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Jensen measures and boundary values of plurisubharmonic functions

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Abstract. We study different classes of Jensen measures for plurisubharmonic functions, in particular the relation between Jensen measures for continuous functions and Jensen measures for upper bounded functions. We prove an approximation theorem for plurisubharmonic functions in B-regular domain. This theorem implies that the two classes of Jensen measures coincide in B-regular domains. Conversely we show that if Jensen measures for continuous functions are the same as Jensen measures for upper bounded functions and the domain is hyperconvex, the domain satisfies the same approximation theorem as above.

The paper also contains a characterisation in terms of Jensen measures of those continuous functions that are boundary values of a continuous plurisubharmonic function.

1. Introduction

If Ω is a bounded domain in $\mathbf{C}^n$, we will use $\mathcal{PSH}^c(\Omega)$ to denote the set of plurisubharmonic functions on Ω which are continuous on $\bar{\Omega}$ as functions into the extended real line $[-\infty, \infty)$.

Let u be a real-valued upper bounded function on the bounded domain Ω. We define $u^*: \bar{\Omega} \to \mathbf{R}$ as the upper semi-continuous regularisation of u, i.e. if $z \in \bar{\Omega}$,

$$u^*(z) = \varlimsup_{\Omega \ni \zeta \to z} u(\zeta).$$

If u is plurisubharmonic on Ω, then $u^* = u$ on Ω, and it is reasonable to call $u^*|_{\partial\Omega}$ the boundary values of u.

Definition 1.1. Let Ω be a bounded domain in $\mathbf{C}^n$, and let μ be a positive, regular Borel measure on $\bar{\Omega}$. We say that μ is a *Jensen measure* with *barycentre* $z \in \bar{\Omega}$ for continuous plurisubharmonic functions, if

$$u(z) \le \int_{\bar{\Omega}} u \, d\mu$$

for every function $u \in \mathcal{PSH}^c(\Omega)$. We denote by $\mathcal{J}_z^c$ the set of Jensen measures for continuous plurisubharmonic functions having barycentre z. Similarly, if

$$u^*(z) \leq \int_{\bar{\Omega}} u^* \, d\mu$$

for every upper bounded function $u \in \mathcal{PSH}(\Omega)$, we say that μ is a Jensen measure with barycentre z for upper bounded plurisubharmonic functions. We write $\mathcal{J}_z$ for the set of all such measures. Clearly, $\mathcal{J}_z \subset \mathcal{J}_z^c$.

This paper is devoted to studying the relation between $\mathcal{J}$ and $\mathcal{J}^c$. In Section 4 we prove that on B-regular domains, upper bounded plurisubharmonic functions can be approximated from above on the closure of the domain using functions in $\mathcal{PSH}^c$. This implies that for a B-regular domain Ω, $\mathcal{J}_z = \mathcal{J}_z^c$ for all $z \in \bar{\Omega}$. Conversely, if Ω is a bounded hyperconvex domain such that $\mathcal{J}_z = \mathcal{J}_z^c$ for all z, Ω satisfies the above mentioned approximation property. At this point, it is unknown to the author whether $\mathcal{J} = \mathcal{J}^c$ holds for every hyperconvex domain. We give an example showing that this equality is not valid for every pseudoconvex domain.

In Section 3 we give an exact characterisation of those continuous functions on $\partial\Omega$, Ω being a bounded domain, that can be extended to a function in $\mathcal{PSH}^c(\Omega)$. If Ω is hyperconvex, the necessary and sufficient condition on $\phi \in C(\partial\Omega)$ for this to hold is that

$$\phi(z) = \inf\left\{\int_{\partial\Omega} \phi \, d\mu : \mu \in \mathcal{J}_z^c\right\}$$

for every $z \in \partial\Omega$. As an easy corollary of this, we show that $\mathcal{PSH}^c(\Omega)|_{\partial\Omega}$ is uniformly closed if Ω is hyperconvex.

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2. The basic duality theorem

The main reason for introducing Jensen measures is that upper envelopes of plurisubharmonic functions can be expressed as lower envelopes of integrals with respect to Jensen measures. This section is devoted to a proof of this result, which goes back to Edwards [6]. The result is little more than a thinly disguised version of the Hahn–Banach theorem, but for convenience we develop the necessary ideas here. This section closely follows the presentation in Chapter III of the monograph by Cegrell [3].

Let X be a compact metric space, and let $\mathcal{F}$ be a cone of upper bounded, upper semicontinuous functions on X containing all the constants. If g is a real-valued function on X, we define

$$Sg(z) = \sup\{u(z) : u \in \mathcal{F},\ u \le g\}.$$

Let $z \in X$ and define a class of positive measures by

$$M_z^{\mathcal{F}} = \left\{\mu : u(z) \le \int_X u\, d\mu \text{ for all } u \in \mathcal{F}\right\}.$$

It is not difficult to verify that $M_z^{\mathcal{F}}$ is a convex, weak-* compact set. If g is a bounded function on X, we define $Ig(z)=\inf\{\int_X g\, d\mu : \mu \in M_z^{\mathcal{F}}\}$. Note that every measure in $M_z^{\mathcal{F}}$ is a probability measure.

Theorem 2.1. (Edwards' theorem) *With $\mathcal{F}$ as above, and if g is a bounded Borel function on X, then $Sg(z) \le Ig(z)$. If g is lower semicontinuous, then $Sg=Ig$.*

Proof. For the inequality $Sg \le Ig$, just note that if $u \in \mathcal{F}$, $u \le g$, and $\mu \in M_z^{\mathcal{F}}$ is arbitrary then

$$u(z) \le \int_X u\, d\mu \le \int_X g\, d\mu.$$

Hence $Sg(z) \le Ig(z)$. For the second part, first assume that $g \in C(X)$. Also, without loss of generality, we may assume that $g \le 0$. The functional S satisfies the following properties:

(i) $S(\alpha g) = \alpha Sg$, $\alpha \ge 0$;

(ii) $S(g_1+g_2) \ge S(g_1)+S(g_2)$;

(iii) if $g_1 \le g_2 \le 0$, then $S(g_1) \le S(g_2)$.

Take any $z \in X$. By the Hahn–Banach theorem and Riesz' representation theorem, we can find a (real) measure s on X, such that $\int_X g\, ds = Sg(z)$ and $\int_X \phi\, ds \ge S\phi(z)$ for every $\phi \in C(X)$.

