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Orthogonality in normed linear spaces

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1. Introduction

Let B be a real normed linear space. We will say that B is Euclidean if there is a symmetric bilinear functional (u, v) (called the inner product of u and v) defined for $u, v \in B$, such that $(u, u) = \|u\|^2$ for every $u \in B$. In a Euclidean space we have the customary definition of orthogonality, viz. an element u is orthogonal to an element v (in notation $u \perp v$) if and only if $(u, v) = 0$. It is an immediate consequence of this definition that orthogonality has the following properties:

(a) If $u \perp v$ then $v \perp u$ (Symmetry);
(b) If $u \perp v$, then $\lambda u \perp v$ for all real λ (Homogeneity);
(c) If $u \perp w$ and $v \perp w$, then $(u+v) \perp w$ (Additivity);
(d) For every pair $u, v \in B$ there is a number a such that $u \perp (a u + v)$.

If a_ν, b_ν, c_ν, $\nu = 1, 2, \ldots, m$, are real numbers satisfying

$$\sum_{\nu=1}^{m} a_\nu b_\nu^2 = \sum_{\nu=1}^{m} a_\nu c_\nu^2 = 0, \quad \sum_{\nu=1}^{m} a_\nu b_\nu c_\nu = 1, \tag{1.1}$$

we have the following identity

$$2(u, v) = \sum_{\nu=1}^{m} a_\nu \|b_\nu u + c_\nu v\|^2 \text{ for } u, v \in B.$$

This means that the definition of orthogonality in a Euclidean space may be reformulated in the following way:

$$u \perp v \text{ if and only if } \sum_{\nu=1}^{m} a_\nu \|b_\nu u + c_\nu v\|^2 = 0.$$

Since this new definition makes no use of the inner product but only of the norm and linear structure of B, it is applicable even in the case of an arbitrary normed linear space B. For this reason we make the following definition:

Definition 1.1. Let B be a normed linear space and a_ν, b_ν, c_ν, $\nu = 1, 2, \ldots, m$, a fixed collection of real numbers satisfying relations (1.1). An element u of B is said to be orthogonal to an element v of B (in notation $u \perp v$) if

$$\sum_{\nu=1}^{m} a_\nu \|b_\nu u + c_\nu v\|^2 = 0$$

Now the question arises, whether the orthogonality so defined has any of the properties (a), (b), (c) or (d). It may have property (a), as illustrated by two special cases considered by R. C. James [7]. He defined two types of orthogonality, viz.

$$\text{"Pythagorean orthogonality": } u \perp v \text{ if } \|u-v\|^2 = \|u\|^2 + \|v\|^2$$

and

$$\text{"Isosceles orthogonality": } u \perp v \text{ if } \|u+v\| = \|u-v\|.$$

James showed that these two types of orthogonality always have property (d), while if one of them has property (b) or (c), then B must be a Euclidean space. The main result of this paper is that these propositions remain true in the case of the more general type of orthogonality defined above.

Our plan of investigation is as follows. In Section 2 we collect some definitions and lemmas which will be needed later.

In Section 3 we first show that orthogonality always has property (d). Then we study a normed linear space in which orthogonality satisfies a certain condition, apparently weaker than homogeneity and additivity. We show that this condition implies that orthogonality has properties (a), (b) and (c). Furthermore, it implies several properties of B, which, if the dimension of B is at least three, permit us to conclude that B is Euclidean.

There remains now an investigation of the two-dimensional case. This is prepared in Section 4, where we remark that the problem it essentially equivalent to proving the uniqueness under certain conditions of a solution of a functional equation

$$\sum_{\nu=1}^{r} p_\nu F(q_\nu x) = C_1 + C_2 x^2.$$

In Section 5 we remind the reader of the definition and fundamental properties of the F-series of a function $f(x)$ associated with a function

$$h(t) = \sum_{\mu=1}^{N} d_\mu e^{\alpha_\mu t}.$$

By means of the results of Section 5, we show in Section 6 that if orthogonality is homogeneous in a normed linear space B, then B is Euclidean.

In Section 7 we use the same method as in Section 6 to prove a certain generalization of the well-known Jordan–von Neumann characterization of Euclidean spaces.

2. Preliminaries

We state without proof the following lemma by James ([7], Lemma 4.4):

Lemma 2.1. *If u and v are elements of a normed linear space then*

$$\lim_{x \to +\infty} [\|(x+a)u+v\| - \|xu+v\|] = a\|u\|.$$

With the aid of Lemma 2.1 the following result is immediately verified:

Lemma 2.2. *If u and v are elements of a normed linear space, then*

$$\lim_{x\to\pm\infty} x^{-1}\,[\|(x+a)\,u+v\|^2-\|x\,u+v\|^2]=2\,a\,\|u\|^2.$$

There is another type of orthogonality which was studied by Birkhoff [1], Fortet [5], [6] and James [8], [9]. Following Fortet we will here call it normality.

Definition 2.3. An element u of a normed linear space is normal to an element v (in notation $u\,N\,v$) if

$$\|u+\lambda v\|\geqslant\|u\| \text{ for every } \lambda.$$

Birkhoff [1] gave the following characterization of Euclidean spaces:

Lemma 2.4. *Let B be a normed linear space, whose dimension is greater than two. If normality is symmetric and unique in B, i.e. if*

(a) *$u\,N\,v$ implies $v\,N\,u$, $u\in B$, $v\in B$, and*
(b) *to every pair of elements u, $v\in B$ there is a unique number a such that $u\,N\,(a\,u+v)$,*

then B is Euclidean.

The assumption concerning the dimension of B is essential in Lemma 2.4. On the other hand, M.M. Day [3] and James [9] have shown that the assumption of uniqueness is superfluous.

We now state the definition of Gateaux differentiability.

Definition 2.5. The norm of a normed linear space B is said to be Gateaux differentiable if, for every pair of elements u, $v\in B$, $u\neq 0$, the limit

$$\lim_{h\to 0} h^{-1}\,[\|u+h\,v\|-\|u\|]$$

exists. In this case the limit is denoted by $N(u;\,v)$ and called the Gateaux differential at u in the direction of v.

It is an immediate consequence of the convexity of the norm that $N(u;\,v)$, when it exists, is a linear functional in v. Even if $N(u;\,v)$ does not exist, the corresponding right and left limits exist. We will denote them by $N_+(u;\,v)$ and $N_-(u;\,v)$ respectively.

Lemma 2.6. *For $\lambda\mu>0$ we have*

$$N_+(\lambda u;\,\mu v)=|\mu|\,N_+(u;\,v),\quad N_-(\lambda u;\,\mu v)=|\mu|\,N_-(u;\,v)$$

and for $\lambda\mu<0$

$$N_+(\lambda u;\,\mu v)=-|\mu|\,N_-(u;\,v),\quad N_-(\lambda u;\,\mu v)=-|\mu|\,N_+(u;\,v).$$

Proof. For $\lambda\mu>0$ we have by definition

$$N_+(\lambda u;\, \mu v) = \lim_{h\to+0} h^{-1}[\|\lambda u + h\mu v\| - \|\lambda u\|]$$

$$= |\lambda| \lim_{h\to+0} h^{-1}\left[\left\|u + h\frac{\mu}{\lambda} v\right\| - \|u\|\right]$$

$$= \frac{\mu}{\lambda} |\lambda| \lim_{h\to-0} h^{-1}[\|u + hv\|] - \|u\|]$$

$$= |\mu|\, N_+(u;\, v).$$

In the same way the second relation is proved. The two remaining relations then follow if we observe that $N_-(u;\, -v) = -N_-(u;\, v)$.

