On the spectral gap for fixed membranes

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Abstract. The distance between the second and first eigenvalues for the Dirichlet Laplacian of a domain is called its (spectral) gap. We show that the gap of a convex planar domain D symmetric about both the x and y axes is no smaller than the gap of an oriented rectangle which contains D.

1. Introduction

In this paper we study the Dirichlet Laplacian in certain bounded planar domains D. We denote the normalized solutions of $-\Delta u = \lambda u$, which vanish at the boundary of D, by ϕ_i^D, $i \geq 0$, and the corresponding eigenvalues by $0 < \lambda_0^D < \lambda_1^D \leq \dots$. The difference $\lambda_1^D - \lambda_0^D$ is called the spectral gap of D. To quote [SWYY], the gap is obviously interesting. It is the rate at which heat kernels in D assume the shape of the ground state eigenfunction. See [Sm] and [D].

van den Berg notes in [Be] that in convex domains (in any dimension) for which the gap can be computed, for example parallelepipeds, the gap always exceeds $3\pi^2/d^2$, where d is the diameter of the domain. Although there are a number of bounds on the gap for convex domains in the literature—sometimes extended to Schrödinger operators, which are not considered here—no one has come even close to $3\pi^2/d^2$. See [Sm], [AB], [L], [SWYY] and [YZ]. Essentially the best that has been done is to show that the gap exceeds π^2/d^2. Here we prove the following.

Theorem 1. *If a bounded domain D is symmetric about both the x and y axes and is convex in both x and y, then the gap of D is no smaller than the gap of the smallest oriented rectangle containing D.*

Here, D convex in x means as usual that horizontal lines intersect D in an interval or not at all, and oriented means the sides are parallel to the coordinate axes. The gap of a rectangle is $3\pi^2/b^2$, where b is the length of its longest side. We remark that dumbbell shaped domains of diameter 1 can have arbitrarily small gaps, so a condition on a domain involving something other than its diameter is necessary

to bound its gap from below. The proof of Theorem 1 is based on an extremal inequality for ratios of heat kernels, an inequality that holds in all dimensions. Despite this, we only prove a poor relation of Theorem 1 in higher dimensions, in the last section of this paper. We remark that the conclusion of Theorem 1 still holds if the hypotheses of convexity in x and y are replaced by the essentially weaker hypotheses that the domain D is convex in x and that the nodal line is the y axis.

2. Preliminaries

The Courant nodal line theorem implies that the second eigenfunction of a bounded planar domain Γ is positive on Γ_1 and negative on Γ_2, where Γ_1 and Γ_2 are those parts of Γ lying on either side of a curve, called the nodal line, which divides the part of Γ not on the nodal line into two connected regions. The first (smallest) eigenvalue of both Γ_1 and Γ_2 is the second eigenvalue of Γ. Second eigenfunctions are in general even less knowable than first ones, since often little can be said about the location of nodal lines. Here we make use of a theorem of Larry Payne [Pa], which implies that if D is as in the statement of Theorem 1 and is strictly convex in one variable, then the nodal line is either the intersection of the x axis with D or the intersection of the y axis with D. This together with both the monotony theorem, which says that if one domain is contained in another then the nth eigenvalue of the smaller domain is no smaller than the nth eigenvalue of the larger for each n, and scaling, show that the truth of Theorem 1 under the additional restriction that D is strictly convex in one variable implies Theorem 1 in full generality. For, given any domain D satisfying the conditions of Theorem 1 and any $\varepsilon > 0$, there is a strictly convex in one variable doubly symmetric domain U contained in D such that $(1+\varepsilon)U$ contains D.

Let $p_t^D(x, y)$ be the heat kernel for D. For any domain U, and every u and v in U, $p_t^U(u, v)e^{\lambda_0^U t}/\phi_0^U(u)\phi_0^U(v)$ approaches 1 as t approaches infinity, as may be seen immediately from the eigenfunction expansion of the heat kernel. Theorem 1 thus follows from the following proposition, upon letting t approach infinity. More specifically, it follows from the truth of this proposition for strictly convex in one variable doubly symmetric domains, which without loss of generality may according to Payne's theorem be assumed to have nodal line along the y axis. We let w^* stand for the reflection of w across the y axis, and if A is a planar set we designate by A^+ those points of A with positive x coordinate.