Clearly, if $\phi \ge 0$, $\int_X \phi\, ds \ge S\phi \ge 0$, and hence s is a positive measure. Now take any $u \in \mathcal{F}$. Since u is upper semicontinuous and upper bounded, we can find a decreasing sequence $u_j \in C(X)$ such that $u_j \searrow u$. Then

$$\int_X u\, ds = \lim_{j\to\infty} \int_X u_j\, ds \ge \lim_{j\to\infty} Su_j(z) \ge Su(z) \ge u(z).$$

Hence $s \in M_z^{\mathcal{F}}$ and thus $Ig(z) = Sg(z)$.

If g is lower semicontinuous, take a sequence $g_j \in C(X)$, such that $g_j \nearrow g$. Then, for every $\varepsilon>0$, for every j we can find μ_j, such that for every fixed k,

$$
\begin{aligned}
Sg(z) &\geq \lim_{j\to\infty} Sg_j(z) = \lim_{j\to\infty} Ig_j(z) \geq \lim_{j\to\infty} \int_X g_j\, d\mu_j - \varepsilon \\
&\geq \lim_{j\to\infty} \int_X g_k\, d\mu_j - \varepsilon = \int_X g_k\, d\mu - \varepsilon,
\end{aligned} \tag{2.1}
$$

where μ is a weak-* limit of μ_j. Letting $k\to\infty$, we get $Sg(z)\geq\int_X g\,d\mu-\varepsilon$, and hence $Sg(z)\geq Ig(z)-\varepsilon$. But $\varepsilon>0$ was arbitrary, and it follows that $Sg(z)\geq Ig(z)$. □

It is straightforward to verify that $\mathcal{PSH}$ and $\mathcal{PSH}^c$ are cones satisfying the conditions in Theorem 2.1, hence we obtain the following corollary.

Corollary 2.2. *Let Ω be a bounded domain in $\mathbf{C}^n$, and let ϕ be a real-valued lower semicontinuous function on $\bar\Omega$. Then, for every $z\in\bar\Omega$,*

$$\sup\{u^*(z) : u\in\mathcal{PSH}(\Omega),\ u^*\leq\phi\} = \inf\left\{\int_{\bar\Omega}\phi\,d\mu : \mu\in\mathcal{J}_z\right\},$$

and

$$\sup\{u(z) : u\in\mathcal{PSH}^c(\Omega),\ u\leq\phi\} = \inf\left\{\int_{\bar\Omega}\phi\,d\mu : \mu\in\mathcal{J}_z^c\right\}.$$

Remark. Poletsky has in a series of papers studied similar methods of constructing plurisubharmonic functions as lower envelopes of "disc functionals" [10], [11]. His methods have recently been expanded and generalised by Lárusson and Sigurdsson [9]. Their approach shows that if ϕ is *upper* semicontinuous on Ω, then

$$\sup\{u(z) : u\in\mathcal{PSH}(\Omega),\ u\leq\phi\} = \inf\left\{\int_{\Omega}\phi\,d\mu : \mu\in\mathcal{J}_z\right\}.$$

In fact, Poletsky showed that it is enough to take the infimum over Jensen measures that are push-forwards of the Lebesgue measure on the circle under closed analytic discs. His approach, however, does not allow for boundary values in the same way as Edwards' theorem does.

Note that we cannot in general expect Edwards' theorem to hold for upper semicontinuous functions ϕ. For example, if $\mathcal{F}$ only contains continuous functions, and ϕ is a discontinuous function which is the pointwise limit of a decreasing sequence $\{\phi_n\}$ of functions in $\mathcal{F}$, then clearly $I\phi_n=\phi_n$ for all n, and hence $I\phi=\phi$. On the other hand, $S\phi$ is a supremum of a family of continuous functions, so $S\phi$ is lower semicontinuous. Hence, we cannot expect that $S\phi=I\phi$ if $\mathcal{F}$ only contains continuous functions and ϕ is not lower semicontinuous.

3. Boundary values of plurisubharmonic functions

Often in pluripotential theory, hyperconvex domains is the natural class of domains to study.

Definition 3.1. Let Ω be a domain in $\mathbf{C}^n$. We say that Ω is *hyperconvex* if there exists a negative plurisubharmonic function $h \in \mathcal{PSH}(\Omega)$, such that for every $\varepsilon > 0$, the set $\{z \in \Omega : h(z) < -\varepsilon\}$ is relatively compact in Ω. Such a function is called a bounded plurisubharmonic exhaustion function for Ω.

If Ω is hyperconvex, it is always possible to find a bounded plurisubharmonic exhaustion function h for Ω, which is continuous on $\bar{\Omega}$. In fact, it is even possible to take $h \in \mathcal{PSH}(\Omega) \cap C^\infty(\Omega)$. (See Błocki [1] for details.) Clearly, every hyperconvex domain is pseudoconvex, and every pseudoconvex domain with Lipschitz boundary [5] is hyperconvex.

Even if Ω is a hyperconvex domain, it can happen that some continuous functions on $\partial\Omega$ are not the boundary values of any plurisubharmonic function. Take for example the (unit) bidisc in $\mathbf{C}^2$ and let ϕ be a continuous function on $\partial\Delta^2$, such that $\phi|_{\partial\Delta \times \partial\Delta} \equiv 0$, and $\phi(0,1) = 1$. The maximum principle shows that ϕ is not the boundary values of a plurisubharmonic function. Using Jensen measures and the duality results from Section 2, it is possible to give an exact characterisation of the functions on $\partial\Omega$ that are boundary values of plurisubharmonic functions.

Lemma 3.2. *Let Ω be a bounded domain in $\mathbf{C}^n$. Let $\{z_j\} \subset \bar{\Omega}$ be a sequence of points converging to z. For each j, let $\mu_j \in \mathcal{J}_{z_j}$. Then there is a subsequence μ_{j_k} and a measure $\mu \in \mathcal{J}_z^c$, such that μ_{j_k} converges weak-* to μ.*

Proof. With the help of the Banach–Alaoglu theorem, by passing to a subsequence, we may assume that μ_j converges to some probability measure μ supported on $\bar{\Omega}$. We claim that $\mu \in \mathcal{J}_z^c$, since if $u \in \mathcal{PSH}^c(\Omega)$, then

$$\int_{\bar{\Omega}} u \, d\mu = \lim_{j\to\infty} \int_{\bar{\Omega}} u \, d\mu_j \geq \lim_{j\to\infty} u(z_j) = u(z).$$

This shows that $\mu \in \mathcal{J}_z^c$. □

Remark. In general, it is not true that $\mu \in \mathcal{J}_z$. See Example 4.6.

Lemma 3.3. *Let Ω be a bounded domain in $\mathbf{C}^n$ and let $\phi \in C(\partial\Omega)$. Then there exists a function $u \in \mathcal{PSH}^c(\Omega)$ such that $u|_{\partial\Omega} = \phi$ if and only if there exists a continuous extension of ϕ (also denoted ϕ) to $\bar{\Omega}$ such that*

$$\phi(z) = \inf\left\{ \int_{\bar{\Omega}} \phi \, d\mu : \mu \in \mathcal{J}_z^c \right\} \tag{3.1}$$

for every $z\in\partial\Omega$.

