setminus U.$$

In Section 1 we explore the first properties of algebraically stable endomorphism of $\mathbf{C}^k$. We show that one can define a Green function G and prove that $\{p\mid G(p)>0\}\subset U$. In particular U is of infinite Lebesgue measure and has non-empty fine interior (Theorem 1.7).

In general the function G is not continuous (Example 1.11) and $\mathcal{K}\subset\{p|G(p)=0\}$ is different from the set K^+ of points with bounded forward orbit.

We say that an endomorphism is *weakly regular* if $X\cap I=\emptyset$. This is the case of the elements of $\mathcal{H}$ in $\mathbf{C}^2$. We show (Theorem 2.2) that for a weakly regular endomorphism $\{p|G(p)=0\}=\mathcal{K}$, $\overline{\partial\mathcal{K}}\cap\{[z:0]\}=I$ and $\dim I+\dim X=k-2$. The proof uses heavily the theory of positive closed currents.

The rest of the paper is concerned with algebraically stable automorphisms. When f is such an automorphism, we define $U^\pm=\{z|\lim_{n\to+\infty} f^{\pm n}(z)\in X^\pm\}$, $\mathcal{K}^\pm=\mathbf{C}^k\setminus U^\pm$ and

$$K^\pm=\{z\in\mathbf{C}^k \mid (f^{\pm n}(z))_{n=0}^\infty \text{ is bounded}\}.$$

In general K^+ is not closed and could be empty (Example 1.5). We always have $X^+\subset I^-$ and $X^-\subset I^+$. Chapter 2 of [S] is devoted to the study of *regular* automorphisms, i.e. automorphisms such that $I^+\cap I^-=\emptyset$. Here we study more general cases and find results that are new even for regular automorphisms. Let $T_+=\lim(1/d_+^n)(f^n)^*\omega$ and $T_-=\lim(1/d_-^n)(f^{-n})^*\omega$. Set

$$r=\dim X^+ +1 \quad\text{and}\quad s=\dim X^- +1,$$

when f and f^{-1}, respectively, are algebraically stable.

Assuming that f^{-1} is weakly regular ($I^-\cap X^-=\emptyset$) and that I^- is attracting for f, we show (Theorem 2.13) that K^+ is the complement of the basin of attraction of I^-, that $K=K^+\cap K^-$ is compact and $W^s(K)=K^+$, $W^u(K)=K^-$, where W^s and W^u denote the stable and unstable sets, respectively. In particular when f and f^{-1} are both weakly regular without being regular and I^- is f-attracting, then the basin $\mathcal{B}(I^+\cap I^-)$ of $I^+\cap I^-$ is not empty.

When f is an algebraically stable automorphism, the current T_+ is extremal in the cone of positive closed currents of bidegree $(1,1)$ on $\mathbf{P}^k$ (Theorem 3.6). This property is crucial to establish dynamical properties of f. When $\dim X^+=0$ and f is weakly regular, then the support of T_+ is equal to $\partial\mathcal{K}^+$ and any positive closed current supported on $\overline{\mathcal{K}}^+$ is proportional to T_+ (Theorem 2.4). This implies in particular that $\partial\mathcal{K}^+$ is connected. When $\dim X^+=r-1$, the current T_+^r is supported on $\partial\mathcal{K}^+$.

In Section 3 we construct a dynamically interesting positive closed current supported on K^+. More precisely if f^{-1} is weakly regular and I^- is f-attracting then the sequence

$$\frac{1}{d_-^{ns}}(f^n)^*(\omega^{k-s}),\quad \dim X^-=s-1,$$

converges to a positive closed current σ_s supported on K^+ (Theorem 3.1). Moreover σ_s satisfies $f^*\sigma_s=d_-^s\sigma_s$. This allows us to construct an interesting invariant probability measure $\mu=\sigma_s\wedge T_-^s$. When f is regular then $s+r=k$, $d_-^s=d_+^r$ and $\sigma_s=T_+^r$ [S].

We show that when $s=1$ (i.e. $\dim X^-=0$), then any stable manifold of dimension 1 is dense in the support of σ_1 (Corollary 3.8). We show in Section 4 that the measure μ is mixing (Theorem 4.1). We also give another construction of σ_s using a partial Green function (Theorem 4.5). Under appropriate assumptions, there is a function h on the support of T_+^r defined by

$$h(z)=\lim \frac{1}{\delta^n}\log^+\|f^n(z)\|, \quad \delta=\frac{d_-^s}{d_+^r}>1.$$

The function h satisfies the functional equation

$$h\circ f(z)=\delta h(z)$$

and describes the rate of escape to infinity in $\mathcal{B}(I^+\cap I^-)$. The measure μ can be constructed using the function h in that case (Theorem 4.6).

In Section 5 we give examples where the non-trivial hypotheses we make are satisfied: when is I^- an f-attracting set (Section 5.2), and estimates on the growth of f on $\mathcal{K}^+\cap\mathcal{K}^-$ (Section 5.3).

It is clear that we are concerned with the first steps of the dynamics of polynomial automorphisms in $\mathbf{C}^k$, $k\geq 3$, and that the subject will be developed in the future.

We end this introduction with a list of the most frequently used notation:

- $z:=(z_1,\dots,z_k)=$ the canonical coordinates in $\mathbf{C}^k$;
- $[z:t]:=[z_1:\dots:z_k:t]=$ the homogeneous coordinates in $\mathbf{P}^k$;
- $\{[z:0]\}:=$ the hyperplane at infinity in $\mathbf{P}^k$;
- $\mathrm{End}(\mathbf{C}^k):=$ the set of polynomial endomorphisms $f=(f_1,\dots,f_k)$ of $\mathbf{C}^k$;
- $\mathrm{Aut}(\mathbf{C}^k):=$ the set of polynomial automorphisms of $\mathbf{C}^k$;
- $\deg(f):=$ degree of $f=\max_{1\leq j\leq k}\deg(f_j)$ when $f\in\mathrm{End}(\mathbf{C}^k)$;
- $d_+:=\deg(f)$ when $f\in\mathrm{Aut}(\mathbf{C}^k)$ and $d_-:=\deg(f^{-1})$;
- algebraically stable: see Definition 1.1;
- weakly regular: see Definition 2.1;
- q-regular: see Definition 2.6;
- $G^+(z):=$ the Green function of $f\in\mathrm{Aut}(\mathbf{C}^k)=\lim(1/d_+^n)\log^+\|f^n(z)\|$;
- $\widetilde{G}^+(z,t):=$ the homogeneous Green function $=G^+(z/t)+\log|t|$;
- $T_+:=$ the Green current of f (it satisfies $T_+=dd^cG^+$ in $\mathbf{C}^k$);
- $\sigma_s:=$ the f^*-invariant current supported on $\overline{K}^+$ (see Theorem 3.1);
- $\mu:=\sigma_s\wedge T_-^s=$ the invariant measure (Section 4);
- $I^+:=$ the indeterminacy set of $f=\{p\in\{[z:0]\}\,|\,f$ is not holomorphic at $p\}$;
- $X^+:=$ the limit set of f at infinity$=f^k(\{[z:0]\}\setminus I_{f^k})$;
- $U^+:=$ the basin of attraction of $X^+=\{p\in\mathbf{C}^k\,|\lim_{n\to+\infty}f^n(p)\in X^+\}$;

- $\mathcal{K}^+ := \mathbf{C}^k \setminus U^+$;
- $K^+ := \{z \in \mathbf{C}^k \mid (f^n(z))_{n=0}^{\infty} \text{ is bounded}\} \subset \mathcal{K}^+$;
- $K := \{z \in \mathbf{C}^k \mid (f^n(z))_{n=-\infty}^{\infty} \text{ is bounded}\} \subset K^+$;
- $r := \dim X^+ + 1$;
- $s := \dim X^- + 1$;
- $l' := \dim I^+ + 1$;
- $l := \dim I^- + 1$;
- $q := \dim(I^+ \cap I^-) + 1$.

1. Algebraically stable endomorphisms

Let $f \in \mathrm{End}(\mathbf{C}^k)$. We still denote by f the rational extension of f to $\mathbf{P}^k$, in homogeneous coordinates $F = (F_1(z,t), \dots, F_k(z,t), t^d)$ in $\mathbf{C}^{k+1}$. Let I denote the indeterminacy set of f at infinity, this is the set of points $[z:0]$ in $\{[z:0]\}$ such that $F_1(z,0) = \dots = F_k(z,0) = 0$. Let I_n denote the indeterminacy set of f^n.

Definition 1.1. We say that f is algebraically stable if and only if for all $n > 0$, $f^n(\{[z:0]\} \setminus I_n)$ is not contained in I.

Let f be an algebraically stable endomorphism of $\mathbf{C}^k$ of degree $d \geq 2$. We define $G(z) = \lim d^{-n} \log^+ \|f^n(z)\|$. The existence of the log-homogeneous Green function $\widetilde{G}(z,t) = \lim d^{-n} \log \|F^n(z,t)\|$ was shown in [S]. It satisfies $\widetilde{G}(z,1) = G(z)$, $\widetilde{G} \circ F(z,t) = d\widetilde{G}(z,t)$ and is not identically $-\infty$. The current $T = dd^c G$ is well defined on $\mathbf{P}^k$ and satisfies $f^*T = d \cdot T$.

Remark 1.2. One should observe that the notion of algebraic stability is not invariant under conjugacy. It also might happen that f is not algebraically stable but f^2 is (see Example 1.4.6.2 in [S]). But clearly the dynamical consequences that can be deduced from the study of T are invariant under conjugacy. When a power of f is algebraically stable, we only consider iterates of that power. This does not change the dynamical behavior much.

In this section we show that the set $\{p \mid G(p) > 0\}$ of orbits converging to infinity with maximal speed is rather big (Proposition 1.3) and consists of orbits attracted by the limit set X of f at infinity (Theorem 1.7). In contrast with the two-dimensional situation, the set K^+ of points with bounded forward orbit is not necessarily closed (Example 1.5) and the Green function G^+ is not necessarily continuous (Example 1.11).

Proposition 1.3. *Let $f\in\operatorname{End}(\mathbf{C}^k)$ be an algebraically stable endomorphism. Let G denote the Green function associated to f. Then*

$$\limsup_{|z|\to\infty}\frac{G(z)}{\log|z|}=1.$$

Moreover the set $\{p\,|\,G(p)>0\}$ is an F_σ set, connected and of infinite measure on any complex line where G is not identically zero. Therefore the set $\{z\,|\lim f^n(z)=\infty\}$ is of infinite measure.

Proof. The Green current T associated to f does not have mass on the hyperplane at infinity $\{[z{:}0]\}$ (Theorem I.8.1, p. 22 [S]). Assume there is $\varepsilon>0$ and $C>0$ such that

$$G(z)\le(1-\varepsilon)\log^+|z|+C.$$

Then the plurisubharmonic log-homogeneous Green function will satisfy

$$\widetilde{G}(z,t)=\log|t|+G(z/t)\le(1-\varepsilon)\max\{\log|z|,\log|t|\}+\varepsilon\log|t|+C.$$

Thus T will have mass at least ε on the hyperplane $\{[z{:}0]\}$, a contradiction. We also know [S] that $G(z)\le\log^+|z|+O(1)$, so we have proved that the lim sup cannot be strictly less than 1.

We assume, for simplicity, that $G(0)=1$ and that G is not identically zero on the line $L=\{(\zeta,0,\dots,0)\,|\,\zeta\in\mathbf{C}\}$. Let $m(r)$ denote the Lebesgue measure of the set $\{e^{i\theta}\,|\,G(re^{i\theta},0,\dots,0)>0\}$. By the submean value property,

$$1=G(0)\le\frac{1}{2\pi}\int_0^{2\pi}G(re^{i\theta},0,\dots,0)\,d\theta\le\frac{1}{2\pi}(\log^+r+C)m(r).$$

So the measure of $\{\zeta\,|\,G(\zeta,0,\dots,0)>0\}$ is infinite. It is crucial in this argument that G is of slow growth. The claim of connectedness of $\{p\,|\,G(p)>0\}$ follows easily from similar statement for subharmonic functions in $\mathbf{C}$ not growing too rapidly, see [He]. □

Proposition 1.4. *Let $f\in\operatorname{End}(\mathbf{C}^k)$. Define*

$$K^+=\{z\in\mathbf{C}^k\mid(f^n(z))_{n=0}^\infty\ \textit{bounded}\}.$$

The set K^+ is an F_σ set (not necessarily closed). If $f\in\operatorname{Aut}(\mathbf{C}^k)$ and $a=|\operatorname{Jac}f|\neq1$ then K^+ is of zero or infinite measure, both cases occur.

Proof. For $M>0$ define $K_M^+=\{z\,|\,|f^n(z)|\le M,\ n\ge0\}$. Then $K^+=\bigcup_{M>0}K_M^+$ so K^+ is an F_σ and an increasing union of polynomially convex sets. The set K^+ is clearly invariant under f.

When $f\in\operatorname{Aut}(\mathbf{C}^k)$, we let $\lambda(K^+)$ denote the Lebesgue measure of K^+. We have $\lambda(K^+)=|a|^{2k}\lambda(K^+)$. If $|a|\neq1$, this implies that $\lambda(K^+)$ is zero or infinite. □

Example 1.5. There are algebraically stable biholomorphisms of $\mathbf{C}^3$ with one of the following properties:

(1) K^+ is empty;

(2) K^+ is non-empty and non-closed with $\overline{K}^+ = \mathbf{C}^3 \setminus U^+$, where U^+ is the basin of attraction of an attractive fixed point at infinity.

