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A theorem on duality mappings in Banach spaces

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1. Introduction

The problem considered in this paper originates in the theory of Fourier series of functions belonging to a Lebesgue space L^p, where L^p denotes the space of measurable functions with period 2π and with norm

$$\|f\|_p = \left\{\int_0^{2\pi} |f(x)|^p\, dx\right\}^{1/p}.$$

For the Fourier coefficients of f, we shall use the notation

$$c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)\, dx.$$

A classical theorem asserts that if $\{a_n\}_{-\infty}^{\infty}$ is a given sequence of numbers such that

$$\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty,$$

then there is a unique element $f \in L^2$ with the property that

$$c_n(f) = a_n$$

for $n = 0, \pm 1, \pm 2, \ldots$.

The above theorem admits the following extension. Let S_α, $0 < \alpha < \infty$, denote the operator taking the complex number z into $|z|^{\alpha-1} z$. We have

$$S_\alpha S_\beta = S_\beta S_\alpha = S_{\alpha\beta},$$

so that the collection $\{S_\alpha\}$ forms a group. Furthermore, S_α performs a one-to-one mapping of the finite complex plane onto itself. Applied to a space L^p, we see that $S_\alpha f$ will take L^p onto $L^{p/\alpha}$, $0 < \alpha \leq p$. In particular, $S_{p-1} f$ will map L^p onto its dual space L^q, $q = p/(p-1)$. We then have

Theorem 1. *Let the integers be partitioned into two disjoint sets A and A', neither of which is empty. Let p be a given exponent such that $1 < p < \infty$. Let $\{a_n; n \in A\}$ and $\{b_n; n \in A'\}$ be given sets of numbers such that, for some $h \in L^p$ and for some $k \in L^q$, $q = p/(p-1)$, we have*

$$c_n(h) = a_n,\ n \in A,$$

$$c_n(k) = b_n,\ n \in A'.$$

Then there is a unique element $f \in L^p$ *such that*

$$c_n(f) = a_n,\ n \in A, \qquad c_n(S_{p-1} f) = b_n,\ n \in A'. \tag{1}$$

For $p = 2$, the above statement reduces to the cited classical result, since, in this case, $S_{p-1} f = f$.

The proof of the uniqueness is quite easy. If there were two solutions f_1 and f_2 of (1), then the difference $F = f_1 - f_2$ would be an element of L^p and would have Fourier coefficients vanishing on A. Similarly, $G = S_{p-1} f_1 - S_{p-1} f_2$ would be in L^q and have Fourier coefficients zero on A'. Consequently, the Fourier coefficients of the continuous function

$$\frac{1}{2\pi}\int_0^{2\pi} F(x+t)\,\bar{G}(t)\,dt,$$

which are given by $c_n(F)\,\bar{c}_n(G)$, are zero for each n. Therefore,

$$0 = \int_0^{2\pi} F(t)\,\bar{G}(t)\,dt = \int_0^{2\pi} (f_1 - f_2)\left(\frac{|f_1|^p}{f_1} - \frac{|f_2|^p}{f_2}\right) dt. \tag{2}$$

Setting $f_1 = |f_1| e^{i\theta_1}$, $f_2 = |f_2| e^{i\theta_2}$, we find that the real part of the integrand in (2) is

$$|f_1|^p + |f_2|^p - (|f_1|\,|f_2|^{p-1} + |f_1|^{p-1}|f_2|)\cos(\theta_1 - \theta_2) \geq (|f_1| - |f_2|)(|f_1|^{p-1} - |f_2|^{p-1}) \geq 0.$$

The vanishing of (2) therefore requires that almost everywhere $|f_1| = |f_2|$ and $\cos(\theta_1 - \theta_2) = 1$, and the uniqueness follows.