Proof. Assume that $\phi=u|_{\partial\Omega}$ for some $u\in\mathcal{PSH}^c(\Omega)$. Take $z\in\partial\Omega$ and let $\mu\in\mathcal{J}_z^c$. Then

$$\phi(z)=u(z)\leq\int_{\bar\Omega}u\,d\mu$$

which implies that $\phi(z)\leq\inf\{\int_{\bar\Omega}u\,d\mu:\mu\in\mathcal{J}_z^c\}$. Taking $\mu=\delta_z$ shows that this inequality is in fact an equality. Hence u is a continuous extension of ϕ to $\bar\Omega$ satisfying (3.1).

Conversely, extend ϕ to a continuous function on $\bar\Omega$, satisfying (3.1) and let $S\phi=\sup\{u(z):u\in\mathcal{PSH}(\Omega),\ u^*\leq\phi\}$. Edwards' theorem implies that

$$S\phi(z)=\inf\left\{\int_{\bar\Omega}\phi\,d\mu:\mu\in\mathcal{J}_z\right\}.$$

Assume that $\underline{\lim}_{\zeta\to z}S\phi(\zeta)<\phi(z)$ for some $z\in\partial\Omega$. Then we can find $\varepsilon>0$ and a sequence $\zeta_j\to z$ such that $S\phi(\zeta_j)<\phi(z)-\varepsilon$ for every j. Hence, there is a measure $\mu_j\in\mathcal{J}_{\zeta_j}$ such that $\int_{\bar\Omega}\phi\,d\mu_j<\phi(z)-\varepsilon$. By passing to a subsequence and using Lemma 3.2, we can assume that μ_j converges weak-* to some $\mu\in\mathcal{J}_z^c$. Hence

$$\int_{\bar\Omega}\phi\,d\mu=\lim_{j\to\infty}\int_{\bar\Omega}\phi\,d\mu_j\leq\phi(z)-\varepsilon.$$

This contradicts the assumption that $\phi(z)=\inf\{\int_{\bar\Omega}\phi\,d\mu:\mu\in\mathcal{J}_z^c\}$. Therefore, we have $\underline{\lim}_{\zeta\to z}S\phi(\zeta)\geq\phi(\zeta)$. Clearly, since ϕ is continuous, $\overline{\lim}_{\zeta\to z}S\phi(\zeta)\leq\phi(z)$. Hence $(S\phi)_*=(S\phi)^*=\phi$ on $\partial\Omega$. By a theorem of Walsh [13], $(S\phi)^*$ is a continuous plurisubharmonic function with boundary value ϕ. □

If we assume is addition that the domain is hyperconvex, then the situation is more satisfactory, since we do not require an extension of the boundary function. To prove this, we will require (a part of) a theorem from [2].

Theorem 3.4. *Let* Ω *be a bounded domain in* $\mathbf{C}^n$. *Then* Ω *is hyperconvex if and only if, for every* $z\in\partial\Omega$, *every Jensen measure* $\mu\in\mathcal{J}_z^c$ *is supported on* $\partial\Omega$.

Proof. Let h be a continuous bounded plurisubharmonic exhaustion function for Ω. Let $z\in\partial\Omega$ and take any $\mu\in\mathcal{J}_z^c$. Then

$$0=h(z)\leq\int_{\bar\Omega}h\,d\mu.$$

But, $h\leq0$ on $\bar\Omega$ and μ is a positive measure, thus $h=0$ μ-a.e. Since $h<0$ in Ω, this implies that μ is supported on $\partial\Omega$. For the converse, we refer to [2]. □

Theorem 3.5. *Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$ and let $\phi\in C(\partial\Omega)$. Then there exists a function $u\in\mathcal{PSH}^c(\Omega)$ such that $u|_{\partial\Omega}=\phi$ if and only if*

$$\phi(z)=\inf\left\{\int_{\partial\Omega}\phi\,d\mu:\mu\in\mathcal{J}_z^c\right\} \tag{3.2}$$

for every $z\in\partial\Omega$.

Proof. If $z\in\partial\Omega$ and $\mu\in\mathcal{J}_z^c$, by Theorem 3.4, μ is supported on $\partial\Omega$. Hence the integral $\int_{\bar\Omega}\phi\,d\mu$ only depends on the values of ϕ on $\partial\Omega$, and the theorem follows from Lemma 3.3. □

Example 3.6. As a straight-forward consequence of Theorem 3.5 we may conclude that $\phi\in C(\partial\Delta^2)$ extends to a continuous plurisubharmonic function on Δ^2 if and only if ϕ is subharmonic on each analytic disc in the boundary, that is if and only if the functions $\zeta\mapsto\phi(e^{i\theta},\zeta)$ and $\zeta\mapsto\phi(\zeta,e^{i\theta})$ are subharmonic in ζ for every real θ.

It is easy to see that the subharmonicity of the slice functions is a necessary condition. Assume that $u\in\mathcal{PSH}^c(\Delta^2)$ is an extension of ϕ. Then $u_r=u(rz_1,rz_2)$ converges uniformly to u on the closed bidisc as $r\nearrow 1$, and each u_r is subharmonic along each analytic disc in the boundary of Δ^2. Hence, the same is true for ϕ.

Conversely, let us first note that if $z\in\partial\Delta^2$, say $z=(e^{i\theta},\zeta)$ for some $\theta\in\mathbf{R}$ and $\zeta\in\Delta$, every $\mu\in\mathcal{J}_z^c$ must be supported on $\{e^{i\theta}\}\times\Delta$. The reason for this is that the function $v(z)=|z_1+e^{i\theta}|-2$ is in $\mathcal{PSH}^c(\Delta^2)$, $v\le 0$, and $\{z:v(z)=0\}=\{e^{i\theta}\}\times\bar\Delta$. Hence, if $\mu\in\mathcal{J}_z^c$, we have that

$$0=v(z)\le\int_{\bar\Omega}v\,d\mu$$

which implies that μ must put zero mass on the set where $v<0$. In a similar fashion, we can show that if $z\in\partial\Delta\times\partial\Delta$ and $\mu\in\mathcal{J}_z^c$, it follows that $\mu=\delta_z$.

Hence, any Jensen measure for a boundary point z can be viewed as a Jensen measure for subharmonic functions on Δ after a canonical projection, and conversely, any Jensen measure on Δ can be lifted to a Jensen measure for a boundary point in Δ^2. Thus, if ϕ is a continuous function on $\partial\Delta^2$ such that every slice function is subharmonic, then condition (3.2) is satisfied.