Proposition 2. *Let F be a bounded domain which is symmetric about the y axis and convex in x, and let R be the smallest oriented rectangle which contains F.*

Let z and w be points in F^+. Then

$$(1) \qquad \frac{p_t^{F^+}(z,w)}{p_t^F(z,w)+p_t^F(z,w^*)} \leq \frac{p_t^{R^+}(z,w)}{p_t^R(z,w)+p_t^R(z,w^*)}, \quad t>0.$$

Proposition 2 will be proved by first proving its analog for a discrete heat kernel, and then proving a local limit theorem connecting the usual heat kernel with the discrete one. Studying partial differential equations by discretizing them, even in a probabilistic setting, goes back to [CFL]. Let L be the two-dimensional integer lattice.

Let X_n, $n \geq 0$, and Y_n, $n \geq 0$, be independent sequences such that both $X_{i+1} - X_i$, $i \geq 0$, and $Y_{i+1} - Y_i$, $i \geq 0$, are sequences of independent identically distributed random variables, each taking on 0, 1, and -1 with probability $\frac{1}{3}$. Let $Z_n = (X_n, Y_n)$. Then we will call Z_n, $n \geq 0$, planar random walk started at Z_0. If Γ is a set of points of L containing x and y, we designate by $q_n^\Gamma(x,y)$ the probability that planar random walk started at x remains in Γ up to time n and equals y at time n. We will prove the following proposition. In the statement of this proposition, *connected* means that it is possible for random walk to get from any point in the set to any other.

Proposition 3. *Let U be a connected subset of L which is symmetric about the y axis and which is convex in x. Let z and w be points in U^+. If U is contained in an oriented rectangle R of points of L which is symmetric about the y axis, and the two denominators in (2) below are positive, and $n > 0$, then*

$$(2) \qquad \frac{q_n^{U^+}(z,w)}{q_n^U(z,w)+q_n^U(z,w^*)} \leq \frac{q_n^{R^+}(z,w)}{q_n^R(z,w)+q_n^R(z,w^*)}.$$

3. Proof of Proposition 3

Let Θ be a connected subset of L. Let $y = (y_0, y_1, \ldots, y_n)$ be a finite sequence of integers such that $y_0 = z_2$, $y_n = w_2$, $|y_i - y_{i-1}| \leq 1$, $1 \leq i \leq n$, and more restrictively such that $P(Y_i = y_i, \ Z_i \in \Theta, \ 0 \leq i \leq n \,|\, Z_0 = z) > 0$, where $z = (z_1, z_2)$. Put

$$q_n^{y,\Theta}(z,w) = P(Z_n = w, \ Z_i \in \Theta, \ 0 \leq i \leq n \,|\, Z_0 = z, \ Y_i = y_i, \ 0 \leq i \leq n).$$

We note that if Θ is an oriented rectangle then $q_n^{y,\Theta}(z,w)$ is the same for all y with the properties listed just above. We will prove (2) by showing that for every sequence y, the expression arrived at upon replacing q_n^A by $q_n^{y,A}$ in (2) for $A = U, U^+, R, R^+$, which we will call conditional (2), holds. Since, as just noted, the right-hand side of conditional (2) does not depend on y, conditional (2) implies (2).

Since the x and y components of Z_n are independent, conditional (2) can and will now be phrased in terms of one-dimensional random walk. Let $S_{i+1}-S_i$, $i\geq 0$, b independent and identically distributed, taking on -1, 0, and 1 each with probability $\frac{1}{3}$. If y is fixed, and $\gamma(k)$ is the largest j such that (j, y_k) belongs to U, then

$$q_n^{y,U}(z,w) = P(S_n = w_1, \ |S_i| \leq \gamma(i), \ 0 \leq i \leq n \,|\, S_0 = z_1),$$

with similar versions for the other subsets of L involved with conditional (2). Fix n and positive integers m_0 and m_1 for the rest of the proof of Proposition 3. If $0 < f(k)$, $0 \leq k \leq n$, $m_0 \leq f(0)$, and $m_1 \leq f(n)$, we put

$$Q(f) := \frac{P_0(S_n = m_1, \ 0 < S_i \leq f(i), \ 0 \leq i \leq n)}{P_0(|S_n| = m_1, \ |S_i| \leq f(i), \ 0 \leq i \leq n)},$$

where the subscripts on P designate $S_0 = m_0$. We always assume that f is such that the probability in the denominator is positive. Conditional (2) is implied by the fact that if $g(i) \leq c$, $0 \leq i \leq n$, ($g \leq c$ for short) for an integer c, then $Q(g) \leq Q(c)$. This in turn is implied by $Q(h) \leq Q(f)$, if $h \leq f$. Thus to prove Proposition 3 it suffices to prove the following lemma.