We consider an algebraically stable biholomorphism of $\mathbf{C}^3$ constructed from a Hénon map in $\mathbf{C}^2$. Define for $d \geq 2$, $h(x,y) = (x^d + ay, x)$. Consider

$$f(x,y,z) = (x^d + ay, x, A(x) + y + z),$$

where A is a polynomial of degree d. We have $I^+ = \{[0{:}y{:}z{:}0]\}$, $X^+ = f(\{[z{:}0]\} \setminus I^+) = [1{:}0{:}\alpha{:}0]$, $\alpha \neq 0$, thus $X^+ \cap I^+ = \emptyset$. Hence f is algebraically stable. Similarly

$$f^{-1}(x,y,z) = \left(y, \frac{1}{a}(x - y^d), z - A(y) - \frac{1}{a}(x - y^d)\right),$$

thus $I^- = \{[x{:}0{:}z{:}0]\}$, $X^- = f^{-1}(\{[z{:}0]\} \setminus I^-) = [0{:}1{:}a\alpha + 1{:}0]$ and f^{-1} is also algebraically stable. If $(x_n, y_n)_{n=0}^{\infty}$ denotes the orbit of (x,y) under h in $\mathbf{C}^2$, then

$$f^n(x,y,z) = \left(x_n, y_n, z + \sum_{j=0}^{n-1} (A(x_j) + x_{j-1})\right).$$

Let $K_h^+ = \{(x,y) \in \mathbf{C}^2 \mid (x_n)_{n=0}^{\infty} \text{ bounded}\}$. It is easy to check that X^+ is an attractive fixed point for f. Let U^+ denote the basin of attraction of X^+. Then $\mathbf{C}^3 \setminus U^+ := \mathcal{K}^+ = K_h^+ \times \mathbf{C}$. It is known [FM] that the orbits of points in K_h^+ cluster on $K_h = K_h^+ \cap K_h^-$ which is compact in $\mathbf{C}^2$. If $\operatorname{Re} A \geq c \gg 1$ on K_h then clearly K^+ is empty and in particular f has no periodic point.

We now show that it is possible to choose the polynomial A so that K^+ is dense in $\mathcal{K}^+$ and $\mathcal{K}^+ \setminus K^+$ is also dense in K^+. Let p be a saddle fixed point for h. Assume $|a| = |Jf| > 1$ and that $Q(x,y) = A(x) + y$ vanishes at p. Let $W^s(p)$ be the stable manifold at p, which is dense in $K_h^+ = \partial K_h^+$ [BS1]. Then $W^s(p) \times \mathbf{C}$ is dense in $\mathcal{K}^+$ and is contained in K^+. Indeed $\sum_{j=0}^n |Q(x_j, x_{j-1})| \leq C \sum_{j=0}^n |(x_j, y_j) - p| \leq C' \sum_{j=0}^n \varepsilon^j$, where $\varepsilon < 1$. If p' is another saddle fixed point of h where $Q(p') \neq 0$, one checks that no point in $W^s(p') \times \mathbf{C}$ is in K^+. Observe that there is a constant C such that for any $(x,y,z) \in \mathcal{K}^+$ one has $|f^n(x,y,z)| \leq Cn$.

Remark 1.6. It is easy to check for the previous example that

$$G^+(x,y,z) = G_h^+(x,y).$$

Observe that $\{p \mid G^+(p) = 0\} = \mathcal{K}^+$ might be different from K^+. Note also that $\deg f = \deg f^{-1}$.

For an algebraically stable endomorphism f of $\mathbf{C}^k$, we define

$$U := \left\{ z \in \mathbf{C}^k \;\middle|\; \lim_{n \to +\infty} f^n(z) \in X \right\} \quad \text{and} \quad \mathcal{K} := \mathbf{C}^k \setminus U.$$

Theorem 1.7. *Let $f\in \mathrm{End}(\mathbf{C}^k)$ be algebraically stable. Then*

$$\mathcal{K}\subset\{p\,|\,G(p)=0\}.$$

In particular U is of infinite measure and of non-empty fine interior.

Proof. Define $\widetilde{\varphi}=\log\max_j|R_j|^{1/D}$, where R_j are homogeneous polynomials of degree D such that $X=\bigcap_j R_j^{-1}(0)$. Since $X\subset\{[z:0]\}$, we can fix $R_1=t^D$ so that if we identify $\mathbf{C}^k$ with $\{[z:1]\}$, we get $\varphi=\widetilde{\varphi}|_{\mathbf{C}^k}\geq 0$.

Recall that that the Green function $\widetilde{G}$ is the decreasing limit of $d^{-n}\log\|F^n\|$, where $F\colon\mathbf{C}^{k+1}\to\mathbf{C}^{k+1}$ is a homogeneous representative of the extension of f to $\mathbf{P}^k$, normalized so that $\|F(Z)\|\leq\|Z\|^d$. Since $d^{-1}\log\|F\|$ has positive Lelong number at every point of $\pi^{-1}(I)$, so has $\widetilde{G}$. Hence there exists $0<\gamma\ll 1$ such that $\widetilde{G}\leq\gamma\widetilde{\varphi}$ in a neighborhood of $\pi^{-1}(I\setminus B(I\cap X,\varepsilon))\cap\partial B_{k+1}$. Here $B(I\cap X,\varepsilon)=\{p\in\mathbf{P}^k\,|\,\mathrm{dist}(p,X\cap I)<\varepsilon\}$ and B_{k+1} denotes the unit ball in $\mathbf{C}^{k+1}$. Since $\log\|Z\|$ is smooth outside the origin, we get from the log-homogeneity of $\widetilde{G}$ that

$$(*)\qquad \widetilde{G}\leq\gamma\widetilde{\varphi}+(1-\gamma)\log\|Z\|+C_V\quad\text{in }\pi^{-1}(V_\varepsilon),$$

where V_ε is a neighborhood of $I\setminus B(I\cap X,\varepsilon)$ in $\mathbf{P}^k$.

We can assume that $\widetilde{\varphi}\leq\log\|Z\|$, and thus the sequence $d^{-n}\widetilde{\varphi}\circ F^n$ is uniformly bounded from above by $\log\|Z\|$. Thus we can extract a subsequence which converges towards a function $\widetilde{\Psi}$ which is either identically $-\infty$ or plurisubharmonic (see [Hö]). Since $\varphi=\widetilde{\varphi}|_{\mathbf{C}^k}\geq 0$ we get $\widetilde{\Psi}\not\equiv-\infty$. We infer from the logarithmic growth of φ that $\psi=\widetilde{\Psi}|_{\mathbf{C}^k}\leq G$ in $\mathbf{C}^k$. Now we claim that $G\leq\psi$ on $\mathcal{K}$. Indeed let $p\in\mathcal{K}$. If $(f^n(p))_{n=0}^{\infty}$ admits a bounded subsequence, then $G(p)=\psi(p)=0$. Therefore we can assume that $f^n(p)\to\infty$. Since $p\in\mathcal{K}$, $f^{n_i}(p)\to I\setminus X$ for some subsequence $n_i\to\infty$. Thus $f^{n_i}(p)\in V_\varepsilon$ for ε small and i large enough. Hence $(*)$ yields

$$G(p)=\frac{1}{d^{n_i}}G\circ f^{n_i}(p)\leq\gamma\frac{1}{d^{n_i}}\varphi\circ f^{n_i}(p)+(1-\gamma)\frac{1}{d^{n_i}}\log^+\|f^{n_i}(p)\|+\frac{C_V}{d^{n_i}}.$$

Thus $G(p)\leq\psi(p)$.

We show hereafter that $\psi\leq(1-\alpha)\log^+\|z\|+C$ for some constants $C>0$ and $0<\alpha<1$. Assuming this we obtain, since $\mathcal{K}$ is f-invariant, that

$$G(p)=\frac{1}{d^n}G\circ f^n(p)\leq(1-\alpha)\frac{1}{d^n}\log^+\|f^n(p)\|+\frac{C}{d^n}\quad\text{for every }p\in\mathcal{K}.$$

Hence $G(p)\leq(1-\alpha)G(p)$, i.e. $G(p)=0$.

It remains to show that $\psi\leq(1-\alpha)\log^+\|z\|+C$ in $\mathbf{C}^k$. By a result of Siu [Si], this is equivalent to saying that the current S defined by $\widetilde{\Psi}$ on $\mathbf{P}^k$ has positive mass on the hyperplane at infinity $\{[z:0]\}$. Now $S=\lim d^{-n_i}(f^{n_i})^*(\sigma)$, where σ is the current defined by $\widetilde{\varphi}$. Note that the Lelong number $\nu(\sigma,q)$ is positive at every point $q\in X$. It is a well-known (and simple) fact that Lelong number increase by taking pull-back (see e.g. [Fa]). Without loss of generality we can assume that $f(\{[z:0]\}\setminus I)\subset X$, thus

$$\nu(f^*(\sigma),p)\geq\nu(\sigma,f(p))>0 \quad \text{at every point } p\in\{[z:0]\}\setminus I.$$

Since $\operatorname{codim}_{\mathbf{C}} I_f\geq 2$, we infer that $d^{-1}f^*(\sigma)=\sigma'+\alpha[\{z:0\}]$ for some $\alpha>0$. The invariance $f^*[\{z:0\}]=d[\{z:0\}]$ thus yields $S\geq\alpha[\{z:0\}]$.

We just showed that $\{p|G(p)>0\}\subset U$, so Proposition 1.3 says that U is of infinite measure. □

Corollary 1.8. *Let $f\in\operatorname{End}(\mathbf{C}^k)$ be an algebraically stable endomorphism. The basin of any attractive fixed point has complement of infinite measure which is open in the fine topology. When f is a biholomorphism, such a basin is biholomorphic to $\mathbf{C}^k$.*

Proof. Such a basin is contained in $\mathcal{K}$, and hence in $\{p|G(p)=0\}$. The set $\{p|G(p)>0\}$ is open in the fine topology and has infinite measure. □

Remarks 1.9. (i) When X is an attracting set then U is its basin of attraction, and hence is open. This happens e.g. when $X\cap I=\emptyset$ (f is "weakly regular") and in this case $U=\{p|G(p)>0\}$ (see Theorem 2.2). Note however that $\{p|G(p)>0\}$ might be different from U (see Example 1.11 below when $|b|>1$).

(ii) The set X is not necessarily attracting: $f(0,y,0)=(0,by,0)$ in Example 1.11 below, thus $X=\{[x:y:0:0]\}$ is not attracting if $|b|<1$.

There might be unbounded orbits in $\mathcal{K}$ (see Example 1.5). However they have slower growth. Moreover in the biholomorphism case we have the following result.

Proposition 1.10. *Let $f\in\operatorname{Aut}(\mathbf{C}^k)$ be an algebraically stable biholomorphism. Assume that f^{-1} is weakly regular (i.e. that $X^-\cap I^-=\emptyset$). Then $\bar{f}(I^+\setminus X^-)\subset I^-$ and unbounded orbits cluster in $\{[z:0]\}$ only on I^-.*

Proof. Let $z_n\to p\in I^+\setminus X^-$ be such that $f(z_n)\to q$, as $n\to\infty$. If $q\notin I^-$, then $z_n=f^{-1}(f(z_n))\to X^-$, a contradiction. So $f(I^+\setminus X^-)\subset I^-$.

Similarly, if $z_{n_i}=f^{n_i}(z)\to q\in\{[z:0]\}\setminus I^-$, where $z\in\mathbf{C}^k$, then $z_{n_i-1}\to f^{-1}(q)\in X^-$. Now X^- is an attracting set for f^{-1} since $X^-\cap I^-=\emptyset$, so $z=f^{-n_i}(z_{n_i})\to X^-$, a contradiction. □

We now give an example where G^+ is discontinuous on a thick set of $\mathbf{C}^3$.

Example 1.11. Let $P(x,y)$ be a homogeneous polynomial of degree $d\geq 2$. Define $f(x,y,z)=(xP(x,y)+z, x^{d+1}+by, x)$. Then

$$f^{-1}(x,y,z)=\left(z, \frac{1}{b}(y-z^{d+1}), x-zP(z, b^{-1}(y-z^{d+1}))\right).$$

If $\deg_y P=d$ then $d_+=d+1$, $d_-=d^2+d+1$, $I^+=\{[0:y:z:0]\}$, $X^+=I^-=\{[x:y:0:0]\}$ and $X^-=\{[0:0:1:0]\}$. If $|b|>1$, then I^- is an attracting set for f (see Lemma 5.8). Consequently the map f^{-1} is normal in $\mathbf{C}^3$, the function G^- is Hölder continuous (Theorem 1.7.1, p. 115 [S]) and $K^-=\{p|G^-(p)=0\}$ (recall that a map g is normal at a point p, if there is a neighborhood V of p such that $\overline{\bigcup_{n>0} g^n(V)}\cap I_g=\emptyset$).

The action of f on X^+ is given by $f_0[x:y]=[P(x,y):x^d]$. We choose P such that the Julia set for f_0 coincides with $\mathbf{P}^1$ (take e.g. $P(x,y)=(x-2y)^d$ in which case the map f_0 is subhyperbolic [CG]). For such a choice we get $E^+=\{[z:0]\}$, where E^+ denotes the closure of $I^+_\infty:=\bigcup_{j=1}^\infty I_{f^j}$.

Let $\{q\}=I^+\cap I^-=\{[0:1:0:0]\}$. The preimages of q are dense on the hyperplane at infinity, and hence the log-homogeneous Green function $\widetilde{G}^+$ is equal to $-\infty$ on a dense subset of $\{[z:0]\}$. Let $p=[x_0:y_0:0:0]$ be a periodic point for f_0, it is repelling in one direction and the other eigenvalues are zero so the stable manifold is two-dimensional. The restriction of $\widetilde{G}^+$ to $W^s(p)$ has to be pluriharmonic as it is the case on any complex manifold M where $f^n|_M$ is equicontinuous (see [FS2]). The local stable manifolds are graphs over (z,t), we can get a sequence M_j of such graphs converging to a graph M_0 through q. If $\widetilde{G}^+$ were continuous then $\widetilde{G}^+|_{M_j}\to \widetilde{G}^+|_{M_0}$ and the function $\widetilde{G}^+|_{M_0\cap\mathbf{C}^3}$ would be pluriharmonic. This is impossible since a pluriharmonic function on a two-dimensional shell extends as a pluriharmonic function in the ball, but we know that $\widetilde{G}^+(q)=-\infty$.