The class of domains admitting a strong plurisubharmonic barrier function at every boundary point was introduced and studied by Sibony [12]. These domains, known as B-regular domains, are in some situations natural. For example the Dirichlet problem for the complex Monge–Ampère operator is always solvable in B-regular domains. (With continuous data and continuous solution.) We refer to Błocki [1] for details. We will use the following (equivalent) definition of B-regularity.

Definition 3.7. Let Ω be a bounded domain in $\mathbf{C}^n$. If every real-valued function $\phi \in C(\partial\Omega)$ can be extended to a plurisubharmonic function $u \in \mathcal{PSH}^c(\Omega)$, we say that Ω is *B-regular.*

In [2], hyperconvexity was characterised in terms of Jensen measures for boundary points. As a corollary to Theorem 3.5 we obtain a similar characterisation of B-regularity. (This fact was already proven by Sibony in [12].)

Corollary 3.8. *A bounded domain $\Omega \subset \mathbf{C}^n$ is B-regular if and only if for every boundary point $z \in \partial\Omega$, $\mathcal{J}_z^c = \{\delta_z\}$, where δ_z denotes the Dirac measure at z.*

Proof. Assume that Ω is B-regular, take a boundary point $z \in \partial\Omega$ and let $\mu \in \mathcal{J}_z^c$. Since Ω is hyperconvex, $\operatorname{supp} \mu \subset \partial\Omega$. Construct a continuous function ϕ on $\partial\Omega$ such that ϕ attains a strict maximum at z. Since Ω is B-regular, we can extend ϕ to a function in $\mathcal{PSH}^c(\Omega)$. Hence

$$\phi(z) \leq \int_{\bar{\Omega}} \phi \, d\mu \leq \bigl(\max_{\operatorname{supp} \mu} \phi\bigr) \int_{\bar{\Omega}} d\mu = \phi(z).$$

Consequently, $\phi = \phi(z)$ μ-a.e., which implies that $\operatorname{supp} \mu = \{z\}$. Hence $\mu = \delta_z$. Conversely, assume that $\mathcal{J}_z^c = \{\delta_z\}$ for every $z \in \partial\Omega$. Theorem 3.5 then implies that every continuous function on $\partial\Omega$ is the boundary value of a function in $\mathcal{PSH}^c(\Omega)$. $\square$

It is also possible to use Theorem 3.5 to show that on hyperconvex domains, the set of boundary values of continuous plurisubharmonic functions is closed under uniform limits.

Corollary 3.9. *Let $\Omega \subset \mathbf{C}^n$ be a bounded hyperconvex domain. Then the set $E = \mathcal{PSH}^c(\Omega)|_{\partial\Omega}$ is uniformly closed.*

Proof. Let $\phi_j \in C(\partial\Omega)$ be a sequence of functions in E converging uniformly to some $\phi \in C(\partial\Omega)$. Assume that

$$\phi(z) > \inf\left\{ \int_{\partial\Omega} \phi \, d\mu : \mu \in \mathcal{J}_z^c \right\}$$

for some $z \in \partial\Omega$. Then $\phi(z) > \int_{\partial\Omega} \phi \, d\mu$ for some $\mu \in \mathcal{J}_z^c$ and consequently $\phi_j(z) > \int_{\partial\Omega} \phi_j \, d\mu$ for j sufficiently large. This contradicts the assumption that ϕ_j is the boundary value of a function in $\mathcal{PSH}^c(\Omega)$.

On the other hand, assume that

$$\phi(z) < \inf\left\{ \int_{\partial\Omega} \phi \, d\mu : \mu \in \mathcal{J}_z^c \right\}$$

for some $z\in\partial\Omega$. Then there is an $\varepsilon>0$ such that $\phi(z)\leq\int_{\partial\Omega}\phi\,d\mu-\varepsilon$ for every $\mu\in\mathcal{J}_z^c$. Choose j so large that $\sup_{\partial\Omega}|\phi-\phi_j|<\frac{1}{3}\varepsilon$. Then

$$\phi_j(z)\leq\phi(z)+\tfrac{1}{3}\varepsilon\leq\int_{\partial\Omega}\phi\,d\mu-\tfrac{2}{3}\varepsilon\leq\int_{\partial\Omega}\phi_j\,d\mu-\tfrac{1}{3}\varepsilon$$

for all $\mu\in\mathcal{J}_z^c$. Using Theorem 3.5, this contradicts the assumption on ϕ_j. Hence,

$$\phi(z)=\inf\left\{\int_{\partial\Omega}\phi\,d\mu:\mu\in\mathcal{J}_z^c\right\}$$

for all $z\in\partial\Omega$, which by Theorem 3.5 implies that $\phi\in E$. □

Remark. This corollary can also be proved by extending the boundary functions to maximal plurisubharmonic functions, and taking a limit of these. This argument requires some theory of solving the complex Monge–Ampère equation on hyperconvex domains, whereas the approach taken here is more self-contained.

4. Global approximation of plurisubharmonic functions on B-regular domains

In this section we will show that upper bounded plurisubharmonic functions can be approximated from above with plurisubharmonic functions continuous up to the boundary on B-regular domains. This generalises a result by Cegrell [4].

Theorem 4.1. *Let $\Omega\subset\mathbf{C}^n$ be a bounded B-regular domain and let u be an upper bounded plurisubharmonic function on Ω. Then there exists a decreasing sequence $u_j\in\mathcal{PSH}^c(\Omega)$, such that $u_j\searrow u^*$ on $\overline{\Omega}$.*

Remark. If we only assume that Ω is pseudoconvex, then Theorem 4.1 is no longer valid. For an example, take Hartogs' triangle $\Omega=\{(z_1,z_2)\in\mathbf{C}^2:|z_1|<|z_2|<1\}$, and let $u(z_1,z_2)=|z_1|/|z_2|$. Then $u\in\mathcal{PSH}(\Omega)$ and $u<1$. Also, note that $u^*(0,0)=1$. Assume that there is a sequence $u_j\in\mathcal{PSH}^c(\Omega)$, such that $u_j\searrow u^*$ on $\overline{\Omega}$. Let $K=\{0\}\times\partial\Delta_{1/2}\subset\Omega$. Note that u is identically 0 on K, and hence in particular, u is continuous on K. Consequently, by Dini's theorem, u_j converges to 0 uniformly on K. Choose J so large that $u_j<\frac{1}{2}$ on K for every $j\geq J$. By applying the maximum principle to u_j on $\{0\}\times\Delta_{1/2}$, we must have that $u_j(0,0)\leq\frac{1}{2}$ for $j\geq J$. This contradicts the assumption that $u_j(0,0)\searrow u^*(0,0)=1$.