Lemma 2.7. *If B is a normed linear space and there exist two real numbers λ, μ with $\lambda + \mu \neq 0$, such that $\lambda N_+(u;\, v) + \mu N_-(u;\, v)$ is a continuous function of u, $v \in B$, $u \neq 0$, then the norm of B is Gateaux differentiable.*

Proof. We accept without proof the fact that if $\varphi(x)$ is a continuous, convex function for $-\infty < x < +\infty$ and $\varphi'_+(x)$, $\varphi'_-(x)$ its right and left derivatives respectively, then, for every x_0,

$$\lim_{x\to x_0-0} \varphi'_\pm(x) = \varphi'_-(x_0) \quad \text{and} \quad \lim_{x\to x_0+0} \varphi'_\pm(x) = \varphi'_+(x_0).$$

Let u and v be linearly independent elements of B, $u \neq 0$, and consider the function $\varphi(x) = \|u + xv\|$, $-\infty < x < +\infty$. This function is clearly continuous and convex for all x. Now we have

$$\varphi'_+(x) = \lim_{h\to+0} h^{-1}[\|u + xv + hv\| - \|u + xv\|] = N_+(u + xv;\, v)$$

and

$$\varphi'_-(x) = \lim_{h\to-0} h^{-1}[\|u + xv + hv\| - \|u + xv\|] = N_-(u + xv;\, v).$$

Our hypothesis then implies that $\lambda\varphi'_+(x) + \mu\varphi'_-(x)$ is a continuous function of x. Thus we have

$$(\lambda + \mu)\,\varphi'_-(x_0) = \lim_{x\to x_0-0} (\lambda\varphi'_+(x) + \mu\varphi'_-(x))$$

$$= \lim_{x\to x_0+0} (\lambda\varphi'_+(x) + \mu\varphi'_-(x))$$

$$= (\lambda + \mu)\,\varphi'_+(x_0).$$

Since $\lambda + \mu \neq 0$, we get

$$\varphi'_+(x_0) = \varphi'_-(x_0)$$

for every x_0. Putting $x_0 = 0$ we see that

$$N_+(u;\, v) = N_-(u;\, v).$$

If u and v are linearly dependent, i.e. $v = ku$, it follows directly from the definition that $N(u;\, v)$ exists and is equal to $k\|u\|$. Hence the lemma is proved.

The following theorem establishes a connection between the Gateaux differential and the notion of normality.

Theorem 2.8. *If the norm of B is Gateaux differentiable, then an element $u \in B$ is normal to an element $v \in B$ if and only if $N(u; v) = 0$* (cf. James [8]).

We omit the proof of Theorem 2.8, since it is a simple consequence of the definitions.

In Section 6 we will need the following characterization of Euclidean spaces due to F. A. Ficken [4].

Lemma 2.9. *Let B be a real normed linear space satisfying the following condition: If $u \in B$, $v \in B$ and $\|u+v\| = \|u-v\|$, then for all real x*

$$\|u + xv\| = \|u - xv\|.$$

Then B is a Euclidean space.

3. Property (H) and its consequences

When we speak of orthogonality in the following, we will always mean the orthogonality of Definition 1.1. We begin by proving that orthogonality has property (d) of the Introduction.

Theorem 3.1. *If u and v are elements of a normed linear space B, there is a number a such that $u \perp (au+v)$.*

Proof. Put

$$f(x) = \sum_{\nu=1}^{m} a_\nu \|b_\nu u + c_\nu (xu+v)\|^2$$

for $-\infty < x < +\infty$. Then we have to show that $f(a) = 0$ for some a. Obviously, $f(x)$ is a continuous function of x.

If we put $E = \{\nu \mid 1 \leqslant \nu \leqslant m,\ c_\nu \neq 0\}$ and $F = \{\nu \mid 1 \leqslant \nu \leqslant m,\ c_\nu = 0\}$, we have

$$\begin{aligned} f(x)x^{-1} &= x^{-1} \sum_{\nu=1}^{m} a_\nu \|b_\nu u + c_\nu(xu+v)\|^2 \\ &= x^{-1} \sum_{\nu=1}^{m} a_\nu [\|b_\nu u + c_\nu(xu+v)\|^2 - \|xc_\nu u + c_\nu v\|^2] \\ &= x^{-1} \sum_{\nu \in E} a_\nu [\|(x + b_\nu c_\nu^{-1}) c_\nu u + c_\nu v\|^2 - \|xc_\nu u + c_\nu v\|^2] + x^{-1} \sum_{\nu \in F} a_\nu b_\nu^2 \|u\|^2. \end{aligned}$$

Here we have used the assumption $\sum_{\nu=1}^{m} a_\nu c_\nu^2 = 0$. Applying Lemma 2.2, we see that

$$\lim_{x \to \pm\infty} f(x)x^{-1} = \sum_{\nu \in E} 2a_\nu b_\nu c_\nu^{-1} \|c_\nu u\|^2 = 2\|u\|^2 \sum_{\nu=1}^{m} a_\nu b_\nu c_\nu = 2\|u\|^2.$$

Hence it follows that $f(x)$ is positive for large positive values of x and negative for large negative values of x. Being a continuous function $f(x)$ must be zero for some value of x, which was to be proved.

Definition 3.2. Orthogonality is said to have property (H) in a normed linear space B if $u \perp v$ implies that

$$\lim_{n \to +\infty} n^{-1} \sum_{\nu=1}^{m} a_\nu \| n b_\nu u + c_\nu v \|^2 = 0 \quad (n \text{ positive integer}). \tag{3.1}$$

Clearly, if orthogonality is homogeneous or additive in B, then it has property (H) in B.

We now proceed to show that if orthogonality has property (H) in B, the number a in Theorem 3.1 is uniquely determined and may be expressed as a linear combination of $N_+(u; v)$ and $N_-(u; v)$.

Theorem 3.3. *If orthogonality has property* (H) *in* B *and if* $u \perp (a u + v)$, $u \neq 0$, *then*

$$a = -\| u \|^{-1} [p N_+(u; v) + q N_-(u; v)], \tag{3.2}$$

where

$$p = \sum_{\substack{1 \leq \nu \leq m \\ b_\nu c_\nu > 0}} a_\nu b_\nu c_\nu \quad \text{and} \quad q = \sum_{\substack{1 \leq \nu \leq m \\ b_\nu c_\nu < 0}} a_\nu b_\nu c_\nu .$$

Proof. Our assumptions imply that

$$\lim_{n \to +\infty} n^{-1} \sum_{\nu=1}^{m} a_\nu \| n b_\nu u + c_\nu (a u + v) \|^2 = 0. \tag{3.3}$$

If $b_\nu \neq 0$ we have by Lemma 2.2

$$\| n b_\nu u + c_\nu (a u + v) \|^2 = \| n b_\nu u + c_\nu v \|^2 + 2 n a b_\nu c_\nu \| u \|^2 + n \varepsilon_\nu(n), \tag{3.4}$$

where $\varepsilon_\nu(n) \to 0$ when $n \to +\infty$. The same holds trivially when $b_\nu = 0$. Taking the sum of the relations (3.4) multiplied by a_ν for $\nu = 1, 2, \dots, m$, dividing by n, and letting n tend to infinity, we get, by (3.3) and (1.1)

$$2 a \| u \|^2 = - \lim_{n \to +\infty} n^{-1} \sum_{\nu=1}^{m} a_\nu \| n b_\nu u + c_\nu v \|^2. \tag{3.5}$$

The limit in the right-hand member of (3.5) may be evaluated in the following way. Because of (1.1) we may write

$$n^{-1} \sum_{\nu=1}^{m} a_\nu \| n b_\nu u + c_\nu v \|^2 = n^{-1} \sum_{\nu=1}^{m} a_\nu [\| n b_\nu u + c_\nu v \|^2 - \| n b_\nu u \|^2]$$

$$= \sum_{\nu=1}^{m} a_\nu [\| n b_\nu u + c_\nu v \| - \| n b_\nu u \|] \, n^{-1} [\| n b_\nu u + c_\nu v \| + \| n b_\nu u \|].$$