Lemma 4. *If $0 < h(i) \leq f(i)$, $0 \leq i \leq n$, and if there is an integer $0 < j_0 < n$ such that $f(i) = h(i)$ unless $i = j_0$ in which case $f(j_0) = h(j_0) + 1$, then $Q(h) \leq Q(f)$.*

Proof. Lemma 4 follows from

(3)
$$\begin{aligned} P_0(S_{j_0} = f(j_0) \,|\, S_n = m_1, \ 0 < S_i \leq f(i), \ 0 \leq i \leq n) \\ \geq P_0(|S_{j_0}| = f(j_0) \,|\, |S_n| = m_1, \ |S_i| \leq f(i), \ 0 \leq i \leq n). \end{aligned}$$

For this inequality implies, if we define Q by a quotient as above, that the numerator of $Q(h)$ divided by the numerator of $Q(f)$ is no larger than the denominator of $Q(h)$ divided by the denominator of $Q(f)$, since the paths removed in going, say, from the numerator of $Q(f)$ to the numerator of $Q(h)$ are those which go through $(j_0, f(j_0))$.

Now let a, b, α, and β be integers satisfying $0 \leq a < b \leq n$, $0 < \alpha \leq f(a)$, and $0 < \beta \leq f(b)$. Define the probability measures $P_{a,b}^{\alpha,\beta}$ on the set of all finite sequences $(i_a, i_{a+1}, \ldots, i_b)$ of integers by

$$P_{a,b}^{\alpha,\beta}(i_a, i_{a+1}, \ldots, i_b) = P(S_k = i_k, \ a \leq k \leq b \,|\, S_a = \alpha, \ S_b = \beta, \ 0 < S_i \leq f(i), \ a \leq i \leq b).$$

Let π_j be the coordinate maps: $\pi_j(i_a, i_{a+1}, \ldots, i_b) = i_j$. Under $P_{a,b}^{\alpha,\beta}$, the finite sequence of random variables $\pi_a, \pi_{a+1}, \ldots, \pi_b$ is a Markov chain started at α with (nonstationary) transition probabilities, which do not depend on α, given by

$$P_{a,b}^{\alpha,\beta}(\pi_{k+1} = j \,|\, \pi_k = i) = \frac{q_j^{k+1}}{\sum_{s=i-1}^{i+1} q_s^{k+1}} \quad \text{for } j = i-1, i, i+1,$$

where
$$q_s^{k+1} = P(S_b = \beta, \ 0 < S_y \le f(y), \ k+1 \le y \le b \mid S_{k+1} = s).$$

Let $\lambda_a, \ \lambda_{a+1}, \dots, \ \lambda_b = \lambda$ stand for a Markov chain which moves according to these transition probabilities. If η is a probability distribution on $\{1, 2, \dots, f(a)\}$, we let P_η denote probabilities for λ given that its initial distribution, that is the distribution of λ_a, is η.

We will prove that if μ and ν are probability distributions on $\{1, \dots, f(a)\}$, then

$$(4) \quad \nu(x, \infty) \le \mu(x, \infty), \ x \ge 0, \quad \Longrightarrow \quad P_\nu(\pi_k \ge x) \le P_\mu(\pi_k \ge x), \ x \ge 0, \ a \le k \le b.$$