On the other hand, if Ω is pseudoconvex and $u\in\mathcal{PSH}(\Omega)$ (u not necessarily upper bounded), we can always find a sequence (see Fornæss and Narasimhan [7] for a proof) $u_j\in\mathcal{PSH}(\Omega)\cap C^\infty(\Omega)$, such that $u_j\searrow u$ on Ω. But as the example above shows, these functions can in general not be extended to continuous functions on $\bar{\Omega}$ such that the extensions decrease to u^* on $\partial\Omega$.

Furthermore, if we do not even assume that Ω is pseudoconvex, there are examples showing (see e.g. Fornæss and Stensønes [8]) that there is a domain Ω and a function $u\in\mathcal{PSH}(\Omega)$ such that there is no sequence of continuous plurisubharmonic functions u_j such that $u_j\searrow u$ on Ω.

These examples show that for Theorem 4.1 to hold, we must assume that the domain has some kind of "convexity", and that pseudoconvexity itself is not sufficient. At this point it is unknown to the author whether Theorem 4.1 holds in every hyperconvex domain.

Proof of Theorem 4.1. Since Ω is hyperconvex, we can find a smooth plurisubharmonic exhaustion function v for Ω such that $v|_{\partial\Omega}\equiv 0$. (See Błocki [1].) First note that u^* is upper semicontinuous on the compact set $\bar{\Omega}$. Hence, we can find a sequence $\phi_j\in C(\partial\Omega)$ such that each $\phi_j>u^*$ and $\phi_j\searrow u^*$ on $\partial\Omega$.

Because Ω is B-regular, we may extend ϕ_j to a maximal plurisubharmonic function on Ω, which is continuous on $\bar{\Omega}$. (We will use the same notation, ϕ_j, to denote this extension.)

For each j, the function $u^*-\phi_j<0$ is upper semicontinuous on $\partial\Omega$ and thus attains a maximum value. In other words, we can find $\varepsilon_j>0$, such that $u^*-\phi_j\leq -\varepsilon_j<0$ on $\partial\Omega$. By the maximality of ϕ_j, we must have

$$u^*(z)\leq\phi_j(z)-\varepsilon_j,\quad z\in\bar{\Omega}. \tag{4.1}$$

For each j, choose $r_j>0$ so small that

$$r_j<d_j:=\operatorname{dist}\left(\left\{z:v(z)<-\frac{1}{2j^2}\right\},\partial\Omega\right),$$

and so that $u_{r_j}\leq\phi_j$ on Ω_{r_j}. By shrinking r_j further, we may also assume that $\{r_j\}$ is a decreasing sequence. Here u_{r_j} denotes the convolution of u with a standard regularising kernel ψ_{r_j} with support in $B(0,r_j)$ and $\Omega_{r_j}=\{z\in\Omega:\operatorname{dist}(z,\partial\Omega)>r_j\}$. The second condition on r_j can be fulfilled because equation (4.1) implies that

$$u*\psi_\delta\leq(\phi_j-\varepsilon_j)*\psi_\delta=\phi_j*\psi_\delta-\varepsilon_j\leq\phi_j$$

if δ is sufficiently small. (We recall that $\phi_j*\psi_\delta$ converges uniformly to ϕ_j as $\delta\to 0$.)

Define

$$\tilde{u}_m(z)=\max\left\{u_{r_m}(z)-\frac{1}{m},mv(z)+\phi_m(z)\right\}, \tag{4.2}$$

and let $u_j(z)=\sup_{m\geq j}\{\tilde{u}_m(z)\}$. In (4.2), note that u_{r_m} is not defined if z is close to $\partial\Omega$. The definition of $\tilde{u}_m$ should be taken as $mv(z)+\phi_m(z)$ for such z.

We claim that $u_j\in\mathcal{PSH}^c(\Omega)$. First, note that if z is a point in Ω such that $\operatorname{dist}(z,\partial\Omega)\leq d_m$, then $v(z)\geq -1/2m^2$ and hence $mv(z)\geq -1/2m$. By the construction of r_m in the previous paragraph, we also have that $u_{r_m}(z)\leq\phi_m(z)$. Hence, for such a z, we see that

$$u_{r_m}(z)-\frac{1}{m}<\phi_m(z)-\frac{1}{2m}\leq mv(z)+\phi_m(z).$$

This implies that each $\tilde{u}_m$ is plurisubharmonic on Ω, continuous on $\bar{\Omega}$ and equal to ϕ_m on $\partial\Omega$. Thus u_j, being the upper bound of a family of continuous functions, is lower semicontinuous. To prove the claim that u_j is a continuous plurisubharmonic function, all that remains is to show that u_j is upper semicontinuous.

Rewriting u_j, we obtain for any $K\geq j$,

$$\begin{aligned}(4.3)\qquad u_j(z)&=\sup_{m\geq j}\left\{\max\left\{u_{r_m}(z)-\frac{1}{m},mv(z)+\phi_m(z)\right\}\right\}\\&=\sup_{m\geq j}\left\{\max\left\{u_{r_m}(z),mv(z)+\frac{1}{m}+\phi_m(z)\right\}-\frac{1}{m}\right\}\\&\leq\max\left\{\max_{K\geq m\geq j}\left\{\max\left\{u_{r_m}(z),mv(z)+\frac{1}{m}+\phi_m(z)\right\}-\frac{1}{m}\right\},\right.\\&\qquad\left.\left\{u_{r_K}(z),Kv(z)+\frac{1}{K}+\phi_K(z)\right\}\right\}.\end{aligned}$$

The inequality in (4.3) follows from the estimate:

$$(4.4)\qquad \max\left\{u_{r_m},mv+\frac{1}{m}+\phi_m\right\}-\frac{1}{m}\leq\max\left\{u_{r_K},Kv+\frac{1}{K}+\phi_K\right\},\quad m\geq K.$$

To prove (4.4), just note that $\max\{u_{r_K},Kv+1/K+\phi_K\}$ is decreasing in K. (Each term is decreasing.)

To finish off, we observe that

$$\begin{aligned}\max\Biggl\{&\max_{K\geq m\geq j}\left\{\max\left\{u_{r_m}(z),mv(z)+\frac{1}{m}+\phi_m(z)\right\}-\frac{1}{m}\right\},\\&\max\left\{u_{r_K}(z),Kv(z)+\frac{1}{K}+\phi_K(z)\right\}\Biggr\}\end{aligned}$$

is a sequence of continuous functions, decreasing to u_j as $K\to\infty$. Hence u_j is upper semicontinuous. This completes the proof. $\square$

To make the following discussion clearer, let us introduce a piece of terminology.