Letting n tend to infinity we get

$$\lim_{n \to +\infty} n^{-1} \sum_{\nu=1}^{m} a_\nu \| n b_\nu u + c_\nu v \|^2 = \sum_{\nu=1}^{m} a_\nu N_+(b_\nu u; c_\nu v) \, 2 \| b_\nu u \|$$

and by (3.5)
$$a = -\|u\|^{-1}\sum_{\nu=1}^{m} a_\nu |b_\nu| N_+(b_\nu u;\ c_\nu v). \tag{3.6}$$

If we now put
$$p = \sum_{\substack{1\leqslant \nu\leqslant m\\ b_\nu c_\nu>0}} a_\nu b_\nu c_\nu,\quad q = \sum_{\substack{1\leqslant \nu\leqslant m\\ b_\nu c_\nu<0}} a_\nu b_\nu c_\nu,$$

and make use of Lemma 2.6, it follows that

$$a = -\|u\|^{-1}\sum_{\substack{1\leqslant \nu\leqslant m\\ b_\nu c_\nu>0}} a_\nu |b_\nu|\,|c_\nu| N_+(u;\ v) + \|u\|^{-1}\sum_{\substack{1\leqslant \nu\leqslant m\\ b_\nu c_\nu<0}} a_\nu |b_\nu|\,|c_\nu| N_-(u;\ v)$$
$$= -\|u\|^{-1}[p N_+(u;\ v) + q N_-(u;\ v)],$$

which was to be proved.

From Theorems 3.1 and 3.3 it follows that to every pair of elements u and v of B, with $u \neq 0$, there is a uniquely determined number a such that $u \perp (a u + v)$. We will denote this number by $a(u;\ v)$.

Lemma 3.4. *If orthogonality has property (H) in B, then $a(u;\ v)$ is a continuous function of u and v, $u \neq 0$.*

Proof. Let $(u_n)_1^\infty$ and $(v_n)_1^\infty$ be two sequences of elements of B such that $u_n \neq 0$ for all n and $u_n \to \bar{u} \neq 0$, $v_n \to \bar{v}$ when $n \to \infty$. We have to show that $a(u_n;\ v_n) \to a(\bar{u};\ \bar{v})$ when $n \to \infty$.

It is an immediate consequence of the definition of $N_+(u;\ v)$ and $N_-(u;\ v)$ that

$$|N_+(u;\ v)| \leqslant \|v\|,\quad |N_-(u;\ v)| \leqslant \|v\| \quad \text{for } u \in B,\ u \neq 0,\ v \in B.$$

Using this fact and (3.2) we see that the sequence of real numbers $(\alpha_n)_1^\infty = (a(u_n;\ v_n))_1^\infty$ is bounded. Let $(\beta_k)_1^\infty$ be an arbitrary subsequence of $(\alpha_n)_1^\infty$. Then we may select from $(\beta_k)_1^\infty$ a subsequence $(\alpha_{n_l})_1^\infty$ which converges to a number c. From the definition of orthogonality it follows that if $u_n \perp v_n$, $n = 1, 2, \ldots$, and $u_n \to u$, $v_n \to v$ when $n \to \infty$, then $u \perp v$. Consequently, since $u_{n_l} \perp (\alpha_{n_l} u_{n_l} + v_{n_l})$, $l = 1, 2, \ldots$, we may take the limit as $l \to \infty$ of each side of this relation and get

$$\bar{u} \perp c\,\bar{u} + \bar{v}.$$

But then we must have $c = a(\bar{u};\ \bar{v})$. Thus we may select from every subsequence of $(a(u_n;\ v_n))_1^\infty$ a subsequence which converges to $a(\bar{u};\ \bar{v})$, which implies that $a(u_n;\ v_n) \to a(\bar{u};\ \bar{v})$ when $n \to \infty$. Hence the lemma is proved.

If $a(u;\ v)$ is continuous in u, v for $u \neq 0$, then, by (3.2), the same holds for $p N_+(u;\ v) + q N_-(u;\ v)$. An application of Lemma 2.7 then shows that the norm of B is Gateaux differentiable. Furthermore, we see that $u \perp v$ if and only if $N(u;\ v) = 0$, i.e. if and only if u is normal to v. We collect these results in the following theorem.

Theorem 3.5. *If orthogonality has property (H) in B, then the norm of B is Gateaux differentiable and $u \perp v$ holds if and only if $N(u;\ v) = 0$.*

Let us say, for a moment, that u is anti-orthogonal to v (in notation $u\,T\,v$) if v is orthogonal to u. I.e. we have $u\,T\,v$ if and only if $v \perp u$ or if and only if

$$\sum_{\nu=1}^{m} a_\nu \| c_\nu u + b_\nu v \|^2 = 0.$$

We have just shown that if orthogonality has property (H) then it is equivalent to normality and therefore homogeneous, i.e. $u \perp v$ implies $\lambda u \perp \mu v$ for all λ, μ. Hence it follows in particular that anti-orthogonality has property (H). Now it is obvious that Theorems 3.1, 3.3 and 3.5 remain true if we replace orthogonality by anti-orthogonality. Assuming that $N(u;\, v) = 0$ we have $u \perp v$ or $v\,T\,u$ and hence, by the analogue of Theorem 3.5 for anti-orthogonality, that $N(v;\, u) = 0$. This means that the relation of normality is symmetric in B. Thus, we have

Theorem 3.6. *If orthogonality has property* (H) *in* B, *then it is symmetric and equivalent to normality in* B.

An application of Lemma 2.4 and Theorems 3.3 and 3.6 gives

Corollary 3.7. *If the dimension of* B *is greater than two and orthogonality has property* (H) *in* B, *then* B *is Euclidean.*

Since homogeneity and additivity of orthogonality each implies property (H), we also have the following corollary.

Corollary 3.8. *If the dimension of* B *is greater than two and orthogonality is homogeneous or additive in* B, *then* B *is Euclidean.*

4. The two-dimensional problem

We have defined $u \perp v$ to mean that

$$\sum_{\nu=1}^{m} a_\nu \| b_\nu u + c_\nu v \|^2 = 0.$$

Now we change our notation a little by introducing constants p_ν, q_ν, $\nu = 1, 2, \ldots, r$, and C_1, C_2 so that

$$\sum_{\nu=1}^{m} a_\nu \| b_\nu u + c_\nu v \|^2 = \sum_{\nu=1}^{r} p_\nu \| u + q_\nu v \|^2 - C_1 \| u \|^2 - C_2 \| v \|^2$$

and

$$\sum_{\nu=1}^{r} p_\nu = C_1, \quad \sum_{\nu=1}^{r} p_\nu q_\nu^2 = C_2, \quad \sum_{\nu=1}^{r} p_\nu q_\nu = 1, \quad q_\nu \neq 0. \tag{4.1}$$

Let B be a two-dimensional normed linear space in which orthogonality is homogeneous. Taking two elements u and v of B, such that $u \perp v$, $\| u \| = \| v \| = 1$, we then have

$$\sum_{\nu=1}^{r} p_\nu \| u + x q_\nu v \|^2 = C_1 + C_2 x^2 \text{ for } -\infty < x < +\infty.$$

This means that the function $\varphi(x)=\|u+xv\|^2$ is a solution of the functional equation

$$\sum_{\nu=1}^{r} p_\nu F(x q_\nu)=C_1+C_2 x^2 \text{ for } -\infty<x<+\infty \tag{4.2}$$

We know from Section 3 that $\varphi(x)$ is continuously differentiable for $-\infty<x<+\infty$. The behaviour of $\varphi(x)$ for large and small values of $|x|$ is given by the following lemma.