The above proof of the uniqueness is satisfactory, but it gives no indication of the nature of the problem. In particular, the role played by the operator S_{p-1} is not at all clear. We shall see in the next section that the operator S_{p-1} is only one of a large class of operators on L^p to $L^{p/(p-1)}$ which arise quite naturally from a consideration of what we call duality maps of a Banach space onto its first conjugate space.

2. Duality mappings

We will be dealing with a complex (or real) Banach space B and its conjugate space B^*. The null element of B will be denoted by θ, of B^* by θ^*. The norm of an element $x \in B$ is $\|x\| = \|x\|_B$, of $u \in B^*$ is $\|u\| = \|u\|_{B^*}$. The unit-spheres in B and B^* will be denoted by S and S^*, respectively.

Let (x, u) be the bilinear functional defined for $x \in B$ and $u \in B^*$ and having the properties:

(a) If λ is a complex (or real) number, then

$$(\lambda x, u) = (x, \lambda u) = \lambda (x, u);$$

(b)
$$(x_1 + x_2, u) = (x_1, u) + (x_2, u),$$
$$(x, u_1 + u_2) = (x, u_1) + (x, u_2);$$

(c)
$$\|u\|_{B^*} = \sup_{x \in S} |(x, u)|.$$

Two elements $x \in S$ and $x^* \in S^*$ are said to be conjugate if $(x, x^*) = 1$. The sets $\{\lambda x;\ 0 \le \lambda < \infty\}$ and $\{\mu x^*;\ 0 \le \mu < \infty\}$ will then be called conjugate rays.

Under the assumption that each element on S (or S^*) has a unique conjugate on S^* (or S), we will consider duality maps of B onto B^* characterized by the following properties: T is one-to-one and takes each ray in B onto the conjugate ray in B^* and each sphere $\|x\| = r$ in B onto a sphere $\|u\| = \varrho$ in B^* in such a way that $r_1 < r_2$ implies $\varrho_1 < \varrho_2$. From this definition, it follows that

$$T(\lambda x) = \phi(\lambda) x^*, \quad \lambda \ge 0,$$

x and x^* being conjugate elements on the respective unit spheres and $\phi(\lambda)$ a continuous function that is strictly increasing from 0 to ∞ with λ. (A special type of duality map has previously been considered by E. R. Lorch [4].)

With regard to such mappings, we have

Lemma 1. *Let there exist a duality map of B onto B^*. If $x, y \in B$, then the relation*

$$(x - y, Tx - Ty) = 0 \tag{3}$$

implies that $x = y$.

To prove this, we observe that $z \in B$ and $u \in B^*$ will belong to conjugate rays if and only if $(z, u) = \|z\| \|u\|$. Consequently, the assumption $x \ne \lambda y$, $\lambda > 0$, will imply that

$$Re(x, Ty) < \|x\| \|Ty\|,$$
$$Re(y, Tx) < \|y\| \|Tx\|,$$

so that upon taking the real part of (3), we will have

$$0 > (\|x\| - \|y\|)(\|Tx\| - \|Ty\|) \ge 0.$$

Therefore $x = \lambda y$, and $\lambda = 1$.

We want next to find conditions on the space B in order that a duality map shall exist. For this purpose, we need to recall certain definitions.

The space B is said to be uniformly convex [1] if there corresponds to each ε, $0 < \varepsilon < 1$, a positive number $\delta(\varepsilon)$ tending to zero with ε and such that

$$\|x\| \le 1, \ \|y\| \le 1, \ \left\|\frac{x+y}{2}\right\| \ge 1 - \varepsilon$$

imply that

$$\left\|\frac{x-y}{2}\right\| \leq \delta(\varepsilon).$$

In uniformly convex Banach spaces, each closed convex subset possesses a unique element of minimal norm.