We get that $\widetilde{G}^+$ has a point of discontinuity in any open set intersecting $\{[z:0]\}$ and actually in any shell of $f^{-j}(M_0)$. Observe also that the set of points of discontinuity of G^+ is totally invariant because $G^+\circ f=d_+G^+$. However since G^+ is a non-negative upper semicontinuous function, it is continuous at any point where G^+ vanishes, for example on $(0,y,0)$. Note that $\{(0,y,0)|y\in\mathbf{C}^*\}$ is in the basin of attraction U^+ of X^+ when $|b|>1$, thus G^+ might vanish in U^+.

When $|b|>1$, the set of periodic points in $\mathbf{C}^3$ is not empty. We also have in this case that the map f is volume expanding so for any open set V, $\bigcup_{n\geq 0} f^n(V)$ clusters on $\{[z:0]\}=E^+$. Hence for such a map the set of normal points is empty.

2. Weakly regular endomorphisms

In this section we introduce the notions of weakly-regular endomorphism (Definition 2.1) and q-regular automorphism (Definition 2.6) and derive properties of their Green currents (Theorem 2.2 and Proposition 2.9). When I^- is assumed to be an f-attracting set (a non-trivial hypothesis which we check on some examples given in Section 5), we get a good understanding of the sets K^+, K^- and K (Theorem 2.13).

Definition 2.1. An endomorphism $f\in\mathrm{End}(\mathbf{C}^k)$ is called *weakly regular* when $X\cap I=\emptyset$.

It follows from the definition that a weakly regular endomorphism is algebraically stable. Moreover X is an attracting set for f, i.e. there exists an open neighborhood V of X such that $f(V)\Subset V$ and $\bigcap_{j=1}^{\infty} f^j(V)=X$. It is enough to compute the derivative of f around X.

Theorem 2.2. *Let $f\in\mathrm{End}(\mathbf{C}^k)$ be a weakly regular endomorphism. Set $r=\dim_{\mathbf{C}} X+1$ and $l'=\dim_{\mathbf{C}} I+1$. Then the following are true.*

(i) *We have $\mathcal{K}=\{p\,|\,G(p)=0\}$. The Green function G is continuous in $\mathbf{C}^k$.*

(ii) *The current T^r is supported on $\partial\mathcal{K}$ and $\overline{\partial\mathcal{K}}\cap\{[z:0]\}=\overline{\mathcal{K}}\cap\{[z:0]\}=I$. The current T^r is of total mass in $\mathbf{C}^k$. For $j\le r$, $f^*T^j=d^jT^j$.*

(iii) *The numbers r and l' satisfy $l'=k-r$ so*

$$\dim_{\mathbf{C}} X+\dim_{\mathbf{C}} I=k-2.$$

(iv) *The current $T^{r+1}=0$ in $\mathbf{C}^{k+1}$, more precisely* $\operatorname{supp} T^{r+1}=I$.

(v) *When $f\in\mathrm{Aut}(\mathbf{C}^k)$, then $d_+^r\le d_-^{k-r}$.*

Proof. We already know $\mathcal{K}\subset\{p\,|\,G(p)=0\}$ from Theorem 1.7. Let V be a small neighborhood of X which does not intersect I. There exists a constant $C_V>0$ such that

$$\log^+|z|-C_V\le G(z)\le\log^+|z|+C_V\quad\text{in } V\cap\mathbf{C}^k.$$

Indeed $\widetilde{G}$ is bounded near X, so we only use log-homogeneity. Therefore $G>0$ in U and it follows from the upper semicontinuity that G is continuous, even Hölder continuous in U, since U is a normal component [S].

As X is an attracting analytic set of dimension $r-1$, it follows from Lemma 2.3 below that $T^r=0$ in U. So T^r is supported on $\partial\mathcal{K}$ and $GT^r=0$ in $\mathbf{C}^k$. Hence $T^{r+1}=0$ in $\mathbf{C}^k$.

Since $I\cap X=\emptyset$ in $\{[z:0]\}=\mathbf{P}^{k-1}$ we get $(r-1)+(l'-1)\le k-2$, so $r+l'\le k$. The current T admits continuous potentials out of I. Since I has dimension $l'-1\le k-(r+1)$, the currents T^j are well defined on $\mathbf{P}^k$ for $j\le r+1$ (see Corollary 3.6

in [FS3]) and satisfy $f^*T^j = d^j T^j$. Moreover T^r has no mass on I [HP], and hence is of total mass in $\mathbf{C}^k$. The current T^{r+1} has support in I. It follows from the support theorem of Federer (see [Fe]) that $\dim I \geq k-(r+1)$. Consequently $r+l'=k$.

We have $\widetilde{G} \leq d^{-1} \log |F|$ and $\pi^{-1}(I)=\{p \mid F(p)=0\}$ in $\mathbf{C}^{k+1}$. Hence the current $T^{k-l'+1}$ has some mass on each branch of I. Therefore $T^{k-l'+1}=T^{r+1}$ is an $\mathbf{R}$-cycle whose support is I. This proves that any point of I is a limit of points in $\partial\mathcal{K}$.

Observe that $f^*(\omega^r)$ has no mass on $\{[z:0]\}$ since $\dim I = k-r-1$. Thus if $f \in \operatorname{Aut}(\mathbf{C}^k)$, we get

$$d_+^r = \int_{\mathbf{C}^k} f^*(\omega^r) \wedge \omega^{k-r} = \int_{\mathbf{C}^k} \omega^r \wedge (f^{-1})^*(\omega^{k-r}) \leq d_-^{k-r}. \quad \square$$

Lemma 2.3. *Let $A \subset \{[z:0]\}$ be an analytic subset of dimension $a-1$. If A is attracting for f, then $T^a=0$ in the basin of attraction of A.*

Proof. Assume that $\{z \in A \mid z_1 = ... = z_a = 0\} = \emptyset$, then, in these coordinates,

$$G = \lim_{n \to +\infty} \frac{1}{2d_+^n} \log^+(|f_1^n|^2 + ... + |f_a^n|^2).$$

The convergence is locally uniform in the basin of attraction of A, therefore $T^a = 0$ (see Theorem 2.5.2 in [S]). $\square$

The rest of the paper is concerned with polynomial automorphisms. If $f \in \operatorname{Aut}(\mathbf{C}^k)$ is weakly regular, we have just seen that G^+ is comparable to $\log^+|z|$ in U^+ and $G^+=0$ on $\mathcal{K}^+$. This allows us to show a convergence result towards T_+ similar to Theorem 2.2.12 of [S]. This yields in particular a rigidity property of $\overline{\mathcal{K}}^+$.

Theorem 2.4. *Assume that $f \in \operatorname{Aut}(\mathbf{C}^k)$ is weakly regular.*

If there exists a non-trivial positive closed current S of bidegree $(1,1)$ on $\mathbf{P}^k$ whose support is contained in $\overline{\mathcal{K}}^+$, then S is proportional to T_+. In this case $r=1$.

Conversely when $r=1$, T_+ is the only positive closed current of bidegree $(1,1)$ and of mass 1 with support on $\overline{\mathcal{K}}^+$.

Example 2.5. Consider $f(x,y,z)=(yx^d+z, y^{d+1}+x, y)$. Then $f \in \operatorname{Aut}(\mathbf{C}^3)$ with $X^+=\{[x:y:0:0]\}$ and $I^+=\{[x:0:z:0]\}$. So f is not weakly regular since $X^+ \cap I^+ = \{[1:0:0:0]\} \neq \emptyset$. On the other hand $f^{-1}(x,y,z)=(y-z^{d+1}, z, x-z(y-z^{d+1})^d)$, so $X^- = \{[0:0:1:0]\}$ and $I^- = \{[x:y:0:0]\}$. Hence f^{-1} is weakly regular.

Note that $X^+ \cap I^+$ is a (super)attractive fixed point for $f_0 := f|_{X^+}$.

Definition/Notation 2.6. Let $f \in \mathrm{Aut}(\mathbf{C}^k)$ be an algebraically stable biholomorphism such that f^{-1} is also algebraically stable. We set

$$\dim X^+ = r-1, \quad \dim I^+ = l'-1, \quad \dim I^+ \cap I^- = q-1,$$
$$\dim X^- = s-1, \quad \dim I^- = l-1.$$

We say that f is *q-regular* if $X^\pm \cap I^\pm = \emptyset$ and

$$\operatorname{codim} I^+ \cap I^- = \operatorname{codim} I^+ + \operatorname{codim} I^- \text{ in } \{[z:0]\}, \quad \text{with } \dim(I^+ \cap I^-) = q-1.$$

So in this case we get the relations

$$r+l' = s+l = k \quad \text{and} \quad q+r+s = k.$$

Remarks 2.7. (i) With the convention $\dim \emptyset = -1$, 0-regular biholomorphisms are precisely the "regular automorphisms" studied in [S]. Observe that f is q-regular if and only if f^{-1} is q-regular.

(ii) If I^- is biholomorphically equivalent to $\mathbf{P}^{l-1}$ (or to any compact complex manifold whose cohomology is one-dimensional) and $X^+ \cap I^+ = \emptyset$, then X^+ and $I^+ \cap I^-$ are disjoint analytic subsets of $I^- \simeq \mathbf{P}^{l-1}$ so $\dim X^+ + \dim(I^+ \cap I^-) \le l-2$, and hence $r+q \le l$. This yields $r+s+q \le k$ if $X^- \cap I^- = \emptyset$. Now $T_+^{r+1} \wedge T_-^s$ is a well-defined current with support in $I^+ \cap I^-$ (see [FS3] and Theorem 2.2) so $\dim(I^+ \cap I^-) \ge k-(r+s+1)$ by the support theorem [Fe], i.e. $r+s+q \ge k$. So in this case the condition $\operatorname{codim}_{\{[z:0]\}} I^+ \cap I^- = \operatorname{codim}_{\{[z:0]\}} I^+ + \operatorname{codim}_{\{[z:0]\}} I^-$ of Definition 2.6 is automatically satisfied. We do not know any example of an automorphism $f \in \mathrm{Aut}(\mathbf{C}^k)$ such that f and f^{-1} are weakly regular and for which $I^+ \cap I^-$ does not have the expected dimension.

Examples 2.8. (i) Let $f(x,y,z) = (x^d + \alpha y^d + z, x^d + y, x)$. Then $f \in \mathrm{Aut}(\mathbf{C}^3)$ with $f^{-1}(x,y,z) = (z, y - z^d, x - z^d - \alpha(y - z^d)^d)$. If $\alpha \ne 0$ we obtain $I^+ \cap I^- = \emptyset$ so f is 0-regular, while f is 1-regular if $\alpha = 0$.

(ii) Consider $f(x,y,z,w) = (h(x,y), g(z,w))$, where $h, g: \mathbf{C}^2 \to \mathbf{C}^2$ are Hénon mappings. Then $f \in \mathrm{Aut}(\mathbf{C}^4)$ is 0-regular if $\deg(h) = \deg(g)$ and 2-regular if $\deg(h) \ne \deg(g)$.

Proposition 2.9. *Let $f \in \mathrm{Aut}(\mathbf{C}^k)$ be a q-regular biholomorphism. Define $G = \max\{G^+, G^-\}$ and let τ be the current defined by G on $\mathbf{P}^k$. Then*

(i) $(dd^c G)^{r+s} = \tau^{r+s} = T_+^r \wedge T_-^s$ *in* $\mathbf{C}^k$;

(ii) $\operatorname{supp} \tau^{r+s+1} = I^+ \cap I^-$;

(iii) *the current* $\tau^{r+s} = T_+^r \wedge T_-^s$ *is of total mass in* $\mathbf{C}^k$;

(iv) $\overline{\partial \mathcal{K}^+ \cap \partial \mathcal{K}^-} \cap \{[z:0]\} = I^+ \cap I^-$;

(v) *if I^- is an attracting set for f, then* $d_+^r \le d_-^s \le d_+^{q+r}$.

Proof. Note that G^+ and G^- are continuous (Theorem 2.2). Since $(dd^cG^+)^r=0$ in $U^+=\{p|G^+(p)>0\}$ and $(dd^cG^-)^s=0$ in $U^-=\{p|G^-(p)>0\}$ (by Theorem 2.2 again), the first claim is a consequence of Lemma 2.12 below.

Since f is q-regular, $I^+\cap I^-$ has dimension $q-1=k-(r+s)-1$. So τ^{r+s}, which clusters only on $I^+\cap I^-$ in $\{[z:0]\}$, has total mass in $\mathbf{C}^k$ (see [HP]). The current τ^{r+s} is supported on $\mathcal{K}^+\cap\mathcal{K}^-=\{p|G(p)=0\}$, therefore $(dd^cG)^{r+s+1}=0$ in $\mathbf{C}^k$. Since $\{[z:0]\}\setminus I^+\subset U^+$ and $\{[z:0]\}\setminus I^-\subset U^-$, it follows that τ^{r+s+1} is supported on $I^+\cap I^-$. Now $G\leq\max\{d_+^{-1}\log|F|, d_-^{-1}\log|F^{-1}|\}$ in $\mathbf{C}^{k+1}$, so τ^{r+s+1} has some mass on each branch of $I^+\cap I^-$. Therefore every point of $I^+\cap I^-$ is a limit of points in $\partial\mathcal{K}^+\cap\partial\mathcal{K}^-$.

Assume that I^- is f-attracting, i.e. there exists $C>1$ such that $1+\|f(z)\|\geq C(1+\|z\|)$ for every point z in a small neighborhood V of I^- with $f(V)\Subset V$. Thus the function $\log^+\|f(z)\|$ grows at least like $\log^+\|z\|$ in V. We recall here below (Lemma 2.11) a comparison principle for plurisubharmonic functions with logarithmic growth. Since $\log\|f\|\geq\log(1+\|z\|)$ on the support of $T_+^r\wedge T_-^s$ and since $T_+^r\wedge T_-^s$ puts no mass on $\{[z:0]\}$, one gets by Lemma 2.11 that

$$1\leq\int_{\mathbf{C}^k}T_+^r\wedge T_-^s\wedge f^*\omega^{k-r-s}=\int_{\mathbf{C}^k}(f^{-1})^*(T_+^r\wedge T_-^s)\wedge\omega^{k-r-s}=\frac{d_-^s}{d_+^r}.$$

There might be equality as follows from Remark 1.6. The last inequality follows from Theorem 2.2(v): if $X^-\cap I^-=\emptyset$, then $d_-^s\leq d_+^{k-s}$. $\square$

Remark 2.10. When $q=0$ (i.e. f is a regular automorphism), then $I^-=X^+$ is always an attracting set for f and we get $d_+^r=d_-^s$ (see also Proposition 2.3.2 in [S]).