Lemma 4.1. *The function $\varphi(x)$ satisfies the conditions*

$$\varphi(x)=1+O(x) \text{ when } x\to 0 \tag{4.3}$$

and

$$\varphi(x)=x^2+O(x) \text{ when } x\to\pm\infty. \tag{4.4}$$

Proof. For every x we have

$$\begin{aligned}|\varphi(x)-1|&=|\|u+xv\|^2-\|u\|^2|\\&=|\|u+xv\|-\|u\||(\|u+xv\|+\|u\|)\leqslant|x|(2+|x|),\end{aligned}$$

which proves (4.3). We also have

$$\begin{aligned}|\varphi(x)-x^2|&=|\|u+xv\|^2-\|xv\|^2|\\&=|\|u+xv\|-\|xv\||(\|u+xv\|+\|xv\|)\leqslant 1+2|x|,\end{aligned}$$

from which (4.4) follows.

If u is suitably chosen, it is possible to strengthen the result (4.3) in the following way.

Lemma 4.2. *Let B be a two-dimensional normed linear space and denote by C the set $\{u\,|\,u\in B,\ \|u\|=1\}$. Then there is a dense subset D of C such that if $u\in D$ and $u\,N\,v$, the function $\varphi(x)=\|u+xv\|^2$ satisfies*

$$\varphi(x)=1+O(x^2) \text{ when } x\to 0. \tag{4.5}$$

Proof. Let us introduce polar coördinates (r,ψ) so that the closed convex curve C has the equation $r=r(\psi)$, $0\leqslant\psi<2\pi$. For all ψ, except at most countably many, C has a unique tangent at the point $(r(\psi),\psi)$. This tangent makes an angle $\theta(\psi)$ with the polar axis. If ψ_0 is an exceptional value of ψ, we define $\theta(\psi_0)$ so that $\theta(\psi)$ is continuous to the left for $\psi=\psi_0$. Then $\theta(\psi)$ is defined and non-decreasing for $0\leqslant\psi<2\pi$. From a well-known theorem it follows that $\theta(\psi)$ has a derivative $\theta'(\psi)$ for almost all ψ in $(0,2\pi)$.

We define D to be the set of all $u\in C$ such that $\theta'(\psi)$ exists where ψ is the polar angle of u. It is obvious that D is a dense subset of C. Now let u and v be elements of B such that $u\in D$ and $u\,N\,v$. Let the endpoint P of the vector u have coordinates $(r(\psi_0),\psi_0)$. The curve C has a unique tangent at P and from the definition of normality it follows that v lies along this tangent. Let the endpoint Q of the vector $u+xv$ have coordinates $(\varrho(\delta),\psi_0+\delta)$ and let

R be the point $(r(\psi_0+\delta), \psi_0+\delta)$. In the triangle PQR we denote the angle QRP by ξ and the angle RPQ by η. Then as $x \to 0$ we have

$$\delta = O(x)$$

and
$$(\sin \xi)^{-1} = O(1).$$

Since $\theta'(\psi_0)$ exists, it is obvious that

$$\eta = O(\delta) \text{ when } \delta \to 0$$

and consequently
$$\eta = O(x) \text{ when } x \to 0.$$

Finally, we have when $x \to 0$

$$\begin{aligned} \|u + xv\| - 1 &= \frac{\varrho(\delta)}{r(\psi_0+\delta)} - 1 = \frac{\varrho(\delta) - r(\psi_0+\delta)}{r(\psi_0+\delta)} \\ &= O(1)\,[\varrho(\delta) - r(\psi_0+\delta)] = O(1)\,\frac{|x|\,\|v\|\sin\eta}{\sin\xi} \\ &= O(x^2). \end{aligned}$$

Applying this result we also have

$$\begin{aligned} \varphi(x) - 1 &= \|u+xv\|^2 - 1 = (\|u+xv\| - 1)(\|u+xv\| + 1) \\ &= O(x^2) \text{ when } x \to 0. \end{aligned}$$

Hence Lemma 4.2 is proved.

In our treatment of equation (4.2) in Section 6 we will distinguish between two cases; we say that the equation is symmetrical if it may be written in the form

$$\sum_{\nu=1}^{s} k_\nu F(l_\nu x) - \sum_{\nu=1}^{s} k_\nu F(-l_\nu x) = C_1 + C_2 x^2.$$

Otherwise the equation is non-symmetrical. We observe that if (4.2) is symmetrical, then it follows from the relations (4.1) that $C_1 = C_2 = 0$.

We are interested in those solutions $\varphi(x)$ of (4.2) which are continuously differentiable and satisfy the conditions (4.4) and (4.5). It turns out that, in the case of a non-symmetrical equation (4.2), there is only one such solution, viz. $\varphi(x) = 1 + x^2$. Of course, this is no longer true if the equation is symmetrical because then every even function is a solution. However, in this case we prove that, conversely, every solution of (4.2) satisfying (4.4) and (4.5) must be even. These results, together with the fact that they imply that B is Euclidean, will be obtained in Section 6.

5. *F*-series

Let us consider the functional equation

$$\sum_{\mu=1}^{N} d_\mu f(x + \alpha_\mu) = 0,$$

where d_μ, α_μ are complex numbers. Clearly, this equation admits as solutions the functions

$$x^m e^{t_s x}, \quad m=0, 1, 2, \ldots, m_s-1, \quad s=1, 2, \ldots, \tag{5.1}$$

where $t_1, t_2, \ldots$ are the zeros of the function $h(t)=\sum_{\mu=1}^{N} d_\mu e^{\alpha_\mu t}$ and m_s is the multiplicity of the zero t_s. The theory of F-series (see Moore [12], [13]) investigates the possibility of expanding a function $f(x)$ in a series of the functions (5.1).

For our purpose it is sufficient to assume that d_μ and α_μ are real numbers, $\mu=1, 2, \ldots, N$. In this case the F-series is a special case of Kitagawa's Cauchy series. (See Kitagawa [11].) Let us also assume that $\alpha_1<\alpha_2<\ldots<\alpha_N$.

Carmichael [2] has proved the existence of contours C_s, $s=1, 2, \ldots$, about the origin in the complex plane, with the properties

(a) there exists an $\varepsilon>0$ such that

$$|h(t)\, e^{-xt}|>\varepsilon \text{ for } \alpha_1 \leqslant x \leqslant \alpha_N, \; t \in C_s, \; s=1, 2, \ldots, \tag{5.2}$$

(b) C_s lies along the circle with radius s and centre at the origin, except for portions of bounded length lying within a bounded distance of the imaginary axis,

(c) no point of C_s lies outside C_{s+1}.

Let us denote by t_{ks}, $k=1, 2, \ldots, \sigma_s$, the zeros of $h(t)$ lying between C_{s-1} and C_s (for $s=1$ inside C_1). Let C_{ks} be a small circle passing through no zero of $h(t)$ and containing only the zero t_{ks} of $h(t)$ in its interior. If $f(x)$ is integrable over the interval (α_1, α_N) we define its F-series in the following way.

Definition 5.1 (Moore [12]). The F-series for $f(x)$ associated with $h(t)$ is the series

$$\sum_{s=1}^{\infty} \sum_{k=1}^{\sigma_s} \frac{1}{2\pi i} \sum_{\mu=1}^{N} d_\mu \int_a^{\alpha_\mu} f(x_1) \int_{C_{ks}} e^{(\alpha_\mu+x-x_1)t} \{h(t)\}^{-1}\, dt\, dx_1, \tag{5.3}$$

where $\alpha_1 \leqslant a \leqslant \alpha_N$.