Definition 4.2. Let Ω be a bounded domain in $\mathbf{C}^n$. If every upper bounded plurisubharmonic function on Ω can be approximated from above on $\bar{\Omega}$ by functions in $\mathcal{PSH}^c(\Omega)$ as in Theorem 4.1, we say that Ω has the *approximation property.*

Corollary 4.3. *Let $\Omega \subset \mathbf{C}^n$ be a domain having the approximation property, and let $z \in \bar{\Omega}$. Then $\mathcal{J}_z^c = \mathcal{J}_z$.*

Proof. Clearly $\mathcal{J}_z \subset \mathcal{J}_z^c$. Conversely, let $\mu \in \mathcal{J}_z^c$ be arbitrary and take any upper bounded plurisubharmonic function u on Ω. Since Ω has the approximation property, there is a sequence $u_j \in \mathcal{PSH}^c(\Omega)$, such that $u_j \searrow u^*$. Hence, using the monotone convergence theorem

$$\int_{\bar{\Omega}} u^* \, d\mu = \lim_{j\to\infty} \int_{\bar{\Omega}} u_j \, d\mu \geq \lim_{j\to\infty} u_j(z) = u^*(z).$$

This means that μ is in fact a Jensen measure for every upper bounded plurisubharmonic function on Ω. $\square$

In particular, the two classes of Jensen measures coincide for B-regular domains. Conversely, if $\mathcal{J}_z = \mathcal{J}_z^c$ for every $z \in \bar{\Omega}$ and if Ω is hyperconvex, then Ω has the approximation property.

Theorem 4.4. *Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$ and assume that $\mathcal{J}_z = \mathcal{J}_z^c$ for every $z \in \bar{\Omega}$. Let u be an upper bounded plurisubharmonic function on Ω, with $u^*|_{\partial\Omega} = \phi$. Then there exists a sequence $\psi_j \in C(\partial\Omega)$, such that $\psi_j \searrow \phi$ and $\psi_j \in \mathcal{PSH}^c(\Omega)|_{\partial\Omega}$.*

Proof. Since ϕ is upper semicontinuous on $\partial\Omega$, we can find a sequence of continuous functions $\phi_j \in C(\partial\Omega)$, such that $\phi_j \searrow \phi$. For each j, extend ϕ_j to a continuous function on $\bar{\Omega}$ in such a way that the extensions still form a decreasing sequence of functions. For each j, define

$$S^c\phi_j(z) = \sup\left\{ v(z) : v \in \mathcal{PSH}^c(\Omega),\ v \leq \phi_j \text{ on } \bar{\Omega} \right\},$$
$$S\phi_j(z) = \sup\left\{ v(z) : v \in \mathcal{PSH}(\Omega),\ v^* \leq \phi_j \text{ on } \bar{\Omega} \right\}.$$

Using Edward's theorem and the assumption that $\mathcal{J}_z = \mathcal{J}_z^c$, we have that

$$S^c\phi_j(z) = \inf\left\{ \int_{\bar{\Omega}} \phi_j \, d\mu : \mu \in \mathcal{J}_z^c \right\} = \inf\left\{ \int_{\bar{\Omega}} \phi_j \, d\mu : \mu \in \mathcal{J}_z \right\} = S\phi_j(z)$$

for every $z \in \bar{\Omega}$. Hence $S^c\phi_j \equiv S\phi_j$. On the other hand $S\phi_j \leq \phi_j$ and ϕ_j is continuous. This implies that $(S\phi_j)^* \leq \phi_j$, and since $(S\phi_j)^*$ is plurisubharmonic, it follows that

$(S\phi_j)^* = S\phi_j$. In particular, $S\phi_j$ is upper semicontinuous on $\bar{\Omega}$. However, $S\phi_j = S^c\phi_j$ and $S^c\phi_j$ is lower semicontinuous, being the supremum of continuous functions. Consequently, $S\phi_j = S^c\phi_j \in \mathcal{PSH}^c(\Omega)$.

Define $\psi_j = S^c\phi_j|_{\partial\Omega}$. Clearly, $\psi_j \le \phi_j$. Furthermore, if $z \in \partial\Omega$, and $\mu \in \mathcal{J}_z = \mathcal{J}_z^c$, then

$$\phi(z) = u^*(z) \le \int_{\bar{\Omega}} u^* \, d\mu = \int_{\partial\Omega} \phi \, d\mu,$$

since Ω is hyperconvex, and thus μ must be supported on $\partial\Omega$ by Theorem 3.4. Hence,

$$\begin{aligned}\phi(z) &\le \inf\left\{\int_{\bar{\Omega}} \phi \, d\mu : \mu \in \mathcal{J}_z\right\} = \inf\left\{\int_{\bar{\Omega}} \phi \, d\mu : \mu \in \mathcal{J}_z^c\right\} \\ &\le \inf\left\{\int_{\bar{\Omega}} \phi_j \, d\mu : \mu \in \mathcal{J}_z^c\right\} = S^c\phi_j(z) = \psi_j(z)\end{aligned}$$

for every $z \in \partial\Omega$. It is clear that ψ_j decreases, and from the calculation above, it follows that ψ_j converges to $u^* = \phi$ on $\partial\Omega$. □

Corollary 4.5. *Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$ such that $\mathcal{J}_z = \mathcal{J}_z^c$ for all $z \in \bar{\Omega}$. Then Ω has the approximation property.*

Proof. Let u denote an arbitrary upper bounded plurisubharmonic function on Ω.

Examining the proof of Theorem 4.1, we see that the only thing that is required for the proof to go through in hyperconvex domains, is the existence of a decreasing sequence ϕ_j of continuous functions on $\partial\Omega$ tending to u^* such that each ϕ_j can be extended to a maximal plurisubharmonic function.

Theorem 4.4 provides us with a sequence $\phi_j \in \mathcal{PSH}^c(\Omega)|_{\partial\Omega}$ decreasing to u^*. Since Ω is hyperconvex these functions can always be extended to *maximal* plurisubharmonic functions on Ω (see Błocki [1]). □

In general $\mathcal{J}_z \subsetneq \mathcal{J}_z^c$ as shown by the following example.

Example 4.6. Define $h: \mathbf{C}^2 \setminus \{z : z_1 = 0\} \to \mathbf{R}$ by $h(z) = |z_1|^{\log|z_1|}|z_2|$. Then h is plurisubharmonic where it is defined, because $\log h(z) = (\log|z_1|)^2 + \log|z_2|$ which shows that $\log h$ is plurisubharmonic on $z_1 \neq 0$. Hence the same is true for h. Let $\Omega = \{z \in \mathbf{C}^2 : h(z) < 1,\ 0 < |z_1| < 1,\ |z_2| < 1\}$. It is easy to verify that Ω is pseudoconvex, but since Ω is Reinhardt and $0 \in \partial\Omega$, Ω is not hyperconvex (see [2]) and h is plurisubharmonic and upper bounded on Ω.