When $q\geq1$, then $\partial\mathcal{K}^+\cap\partial\mathcal{K}^-$ is not compact. We give examples in Section 5.2 such that I^- is an attracting set for f. Observe that if $\|f(p)\|\geq C(1+\|p\|)^\gamma$ for $\|p\|\gg1$ on $\partial\mathcal{K}^+\cap\partial\mathcal{K}^-$, then since $T_+^r\wedge T_-^s$ is supported on $\partial\mathcal{K}^+\cap\partial\mathcal{K}^-$, we get with the same proof that $d_-^s/d_+^r\geq\gamma$. This is of interest when $\gamma>1$ (see Remark 3.2).

Lemma 2.11. ([T]) *Let S be a positive closed current of bidimension (s,s) in $\mathbf{C}^k$. Let u and v be locally bounded plurisubharmonic functions in a neighborhood of* $\operatorname{supp} S$ *in $\mathbf{C}^k$. Assume that $v>0$ and $u(z)<v(z)+o(v(z))$, $\|z\|\to+\infty$. Then*

$$\int_{\mathbf{C}^k}S\wedge(dd^cu)^s\leq\int_{\mathbf{C}^k}S\wedge(dd^cv)^s.$$

The corresponding lemma when $s=k$ is given in [T], p. 322. The proof is an integration by part argument.

Lemma 2.12. *Let u and v be continuous non-negative plurisubharmonic functions in $\mathbf{C}^k$ such that $(dd^cu)^r=0$ in $\{p|u(p)>0\}$ and $(dd^cv)^s=0$ in $\{p|v(p)>0\}$. Set $w=\max\{u,v\}$. Then*

$$(dd^cw)^{r+s}=(dd^cu)^r\wedge(dd^cv)^s.$$

Proof. Fix ε and consider $u_\varepsilon=\max\{u+\varepsilon, v\}$ and $v_\varepsilon=\max\{u, v+\varepsilon\}$. Since u_ε and v_ε decrease toward w as $\varepsilon\to 0$, we have

$$(dd^c u_\varepsilon)^r \wedge (dd^c v_\varepsilon)^s \to (dd^c w)^{r+s}.$$

We can assume without loss of generality that $r\geq s$. We have $(dd^c v)^r=0$ in $\{p\mid v(p)>u(p)+\varepsilon\}\subset\{p\mid v(p)>0\}$. Moreover $v_\varepsilon\equiv v+\varepsilon$ near $\{p\mid v(p)=u(p)+\varepsilon\}$ and $v>0$, therefore $(dd^c v_\varepsilon)^s=(dd^c v)^s=0$ near $\{p\mid v(p)=u(p)+\varepsilon\}$. Thus $(dd^c u_\varepsilon)^r\wedge(dd^c v_\varepsilon)^s$ has support in the open set $\{p\mid v(p)<u(p)+\varepsilon\}$. Hence

$$(dd^c u_\varepsilon)^r \wedge (dd^c v_\varepsilon)^s = (dd^c u)^r \wedge (dd^c v_\varepsilon)^s.$$

Now $\operatorname{supp}(dd^c u)^r\subset\{p\mid u(p)=0\}$, thus $v_\varepsilon\equiv v+\varepsilon$ near $\operatorname{supp}(dd^c u)^r$, which yields

$$(dd^c u_\varepsilon)^r \wedge (dd^c v_\varepsilon)^s = (dd^c u)^r \wedge (dd^c v)^s. \quad \square$$

Theorem 2.13. *Let $f\in\operatorname{Aut}(\mathbf{C}^k)$. Assume that f^{-1} is weakly regular and I^- is an attracting set for f. Then the following holds:*

(i) *f^{-1} is normal on $\mathbf{C}^k$ and $K^-=\mathcal{K}^-=\{p\mid G^-(p)=0\}$ is closed in $\mathbf{C}^k$;*

(ii) *$K^+=\mathbf{C}^k\setminus\mathcal{B}(I^-)$ is closed in $\mathbf{C}^k$, where $\mathcal{B}(I^-)$ denotes the basin of attraction of I^-; $\overline{K}^+\cap\{[z:0]\}=X^-=\overline{\partial K^+}\cap\{[z:0]\}$;*

(iii) *$K:=K^+\cap K^-$ is a compact polynomially convex subset of $\mathbf{C}^k$ which contains the non-wandering set of f;*

(iv) *$W^s(K)=K^+$ and $W^u(K)=K^-$.*

Proof. That I^- is attracting for f means that there exists a neighborhood V of I^- in $\mathbf{P}^k$ such that $f(V\setminus I_f)\Subset V$ and $\bigcap_{j=1}^\infty \overline{f^j(V\setminus I_{f^j})}=I^-$. It follows that if $x_p\to x\in\mathbf{C}^k$, $f^{-n_p}(x_p)$ cannot cluster on I^-. Hence f^{-1} is normal. Since an unbounded orbit for f^{-1} cannot approach I^-, it is necessarily in U^-, the basin of X^- which is attracting for f^{-1} since $I^-\cap X^-=\emptyset$. Therefore $K^-=\mathcal{K}^-=\mathbf{C}^k\setminus U^-$ is closed. The fact that $K^-=\{p\mid G^-(p)=0\}$ follows from Theorem 2.2.

Let $x\in\mathbf{C}^k\setminus\mathcal{B}(I^-)$. If $f^{n_j}(x)$ clusters at infinity, it has to avoid a neighborhood of I^-. Hence f^{-n_j} is well defined and $x=f^{-n_j}(f^{n_j}(x))$ is arbitrarily close to X^-, a contradiction. So $K^+=\mathbf{C}^k\setminus\mathcal{B}(I^-)$ and it is closed. Since $\{[z:0]\}\setminus I^+$ is sent by f into $X^+\subset I^-$ which is attracting for f, K^+ can cluster only on I^+. If $p\in I^+\setminus X^-$ then the blow-up $f(p)$ of f at the point p is an analytic subset of $\{[z:0]\}$ which is included in I^-, otherwise $f^{-1}(f(p)\setminus I^-)=p$ should belong to X^-. Therefore $p\in\mathcal{B}(I^-)$ and K^+ can only cluster on X^-. On the other hand, we will show hereafter (Theorem 3.1) that there exists a non-zero positive closed current σ_s of bidimension (s,s) with support in $\overline{\partial K^+}$—here $\dim X^-=s-1$. Moreover $\sigma_s\wedge\{[z:0]\}$ is a well-defined current of bidimension $(s-1,s-1)$ (see Theorem 3.1) which has support on X^-. Since X^- is irreducible, we have $X^-\subset\operatorname{supp}\sigma_s$. Hence ∂K^+ clusters at every point of X^-.

Similarly K^- clusters on I^-, and hence $K=K^+\cap K^-$ is compact. The polynomial convexity of K follows from the fact that $H_n:=\max\{\log^+\|f^n\|,\log^+\|f^{-n}\|\}$ is bounded exactly on K.

We now prove that the stable set $W^s(K):=\{z\in\mathbf{C}^k\,|\lim_{n\to+\infty}f^n(z)\in K\}$ equals K^+. Indeed for $x\in K^+$, $G^-(f^n(x))=d_-^{-n}G^-(x)$ so if $x_0=\lim f^{n_j}(x)$ then $G^-(x_0)=0$. Thus $x_0\in K^-\cap K^+=K$, i.e. $W^s(K)=K^+$.

Similarly let $x\in K^-$. Assume $f^{-n_i}(x)\to y$. For any neighborhood U of y, $f^{n_i}(U)$ contains x, so $y\notin\mathcal{B}(I^-)$. Therefore $y\in K^+$ and $W^u(K)=K^-$. □

Remark 2.14. The hypotheses of the theorem are satisfied in Example 1.8 when $|b|>1$. We give other examples in Section 5.

Corollary 2.15. *Assume that f and f^{-1} are weakly regular and I^- is f-attracting. Then either f is a regular automorphism (i.e. $I^+\cap I^-=\emptyset$), or $\partial\mathcal{K}^+\setminus K^+$ is non-empty. In the latter case, the basin $\mathcal{B}(I^+\cap I^-)$ contains $\mathcal{K}^+\setminus K^+$, and hence f is not normal.*

Proof. We know from Theorem 2.2 that $\overline{\partial\mathcal{K}^+}\cap\{[z:0]\}=I^+$. On the other hand $\overline{K}^+\cap\{[z:0]\}\subset X^-$ by Theorem 2.13. Since $X^-\cap I^-=\emptyset$, this yields that if $X^-=I^+$ then f is regular, and if $X^-\neq I^+$ then $\partial\mathcal{K}^+\setminus K^+$ is non-empty. Proposition 1.10 implies that orbits in $\mathcal{K}^+\setminus K^+$ cluster only on $I^+\cap I^-$. Hence $\mathcal{K}^+\setminus K^+$ is in the basin $\mathcal{B}(I^+\cap I^-)$ of $I^+\cap I^-$. □

3. Currents supported by K^+

In this section we construct, under suitable assumptions, a canonical current σ_s which is invariant by f and supported on $\overline{K}^+$ (Theorem 3.1). This shows in particular that K^+ is non-empty (compare with Example 1.5). When T_-^s is an extremal point in the cone of positive closed currents of bidegree (s,s) on $\mathbf{P}^k$, we show a strong convergence result towards σ_s (Theorem 3.4) which can be thought of as a "mixing property" of σ_s. We prove the extremality of T_- (Theorem 3.6), so everything works fine when $s=1$: we obtain as a consequence the density of stable manifolds of dimension 1 in the support of σ_1 (Corollary 3.8). It is an interesting open problem to show extremality of currents like T_-^s, $s>1$.

Theorem 3.1. *Let $f\in\operatorname{Aut}(\mathbf{C}^k)$ be such that f^{-1} is weakly regular and I^- is f-attracting. Then $\overline{K}^+$ does not carry any non-zero positive closed current of bidimension $(s+1,s+1)$, where $\dim X^-=s-1$.*

However there is a positive closed current σ_s of bidimension (s,s) supported on $\overline{K}^+$ which satisfies $f^*\sigma_s = d_-^s\sigma_s$ and

$$\int_{\mathbf{P}^k} \sigma_s \wedge \omega^{k-s} = \int_{\mathbf{C}^k} \sigma_s \wedge \omega^{k-s} = 1.$$

More precisely, if $d_-^s > d_+^{k-s-1}$, then

$$\frac{1}{d_-^{ns}} (f^n)^*(\omega^{k-s}) \to \sigma_s,$$

in the weak sense of currents. Moreover for any smooth closed form $\Theta \sim \omega^{k-s}$,

$$\frac{1}{d_-^{ns}} (f^n)^*(\Theta) \to \sigma_s.$$

Proof. Assume that S is a non-zero positive closed current of bidimension $(s+1, s+1)$ with support in $\overline{K}^+$. Then $S\wedge\{[z:0]\}$ is well defined and non-zero (this follows from [FS3], p. 412), since $\overline{K}^+ \cap \{[z:0]\} = X^-$ is of dimension $s-1$. The current $S\wedge\{[z:0]\}$ has support in X^- (Theorem 2.13) and is of bidimension (s,s), this is impossible since $\dim X^- = s-1$.

Define $R_n = (1/d_-^{ns})(f^n)^*(\omega^{k-s})$. The current R_n is positive, closed, of bidimension (s,s) and with mass

$$\int_{\mathbf{C}^k} R_n \wedge \omega^s = \int_{\mathbf{C}^k} \omega^{k-s} \wedge \frac{1}{d_-^{ns}} (f^{-n})^*(\omega^s) = 1.$$

The last equality holds since $\dim I^- = k-s-1$ (Theorem 2.2) so $(f^{-n})^*(\omega^s)$ has no mass on I^-. We still denote by R_n the trivial extension to $\mathbf{P}^k$. Since I^- is an attracting set for f, any cluster point of $(R_n)_{n=1}^\infty$ has support in $\overline{K}^+ = \mathbf{C}^k \setminus \mathcal{B}(I^-)$ (we can argue as in Lemma 2.3 since I^- is attracting and of dimension $k-s-1$) and is of total mass 1 in $\mathbf{C}^k$ since $\dim X^- = s-1$. If we take a limit point of a Cesàro sum, we get the invariant candidate because f^* is continuous on currents in $\mathbf{C}^k$, and the limit current cannot have mass on X^- whose dimension is only $s-1$.