We remark that this definition is correct, that is, the series (5.3) is independent of the choice of a. In fact, we have

$$\begin{aligned}
&\sum_{\mu=1}^{N} d_\mu \int_a^{\alpha_\mu} f(x_1) \int_{C_{ks}} e^{(\alpha_\mu+x-x_1)t} \{h(t)\}^{-1}\, dt\, dx_1 \\
&\qquad - \sum_{\mu=1}^{N} d_\mu \int_{a'}^{\alpha_\mu} f(x_1) \int_{C_{ks}} e^{(\alpha_\mu+x-x_1)t} \{h(t)\}^{-1}\, dt\, dx_1 \\
&= \int_a^{a'} f(x_1) \int_{C_{ks}} e^{(x-x_1)t}\, dt\, dx_1 \\
&= 0,
\end{aligned}$$

by Cauchy's theorem.

The following lemma is a special case of a result due to Carmichael [2] and we state it without proof.

Lemma 5.2. *When* $s \to \infty$ *the integral*

$$\int_{C_s} e^{xt} \{h(t)\}^{-1} t^{-1} \, dt$$

tends uniformly to zero with respect to x *in every interval* $\alpha_1 + \delta' \leqslant x \leqslant \alpha_N - \delta''$, $\delta' > 0, \delta'' > 0$.

Theorem 5.3. *If* $f(x)$ *has a continuous derivative in the interval* $\alpha_1 \leqslant x \leqslant \alpha_N$, *then the series* (5.3) *converges to* $f(x)$ *in the open interval* $\alpha_1 < x < \alpha_N$.

Moore ([12], Theorem 2) has proved a much more general theorem. We give, however, a special proof of Theorem 5.3 since it is particularly simple in this case.

Proof. The sth partial sum of the series (5.3) may be written

$$S_s(f; x) = \frac{1}{2\pi i} \sum_{\mu=1}^{N} d_\mu \int_a^{\alpha_\mu} f(x_1) \int_{C_s} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} \, dt \, dx_1.$$

Put
$$Q_s(x_1) = \int_{C_s} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} \, dt.$$

As we have seen above, we may replace a by x in the expression for $S_s(f; x)$. Doing this and integrating by parts with respect to x_1, we get

$$\begin{aligned}
S_s(f; x) &= \frac{1}{2\pi i} \sum_{\mu=1}^{N} d_\mu \left\{ \left[f(x_1) \int_{\alpha_\mu}^{x_1} Q_s(x_2) \, dx_2 \right]_x^{\alpha_\mu} - \int_x^{\alpha_\mu} f'(x_1) \int_{\alpha_\mu}^{x_1} Q_s(x_2) \, dx_2 \, dx_1 \right\} \\
&= \frac{1}{2\pi i} \sum_{\mu=1}^{N} d_\mu f(x) \left\{ \int_{C_s} e^{\alpha_\mu t} \{h(t)\}^{-1} t^{-1} \, dt - \int_{C_s} e^{xt} \{h(t)\}^{-1} t^{-1} \, dt \right\} \\
&\quad + \frac{1}{2\pi i} \sum_{\mu=1}^{N} d_\mu \int_x^{\alpha_\mu} f'(x_1) \int_{C_s} \{ e^{(\alpha_\mu + x - x_1)t} - e^{xt} \} \{h(t)\}^{-1} t^{-1} \, dt \, dx_1 \\
&= f(x) + \frac{1}{2\pi i} \sum_{\mu=1}^{N} d_\mu \int_x^{\alpha_\mu} f'(x_1) \int_{C_s} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} t^{-1} \, dt \, dx_1 \\
&\quad - \frac{1}{2\pi i} \sum_{\mu=1}^{N} d_\mu f(\alpha_\mu) \int_{C_s} e^{xt} \{h(t)\}^{-1} t^{-1} \, dt.
\end{aligned}$$

The integral
$$\int_{C_s} e^{xt} \{h(t)\}^{-1} t^{-1} \, dt$$

tends to zero for $\alpha_1 < x < \alpha_N$ by Lemma 5.2. Putting

$$U_s(x_1) = f'(x_1) \int_{C_s} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} t^{-1} dt$$

and making use of properties (a), (b) of C_s and Lemma 5.2, we see that, for $\alpha_1 \leqslant x \leqslant \alpha_N$, the sequence $U_s(x_1)$ is uniformly bounded in the interval (x, α_μ) and tends uniformly to zero in any interval whose endpoints are interior points of (x, α_μ). Hence it follows that the second term in the last expression for $S_s(f; x)$ tends to zero when $\alpha_1 \leqslant x \leqslant \alpha_N$. This completes the proof of Theorem 5.3.

Lemma 5.4. *If $f(x)$ satisfies the equation $\sum_{\mu=1}^N d_\mu f(x+\alpha_\mu) = 0$ for $-\infty < x < +\infty$, we have*

$$\sum_{\mu=1}^N d_\mu \int_a^{\alpha_\mu} f(x_1) \int_{C_{ks}} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} dt\, dx_1$$
$$= \sum_{\mu=1}^N d_\mu \int_{a'}^{\alpha_\mu + K} f(x_1) \int_{C_{ks}} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} dt\, dx_1, \qquad (5.4)$$

where a, a' and K are arbitrary real numbers.

Proof. According to the remark following Definition 5.1, the left-hand side of (5.4) is independent of a. Thus, the difference between the two sides of (5.4) may be written

$$\sum_{\mu=1}^N d_\mu \int_{\alpha_\mu}^{\alpha_\mu + K} f(x_1) \int_{C_{ks}} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} dt\, dx_1$$
$$= \sum_{\mu=1}^N d_\mu \int_0^K f(u+\alpha_\mu) \int_{C_{ks}} e^{(x-u)t} \{h(t)\}^{-1} dt\, du,$$

which is equal to zero because $\sum_{\mu=1}^N d_\mu f(u+\alpha_\mu) = 0$.

Theorem 5.5. *If $f(x)$ has a continuous derivative and satisfies the equation $\sum_{\mu=1}^N d_\mu f(x+\alpha_\mu) = 0$ for $-\infty < x < +\infty$, then the series (5.3) converges to $f(x)$ for all x.*

Proof. Suppose $\alpha_1 < x < \alpha_N$ and let K be an arbitrary real number. Put $\bar{f}(x) = f(x+K)$. Then we have using Lemma 5.4

$$S_s(f; x+K) = \frac{1}{2\pi i} \sum_{\mu=1}^N d_\mu \int_a^{\alpha_\mu} f(x_1) \int_{C_s} e^{(\alpha_\mu + x + K - x_1)t} \{h(t)\}^{-1} dt\, dx_1$$
$$= \frac{1}{2\pi i} \sum_{\mu=1}^N d_\mu \int_{a-K}^{\alpha_\mu - K} f(x_1 + K) \int_{C_s} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} dt\, dx_1$$
$$= \frac{1}{2\pi i} \sum_{\mu=1}^N d_\mu \int_a^{\alpha_\mu} \bar{f}(x_1) \int_{C_s} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} dt\, dx_1$$
$$= S_s(\bar{f}; x),$$

which tends to $\bar{f}(x) = f(x+K)$ by Theorem 5.3. Thus we have

$$\lim_{s\to\infty} S_s(f; x+K) = f(x+K) \text{ for every } K,$$

which proves the theorem.

Lemma 5.6. (Moore [12], Theorem 1.) *The functions $x^m e^{t_{ks}x}$, $m=0, 1, \dots, m_{ks}-1$, $k=1, 2, \dots, \sigma_s$, $s=1, 2, \dots$, where m_{ks} is the multiplicity of t_{ks}, satisfy the following "biorthogonality relations"*

$$\frac{1}{2\pi i}\sum_{\mu=1}^{N} d_\mu \int_a^{\alpha_\mu} x_1^m e^{t_{ks}x_1} \int_{C_{lr}} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} dt\, dx_1 = \begin{cases} 0 \text{ if } t_{ks} \neq t_{lr} \\ x^m e^{t_{ks}x} \text{ if } t_{ks} = t_{lr}. \end{cases}$$

6. Solution of the two-dimensional problem

We now apply the results of Section 5 to prove the following theorem.