We prove (4) by induction on k. The inductive step is the same for all k, so we just prove the special case of (4), in which $a \le k \le b$ is replaced by $k = a+1$. There is a finite sequence η_i, $1 \le i \le n$, of probability distributions on $\{1, \dots, f(a)\}$ such that $\eta_1 = \nu$, $\eta_n = \mu$, $\eta_{i-1}(x, \infty) \le \eta_i(x, \infty)$, $x \ge 0$, $2 \le i \le n$, and $\eta_{i-1}\{j\} = \eta_i\{j\}$ for every integer j except perhaps for two consecutive j. Thus our special case of (4) follows from the still more special case where $\nu\{j\} = \mu\{j\}$ for all but perhaps two consecutive j, and the proof of this special case easily reduces to the (even more) special case where $\nu\{\theta\} = \mu\{\theta+1\} = 1$ for some integer θ. This final case is easy because of the particular form of the transition probabilities of λ, given above. We remark that just the fact that λ makes no jumps of magnitude exceeding 1 is not sufficient information to imply the conclusion desired here.

We need the following inequalities. If $0 < \alpha_0 \le \alpha_1 \le f(a)$ and $0 < \beta_0 \le \beta_1 \le f(b)$, then

$$(5) \qquad P_{a,b}^{\alpha_0, \beta_0}(\pi_j = f(j)) \le P_{a,b}^{\alpha_1, \beta_1}(\pi_j = f(j)), \quad a \le j \le b.$$

To see (5), note that the special case $\beta_0 = \beta_1$ follows immediately from (4). Also, since everything is reversible (we are just counting paths), the special case $\alpha_0 = \alpha_1$ also follows immediately from (4). And (5) in its entirety follows from these two special cases.

Now we use (5) to complete the proof of Lemma 4 and thus of Proposition 3. Let p_0 be the conditional probability on the left-hand side of (3). We will prove (3) by exhibiting a finite partition Δ of $\Gamma := \{S_0 = m_0, \ |S_n| = m_1, \ |S_i| \le f(i), \ 0 \le i \le n\}$ such that

$$P(|S_{j_0}| = f(j_0) \mid A) \le p_0, \ A \in \Delta.$$

Let $M, \ M+1, \dots, \ N$ be the longest string of consecutive integers, if it exists, such that both $S_k \ne 0$, $M \le k \le N$, and $M \le j_0 \le N$. Let Q be the event that there is no such interval, that is, that $S_{j_0} = 0$, and for each $i \in \{0, \dots, j_0\}$ and $k \in \{j_0, \dots, n\}$, let $Q_{i,k} = \{M = i, \ N = k\}$. Clearly, $0 = P(|S_{j_0}| = f(j_0) \mid Q) \le p_0$.

Furthermore, $P(|S_{j_0}|=f(j_0)|Q_{0,n})=p_0$, since $Q_{0,n}=\{0<S_j\leq f(j),\ 0\leq j\leq n,\ S_0=m_0,\ S_n=m_1\}$. Let $0\leq i\leq j_0\leq k\leq n$. Then $Q_{i,k}$ is the set of all the paths of Γ such that $|S_i|=1$, $|S_k|=1$, $|S_j|\geq 1$, $i\leq j\leq k$. Thus the conditional distribution of $(|S_j|,\ i\leq j\leq k)$, given $Q_{i,k}$ is exactly $P_{i,k}^{1,1}$. Again, we are just counting paths here. On the other hand, the conditional distribution of $(S_j,\ i\leq j\leq k)$, given $\{S_0=m_0,\ S_n=m_1,\ 0<S_i\leq f(i),\ 0\leq i\leq n\}$, is a mixture of the distributions $P_{i,k}^{s,t}$ over all the integers s in $[1,f(i)]$ and t in $[1,f(k)]$, which follows by further conditioning on S_i and S_k. Thus (5), or more precisely its analog where a and b are replaced by i and j, implies $P(|S_{j_0}|=f(j_0)|Q_{i,j})\leq p_0$. This completes the proof of Lemma 4 and thus of Proposition 3.

4. Derivation of Proposition 2 from Proposition 3

Throughout this section random walk $Z_n=(X_n,Y_n)$, $n\geq 0$, will be assumed to start at the origin. We put $\phi_\theta(x)=(2\pi\theta)^{-1/2}\exp(-x^2/2\theta)$, $\theta>0$, and $\lambda_\theta(x_1,x_2)=\phi_\theta(x_1)\phi_\theta(x_2)$. The classical local limit theorem implies that if $M>0$, both

$$(6) \qquad \max_{\{|i|,|j|<M\sqrt{n}\}}\left|\frac{P(Z_n=(i,j))}{\lambda_{2n/3}(i,j)}-1\right|\to 0, \quad \text{as } n\to\infty,$$

and

$$(7) \qquad \max_{\{|i|<M\sqrt{n}\}}\left|\frac{P(X_n=i)}{\phi_{2n/3}(i)}-1\right|\to 0, \quad \text{as } n\to\infty,$$

where of course i and j stand for integers. See [DM] for a discussion of local limit theorems. In addition, the following holds.