Consider now Θ being a smooth closed form cohomologous to ω^{k-s} whose support does not intersect I^-. This is possible since $\dim I^- = k-s-1$, so we can find a linear subspace L of dimension s in $\mathbf{P}^k$ which does not intersect I^- and regularize the current of integration $[L]$. Define

$$\Theta_1 := \frac{1}{d_-^s} (f^*\Theta)|_{\mathbf{C}^k}.$$

The current Θ_1 is closed and positive in $\mathbf{C}^k$. The mass of Θ_1 is

$$\int_{\mathbf{C}^k} \Theta_1 \wedge \omega^s = \frac{1}{d_-^s} \int_{\mathbf{C}^k} \Theta \wedge (f^{-1})^*(\omega^s) = \frac{1}{d_-^s} \int_{\mathbf{P}^k} \Theta \wedge (f^{-1})^*(\omega^s) = 1$$

since $I^- \cap \operatorname{supp} \Theta = \emptyset$. We still denote by $\Theta_1 = d_-^{-s} f^* \Theta$ the trivial extension to $\mathbf{P}^k$. Observe that since $f^{-1}(\{[z:0]\} \setminus I^-) \subset X^-$, we get

$$\operatorname{supp} \Theta_1 \cap \{[z:0]\} \subset X^-.$$

So Θ_1 is cohomologous to Θ and it is smooth in $\mathbf{P}^k \setminus X^-$. Hence

$$\frac{1}{d_-^s} f^* \Theta = \Theta + dd^c(S),$$

where S is a current of bidegree $(k-s-1, k-s-1)$ which is smooth in $\mathbf{P}^k \setminus X^-$. Replacing S by $S - A\omega^{k-s-1}$, we can assume further that $S \leq 0$ in $\mathbf{P}^k \setminus V$, where V is a small neighborhood of X^-. We can iterate the previous equation and get

$$\frac{1}{d_-^{ns}} (f^n)^* \Theta = \Theta + dd^c(S_n),$$

where

$$S_n = \sum_{j=0}^{n-1} \frac{1}{d_-^{js}} (f^j)^*(S)$$

is a decreasing sequence of negative currents in $\mathbf{P}^k \setminus V$, since we can assume that $f^{-1}(V) \Subset V$. Fix $C > 0$ so that $-C\omega^{k-s-1} \leq S \leq 0$ on $\mathbf{P}^k \setminus V$. Then

$$-C \frac{1}{d_-^{js}} (f^j)^*(\omega^{k-s-1}) \leq \frac{1}{d_-^{js}} (f^j)^*(S) \leq 0 \quad \text{in } \mathbf{P}^k \setminus V.$$

Then

$$0 \leq S_n - S_{n+p} \leq \frac{C}{\delta^n} \sum_{j=0}^{p-1} \frac{1}{\delta^j} \left(\frac{1}{d_+^{j+n}} (f^{j+n})^* \omega \right)^{k-s-1} \quad \text{in } \mathbf{P}^k \setminus V,$$

where $\delta := d_-^s / d_+^{k-s-1} > 1$. This shows that $(S_n)_{n=1}^{\infty}$ converges towards a current S_∞ in $\mathbf{P}^k \setminus V$, and hence in $\mathbf{P}^k \setminus X^-$, since V was an arbitrarily small neighborhood of X^-. Thus

$$\frac{1}{d_-^{ns}} (f^n)^* \Theta = \Theta + dd^c S_n \to \sigma_s := \Theta + dd^c S_\infty \quad \text{in } \mathbf{P}^k \setminus X^-.$$

Now σ_s extends trivially through X^- for dimension reasons (Harvey's theorem). It follows from the discussion above that the invariant current σ_s has support on $\overline{K}^+$ and is of total mass 1 in $\mathbf{C}^k$.

Note that if Θ' is a smooth form cohomologous to Θ, then $\Theta'=\Theta+dd^c\alpha$, where α is a smooth form of bidegree $(k-s-1,k-s-1)$. Now $\|(f^n)^*(\alpha)\|=O(d_+^{n(k-s-1)})$, so $d_-^{-ns}(f^n)^*(\alpha)\to 0$ since $d_-^s>d_+^{k-s-1}$. Therefore $d_-^{-ns}(f^n)^*(\Theta')\to\sigma_s$, in particular

$$\frac{1}{d_-^{ns}}(f^n)^*(\omega^{k-s})\to\sigma_s. \quad \square$$

Remarks 3.2. (i) When f is 0-regular, we have $d_+^{k-1-s}<d_+^{k-s}=d_-^s$. In this case $I^-=X^+$ is f-attracting and $\sigma_s=T_+^{k-s}$ (see [S]).

(ii) When f is q-regular with $\delta=d_-^s/d_+^r>1$, then for σ_s we can consider a cluster point of the sequence $N^{-1}\sum_{j=1}^N T_+^r\wedge\delta^{-j}(f^j)^*\omega^{k-r-s}$. This will allow us to construct an invariant measure which does not charge pluripolar sets in Section 4.

The next result uses the Cauchy–Schwarz inequality in the style of Ahlfors–Beurling (see [A]) to show convergence of truncated currents towards closed currents (see [BS1] and [S] for similar results in the context of complex dynamics).

Proposition 3.3. *Let $f\in\mathrm{Aut}(\mathbf{C}^k)$ be such that $d_-^s>d_+^{k-s-1}$ for some integer $s\le k-1$. Let $\psi\ge 0$ be a test function with support in a ball B of $\mathbf{C}^k$. Let $u_1,\dots,u_l$ be continuous plurisubharmonic functions in B. Then*

$$S_n^{(l)}:=\frac{1}{d_-^{ns}}(f^{-n})^*(\psi\omega^s)\wedge dd^cu_1\wedge\ldots\wedge dd^cu_l$$

is a bounded sequence of positive currents. Moreover $\|dS_n^{(l)}\|,\|dd^cS_n^{(l)}\|\to 0$. So any cluster point is a closed positive current of bidegree $(s+l,s+l)$.

Proof. We first consider the sequence $S_n:=S_n^{(0)}=d_-^{-ns}(f^n)^*(\psi\omega^s)$. It is clearly bounded. Let θ be a $(0,1)$ test form on $\mathbf{C}^k$. We have

$$\begin{aligned}\left|\int_{\mathbf{C}^k}(f^{-n})^*(\partial\psi\wedge\omega^s)\wedge\theta\wedge\omega^{k-s-1}\right| &= \left|\int_{\mathbf{C}^k}\partial\psi\wedge\omega^s\wedge(f^n)^*\theta\wedge(f^n)^*\omega^{k-s-1}\right|\\ &\le\left(\int_{\mathbf{C}^k}\omega^s\wedge\partial\psi\wedge\bar\partial\psi\wedge(f^n)^*\omega^{k-s-1}\right)^{1/2}\\ &\quad\times\left(\int_{\mathbf{C}^k}\omega^s\wedge(f^n)^*(\theta\wedge\bar\theta\wedge\omega^{k-s-1})\right)^{1/2}\\ &\le O(d_+^{n(k-s-1)/2})O(d_-^{ns/2}).\end{aligned}$$

The mass $\|dS_n\|$ of the currents dS_n thus satisfies

$$\|dS_n\| = O\left(\left(\frac{d_+^{k-s-1}}{d_-^s}\right)^{n/2}\right) \to 0.$$

Similarly one gets $\|dd^c S_n\| = O((d_+^{k-s-1}/d_-^s)^n) \to 0$.

Consider now $S_n^{(1)} = d_-^{-ns}(f^n)^*(\psi\omega^s)\wedge dd^c u_1$. We can use exactly the same inequalities, replacing ω^{k-s-1} by $dd^c u_1 \wedge \omega^{k-s-2}$. So we have again $\|dS_n^{(1)}\| = O((d_+^{k-s-1}/d_-^s)^{n/2})$ if we show that

$$\int_{\operatorname{supp}\psi} \omega^{s+1}\wedge (f^n)^*(dd^c u_1 \wedge \omega^{k-s-2}) = O(d_+^{n(k-s-1)}).$$

Note that we can assume without loss of generality that $u_1 \le 0$ on B. So $\tilde{u}_1 := \max\{u_1, A\log\|z\|\}$ defines a plurisubharmonic function in $\mathbf{C}^k$, where A is chosen large enough so that $\tilde{u}_1 \equiv u_1$ in a neighborhood of $\operatorname{supp}\psi$ and $\tilde{u}_1 \equiv A\log\|z\|$ near $\partial B = \{z \mid \|z\| = 1\}$. We infer that

$$\begin{aligned}\int_{\operatorname{supp}\psi} \omega^{s+1}\wedge (f^n)^*(dd^c u_1 \wedge \omega^{k-s-2}) &\le \int_{\mathbf{C}^k} \omega^{s+1}\wedge (f^n)^*(dd^c \tilde{u}_1 \wedge \omega^{k-s-2}) \\ &= A\int_{\mathbf{C}^k} \omega^{s+1}\wedge (f^n)^*(\omega^{k-s-1}) \le A d_+^{n(k-s-1)}.\end{aligned}$$

Thus $\|dS_n^{(1)}\| \to 0$. One gets similarly that $\|dS_n^{(l)}\|, \|dd^c S_n^{(l)}\| \to 0$ for all l. □

Theorem 3.4. *Let $f \in \operatorname{Aut}(\mathbf{C}^k)$ be such that $X^- \cap I^- = \emptyset$ with I^- being f-attracting. Assume that $d_+^{k-s-1} < d_-^s$, where $s-1 = \dim X^-$, and T_-^s is extremal in the cone of positive closed currents of bidegree (s,s). Let R be a positive closed current of bidimension (s,s) in $\mathbf{C}^k$. We assume that R is smooth or $R = dd^c u_1 \wedge \ldots \wedge dd^c u_{k-s}$, where u_j are continuous plurisubharmonic functions. Let $\varphi \ge 0$ be a test function. Then*

$$\frac{1}{d_-^{ns}}(f^n)^*(\varphi R) \to c\sigma_s,$$

where $c = \int_{\mathbf{C}^k} \varphi R \wedge T_-^s$.

Proof. It is enough to show convergence on a generating family of test forms $\psi\alpha^s$, with α being d, d^c-closed and strictly positive and $0 \le \psi \le 1$. For simplicity we only consider $\psi\omega^s$.

The sequence $S_n = d_-^{-ns}(f^{-n})^*(\psi\omega^s)$ is bounded and all cluster points are closed (Proposition 3.3). We compute the mass of S_n. We infer from Theorem 3.1 that

$$\int_{\mathbf{P}^k} S_n \wedge \omega^{k-s} = \int_{\mathbf{C}^k} \psi\omega^s \wedge \frac{1}{d_-^{ns}}(f^n)^*(\omega^{k-s}) \to \int_{\mathbf{C}^k} \psi\omega^s \wedge \sigma_s =: C_\psi.$$

Let S be a limit point of $(S_n)_{n=1}^\infty$. Clearly $0 \le S \le T_-^s$. Now T_-^s is extremal thus $S = C_\psi T_-^s$, so the sequence $(S_n)_{n=1}^\infty$ actually converges towards $C_\psi T_-^s$. Therefore if R is smooth then

$$\left\langle \frac{1}{d_-^{ns}}(f^n)^*(\varphi R), \psi\omega^s \right\rangle = \langle \varphi R, S_n \rangle \to \int_{\mathbf{C}^k} \psi\omega^s \wedge \sigma_s \langle \varphi R, T_-^s \rangle,$$

and thus $R_n = d_-^{-ns}(f^n)^*(\varphi R) \to c\sigma_s$, with $c = \langle \varphi R, T_-^s \rangle$.

When $R = dd^c u_1 \wedge \ldots \wedge dd^c u_{k-s}$, where the u_j are merely continuous plurisubharmonic functions, we need to go step by step using Proposition 3.3 (as in the proof of Theorem 7.1 in [S]). We first show that $S_n \wedge dd^c u_1$ converges towards $C_\psi T_-^s \wedge dd^c u_1$. Let θ be a test form of bidegree $(k-s-1, k-s-1)$. We have

$$\langle S_n \wedge dd^c u_1, \theta \rangle = \langle dd^c(S_n \wedge \theta), u_1 \rangle = \langle S_n \wedge dd^c\theta, u_1 \rangle + 2\langle d\theta \wedge d^c S_n, u_1 \rangle + \langle \theta \wedge dd^c S_n, u_1 \rangle.$$

The first term converges towards $\langle C_\psi T_-^s \wedge dd^c\theta, u_1 \rangle = \langle C_\psi T_-^s \wedge dd^c u_1, \theta \rangle$, since u_1 is continuous. The last two terms converge to 0 since $\|dS_n\|, \|dd^c S_n\| \to 0$ (Proposition 3.3).

Now set $S_n^{(j)} = S_n \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_j$. It then follows from Proposition 3.3 that $\|dS_n^{(j)}\|, \|dd^c S_n^{(j)}\| \to 0$. So using that u_{j+1} is continuous, we get by induction that $S_n^{(j)} \to C_\psi T_-^s \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_j$. For $j = k-s$ this yields that $R_n \to c\sigma_s$. $\square$

Remarks 3.5. (i) When f is 1-regular, we have $k-1-s = r$, thus the hypothesis $d_+^{k-1-s} < d_-^s$ is equivalent to $d_+^r < d_-^s$. Since I^- is f-attracting, we have showed that $d_+^r \le d_-^s$ always (Proposition 2.9) and $d_+^r < d_-^s$ if $\|f(z)\| \ge (1+\|z\|)^{1+\gamma}$ on $\partial\mathcal{K}^+ \cap \partial\mathcal{K}^-$ for $\|z\|$ large (Remark 2.10).

(ii) When T_-^s is merely extremal in the cone of positive closed currents S of bidegree (s,s) which satisfy $(f^{-1})^* S = d_-^s S$, then the same proof shows convergence of $(1/n)\sum_{j=1}^n (1/d_-^{js})(f^j)^*(\varphi S)$ towards $c\sigma_s$.

When $s=1$ the next theorem asserts that T_- is extremal. So our assumptions become that I^- is f-attracting and $d_- > d_+$ if $k=3$. The latter is necessary to ensure non-trivial dynamics as follows from Example 1.5.

Theorem 3.6. *Let $f\in\mathrm{Aut}(\mathbf{C}^k)$ be an algebraically stable biholomorphism. Then T_+, the Green current of f, is extremal in the cone of positive closed currents of bidegree $(1,1)$. When $r=1$, i.e. $\dim X^+=0$, then $\partial\mathcal{K}^+$ is connected.*

Proof. Let S be a positive closed current of bidegree $(1,1)$ on $\mathbf{P}^k$ such that $S\le T_+$. We want to show that $S=\alpha T_+$, where $0\le\alpha\le 1$. Denote by S_n the trivial extension through $\{[z:0]\}$ of the current $d_+^n(f^{-n})^*S|_{\mathbf{C}^k}$. Since $d_+^n(f^{-n})^*T_+=T_+$ in $\mathbf{C}^k$, we have $S_n\le T_+$ on $\mathbf{P}^k$.

Set now $S_n'=d_+^{-n}(f^n)^*S_n$. Clearly $S_n'\equiv S$ in $\mathbf{C}^k$ and

$$S_n'\le\frac{1}{d_+^n}(f^n)^*T_+=T_+.$$

Since T_+ has no mass on the hyperplane $\{[z:0]\}$, neither have S_n' and S, and hence $S_n'\equiv S$ on $\mathbf{P}^k$. The next lemma yields that $S=S_n'\to\alpha T_+$, where $\alpha=\|S\|=\|S_n\|$.