We shall say that B is differentiable if for any $x \in B$, $x \neq 0$, and any $y \in B$ there is a finite complex (or real) number $D = D(y, x)$ such that

$$\|x+ty\| - \|x\| = Re\{tD\} + o(|t|) \tag{4}$$

as the complex (or real) number t tends to zero. It is known [3] that for differentiable spaces $D(y, x)$ is a linear functional in y of norm unity. We observe that (4) implies

$$|D(y, x)| \leq \|y\|; \; D(x, x) = \|x\|;$$

$$D(y, \lambda x) = D(y, x), \; 0 < \lambda < \infty.$$

An important consequence of uniform convexity and differentiability in a Banach space is the, in principle, known

Lemma 2. *If B is a uniformly convex and differentiable Banach space, then each element on S (or S^*) has a unique conjugate element on S^* (or S).*

Assume first that $x \in S$ is given. We are to show the existence and the uniqueness of a linear functional $L(y) = (y, x^*)$ of norm unity and for which $L(x) = 1$. The existence of $L(y)$ is clear, for we may take $L(y) = D(y, x)$. As for the uniqueness, we observe that for any $y \in B$ and for any complex (or real) scalar t, we have

$$0 \leq \|x+ty\| - Re\{L(x+ty)\} = \|x+ty\| - \|x\| - Re\{tL(y)\}.$$

Combining this relation with (4) yields

$$0 \leq Re\{t[D(y, x) - L(y)]\} + o(|t|)$$

from which it follows that $L(y) = D(y, x)$.

On the other hand, if $x^* \in S^*$ is given, its conjugate $x \in S$ can be characterized as the element of minimal norm in the closed and convex set $\{y; (y, x^*) = 1\} \subset B$. Therefore, x exists and is unique.

(We point out that $x^* \in S^*$ can be shown to have a unique conjugate $x \in S$ if we assume only that B is strictly convex and that the set $\{y; y \in B, \|y\| \leq 1\}$ is weakly compact. These two conditions are known to be weaker than the requirement that B be uniformly convex.)

In the sequel, we will use the following definitions and notations: If A is a closed linear subset of B, then its orthogonal complement in B^* is the set $A^\perp$ of $u \in B^*$ for which $(x, u) = 0$ for every $x \in A$. If y is an element and C a set of elements, then $C + y$ will denote the set $\{x; x = y + z \text{ for } z \in C\}$.

We are now ready to prove

Theorem 2. *Let B be a uniformly convex and differentiable Banach space and T a duality map of B onto its conjugate B^*. Let C be a closed, linear, and proper subset of B and $C^\perp$ its orthogonal complement in B^*. If $H \in B$ and $K \in B^*$ are given elements, then the sets $C^\perp + K$ and $T(C+H)$ have one and only one element in common.*

(Note that, by virtue of Lemma 2, duality maps of B onto B^* do indeed exist.)

Recall that the duality map T defines a continuous function $\phi(\lambda)$, $0 \le \lambda < \infty$, which is strictly increasing from 0 to ∞ with λ. Set

$$\Phi(\lambda) = \int_0^\lambda \phi(u)\,du,$$

and consider the functional

$$F(x) = \Phi(\|x\|) - Re\{(x, K)\}$$

for $x \in C+H$. For $\|x\| = r$, we have

$$F(x) \ge \Phi(r) - r\|K\|_{B^*}.$$

The right-hand member of this inequality is a strictly convex function of r, tends to infinity as $r \to \infty$, and is bounded from below for $r \ge 0$. Hence,

$$M = \inf\{F(x);\ x \in C+H\}$$

is a finite number. Furthermore, $F(x) \le m+1$ implies that $\|x\| \le r_0 < \infty$.