Lemma 5. *Let $\eta>0$. Then*

$$\sup\{nP(Z_k=(i,j)):k\leq\theta n,\ |i|\geq\eta\sqrt{n}\ \text{ or }\ |j|\geq\eta\sqrt{n}\}\to 0, \quad \text{as } \theta\to 0,\ n\to\infty.$$

Proof. Since $P(X_k=i)$ is symmetric in i and nonincreasing as i increases from 0, we have both

$$(8) \qquad \max\{P(Z_k=(i,j)):|i|\geq m\ \text{or}\ |j|\geq m\}=P(X_k=m)P(Y_k=0)$$

and

$$(9) \qquad P(X_k=m)\leq\frac{2P\left(X_k\geq\frac{1}{3}m\right)}{m}$$

for every positive integer m. Now by (7) we have

$$(10) \qquad \sup_{k>0} P(Y_k = 0)\sqrt{k} = c < \infty.$$

Furthermore, using Hoeffding's large deviation inequality (see [Po]), we have

$$(11) \qquad P(X_k \geq j) \leq \exp(-j^2/2k), \quad j \geq 0.$$

Together (8)–(11) give that if $\theta \leq \frac{1}{9}\eta^2$, then

$$\sup\{nP(Z_k = (i,j)) : |i| > \eta\sqrt{n} \text{ or } |j| > \eta\sqrt{n}, \ 1 \leq k \leq \theta n\}$$

$$\leq \sup\{nP(X_k \geq \tfrac{1}{3}\eta\sqrt{n})2\eta^{-1}n^{-1/2}ck^{-1/2} : 1 \leq k \leq \theta n\}$$

$$\leq \sup\{n\exp\left(-\eta^2\frac{n}{18k}\right)2\eta^{-1}n^{-1/2}ck^{-1/2} : 1 \leq k \leq \theta n\}.$$

Calculus shows that if $\theta \leq \frac{1}{9}\eta^2$ this last expression is maximized for $k = \theta n$, from which Lemma 5 follows easily.

Now let $B_t = (C_t, D_t)$, $t \geq 0$, be standard two-dimensional Brownian motion started at 0. Let G be a bounded domain convex in both x and y which contains 0, and in this section let $p_t(z, w)$ be the heat kernel for G. Let $\tau = \inf\{t : B_t \notin G\}$. For any Borel set A,

$$\int_A p_t(0, z)\,dz = P(B_t \in A, t < \tau).$$

Also, since G is simply connected

$$(12) \qquad P(\tau < t, \ B_s \in \bar{G}, \ 0 \leq s \leq t) = 0, \quad t > 0,$$

where the $^-$ denotes closure. See [DY]. Put $Z_j^n = n^{-1/2}Z_j$, $j \geq 0$, and let ν_j^n be the distribution of Z_j^n. There is a sequence of random vectors $(C_{s_i^n}, D_{t_i^n}) := W_i^n$, $i \geq 0$, such that Z_j^n, $j \geq 0$, and W_j^n, $j \geq 0$, have the same distribution for each n, and

$$(13) \qquad \max_{0 \leq i \leq 3n/2} |B_{2i/3n} - W_i^n| \to 0 \quad \text{in distribution, as } n \to \infty.$$

Here and from now on let n be an even integer so that $\frac{3}{2}n$ is an integer. We may take s_i^n to be the times for the standard Skorohod scheme (see [Br]) embedding of the distribution of X_j^n, $j \geq 0$, in the one-dimensional Brownian motion C_t, and take t_i^n to be the independent Skorohod scheme for the embedding of this same distribution in D_t. It is routine to prove (13). We remark that we do not need the full strength of (13) in what follows, only invariance, although (13) makes things conceptually a little easier.