When $r=1$, T^+ has support equal to $\partial\mathcal{K}^+$. Hence extremality of T^+ implies the connectedness of $\partial\mathcal{K}^+$ in $\mathbf{C}^k$. $\square$

Lemma 3.7. *Let $(\sigma_n)_{n=1}^\infty$ be a sequence of positive closed currents of bidegree $(1,1)$ and constant mass $\alpha\in[0,1]$. If $\sigma_n\le T_+$ then*

$$\frac{1}{d_+^n}(f^n)^*(\sigma_n)\to\alpha T_+.$$

Proof. Set $\sigma_n'=T_+-\sigma_n$, a positive closed current of bidegree $(1,1)$ and of mass $1-\alpha$ on $\mathbf{P}^k$. Consider the potentials φ_n and φ_n' of σ_n and σ_n' in $\mathbf{C}^{k+1}$ such that $G^+=\varphi_n+\varphi_n'$,

$$(1)\qquad \varphi_n(z,t)\le\alpha\log\|(z,t)\|\quad\text{and}\quad\varphi_n'(z,t)\le(1-\alpha)\log\|(z,t)\|.$$

Set $v_n:=d_+^{-n}\varphi_n\circ F^n$. Then $(v_n)_{n=1}^\infty$ is a sequence of potentials of $d_+^{-n}(f^n)^*\sigma_n$. It follows from (1) that $(v_n)_{n=1}^\infty$ is locally uniformly bounded from above. We can extract a convergent subsequence, $v_{n_p}\to v$. Since

$$\varphi_n=G^+-\varphi_n'\ge G^+-(1-\alpha)\log\|(z,t)\|,$$

we get $v_n\ge G^+-(1-\alpha)d_+^{-n}\log\|F^n(z,t)\|$. Hence $v\ge\alpha G^+$ is not identically $-\infty$. Now $\varphi_n\le\alpha\log\|(z,t)\|$ gives $v\le\alpha G^+$, so $v=\alpha G^+$. $\square$

Corollary 3.8. *Let $f \in \mathrm{Aut}(\mathbf{C}^3)$ be such that f^{-1} is weakly regular with I^- being f-attracting and $d_- > d_+$. Let p be a periodic saddle point of type $(1,2)$ (one eigenvalue has modulus <1, and two have modulus >1). Then the stable manifold $W^s(p)$ is dense in the support of σ_1.*

Proof. Let D be a holomorphic disk through p in the stable direction. Let $[\widetilde{D}] = \int R_\theta^*[D]\, d\theta$, where R_θ are rotations around p in a cone, such that for each θ in the parameter space, $f^{-n}(R_\theta^* D)$ converges to the stable manifold. Moreover we can assume that the local potential for $[\widetilde{D}]$ is continuous except at the point p where it has a logarithmic singularity. Let φ be a positive test function. We infer from Theorem 3.4 that

$$\frac{1}{d_-^{ns}} (f^n)^* (\varphi[\widetilde{D}]) \to c\sigma_1,$$

where $c = \int \varphi[\widetilde{D}] \wedge T_-$ (the proof of Theorem 3.4 goes through with minor modification in the presence of an isolated logarithmic singularity).

We claim that $c>0$. Otherwise G^- would be harmonic and non-negative on $W^s(p) \simeq \mathbf{C}$, and hence $G^-|_{W^s(p)} \equiv 0$ by the minimum principle. Now $W^s(p) \subset K^+$ which clusters on X^- at infinity. Since X^- is disjoint from I^-, there exists $C>0$ such that

$$\log^+|z| - C \le G^-(z) \le \log^+|z| + C \quad \text{on } K^+.$$

Thus G^- is unbounded on $W^s(p)$, and hence non-constant. Therefore $c>0$, so $W^s(p)$ is dense in the support of σ_1. $\square$

Remark 3.9. When $f \in \mathrm{Aut}(\mathbf{C}^k)$ is as in Theorem 3.4 and p is a periodic saddle point of type $(s, k-s)$, we can similarly show that the stable manifold of p either is dense in $\operatorname{supp} \sigma_s$ or does not intersect $\operatorname{supp} \sigma_s$.

4. Invariant measure

Let $f \in \mathrm{Aut}(\mathbf{C}^k)$ with f^{-1} weakly regular, I^- being f-attracting and $d_-^s > d_+^{k-s-1}$. We set $\mu := \sigma_s \wedge T_-^s$, where σ_s and T_-^s are the invariant currents defined by Theorems 3.1 and 2.2. The wedge product is well-defined since T_- has locally bounded potential near $\overline{K}^+$.

We show in Section 4.1 that μ is mixing if T_-^s is extremal (Theorem 4.1). In Section 4.2 we give, for some q-regular biholomorphisms, an alternative construction of μ in terms of a partial Green function. As a simple application, we show that μ does not charge pluripolar sets (Theorem 4.6).

4.1. Mixing

Theorem 4.1. *Let $f\in\mathrm{Aut}(\mathbf{C}^k)$ be such that $X^-\cap I^-=\emptyset$ with I^- being f-attracting and $d_-^s>d_+^{k-s-1}$. Then $\mu:=\sigma_s\wedge T_-^s$ is an invariant probability measure with support in the compact set $K=\{p\in\mathbf{C}^k\,|\,(f^n(p))_{n=-\infty}^{\infty}$ is bounded$\}$.*

If T_-^s is extremal then μ is mixing.

Proof. The current T_-^s has support in $K^-=\mathcal{K}^-$ by Theorems 2.2 and 2.13, and σ_s has support in K^+ by Theorem 3.1, therefore μ has support in the set $K=K^+\cap K^-$ which is compact (Theorem 2.13). That μ is an invariant probability measure follows from the corresponding invariance of T_-^s and σ_s.

Let φ be a test function. Assuming that T_-^s is extremal and $d_-^s>d_+^{k-s-1}$, we want to show

$$\varphi\circ f^{-n}T_-^s\wedge\sigma_s=\frac{1}{d_-^{ns}}(f^{-n})^*(\varphi T_-^s)\wedge\sigma_s\to c_\varphi T_-^s\wedge\sigma_s,$$

where $c_\varphi=\int\varphi\,d\mu$.

Assume without loss of generality that $0\le\varphi\le1$. Let $R_n=d_-^{-ns}(f^{-n})^*(\varphi T_-^s)$. This is a bounded sequence of positive currents. Any cluster point R is closed (Proposition 3.3), with $0\le R\le T_-^s$. So $R=cT_-^s$ with

$$c=\lim\langle R_n,\Theta\rangle=\lim\left\langle\varphi T_-^s,\frac{1}{d_-^{ns}}(f^n)^*\Theta\right\rangle,$$

where Θ is as in the proof of Theorem 3.1. Since $d_-^{-ns}(f^n)^*\Theta$ converges to σ_s in the sense of positive currents and since $T_-^s=(dd^cG^-)^s$ with G^- continuous, one can show, in the style of Proposition 3.3, that

$$\left\langle\varphi T_-^s,\frac{1}{d_-^{ns}}(f^n)^*\Theta\right\rangle\to\langle\varphi T_-^s,\sigma_s\rangle=c_\varphi.$$

Thus $c=c_\varphi$ is independent of the cluster point, and hence $(R_n)_{n=1}^\infty$ and actually converges towards $c_\varphi T_-^s$.

We now need to show that $R_n\wedge\sigma_s\to c_\varphi T_-^s\wedge\sigma_s$. Let ψ be a test function. Recall from the proof of Theorem 3.1 that $\sigma_s=\Theta+dd^cS_\infty$. Thus

$$\langle R_n\wedge\sigma_s,\psi\rangle=\langle R_n,\psi\Theta\rangle+\langle dd^c(\psi R_n),S_\infty\rangle.$$

The first term converges towards $\langle c_\varphi T_-^s\wedge\Theta,\psi\rangle$ since Θ is smooth, the second can be decomposed as $A_n+B_n+C_n$, where

$$A_n=\langle R_n,dd^c\psi\wedge S_\infty\rangle,\quad B_n=2\langle dR_n,d^c\psi\wedge S_\infty\rangle,\quad C_n=\langle dd^cR_n,\psi S_\infty\rangle.$$

We are going to show that

$$A_n \to c_\varphi \langle T_-^s, dd^c\psi \wedge S_\infty\rangle = c_\varphi \langle T_-^s \wedge dd^c S_\infty, \psi\rangle,$$

and $B_n, C_n \to 0$. This will yield the desired mixing property (see [W]).

Recall from the construction of σ_s (Theorem 3.1) that $S_\infty = \lim_{N\to\infty} S_N$ out of a neighborhood of X^+, with S_N smooth in $\mathbf{C}^k$. Out of a small neighborhood of X^+, we have

$$(\sharp) \qquad 0 \le S_N - S_\infty \le \frac{C}{\delta^N} \sum_{j=0}^{\infty} \frac{1}{\delta^j} (dd^c G_{j+N}^+)^{k-s-1},$$

where $G_j^+ = d_+^{-j} \log^+ \|f^j\| \le \log^+ \|z\|$ is locally uniformly bounded. As S_N is smooth, we have the desired convergence when replacing S_∞ by S_N. So we need to get a control on $\langle R_n, dd^c\psi \wedge (S_\infty - S_N)\rangle$ that is uniform in n. Now this is a straightforward consequence of ($\sharp$),

$$|\langle R_n, dd^c\psi \wedge (S_\infty - S_N)\rangle| \le \frac{C\|\psi\|_2}{\delta^N} \sum_{j=0}^{\infty} \frac{1}{\delta^j} \int_{\operatorname{supp}\psi} T_-^s \wedge \omega \wedge (dd^c G_{j+N})^{k-s-1}$$

and it follows from Chern–Levine–Nirenberg inequalities that the integrals are all bounded by 1. Therefore

$$|\langle R_n, dd^c\psi \wedge (S_\infty - S_N)\rangle| \le \frac{C'}{\delta^N}.$$

This estimate allows us to show that A_n has the right convergence. We show similarly that B_n and C_n both converge to 0 using the fact that $\|dR_n\|, \|dd^c R_n\| \to 0$ (Proposition 3.3). $\square$

Proposition 4.2. *Let $f \in \operatorname{Aut}(\mathbf{C}^k)$ be such that f^{-1} is algebraically stable. Let $\varphi \ge 0$ be a test function in a ball B of $\mathbf{C}^k$. Let R be a positive closed current of bidimension (s,s) and $u_1, \dots, u_l$ be continuous plurisubharmonic functions in B. Set*

$$R_n^{(l)} := \frac{1}{d_-^{ns}} (f^n)^*(\varphi R) \wedge dd^c u_1 \wedge \dots \wedge dd^c u_l.$$

Then $(R_n^{(l)})_{n=1}^{\infty}$ is bounded, $\|dR_n^{(l)}\| = O(d_-^{-n/2})$ and $\|dd^c R_n^{(l)}\| = O(d_-^{-n})$.

Proof. The proof is very similar to the proof of Proposition 3.3, and we therefore only treat the case $l=0$. Recall that $d_-^{-n}(f^{-n})^*\omega = dd^c G_n^-$ in $\mathbf{C}^k$, where $0 \le G_n^- \le \log^+ \|z\| + O(1)$. Hence G_n^- is locally uniformly bounded in $\mathbf{C}^k$. Therefore

$$\left\langle \frac{1}{d_-^{ns}} (f^n)^*(\varphi R), \omega^s \right\rangle = \langle \varphi R, (dd^c G_n^-)^s\rangle \le \|\varphi R\| \, \|G_n^-\|_{L^\infty(B)}^s,$$

by Chern–Levine–Nirenberg inequalities. This shows that $(R_n^{(0)})_{n=1}^{\infty}$ is bounded.

Now let θ be a $(0,1)$ test form. We have

$$\begin{aligned}\left|\int_{\mathbf{C}^k}(f^n)^*(\partial\varphi\wedge R)\wedge\theta\wedge\omega^{s-1}\right| &= \left|\int_{\mathbf{C}^k}\partial\varphi\wedge R\wedge(f^{-n})^*\theta\wedge(f^{-n})^*\omega^{s-1}\right| \\ &\le \left(\int_{\mathbf{C}^k}R\wedge\partial\varphi\wedge\bar\partial\varphi\wedge(f^{-n})^*\omega^{s-1}\right)^{1/2} \\ &\quad\times\left(\int_{\mathbf{C}^k}R\wedge(f^{-n})^*(\theta\wedge\bar\theta\wedge\omega^{s-1})\right)^{1/2} \\ &\le O(d_-^{n(s-1)/2})O(d_-^{ns/2}).\end{aligned}$$

So $\|dR_n\|=O(d_-^{-n/2})\to 0$. Similarly, one shows that $\|dd^cR_n\|=O(d_-^{-n})$. $\square$

Recall that the volume-entropy of f is defined as

$$H(f)=\liminf_{n\to\infty}\max_{1\le j\le k}\frac{\log\varrho_j(f^n)}{n},$$

where $\varrho_j(f)$ denotes the degree of f^*L, more precisely

$$\varrho_j(f)=\int_{\mathbf{C}^k}f^*(L)\wedge\omega^{k-j}=\int_{\mathbf{C}^k}f^*(\omega^j)\wedge\omega^{k-j},$$

where L is a generic linear subspace of codimension j in $\mathbf{P}^k$. Friedland has shown that $H(f)$ always dominate the topological entropy of f and conjectured that they actually coincide (see [Fr]).

Lemma 4.3. *Let $f\in\operatorname{Aut}(\mathbf{C}^k)$ be such that f^{-1} is weakly regular and $d_-^s\ge d_+^{k-s-1}$. Then $H(f)=\log d_-^s$.*

Proof. The jth dynamical degree of f is defined as

$$\lambda_j(f):=\liminf_{n\to\infty}\varrho_j(f^n)^{1/n}.$$

Clearly $\lambda_j(f)=\lambda_{k-j}(f^{-1})$ for every $1\le j\le k-1$ (the kth dynamical degree is nothing but the topological degree of f which equals 1). Now

$$\lambda_1(f^{-1})=d_-<\lambda_2(f^{-1})=d_-^2<\dots<\lambda_s(f^{-1})=d_-^s,$$

because $(f^{-j})^*(\omega^l)$ has no mass at infinity if $l\le s$. On the other hand

$$\lambda_j(f^{-1})=\lambda_{k-j}(f)\le d_+^{k-j}\le d_+^{k-s-1}\le d_-^s\quad\text{for } s+1\le j\le k.$$

This yields that $H(f)=\log d_-^s$. $\square$

Remark 4.4. We can actually show that the measure μ has maximal entropy

$$h_\mu(f)=h_{\mathrm{top}}(f)=H(f)=\log d_-^s.$$

A proof of this fact will appear elsewhere.