Theorem 6.1. *If $f(x)$ is a continuously differentiable solution of the non-trivial functional equation*

$$\sum_{\mu=1}^{N} d_\mu f(x+\alpha_\mu) = 0, \quad -\infty < x < +\infty, \tag{6.1}$$

satisfying (A, B real constants)

$$f(x) = A + O(e^{2x}) \text{ when } x \to -\infty \tag{6.2}$$

and

$$f(x) = B e^{2x} + O(e^{x}) \text{ when } x \to +\infty, \tag{6.3}$$

then $f(x) = A + B e^{2x}$ for $-\infty < x < +\infty$.

Proof. As in Section 5 we put $h(t) = \sum_{\mu=1}^{N} d_\mu e^{\alpha_\mu t}$. Let us denote by $T_{ks}(f; x)$ the term of the F-series for $f(x)$ corresponding to the zero t_{ks} of $h(t)$, i.e. $(\alpha_1 \leqslant a \leqslant \alpha_N)$,

$$T_{ks}(f; x) = \frac{1}{2\pi i}\sum_{\mu=1}^{N} d_\mu \int_a^{\alpha_\mu} f(x_1) \int_{C_{ks}} e^{(\alpha_\mu + x - x_1)t} \{h(t)\}^{-1} dt\, dx_1.$$

Then we know by Theorem 5.5 that

$$f(x) = \sum_{s=1}^{\infty} \sum_{k=1}^{\sigma_s} T_{ks}(f; x) \text{ for } -\infty < x < +\infty. \tag{6.4}$$

We now calculate $T_{ks}(f; x)$ for the various zeros t_{ks} of $h(t)$. We distinguish between two cases.

(1) Suppose that $Re\,(t_{ks})<2$.

By (6.2) we may write $f(x_1)=A+e^{2x_1}\eta(x_1)$, where $\eta(x_1)$ is bounded when $x_1\to-\infty$, say

$$|\eta(x_1)|\leqslant M_1 \text{ for } x_1\leqslant\omega_1. \tag{6.5}$$

Let us put, for the moment, $g(x_1)=A$ and $h(x_1)=e^{2x_1}\eta(x_1)$, so that $f(x_1)=g(x_1)+h(x_1)$, $-\infty<x_1<+\infty$. Then we have

$$T_{ks}(f;x)=T_{ks}(g;x)+T_{ks}(h;x). \tag{6.6}$$

If $A=0$ then $T_{ks}(g;x)=0$. If $A\neq0$, we substitute the expression (6.2) for $f(x_1)$ in equation (6.1) and get, when $x_1\to-\infty$, $\sum_{\mu=1}^{N}d_\mu=0$. This means that in this case $t=0$ is a zero of $h(t)$ and we have, by Lemma 5.6, that

$$T_{ks}(g;x)=A \text{ if } t_{ks}=0,\; T_{ks}(g;x)=0 \text{ if } t_{ks}\neq0.$$

Thus, in any case we have

$$T_{ks}(f;x)=\begin{cases}A+T_{ks}(h;x) & \text{if } t_{ks}=0,\\ \quad\; T_{ks}(h;x) & \text{if } t_{ks}\neq0.\end{cases} \tag{6.7}$$

We next show that $T_{ks}(h;x)=0$ for every x if $Re\,(t_{ks})<2$. Choose the radius r of the circle C_{ks} so small that

$$Re\,(2-t)\geqslant c>0 \text{ for } t\in C_{ks}. \tag{6.8}$$

For fixed x and r there is a constant M_2 such that

$$|e^{(\alpha_\mu+x)t}\{h(t)\}^{-1}|\leqslant M_2 \text{ for } \mu=1,\dots,N,\, t\in C_{ks}. \tag{6.9}$$

Now let ε be an arbitrary positive number and choose ω_2 so that

$$\omega_2<0,\; \omega_2\leqslant\omega_1 \text{ and } e^{cx_1}\leqslant\varepsilon \text{ for } x_1\leqslant\omega_2. \tag{6.10}$$

Since $h(x)=e^{2x}\eta(x)$ is a solution of (6.1) we have by Lemma 5.4

$$T_{ks}(h;x)=\frac{1}{2\pi i}\sum_{\mu=1}^{N}d_\mu\int_{a+K}^{\alpha_\mu+K}e^{2x_1}\eta(x_1)\int_{C_{ks}}e^{(\alpha_\mu+x-x_1)t}\{h(t)\}^{-1}\,dt\,dx_1, \tag{6.11}$$

where K is arbitrary. If we choose K so that $\alpha_N+K\leqslant\omega_2$ we get, using the estimates (6.5), (6.8), (6.9), and (6.10),

$$|T_{ks}(h;x)|\leqslant\frac{1}{2\pi}\sum_{\mu=1}^{N}|d_\mu|\left|\int_{a+K}^{\alpha_\mu+K}M_1M_2\,\varepsilon\,dx_1\right|2\pi r\leqslant\varepsilon M_1M_2\,r\,(\alpha_N-\alpha_1)\sum_{\mu=1}^{N}|d_\mu|.$$

Since ε was arbitrary, we conclude that $T_{ks}(h; x) = 0$ for every x. Thus, we have proved that

$$T_{ks}(f; x) = \begin{cases} A \text{ if } t_{ks} = 0, \\ 0 \text{ if } Re\,(t_{ks}) < 2,\, t_{ks} \neq 0. \end{cases} \quad (6.12)$$

(2) Suppose that $Re\,(t_{ks}) \geqslant 2$.

In this case we have

$$T_{ks}(f; x) = \begin{cases} B e^{2x} \text{ if } t_{ks} = 2 \\ 0 \text{ if } Re\,(t_{ks}) \geqslant 2,\, t_{ks} \neq 2. \end{cases} \quad (6.13)$$

The proof of this differs very little from the proof of (6.12), except that we now use condition (6.3) instead of (6.2). We omit the details.

Combining (6.4), (6.12), and (6.13) we finally have

$$f(x) = A + B e^{2x} \text{ for } -\infty < x < +\infty,$$

which is the desired result.

Corollary 6.2. *If $\varphi(x)$ is a continuously differentiable solution of the non-trivial functional equation*

$$\sum_{\mu=1}^{N} d_\mu F(q_\mu x) = 0, \quad x > 0, \quad (6.14)$$

where $q_\mu > 0$, $\mu = 1, \dots, N$, satisfying

$$\varphi(x) = A + O(x^2) \text{ when } x \to +0 \quad (6.15)$$

and

$$\varphi(x) = B x^2 + O(x) \text{ when } x \to +\infty, \quad (6.16)$$

then $\varphi(x) = A + B x^2$ for $x > 0$.

This result follows at once from Theorem 6.1 if we put $q_\mu = e^{\alpha_\mu}$ and $f(x) = \varphi(e^x)$.

Theorem 6.3. *Let $\varphi(x)$ be a continuously differentiable solution of the non-symmetrical equation*

$$\sum_{\nu=1}^{r} p_\nu F(q_\nu x) = C_1 + C_2 x^2, \quad -\infty < x < +\infty, \quad (6.17)$$

where

$$\sum_{\nu=1}^{r} p_\nu = C_1, \quad \sum_{\nu=1}^{r} p_\nu q_\nu^2 = C_2, \quad \sum_{\nu=1}^{r} p_\nu q_\nu = 1, \quad q_\nu \neq 0, \ \nu = 1, \dots, r. \quad (6.18)$$

If $\varphi(x)$ satisfies

$$\varphi(x) = 1 + O(x^2) \text{ when } x \to 0 \quad (6.19)$$

and
$$\varphi(x)=x^2+O(x) \text{ when } x\to\pm\infty, \tag{6.20}$$

then $\varphi(x)=1+x^2$ *for* $-\infty<x<+\infty$.