Let $\tau_n = \inf\{j : Z_j^n \notin G\}$, and let γ_j^n be the distribution of $(Z_j^n; j < \tau_n)$. Then (12) and (13) imply that if $\alpha > 0$, and if q_n is a sequence of integers such that $q_n/\frac{3}{2}n \to \alpha$, as $n \to \infty$, then if f is a bounded and continuous function on the plane,

$$(14) \qquad \int f\,d\gamma_{q_n}^n \to \int f(z)P_\alpha(0, z)\,dz, \quad \text{as } n \to \infty.$$

Lemma 6. *We have $p_1(0,0) = \lim_{n\to\infty} n\gamma^n_{3n/2}(0)$.*

Proof. Let $\varepsilon > 0$. Let S be an oriented square centered at 0 and contained in G, and T be a number in $(0.9, 1)$, such that

$$(15) \qquad |p_t(0, z) - p_1(0, 0)| < \varepsilon, \quad \text{if } z \in S \text{ and } T \le t \le 1.$$

Let Γ be a number in $[T, 1)$ such that

$$(16) \qquad \int_S \lambda_{1-\Gamma}(z)\, dz \ge 1 - \varepsilon,$$

and which in addition has the property that if Γ_n is the smallest integer such that $\Gamma_n > \frac{3}{2}n\Gamma$, and $\Theta_n = \frac{3}{2}n - \Gamma_n$, then

$$(17) \qquad \nu^n_k(l) < \frac{\varepsilon}{n}, \quad \text{if } l \notin S \text{ and } k \le \Theta_n.$$

This is possible by Lemma 5. Now, if $*$ denotes convolution,

$$(18) \qquad \gamma^n_{3n/2}(0) = \gamma^n_{\Gamma_n} * \nu^n_{\Theta_n}(0) - \sum_{l \notin G} \sum_{1 \le k \le \Theta_n} P\big(\tau_n = \tfrac{3}{2}n - k,\ Z^n_{3n/2-k} = l\big)\nu_k(-l).$$

By (17), the double sum of (18) is smaller than εn^{-1}. In addition (14) and (6) give

$$n\gamma^n_{\Gamma_n} * \nu^n_{\Theta_n}(0) \to \int_G p_\Gamma(0, z)\lambda_{1-\Gamma}(z)\, dz, \quad \text{as } n \to \infty.$$

We may and do assume that $\Gamma > \frac{1}{2}$, so that $p_\Gamma(0, z) < \lambda_\Gamma(z) < 1$. Upon writing $\int_G p_\Gamma(0, z)\lambda_{1-\Gamma}(z)\, dz$ as a sum of integrals over S and S^c, and using this last inequality together with (15) and (16), we get

$$(1-\varepsilon)^2 p_1(0, 0) - \varepsilon < n\gamma^n_{\Gamma_n} * \nu^n_{\Theta_n}(0) < p_1(0, 0) + 2\varepsilon.$$

Together with (18) and the comment just after (18), this establishes Lemma 6.

An essentially identical proof will establish the following lemma.

Lemma 7. *Let u and v be points of G and suppose there is an integer $n > 0$ such that $u - v$ is in the lattice L/n. Let $\Gamma^n_j = u + n^{-1/2}Z_j$, $j \ge 0$, and let $\xi_n = \inf\{j : \Gamma^n_j \notin G\}$. Let ϱ^n_j be the distribution of $(\Gamma^n_j; j < \xi_n)$. Then*

$$p_1(u, v) = \lim_{k\to\infty} 6k^2 n^2 \varrho^{6k^2 n^2}_{18k^2 n^2/2}(v).$$

Together, Proposition 3 and Lemma 7 prove Proposition 2 in the special case that $t = 1$ and both $z - w$ and $z - w^*$ lie in one of the lattices L/k, for some integer k. Clearly the $t = 1$ restriction can be removed. And since the union of all the lattices is dense in the plane, Proposition 2 in its entirety follows easily.

5. Higher dimensions

The analogs of Proposition 2 in all dimensions hold. We state this as a proposition. Points in $\mathbf{R}^n$ are denoted $(x_1, x_2, \ldots, x_n) = x$. The notation A^+ stands for $A \cap \{x_1 > 0\}$, and x^* stands for the reflection of x across $\{x_1 = 0\}$. By an oriented n-dimensional rectangle we mean the product of n intervals.