Let $u(z) = \max\{h(z), |z_1|, |z_2|\} - 1$. Then $u^* = 0$ on $\partial\Omega$. Hence, if $z \in \partial\Omega$ and $\mu \in \mathcal{J}_z$, then

$$0 = u^*(z) \le \int_{\bar{\Omega}} u^* \, d\mu,$$

which implies that $u^*=0$ μ-a.e. Hence μ must be supported on $\partial\Omega$. On the other hand, since Ω is not hyperconvex, there exists a measure $\mu\in\mathcal{J}_0^c$ which is not supported on $\partial\Omega$. (See [2].) This is, of course, another example which shows that Theorem 4.1 is not valid in every bounded pseudoconvex domain.

Inspired by the above example, we introduce the following definition.

Definition 4.7. Let Ω be a bounded domain in $\mathbf{C}^n$. If there is a function $u\in\mathcal{PSH}(\Omega)$, $u\not\equiv 0$, such that $u^*|_{\partial\Omega}\equiv 0$, we say that Ω is *almost hyperconvex.*

As in the example, it follows that if Ω is almost hyperconvex, then for every $z\in\partial\Omega$ and every $\mu\in\mathcal{J}_z$, $\operatorname{supp}\mu\subset\partial\Omega$. We also note that the notion of almost hyperconvexity is *not* biholomorphically invariant. The reason is that a biholomorphism $\phi:\Omega_1\to\Omega_2$ does not necessarily extend to a homeomorphism between the closures. As an example, take $\Omega_1=\Delta^2\setminus\{z:z_2=0\}$. Clearly, if u is plurisubharmonic on Ω_1 and upper bounded, we can extend u to be plurisubharmonic on Δ^2. Hence, if $u^*|_{\partial\Omega_1}\equiv 0$, the maximum principle forces u to vanish identically, which means that Ω_1 is not almost hyperconvex. However, $f(z_1,z_2)=(z_1z_2,z_2)$ is a biholomorphism between Ω_1 and Hartogs' triangle, which is almost hyperconvex.

Recall that Hartogs' triangle is defined by $T=\{(z_1,z_2)\in\mathbf{C}^2:|z_1|<|z_2|<1\}$. To see that T is almost hyperconvex, note that the function $v(z)=\max\{|z_1|/|z_2|,|z_2|\}-1$ is plurisubharmonic and satisfies $v^*\equiv 0$ on ∂T.

Let μ denote the normalised Lebesgue measure on $\{(z_1,z_2):z_1=0,\ |z_2|=\frac{1}{2}\}$. Note that $\mu\in\mathcal{J}_0^c$, but $\mu\notin\mathcal{J}_0$. The reason for this is that if $u\in\mathcal{PSH}$ is upper bounded on Hartogs' triangle, the value $u^*(0)$ is *not* determined by the restriction of u to the disc $\{0\}\times\Delta$, in which $\operatorname{supp}\mu$ is contained. This allows for the existence of a function u in $\mathcal{PSH}(\Omega)$, such that $u^*(0)>u|_{\operatorname{supp}\mu}$. The function v above is an example of such a function.

It is possible to strengthen some of the previous results to a wider class of domains than B-regular ones. In particular, it is possible to show that the bidisc has the approximation property. We begin by stating some preliminary lemmas.

Lemma 4.8. *If $\{\mu_j\}_{j=1}^\infty$ is a sequence of positive measures such that μ_j converges weak-* to μ and if φ is an upper semicontinuous function with compact support, then*

$$\varlimsup_{j\to\infty}\int_{\overline{\Omega}}\varphi\,d\mu_j\le\int_{\overline{\Omega}}\varphi\,d\mu.$$

The lemma follows easily from the monotone convergence theorem. For the details, see Lemma I:1 in Cegrell [3].

Lemma 4.9. *Let Ω be a bounded domain in $\mathbf{C}^n$ and let $\mu\in\mathcal{J}_z^c$, where $z\in\overline{\Omega}$. Assume that, for every sequence $\{z_j\}$, $z_j\in\Omega$ converging to z, there exists a corre-*

sponding sequence of measures $\mu_j \in \mathcal{J}_{z_j}$, *such that* μ_j *converge to* μ *in the weak-* topology. Then* $\mu \in \mathcal{J}_z$.

Proof. Let u be an upper bounded plurisubharmonic function and let $\{z_j\}$ be any sequence in $\bar{\Omega}$ converging to z. Take $\{\mu_j\}$ as in the statement of the lemma. Since each $\mu_j \in \mathcal{J}_{z_j}$, we have that

$$u^*(z_j) \le \int_{\bar{\Omega}} u^* \, d\mu_j.$$

Letting $j \to \infty$, and using Lemma 4.8, we see that

$$\varlimsup_{j\to\infty} u^*(z_j) \le \varlimsup_{j\to\infty} \int_{\bar{\Omega}} u^* \, d\mu_j \le \int_{\bar{\Omega}} u^* \, d\mu.$$

The sequence $\{z_j\}$ was arbitrary and hence $u^*(z) \le \int_{\bar{\Omega}} u^* \, d\mu$. Thus $\mu \in \mathcal{J}_z$. $\square$

For star-shaped domains, we can show that the equality $\mathcal{J}_z^c = \mathcal{J}_z$ holds for every interior point z. Recall that a domain Ω is said to be star-shaped (with respect to 0) if for any $z \in \Omega$, the (real) line segment connecting 0 with z is a subset of Ω.

Theorem 4.10. *Let* $\Omega \ni 0$ *be a bounded star-shaped domain in* $\mathbf{C}^n$. *Then, for every* $z \in \Omega$, $\mathcal{J}_z^c = \mathcal{J}_z$.