4.2. Partial Green function

We now give an alternative construction of the current σ_s and the invariant measure $\mu=\sigma_s\wedge T_-^s$. It relies on a control of the growth of f on $\operatorname{supp} T_+^r$ which needs to be established (see Examples 5.3), but allows us to get extra information on the invariant measure μ.

Theorem 4.5. *Let $f\in\operatorname{Aut}(\mathbf{C}^k)$ be a q-regular biholomorphism such that $\delta:=d_-^s/d_+^r>1$.*

(1) *Assume that on* $\operatorname{supp} T_+^r$,

$$\|f(p)\|\leq C_1\|p\|^\delta \quad \textit{for } \|p\|\gg 1. \tag{1}$$

Then $\delta^{-nq}(f^n)^*\omega^q\wedge T_+^r$ and $d_-^{-ns}(f^n)^*\omega^{k-s}$ converge to the same limit σ_s which is a positive closed current of bidimension (s,s). Moreover $f^*\sigma_s=d_-^s\sigma_s$ and

$$\sigma_s=(dd^c h)^q\wedge T_+^r,$$

where $h=\lim_{n\to\infty}\delta^{-n}\log^+\|f^n\|$ is defined on $\operatorname{supp} T_+^r$. The current σ_s is of total mass 1 in $\mathbf{C}^k$ and has support in $\overline{K}^+$ if I^- is an attracting set for f.

(2) *Assume moreover that in a neighborhood of $I^+\cap I^-$ on $\operatorname{supp} T_+^r$, we have*

$$\|f(p)\|\geq C_2\|p\|^\delta. \tag{2}$$

Then h is continuous and σ_s has support in the set $\{p\,|\,h(p)=0\}$.

Proof. Set $h_n(p)=\delta^{-n}\log^+\|f^n(p)\|$. We see from (1) that $h_n+\sum_{j=n+1}^\infty C_1/\delta^j$ is a decreasing sequence on $\operatorname{supp} T_+^r$. Let h be the limit, it clearly satisfies $h\circ f=\delta h$. We have

$$S_n=\frac{1}{\delta^{nq}}(f^n)^*\omega^q\wedge T_+^r=(dd^c h_n)^q\wedge T_+^r.$$

Since h_n decrease towards $h\geq 0$, we get by induction on q that the sequence $(S_n)_{n=1}^\infty$ has a unique limit σ_s which satisfies $\sigma_s=(dd^c h)^q\wedge T_+^r$. Set $R_n=d_-^{-ns}(f^n)^*\omega^{k-s}$. Then

$$R_n-S_n=\frac{1}{\delta^{nq}}(f^n)^*\omega^q\wedge\left(\frac{1}{d_+^{nr}}(f^n)^*\omega^r-T_+^r\right)=\left(\frac{1}{d_+^n}(f^n)^*\omega-T_+\right)\wedge\tau_n,$$

where $(\tau_n)_{n=1}^\infty$ is a bounded sequence of positive closed currents of bidimension $(s+1,s+1)$. Since the potentials of $d_+^{-n}(f^n)^*\omega$ uniformly converge towards G^+ on compact subsets of $\mathbf{C}^k$, we infer that $R_n-S_n\to 0$. The functional equation satisfied by σ_s follows from $f^*S_n=d_-^s S_{n+1}$ (or equivalently from the invariance of T_+ and the identity $h\circ f=\delta h$).

When I^- is an attracting set for f, it follows from Lemma 2.3 that σ_s has support in $\overline{K}^+$, and hence it is of total mass in $\mathbf{C}^k$. Note that in this case we recover the situation of Theorem 3.1.

When the second inequality (2) holds, we easily get that

$$|h_{n+1}-h_n| \le \frac{c}{\delta^n} \text{ near } I^+\cap I^- \quad \text{on } \operatorname{supp} T_+^r.$$

So h is continuous in a neighborhood W of $I^+\cap I^-$ on $\operatorname{supp} T_+^r$ and there exists $C>0$ such that $\log^+\|p\|-C\le h(p)\le \log^+\|p\|+C$ in W.

Condition (2) implies that $I^+\cap I^-$ is an attracting set for $f|_{\operatorname{supp} T_+^r}$. Denote by $\mathcal{B}(I^+\cap I^-)=\bigcup_{j=0}^{\infty} f^{-j}(W)$ its basin of attraction. We claim that $\operatorname{supp} T_+^r \setminus \mathcal{B}(I^+\cap I^-) \subset K^+$. Indeed if $(f^n(p))_{n=0}^{\infty}$ is unbounded, then it cannot cluster on X^- which is attracting for f^{-1}. So it clusters on $q\in I^+\setminus X^-$ (recall that $\operatorname{supp} T_+^r$ intersects $\{[z:0]\}$ exactly along I^+ by Theorem 2.2(ii)). Now the blow-up $f(q)$ of f at q is included in I^- (otherwise $f^{-1}(f(q)\setminus I^-)=q\in X^-$), so q is sent by $f|_{\operatorname{supp} T_+^r}$ in $I^+\cap I^-$. In other words, we have shown the inclusion $I^+\setminus X^-\subset \mathcal{B}(I^+\cap I^-)$, so forward unbounded orbits on $\operatorname{supp} T_+^r$ actually converge towards $I^+\cap I^-$. Clearly $h=0$ on $K^+\cap \operatorname{supp} T_+^r$ and $h>0$ in $\mathcal{B}(I^+\cap I^-)$ by the functional equation $h\circ f=\delta h$. Thus h is continuous, since it is upper semicontinuous, non-negative and continuous in $\{p\,|\,h(p)>0\}=\mathcal{B}(I^+\cap I^-)$.

It remains to check that $\sigma_s=(dd^c h)^q\wedge T_+^r$ has support in $\{p\,|\,h(p)=0\}\subset K^+$. This follows from an argument similar to Lemma 2.3, using that $I^+\cap I^-$ is an attracting set for $f|_{\operatorname{supp} T_+^r}$ with $\dim I^+\cap I^-=q-1$. □

Theorem 4.6. *Let $f\in \operatorname{Aut}(\mathbf{C}^k)$ be a q-regular biholomorphism which satisfies* (1) *above with $\delta:=d_-^s/d_+^r>1$. Then $\mu:=\sigma_s\wedge T_-^s$ is an invariant probability measure with compact support in K which does not charge pluripolar sets.*

Proof. Since $\sigma_s=(dd^c h)^q\wedge T_+^r$ and T_-^s have locally bounded potentials, it follows from the Chern–Levine–Nirenberg inequalities (and their generalization to the case of pluripositive currents, see [FG]) that the measure $\mu=\sigma_s\wedge T_-^s$ does not charge pluripolar sets. That μ is invariant and has support in the compact set $K=K^+\cap K^-$ follows from Theorem 4.1. □

Remark 4.7. An argument similar to that of Corollary 3.8 shows that any unstable manifold of dimension $k-s$ intersecting the support of σ_s is dense in the support of T_-^s. The crucial point here is that if Δ is an unstable polydisc of dimension $k-s$, then $\sigma_s\wedge[\Delta]$ is well defined (and non-zero) since $\sigma_s=(dd^c h)^q\wedge T_+^r$ has locally bounded potentials.

5. Examples

5.1. The sets X^+ and I^+

Let $f\in\mathrm{Aut}(\mathbf{C}^k)$ be an algebraically stable biholomorphism. Recall that X_j^+ is defined inductively by

$$X_1^+=\overline{f(\{[z:0]\}\setminus I_f)},\quad X_{j+1}^+=\overline{f(X_j^+\setminus I_f)}.$$

This is a decreasing sequence of *irreducible* analytic subsets. Thus it is stationary and we have denoted by X^+ the corresponding limit set. Recall also that the sequence of indeterminacy sets I_{f^j} is increasing since $I_{f^j}=\bigcup_{l=0}^{j-1} f^{-l}(I_f)$. We have denoted by I^+ the set $I_{f^{j_0}}$, where j_0 is the first integer such that $X^+=X_{j_0}^+$.

When f is 0-regular, it was shown in [S] that $X^+=X_1^+$ and $I^+=I_f=X^-$ is irreducible. This is not so in general.

Example 5.1. Let $f(x,y,z)=(x^d+z^d+y, z^d+x, z)$. Then $f\in\mathrm{Aut}(\mathbf{C}^3)$ with $X_1^+=\{[x:y:0:0]\}$, $X_2^+=X^+=\{[1:0:0:0]\}$, $I_f=\{[0:1:0:0]\}$ and $I_{f^2}=I^+=\{[x:y:z:0]\mid x^d+z^d=0\}$. Note that I^+ is not irreducible.

When X^+ is an attracting set for f, the dynamics of f in U^+, the basin of attraction of X^+, is given by that of $f_0:=f|_{X^+}:X^+\to X^+$. It is therefore natural to wonder what kind of pairs (f_0,X^+) arise. When $X^+\cap I^+=\emptyset$, we can find a projective space $\mathbf{P}^{r-1}$ which is disjoint from I^+ and mapped surjectively by f onto X^+. In this case, if X^+ is smooth, it follows from a result of Lazarsfeld [L] that X^+ is isomorphic to a projective space $\mathbf{P}^{r-1}$ and f_0 is an endomorphism of $X^+\simeq\mathbf{P}^{r-1}$ of degree d_+. However it is easy to construct examples with X^+ non-smooth or, when $X^+\cap I^+\neq\emptyset$, with X^+ smooth but non-isomorphic to $\mathbf{P}^{r-1}$.

5.2. When is I^- an attracting set for f?

5.2.1. *The case of q-regular automorphisms.* When f is a 0-regular automorphism of $\mathbf{C}^k$, then $I^-=X^+$ is an attracting set for f (see Proposition 2.5.3 in [S]). We now consider biholomorphisms of $\mathbf{C}^3$ of the form

$$f\colon (x,y,z)\in\mathbf{C}^3\longmapsto (P(x)+A(y)+az, Q(x)+by, x)\in\mathbf{C}^3,$$

where $ab\neq 0$ and P, A and Q are polynomials of degree d, m and d'. We assume that $d\geq d'>m$ so that $d_+=d$, $I^+=\{[0:y:z:0]\}$ and X^+ is a point which does not belong to I^+ (and hence f is weakly regular). The inverse mapping is given by

$$f^{-1}(x,y,z)=\left(z,\frac{y-Q(z)}{b},\frac{1}{a}\left(x-P(z)-A\left(\frac{y-Q(z)}{b}\right)\right)\right).$$

We assume that $md'>d$ so that $d_-=md'>d_+$, $I^-=\{[x:y:0:0]\}$ and $X^-=\{[0:0:1:0]\}$. Note that f is 1-regular.

Lemma 5.2. *Assume that* $d \ge d' \ge m+1 \ge 3$ *and set*

$$V_\varepsilon := \left\{ (x,y,z) \in \mathbf{C}^3 \,\middle|\, \max\{|x|,|y|\} > \frac{1}{\varepsilon} \max\{1,|z|\} \right\}.$$

Then there exists $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$, *then* $f(V_\varepsilon) \subset V_{\varepsilon/2}$. *Therefore* I^- *is an attracting set for* f.

Proof. Pick $(x,y,z) \in V_\varepsilon$ and set $(x',y',z') = f(x,y,z)$.
If $|x| = \max\{|x|,|y|\}$, then

$$|y'| = |Q(x)+by| \ge C_1|x|^{d'} - b|y| \ge \tfrac{1}{2}C_1|x|^{d'} \quad \text{for } 0<\varepsilon<\varepsilon_1.$$

Since $d' \ge 2$, we get $|y'| \ge 2/\varepsilon$ for ε_1 small enough. Moreover $|z'| = |x| < \frac{1}{2}\varepsilon|y'|$, so $(x',y',z') \in V_{\varepsilon/2}$.

We assume now that $|y| = \max\{|x|,|y|\} > 1/\varepsilon$. Suppose first that $|x|^{d'} \ge |y|^{1+t}$ where $0<t<1$ will be chosen later. In this case

$$|y'| \ge C_1|x|^{d'} - b|x|^{d'/(1+t)} \ge \frac{C_1}{2}|x|^{d'} \ge \frac{C_1}{2}|y|^{1+t} > \frac{2}{\varepsilon} \quad \text{for } 0<\varepsilon<\varepsilon_2 \ll 1.$$

Moreover

$$\frac{|z'|}{|y'|} \le \frac{2}{C_1|x|^{d'-1}} \le \frac{C_1'}{|y|^{(1+t)(1-1/d')}}.$$

We choose $t>0$ so that $d' > 1+t > d'/(d'-1)$. This is possible since we assumed that $d' \ge 3$. The first inequality will be used below, the second one ensures that $(1+t)(1-1/d') > 1$. Therefore $|z'| < \frac{1}{2}\varepsilon|y'|$. Hence $(x',y',z') \in V_{\varepsilon/2}$.

Finally suppose $|x|^{d'} \le |y|^{1+t}$. We have no clear control on $|y'|$, however we can control $|x'|$. Indeed observe that $|P(x)| \le C_2 \max\{|x|,1\}^d \le C_2|y|^{(1+t)d/d'}$. Thus

$$|x'| = |P(x)+A(y)+az| \ge C_3|y|^m - C_2|y|^{(1+t)d/d'} - \varepsilon|a|\,|y| \ge \frac{C_3}{2}|y|^m > \frac{2}{\varepsilon}$$

for $0<\varepsilon<\varepsilon_4 \ll 1$, since $d(1+t) < md'$. Moreover

$$\frac{|z'|}{|x'|} \le \frac{2|x|}{C_3|y|^m} \le \frac{2}{C_3}\frac{1}{|y|^{m-(1+t)/d'}} < \frac{2}{\varepsilon}.$$

The latter inequality follows from our choice of t: we have indeed $m-(1+t)/d' > 2-(1+t)/d' > 1$. This shows that $(x',y',z') \in V_{\varepsilon/2}$. □

Remark 5.3. (1) More generally, the set I^- is f-attracting for mappings of the form $f = (x^d + y^m + B(x,y) + az, Q(x)+by, x)$, with appropriate conditions on the degrees of the mixed terms in B.