Proof. Let us define the function $\psi(x)$ by

$$\psi(x)=\varphi(x)-1-x^2 \text{ for } -\infty<x<+\infty. \tag{6.21}$$

Then $\psi(x)$ is a solution of

$$\sum_{\nu=1}^{r} p_\nu F(q_\nu x)=0, \quad -\infty<x<+\infty, \tag{6.22}$$

satisfying
$$\psi(x)=O(x^2) \text{ when } x\to 0 \tag{6.23}$$

and
$$\psi(x)=O(x) \text{ when } x\to\pm\infty. \tag{6.24}$$

Let M be the linear space of all real-valued functions defined for $x>0$. We now define two linear operators $T_1: M\to M$ and $T_2: M\to M$ by

$$(T_1 F)(x)=\sum_{\substack{1\le\nu\le r\\ q_\nu>0}} p_\nu F(q_\nu x), \quad x>0,$$

and
$$(T_2 F)(x)=\sum_{\substack{1\le\nu\le r\\ q_\nu<0}} p_\nu F(-q_\nu x), \quad x>0.$$

Put $\psi_1(x)=\psi(x)$, $\psi_2(x)=\psi(-x)$ for $x\geq 0$. Then the fact that $\psi(x)$ is a solution of (6.22) may be expressed by the two conditions

$$(T_1\psi_1)(x)+(T_2\psi_2)(x)=0, \quad x>0,$$

and
$$(T_1\psi_2)(x)+(T_2\psi_1)(x)=0, \quad x>0,$$

or, equivalently, by
$$(T_1+T_2)(\psi_1+\psi_2)(x)=0, \quad x>0, \tag{6.25}$$

and
$$(T_1-T_2)(\psi_1-\psi_2)(x)=0, \quad x>0. \tag{6.26}$$

The assumption that (6.17) is non-symmetrical implies that $T_1+T_2\neq 0$. The same conclusion concerning the operator T_1-T_2 may be drawn from the fact that $\sum_{\nu=1}^{r} p_\nu q_\nu \neq 0$. Consequently, the equations (6.25) and (6.26) are of the type considered in Corollary 6.2. Furthermore, from (6.23) and (6.24) it follows that the functions $\psi_1(x)-\psi_2(x)$ and $\psi_1(x)+\psi_2(x)$ satisfy

$$\psi_1(x)\pm\psi_2(x)=O(x^2) \text{ when } x\to+0 \tag{6.27}$$

and
$$\psi_1(x)\pm\psi_2(x)=O(x) \text{ when } x\to+\infty. \tag{6.28}$$

Hence, from the same corollary we get

$$\psi_1(x) \pm \psi_2(x) = 0 \text{ for } x > 0$$

and consequently $\psi(x) = 0$ for $-\infty < x < +\infty$. Thus, (6.21) gives

$$\varphi(x) = 1 + x^2 \text{ for } -\infty < x < +\infty,$$

which was to be proved.

Next, we turn to the case of a symmetrical equation (6.17) satisfying the first two of the conditions (6.18). As we remarked in Section 4, we then have $C_1 = C_2 = 0$.

Theorem 6.4. *If $\varphi(x)$ is a continuously differentiable solution of the non-trivial symmetrical equation*

$$\sum_{\nu=1}^{r} p_\nu F(q_\nu x) = 0, \quad -\infty < x < +\infty, \tag{6.29}$$

satisfying

$$\varphi(x) = 1 + O(x^2) \text{ when } x \to 0 \tag{6.30}$$

and

$$\varphi(x) = x^2 + O(x) \text{ when } x \to \pm\infty, \tag{6.31}$$

then $\varphi(x) = \varphi(-x)$ for $-\infty < x < +\infty$.

Proof. Define the function $\psi(x)$ for $x > 0$ by $\psi(x) = \varphi(x) - \varphi(-x)$. Since equation (6.29) is symmetrical it may be written in the form

$$\sum_{\nu=1}^{s} k_\nu F(l_\nu x) - \sum_{\nu=1}^{s} k_\nu F(-l_\nu x) = 0, \quad x > 0,\ l_\nu > 0,\ \nu = 1, \ldots, s. \tag{6.32}$$

Since $\varphi(x)$ is a solution of (6.32) we see that $\psi(x)$ is a solution of the equation

$$\sum_{\nu=1}^{s} k_\nu F(l_\nu x) = 0, \quad x > 0. \tag{6.33}$$

Moreover, from (6.30) and (6.31) it follows that

$$\psi(x) = O(x^2) \text{ when } x \to +0 \tag{6.34}$$

and

$$\psi(x) = O(x) \text{ when } x \to +\infty. \tag{6.35}$$

An application of Corollary 6.2 then gives us $\psi(x) = 0$ or

$$\varphi(x) = \varphi(-x) \text{ for } -\infty < x < +\infty.$$

We are now able to prove that the assumption concerning the dimension of B is superfluous in Corollary 3.7.

Theorem 6.5. *If B is a normed linear space in which orthogonality has property (H), then B is Euclidean.*

Proof. It is sufficient to assume that B has dimension two. Let $C=\{u \mid u\in B, \|u\|=1\}$ and D the subset of C figuring in Lemma 4.2. From the results in Section 3 it follows that orthogonality is homogeneous if it has property (H).

Now let $u\in D$ and $u \perp v$, $\|v\|=1$. Then we know from Sections 3 and 4 that $\varphi(x)=\|u+xv\|^2$ is a continuously differentiable function satisfying an equation

$$\sum_{\nu=1}^{r} p_\nu F(q_\nu x)=C_1+C_2x^2, \quad -\infty<x<+\infty, \tag{6.36}$$

with $$\sum_{\nu=1}^{r} p_\nu=C_1, \ \sum_{\nu=1}^{r} p_\nu q_\nu^2=C_2, \ \sum_{\nu=1}^{r} p_\nu q_\nu=1, \quad q_\nu \neq 0, \nu=1, \ldots, r.$$

Furthermore, we have by Lemmas 4.1 and 4.2

$$\varphi(x)=1+O(x^2) \text{ when } x\to 0 \tag{6.37}$$

and $$\varphi(x)=x^2+O(x) \text{ when } x\to \pm\infty. \tag{6.38}$$

First, let us suppose that equation (6.36) is non-symmetrical. Then $\varphi(x)$ satisfies the hypothesis of Theorem 6.3 and consequently $\varphi(x)=1+x^2$ for $-\infty<x<+\infty$. If we choose u and v as unit vectors of a coordinate system in the plane B and write $w=xu+yv$ we see that $\|w\|=1$ if and only if $x^2+y^2=1$. This means that the curve C has the equation $x^2+y^2=1$ and so is an ellipse. But this is precisely the condition that B be Euclidean.

Now suppose that equation (6.36) is symmetrical. Then it follows from Theorem 6.4 that $\varphi(x)=\varphi(-x)$ for all x or

$$\|u+xv\|=\|u-xv\| \text{ for all real } x. \tag{6.39}$$

The relation (6.39) holds for each pair of elements u and v of B, such that $u\in D$ and $u\perp v$. Since D is a dense subset of C, it is easy to see that (6.39) even holds if $u\in B$ and $u\perp v$. Geometrically speaking, this means that all chords of C parallel to a fixed direction have their midpoints on a straight line through the origin. Now let u and v be two elements of B such that $\|u+v\| =\|u-v\|$. The chords of C parallel to v have their midpoints on a straight line through the origin containing a vector $u^*\neq 0$. This is equivalent to $\|u^*+xv\| =\|u^*-xv\|$ for all x. A special such chord is the one joining the points $(u+v)/\|u+v\|$ and $(u-v)/\|u-v\|$ of C. The midpoint of this chord is $u/\|u+v\|$. Consequently u and u^* are parallel. From this it follows that $\|u+xv\|=\|u-xv\|$ for all x. An application of Lemma 2.9 now shows that B is Euclidean, which completes the proof of Theorem 6.5.