Let $\{x_n\} \subset C+H$ be a minimizing sequence of $F(x)$. Without loss of generality, we may assume that $\|x_n\|$ tends to some finite limit α. (Actually, we need not make this assumption, for the strict convexity of $\Phi(\lambda)$, $0 \le \lambda < \infty$, guarantees that $\|x_n\|$ tends to a finite limit.) Since $C+H$ is a convex set, we have

$$F\left(\frac{x_m + x_n}{2}\right) \ge M.$$

Since $\Phi(r)$, $r \ge 0$, is convex, we obtain

$$0 \le \tfrac{1}{2}\{\Phi(\|x_m\|) + \Phi(\|x_n\|)\} - \Phi\left(\left\|\frac{x_m + x_n}{2}\right\|\right)$$

$$= \tfrac{1}{2}\{F(x_m) + F(x_n)\} - F\left(\frac{x_m + x_n}{2}\right) = \varepsilon_{mn},$$

where $\varepsilon_{mn} \to 0$ as $m, n \to \infty$. Therefore

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \Phi\left(\left\|\frac{x_m + x_n}{2}\right\|\right) = \Phi(\alpha),$$

from which we deduce that

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \left\| \frac{x_m + x_n}{2} \right\| = \alpha$$

But in a uniformly convex space, this implies that $\{x_n\}$ is a Cauchy sequence. Since $C+H$ is a closed subset of B, there is an $x_0 \in C+H$ such that $\|x_n - x_0\| \to 0$ and $F(x_0) = M$. Consequently,

$$F(x_0 + tx) - F(x_0) \geq 0$$

for $x \in C$ and t a complex (or real) scalar.

Since Φ is a differentiable function,

$$\Phi(\|x_0 + tx\|) - \Phi(\|x_0\|) = \Phi'(\|x_0\|)\, Re\{t D(x, x_0)\} + o(|t|)$$

$$= \phi(\|x_0\|)\, Re\{t D(x, x_0)\} + o(|t|).$$

Thus

$$Re\{t[\phi(\|x_0\|) D(x, x_0) - (x, K)]\} + o(|t|) \geq 0,$$

from which we conclude that

$$\phi(\|x_0\|) D(x, x_0) - (x, K) = 0 \tag{5}$$

for $x \in C$. Since $\phi(\|x_0\|) D(x, x_0)$ is a linear functional in x of norm $\phi(\|x_0\|)$, it may be written in the form (x, u_0), where u_0 is an element on the sphere $\|u\|_{B^*} = \phi(\|x_0\|)$. Setting $x = x_0$ gives

$$(x_0, u_0) = \phi(\|x_0\|) D(x_0, x_0) = \|x_0\|\, \phi(\|x_0\|),$$

from which it follows that $u_0 = Tx_0$. Consequently, (5) may be written as

$$(x, Tx_0 - K) = 0 \tag{6}$$

for $x \in C$, which implies that $Tx_0 \in C_\perp + K$.

We have therefore shown that $T(C+H)$ and $C^\perp + K$ have at least one element in common.

If $x \in C+H$ and $Tx \in C_\perp + K$, then (6) implies that

$$(x_0 - x, Tx_0 - Tx) = 0.$$

An application of Lemma 1 shows that $x = x_0$ and, hence, that $T(C+H)$ and $C^\perp + K$ have at most one element in common.

The proof of the theorem is now complete.

We wish to point out that the conclusion of Theorem 2 remains valid if we again assume only that B is strictly convex and that $\{y;\, y \in B,\, \|y\| \leq 1\}$ is weakly compact.

In order to deduce Theorem 1 from Theorem 2, we set $B = L^p$, $B^* = L^q$, and let C be the set of functions in L^p with Fourier coefficients vanishing for $n \in A$. If

$$(x, u) = \int_0^{2\pi} x(t)\, u(t)\, dt$$

for $x \in L^p$, $u \in L^q$, the orthogonal complement $C^\perp$ of C consists of those $u \in L^q$ for which $\bar{u}$ has Fourier coefficients zero on A'.

It is known that L^p, $1 < p < \infty$, is uniformly convex [1] and differentiable [2], and it is clear that $Tx = S_{p-1}\bar{x} = |x|^p/x$ is a duality map of L^p onto L^q. The desired result now follows from Theorem 2 if we set $H = h$ and $K = \bar{k}$.

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