(2) If the leading term in y depends on x or if $m=1$, then some hypothesis on b has to be made to ensure that I^- is attracting. Consider for instance $f=(x^d+x^py^m+az, x^d+by, x)$, where $d>m+p$ and $p\geq 1$. Then f is still 1-regular and I^- is f-attracting if and only if $|b|>1$. The proof of this fact is left to the reader since it is very close to that of Lemma 5.4 below. Observe that $d_-=md+p$.

5.2.2. *Other examples.* Consider $f(x,y,z)=(xP(x,y)+az, x^{d+1}+by, x)$, where P is a homogeneous polynomial of degree $d\geq 1$ and $ab\neq 0$. We assume that $P(0,1)\neq 0$. Then $f\in \mathrm{Aut}(\mathbf{C}^3)$ is an algebraically stable biholomorphism such that $d_-=d^2+d+1>d+1=d_+$. Observe that f is not weakly regular but f^{-1} is, since $I^-=\{[x:y:0:0]\}$ and $X^-=\{[0:0:1:0]\}$. The following lemma completes the assertions of Example 1.11.

Lemma 5.4. *Fix λ such that $0<\lambda<1/(1+d)$ and set*

$$V_\varepsilon=\left\{(x,y,z)\in \mathbf{C}^3 \,\middle|\, \max\{|x|,|y|\}>\max\left\{\frac{1}{\varepsilon},\frac{1}{\varepsilon^\lambda}|z|\right\}\right\}.$$

Assume that $|b|=1+2t>1$. Then there exists $\varepsilon_0>0$ such that if $0<\varepsilon<\varepsilon_0$, then $f(V_\varepsilon)\subset V_{\varepsilon/(1+t)}$. In particular I^- is an attracting set for f.

Proof. Pick $(x,y,z)\in V_\varepsilon$ and set $(x',y',z')=f(x,y,z)$.
If $|x|=\max\{|x|,|y|\}>1/\varepsilon$, then $|y'|=|x^{d+1}+by|\leq \frac{1}{2}|x|^{d+1}\leq(1+t)/\varepsilon$ and

$$\frac{|z'|}{|y'|}\leq\frac{2}{|x|^d}<\left(\frac{\varepsilon}{1+t}\right)^\lambda \quad \text{for } 0<\varepsilon<\varepsilon_1\ll 1.$$

Thus $(x',y',z')\in V_{\varepsilon/(1+t)}$.

Assume now that $|y|=\max\{|x|,|y|\}>1/\varepsilon$. If $|x|^{d+1}<t|y|$, then $|y'|\geq(1+t)|y|>(1+t)/\varepsilon$ and

$$\frac{|z'|}{|y'|}\leq\frac{|x|}{(1+t)|y|}\leq\frac{C}{|y|^{1-1/(d+1)}}<\left(\frac{\varepsilon}{1+t}\right)^\lambda \quad \text{for } \varepsilon \text{ small enough.}$$

Similarly if $|x|^{d+1}>2|b|\,|y|$, we obtain that $(x',y',z')\in V_{\varepsilon/(1+t)}$ by considering $|y'|$. On the other hand if $t|y|<|x|^{d+1}<2|b|\,|y|$, then $|P(x,y)|\geq C|y|^d$ for some constant $C>0$. Hence

$$|x'|\geq C|x|\,|P(x,y)|-|a|\,|z|\geq C'|y|^{d+1/(1+d)}-|a|\varepsilon^\lambda|y|\geq C''|y|^{d+1/(1+d)}.$$

Therefore $|x'|\geq(1+t)/\varepsilon$ and $|z'|\leq(\varepsilon/(1+t))^\lambda|x'|$. In all cases, we get $(x',y',z')\in V_{\varepsilon/(1+t)}$. $\square$

5.3. Growth of f on $\operatorname{supp} T_+^r$

We continue our analysis of the mapping $f(x,y,z)=(P(x)+A(y)+az, Q(x)+by, x)$ and show that they satisfy the growth conditions of Section 4.2.

Proposition 5.5. *Let f be as in Lemma 5.2. Set $\delta=d_-/d_+=md'/d>1$. Then there exists $C>0$ such that*

$$\frac{1}{C}\|p\|^\delta \le \|f(p)\| \le C\|p\|^\delta \quad \textit{for all } p\in V_{\varepsilon_0}\cap\operatorname{supp} T_+,$$

where $\varepsilon_0>0$ is chosen small enough.

Proof. It follows from Lemma 5.2 that $f(V_\varepsilon)\subset V_{\varepsilon/2}$. Since $\operatorname{supp} T_+$ is completely invariant, this yields that $f(V_\varepsilon\cap\operatorname{supp} T_+)\subset V_{\varepsilon/2}\cap\operatorname{supp} T_+$. Note that $V_\varepsilon\cap\operatorname{supp} T_+$ is a neighborhood (in $\operatorname{supp} T_+$) of the point $I^+\cap I^-=[0:1:0:0]$. Thus

$$V_\varepsilon\cap\operatorname{supp} T_+ = \left\{(x,y,z)\in\operatorname{supp} T_+ \;\middle|\; |y|>\frac{1}{\varepsilon}\max\{1,|z|\} \text{ and } |x|<c(\varepsilon)|y|\right\},$$

where $c(\varepsilon)\to0$ as $\varepsilon\to0$.

Fix $(x,y,z)\in V_\varepsilon\cap\operatorname{supp} T_+$ and set $(x',y',z')=f(x,y,z)$. To simplify the notation, we assume that P, A and Q are unitary polynomials. We claim that $|x|^d\ge\frac12|y|^m$ if ε is small enough. Otherwise $|x'|=|x^d+y^m+...+az|\ge\frac14|y|^m$ and $|y'|=|Q(x)+by|\le C_0|y|^{md'/d}$. Hence

$$\frac{|x'|}{|y'|}\ge\frac{1}{4C_0}|y|^{m(1-d'/d)}\ge\frac{1}{4C_0},$$

contradicting that $|x'|<c\left(\frac12\varepsilon\right)|y'|$.

Similarly one gets that $|x|^d\le2|y|^m$ in $V_\varepsilon\cap\operatorname{supp} T_+$. This shows that $|y|^\delta/C_1\le |x|^{d'}\le C_1|y|^\delta$ for some constant $C_1>0$. Since $\delta=md'/d>1$, this yields that

$$\frac{1}{C}|y|^\delta\le|y'|=|Q(x)+by|\le C|y|^\delta. \quad \square$$

5.4. Various examples

5.4.1. We here give examples of algebraically stable biholomorphisms $f\in \operatorname{Aut}(\mathbf{C}^3)$ such that $G^+>0$ on an open set which is attracted by a point of indeterminacy $m\in I^+\cap X^+$.

Proposition 5.6. *Consider $f(x,y,z)=(yx^d+az, y^{d+1}+bx, y)$, where $d\geq 3$ and $ab\neq 0$. Set*

$$W_{t,R,R'} := \{(x,y,z)\in \mathbf{C}^3 \mid R<|x|<R',\ R<|y|<t|x| \text{ and } |z|<t|y|\},$$

where $R'>R>1$ and $0<t<1$. Fix $\varepsilon>0$ such that $(1+\varepsilon)/(1-\varepsilon)<t^{-1}$.

Then there exists $R_0>1$ such that

$$R>R_0 \implies f(W_{t,R,R'})\subset W_{t^{d-1},(1-\varepsilon)R^{d+1},(1+\varepsilon)(R')^{d+1}}.$$

In particular $f^j(W_{t,R,R'})\to[1:0:0:0]=X^+\cap I^+$ and $G^+(p)>0$ for $p\in W_{t,R,R'}$.

Proof. Pick $(x,y,z)\in W_{t,R,R'}$ and set $(x',y',z')=f(x,y,z)$. Then

$$|x'|\leq |y|\,|x|^d+|a|\,|z|\leq (1+\varepsilon)|y|\,|x|^d, \quad \text{if } R \text{ is large enough,}$$

Similarly $|x'|\geq(1-\varepsilon)|y|\,|x|^d$ and $(1-\varepsilon)|y|^{d+1}\leq|y'|\leq(1+\varepsilon)|y|^{d+1}$. Therefore

$$\frac{|z'|}{|y'|}\leq\frac{1}{1+\varepsilon}\frac{1}{|y|^d}<t^{d-1}, \quad \text{if } R \text{ is large enough,}$$

and

$$\frac{|y'|}{|x'|}\leq\frac{1+\varepsilon}{1-\varepsilon}\frac{|y|^d}{|x|^d}<t^{d-1},$$

since $(1+\varepsilon)/(1-\varepsilon)<t^{-1}$. As a consequence $f^j(W_{t,R,R'})\to[1:0:0:0]=X^+\cap I^+$ if $R\geq R_0\gg 1$. Moreover for any $p\in W_{t,R,R'}$, we can find $M>1$ such that $\|f^j(p)\|\geq M^{(d+1)^j}$. Thus $G^+(p)>0$. $\square$

5.4.2. Let $f\in\mathrm{Aut}(\mathbf{C}^3)$ be such that $X^-\cap I^-=\emptyset$ with $d_->d_+$. Assume that I^- is an attracting set for f. Note that X^- is a point ($s=1$) since otherwise $\dim I_{f^{-2}}=\dim I^-=0$ so f would be regular with $d_+=d_-^2$, contradicting our assumption. We have constructed, in Theorem 4.1, an invariant ergodic measure $\mu=\sigma_1\wedge T_-$. It is expected that periodic saddle points of type $(1,2)$ are equidistributed with respect to the measure μ. A first glimpse of the importance of these points was given in Corollary 3.8 (resp. Remark 4.7), where we showed that the stable (resp. unstable) manifolds of such points are dense in the support of σ_1 (resp. T_-). The following example shows that one cannot expect similar properties for the periodic points of type $(2,1)$. Indeed we obtain an unstable manifold of dimension 1 which is closed.

Example 5.7. Consider $f(x,y,z)=(xy^d+az, x^{d+1}+by, x)$, where $d\geq 1$ and $ab\neq 0$. Then $f\in\mathrm{Aut}(\mathbf{C}^3)$ with

$$f^{-1}(x,y,z)=\left(z, \frac{y-z^{d+1}}{b}, \frac{x-b^{-d}z(y-z^{d+1})^d}{a}\right).$$

We easily obtain that $X^+=\{[x:y:0:0]\}$, $X^-=\{[0:0:1:0]\}$, $I^+=\{[0:y:z:0]\}$ and $I^-=\{[x:y:0:0]\}$. Note that $X^-\cap I^-=\emptyset$ and $I^+_\infty=I_{f^2}=I^+\cup\{[x:0:z:0]\}$.

We can check that I^- is an attracting set for f if $|b|>1$. Since $d_-=d^2+d+1>d+1=d_+$, we are in the situation described above. Observe however that 0 is a fixed point with eigenvalues b, $\sqrt{a}$ and $-\sqrt{a}$. So 0 is a saddle fixed point of type $(2,1)$ if $|a|<1$. Since $f(0,y,0)=(0,by,0)$, we get that the unstable manifold of 0 is exactly the line $\{(0,y,0)\}$.

5.4.3. It is interesting to point out that our main results apply to biholomorphisms $f\in\mathrm{Aut}(\mathbf{C}^k)$ which are not necessarily algebraically stable. Consider e.g.

$$f(x,y,z)=(z, y-z^d, x+y^2-2yz^d) \quad \text{with } d\geq 3.$$

We have $I_f=\{(x,0,z)\}\cup\{(x,y,0)\}$ and $f(\{[z:0]\}\setminus I_f)=[0:0:1:0]\in I_f$, so f is not algebraically stable. More precisely f^j is never algebraically stable $(j\geq 1)$.

Lemma 5.8. *The first dynamical degree is given by* $\lambda_1(f)=\frac{1}{2}\big(d+\sqrt{d^2+4d}\big)$.

Proof. One easily gets by induction on j that the dominating term in f^j arises on the third coordinate as $c_jy^{\alpha_j}z^{\beta_j}$, where α_j and β_j satisfy $\alpha_{j+1}=\beta_j$ and $\beta_j=d(\alpha_j+\beta_j)$. We infer that $\deg(f^j)=c\big(\frac{1}{2}\big(d+\sqrt{d^2+4d}\big)\big)^j+c'\big(\frac{1}{2}\big(d-\sqrt{d^2+4d}\big)\big)^j$, where c and c' are constants with $c>0$. This yields $\lambda_1(f)=\lim_{j\to+\infty}\deg(f^j)^{1/j}=\frac{1}{2}\big(d+\sqrt{d^2+4d}\big)$. $\square$

On the other hand $f^{-1}(x,y,z)=(x^{2d}-y^2+z, x^d+y, x)$ is weakly regular with $I^-=\{[0:y:z:0]\}$, $X^-=\{[1:0:0:0]\}$ $(s=1)$ and $d_-=2d>d_+=d+1$. One can check in this case that I^- is an f-attracting set, this ensures the existence of the invariant current σ_1.

Remark 5.9. It is interesting to note that for every $j\geq 1$, f^j is not even conjugated to an algebraically stable biholomorphism. This is clear since $\lambda_1(f^j)=\big(\frac{1}{2}\big[d+\sqrt{d^2+4d}\big]\big)^j\notin\mathbf{N}$. There are polynomial automorphisms g of $\mathbf{C}^3$ with interesting dynamics such that g is not algebraically stable, but g^2 is 0-regular: consider e.g. $g(x,y,z)=(x^d+y^m+z, x^{d'}+y, x)$ with $d'>\max\{d,m\}$. Then g^2 is regular, so $\lambda_1(g)=\sqrt{\lambda_1(g^2)}=\sqrt{md'}$.

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