Since orthogonality has property (H) if it is homogeneous or additive we also have

Corollary 6.6. *If B is a normed linear space in which orthogonality is homogeneous or additive, then B is Euclidean.*

7. A generalization of the Jordan–von Neumann condition

Jordan and von Neumann [10] proved that a normed linear space B must be Euclidean if it satisfies the condition

$$\|u+v\|^2+\|u-v\|^2=2\,(\|u\|^2+\|v\|^2)\ \text{ for all }\ u,v\in B. \tag{7.1}$$

In analogy with our definition of orthogonality in Section 1, it is natural to ask if this remains true when condition (7.1) is replaced by

$$\sum_{\nu=1}^{m} a_\nu\,\|b_\nu\,u+c_\nu\,v\|^2=0\ \text{ for all }\ u,v\in B, \tag{7.2}$$

where $a_\nu \neq 0$, b_ν, c_ν, $\nu=1.\,\dots\,m$, are real numbers. In order to avoid that (7.2) is trivial, we further assume that the vectors (b_ν, c_ν) and (b_μ, c_μ) are linearly independent for $\nu \neq \mu$. We will show that the question thus raised is to be answered in the affirmative. It is sufficient to assume that B is two-dimensional.

First, as is easily seen, condition (7.2) is equivalent to a condition of the form

$$\sum_{\nu=1}^{r} p_\nu\,\|u+q_\nu\,v\|^2=0\ \text{ for all }\ u,v\in B, \tag{7.3}$$

where $p_\nu \neq 0$, $\nu=1,2,\dots,r$, and $q_\nu \neq q_\mu$ for $\nu\neq\mu$. Further, for every real number λ (7.3) is equivalent to

$$\sum_{\nu=1}^{r} p_\nu\,\|u+(q_\nu+\lambda)\,v\,\|^2=0\ \text{ for all }\ u,v\in B, \tag{7.4}$$

as is seen by replacing u by $u+\lambda v$ in (7.3). Now, for every $u\neq 0$ and v in B we have, using (7.4),

$$\sum_{\nu=1}^{r} p_\nu\,n^{-1}\,[\|n\,u+(q_\nu+\lambda)\,v\|^2-\|n\,u\|^2]=0, \tag{7.5}$$

where n is an arbitrary natural number. Letting n tend to infinity in (7.5) we get

$$\sum_{\nu=1}^{r} p_\nu\,N_+\,(u\,;\,(q_\nu+\lambda)\,v)=0. \tag{7.6}$$

From Lemma 2.6 it now follows that

$$p\,(\lambda)\,N_+\,(u\,;v)+q\,(\lambda)\,N_-\,(u\,;v)=0, \tag{7.7}$$

where we have put

$$p\,(\lambda)=\sum_{\substack{1\leq\nu\leq r\\ q_\nu+\lambda>0}} p_\nu\,(q_\nu+\lambda)\ \text{ and }\ q\,(\lambda)=\sum_{\substack{1\leq\nu\leq r\\ q_\nu+\lambda<0}} p_\nu\,(q_\nu+\lambda).$$

If we put $u=0$, $v\neq 0$ in (7.4) we get $\sum_{\nu=1}^{r} p_\nu (q_\nu+\lambda)^2=0$. In the same way $u\neq 0$, $v=0$ gives $\sum_{\nu=1}^{r} p_\nu=0$ and $u=v\neq 0$ gives $\sum_{\nu=1}^{r} p_\nu (1+q_\nu+\lambda)^2=0$. Hence it follows that $\sum_{\nu=1}^{r} p_\nu (q_\nu+\lambda)=0$ or $p(\lambda)+q(\lambda)=0$. We may now choose λ so that the sequence $q_\nu+\lambda$, $\nu=1, \dots, r$, contains exactly one negative term. For this choice of λ we then have $p(\lambda)=-q(\lambda)\neq 0$. Consequently, (7.7) gives

$$N_+(u;v)=N_-(u;v)$$

which means that the norm of B is Gateaux differentiable.

Now choose λ in (7.4) so that $q_\nu+\lambda=q'_\nu>0$ for $\nu=1, \dots, r$. Let $u\in D$, where D is the set introduced in Lemma 4.2, and let uNv, $\|v\|=1$. Then $\varphi(x) =\|u+xv\|^2$ is a continuously differentiable solution of the equation

$$\sum_{\nu=1}^{r} p_\nu F(q'_\nu x)=0, \quad x>0,$$

where $q'_\nu>0$, $\nu=1, \dots, r$, satisfying

$$\varphi(x)=1+O(x^2) \text{ when } x\to +0$$

and

$$\varphi(x)=x^2+O(x) \text{ when } x\to +\infty.$$

Consequently Corollary 6.2 gives

$$\varphi(x)=1+x^2 \text{ for } x>0.$$

Changing v to $-v$ we see in the same way that

$$\varphi(x)=1+x^2 \text{ for } x<0.$$

As in Theorem 6.5 this means that B is Euclidean. We have thus proved the following result.

Theorem 7.1. *Let $a_\nu\neq 0$, b_ν, c_ν, $\nu=1, \dots, m$, be real numbers such that (b_ν, c_ν) and (b_μ, c_μ) are linearly independent for $\nu\neq\mu$. If B is a normed linear space satisfying the condition*

$$\sum_{\nu=1}^{m} a_\nu \|b_\nu u+c_\nu v\|^2=0 \text{ for } u, v\in B,$$

then B is Euclidean.

REFERENCES

1. Birkhoff, G., Orthogonality in linear metric spaces. Duke Math. J. *1*, 169–172 (1935).
2. Carmichael, R. D., Systems of linear difference equations and expansions in series of exponential functions. Trans. Amer. Math. Soc. *35*, 1–28 (1933).
3. Day, M. M., Some characterizations of inner-product spaces. Trans. Amer. Math. Soc. *62*, 320–337 (1947).

4. FICKEN, F. A., Note on the existence of scalar products in normed linear spaces. Ann. of Math. (2) *45*, 362–366 (1944).
5. FORTET, R,, Remarques sur les espaces uniformément convexes. C. R. Acad. Sci. Paris *210*, 497–499 (1940).
6. FORTET, R., Remarques sur les espaces uniformément convexes. Bull. Soc. Math. France *69*, 23–46 (1941).
7. JAMES, R. C., Orthogonality in normed linear spaces. Duke Math. J. *12*, 291–302 (1945).
8. JAMES, R. C., Orthogonality and linear functionals in normed linear spaces. Trans. Amer. Math. Soc. *61*, 265–292 (1947).
9. JAMES, R. C., Inner products in normed linear spaces. Bull. Amer. Math. Soc. *53*, 559–566 (1947).
10. JORDAN, P., and NEUMANN, J. VON, On inner products in linear metric spaces. Ann. of Math. (2) *36*, 719–723 (1935).
11. KITAGAWA, T., On the theory of linear translatable functional equations and Cauchy's series. Jap. J. of Math. *13*, 233–332 (1936).
12. MOORE, M. G., On expansions in series of exponential functions. Amer. J. Math. *62*, 83–90 (1940).
13. MOORE, M. G., Fundamental systems of solutions for linear difference equations. Duke Math. J. *6*, 652–657 (1940).

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