Proposition 8. *Let $n > 2$. If "rectangle" is replaced by "n-dimensional rectangle", and "y axis" is replaced by "$\{x_1 = 0\}$", and "convex in x" is replaced by "convex in x_1" in the statement of Proposition 2, the resulting statement is still true.*

The proof of Proposition 8 is almost identical to that of Proposition 2. We let x_i^1, $i \geq 0$, x_i^2, $i \geq 0, \ldots$, x_i^n, $i \geq 0$, be n independent one-dimensional random walks. The analog of Proposition 3 holds, and is proved by conditioning on $(x_i^2, x_i^3, \ldots, x_i^n)$, $i \geq 0$. What we need to extend Theorem 1 to higher dimensions is, evidently, an extension of Payne's theorem to higher dimensions, an extension which has not yet been shown to hold, although it seems quite likely to be true. What we are left with is the following.

Theorem 2. *Let $D \in \mathbf{R}^n$ be convex in x_1 and symmetric in x_1. Let R be the smallest oriented n-dimensional rectangle centered at 0 which contains D. Then $\lambda_0^{D^+} - \lambda_0^D$ is at least as large as the gap of R.*

References

[AB]   ASHBAUGH, M. S. and BENGURIA, R., Optimal lower bounds for eigenvalue gaps for Schrödinger operators with symmetric single-well potentials and related results, in *Maximum Principles and Eigenvalue Problems in Partial Differential Equations (Knoxville, Tenn., 1987)* (Schaefer, P. W., ed.), pp. 134–145, Longman, Harlow, 1988.

[BM]   BAÑUELOS, R. and MENDEZ, P., Sharp inequalities for heat kernels of Schrödinger operators and applications to spectral gaps, *J. Funct. Anal.* **176** (2000), 368–399.

[Be]   VAN DEN BERG, M., On condensation in the free-boson gas and the spectrum of the Laplacian, *J. Statist. Phys.* **31** (1983), 623–637.

[Br]   BREIMAN, L., *Probability*, Addison-Wesley, Reading, Mass., 1969.

[C]   CHENG, S. Y., Eigenvalue comparison theorems and its geometric applications, *Math. Z.* **143** (1975), 289–297.

[CFL]   COURANT, R., FRIEDRICHS, K. and LEWY, H., On the partial difference equations of mathematical physics, *Math. Ann.* **100** (1928), 32–72.

[D]   DAVIES, E. B., *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, Cambridge, 1989.

[DM]   DAVIS, B. and MCDONALD, D., An elementary proof of the local central limit theorem, *J. Theoret. Probab.* **24** (1995), 693–701.

[DY]   DYNKIN, E. B. and YUSKEVITCH, A., *Markov Processes, Theorems and Problems*, Plenum Press, New York, 1969.

[KS]   KIRSCH, W. and SIMON, B., Comparison theorems for the gap of Schrödinger operators, *J. Funct. Anal.* **73** (1987), 396–410.

[L]   LING, J., A lower bound for the gap between the first two eigenvalues of Schrödinger operators on convex domains in S^n or $\mathbf{R}^n$, *Michigan Math. J.* **40** (1993), 259–270.

[Pa]   PAYNE, L., On two conjectures in the fixed membrane eigenvalue problem, *J. Appl. Math. Phys.* **24** (1973), 720–729.

[Po]   POLLARD, D., *Convergence of Stochastic Processes*, Springer-Verlag, New York, 1984.

[Sa]   SALOFF-COSTE, L., Precise estimates on the rate at which certain diffusions tend to equilibrium, *Math. Z.* **217** (1994), 641–677.

[SWYY]   SINGER, I. M., WONG, B., YAU, S. T. and YAU, S. S. T., An estimate of the gap of the first two eigenvalues in the Schrödinger operator, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **12** (1985), 319–333.

[Sm]   SMITS, R., Spectral gaps and rates to equilibrium in convex domains, *Michigan Math. J.* **43** (1996), 141–157.

[YZ]   YU, Q. and ZHONG, J. Q., Lower bounds of the gap between the first and second eigenvalues of the Schrödinger operator, *Trans. Amer. Math. Soc.* **294** (1986), 341–349.

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