Proof. Let u be any upper bounded plurisubharmonic function on Ω and take any $z \in \Omega$. Let $\mu \in \mathcal{J}_z^c$ and define $u_r(\zeta) = u(r\zeta_1, \dots, r\zeta_n)$. Then for any $0<r<1$, u_r is plurisubharmonic on a neighbourhood of $\bar{\Omega}$. Hence u_r can be approximated monotonically from above on $\bar{\Omega}$ by functions in $\mathcal{PSH}^c(\Omega)$, and consequently

$$u_r(z) \le \int_{\bar{\Omega}} u_r(\zeta) \, d\mu(\zeta)$$

by the monotone convergence theorem. Letting $r \nearrow 1$, and using Fatou's lemma, we obtain

$$\varlimsup_{r\nearrow 1} u_r(z) \le \varlimsup_{r\nearrow 1} \int_{\bar{\Omega}} u_r \, d\mu \le \int_{\bar{\Omega}} \varlimsup_{r\nearrow 1} u_r \, d\mu \le \int_{\bar{\Omega}} u^* \, d\mu.$$

But, since a (real) line segment as a subset of $\mathbf{C}^n$ is not plurithin at its endpoints, we conclude that

$$u^*(z) = \varlimsup_{\zeta\to z} u(\zeta) = \varlimsup_{r\nearrow 1} u_r(z),$$

and hence that $u^*(z) \le \int_{\bar{\Omega}} u^* \, d\mu$, i.e. that $\mu \in \mathcal{J}_z$. $\square$

Remark. Note that this proof fails for $z \in \partial\Omega$. If $z \in \partial\Omega$, we cannot assert that $u^*(z) = \varlimsup_{r\nearrow 1} u_r(z)$.

Using these results, we can prove that a polydisc has the approximation property.

Theorem 4.11. *The unit bidisc $\Delta^2 \subset \mathbf{C}^n$ has the approximation property.*

Proof. Since the bidisc is hyperconvex, by Corollary 4.5, it suffices to show that $\mathcal{J}_z^c = \mathcal{J}_z$ for every $z \in \bar{\Delta}^2$. The bidisc is star-shaped, so from Theorem 4.10 we see that the equality holds for every interior point. It remains to show that $\mathcal{J}_z^c = \mathcal{J}_z$ for every $z \in \partial\Delta^2$. We may assume that $z \in \Delta \times \partial\Delta$, the other case being completely similar.

Let $\mu \in \mathcal{J}_z^c$. From Example 3.6 we know that the support of μ is contained in the analytic disc in $\partial\Delta^2$ determined by z. (In the case where $z \in \partial\Delta \times \partial\Delta$, the disc is not uniquely determined, but that does not matter.) Furthermore, μ is the lifting of a Jensen measure $\tilde{\mu}$ on $\bar{\Delta}$ for subharmonic functions.

Take any sequence $\{z_j\}$, $z_j = (z_j^{(1)}, z_j^{(2)})$ in Δ^2 converging to z. For $a \in \Delta$, define

$$m_a(\zeta) = \frac{\zeta - a}{1 - \bar{a}\zeta},$$

i.e. m_a is the canonical Möbius transformation interchanging a and 0. For each j, we define an analytic disc f_j by

$$f_j(\zeta) = (m_{z^{(1)}}^{-1} \circ m_{z_j^{(1)}}(\zeta), z_j^{(2)}).$$

The first component of f_j is a Möbius transformation interchanging $z_j^{(1)}$ and $z^{(1)}$. Put $\mu_j = (f_j)_* \tilde{\mu}$. Then $\mu_j \in \mathcal{J}_{z_j}^c = \mathcal{J}_{z_j}$ (by Theorem 4.10), and since the first component of f_j converges uniformly to id_Δ as $j \to \infty$, it follows that μ_j converge weak-* to $\mu = (\mathrm{id}, z^{(2)})_* \tilde{\mu}$. Invoking Lemma 4.9, it follows that $\mu \in \mathcal{J}_z$. Hence any measure in $\mathcal{J}_z^c$ is a Jensen measure for upper bounded plurisubharmonic functions. □

5. Different kinds of boundary values for plurisubharmonic functions

Given an (upper bounded) plurisubharmonic function $u \in \mathcal{PSH}(\Omega)$, there are several reasonable ways to define boundary values of u. In this paper, we have used the upper semicontinuous regularisation u^* as a convenient way to extend u to $\bar{\Omega}$. The obvious advantage of using $u^*|_{\partial\Omega}$ as boundary values of u is that u^* is upper semicontinuous on the compact set $\bar{\Omega}$ which simplifies some things. On the other hand, u^*—the unrestricted upper limit of u—is the largest reasonable choice of boundary values for u. If we can use another, smaller, choice of boundary values, many results would (at least formally) be sharper than for u^*. In this final section, we will look into other ways of introducing boundary values for u. To simplify some of the concepts, we will restrict the discussion to the case of the unit ball B in $\mathbf{C}^2$, even though the concepts we will introduce can be adapted to more general situations.

First we will introduce radial boundary values.

Definition 5.1. Let $u\in\mathcal{PSH}(B)$ be an upper bounded plurisubharmonic function. If $z\in\partial B$, we define the *radial boundary value* of u, denoted u^R, by

$$u^R(z)=\overline{\lim_{r\nearrow 1}}\, u(rz).$$

We extend the function u^R to $\overline{B}$ by defining $u^R(z)=u(z)$ if $z\notin\partial B$.

Remark. For a bounded domain Ω with C^1 boundary, one could define radial boundary values by taking the upper limit along the (real) normal to Ω at z. In general u^R will not be upper semicontinuous on $\overline{B}$.

Looking at boundary values of bounded holomorphic functions, the theory is most satisfactory when studying non-tangential approach regions (in one variable) and the even larger Korányi–Stein approach regions (in several variables). With this in mind it is natural to look at boundary values of plurisubharmonic functions in a similar fashion. We recall the definition of a Korányi–Stein region.

Definition 5.2. Let $\alpha>1$ and let $\zeta\in\partial B$. We put

$$D_\alpha(z)=\{z\in B:|1-\langle z,\zeta\rangle|<\alpha(1-|z|^2)\}.$$

Remark. Note that D_α is non-tangential in complex tangential directions, but parabolic in the complex normal direction. In more general domains, one can define the Korányi–Stein regions using the Kobayashi metric. In strictly pseudoconvex domains, the shape of D_α is roughly as in the ball.

Using these approach regions, we define a non-tangential boundary value for plurisubharmonic functions.

Definition 5.3. Let $u\in\mathcal{PSH}(B)$ be an upper bounded plurisubharmonic function. If $z\in\partial B$, we define the *α-admissible boundary value* of u, denoted u^α, by

$$u^\alpha(z)=\overline{\lim_{D_\alpha(z)\ni\zeta\to z}}\, u(\zeta).$$

We extend the function u^α to $\overline{B}$ by defining $u^\alpha(z)=u(z)$ if $z\notin\partial B$.

Even though u^R and u^α are not in general upper semicontinuous on $\overline{\Omega}$, they are Borel functions. Hence it is meaningful to introduce Jensen measures modelled on these boundary values. If $z\in\overline{B}$, we define $\mathcal{J}_z^R$ as the set of regular Borel measures μ such that

$$u^R(z)\le\int_{\overline{\Omega}} u^R\,d\mu$$

for every upper bounded plurisubharmonic function u on B. Similarly, for $\alpha>1$, we define $\mathcal{J}_z^\alpha$ in the same way, with u^R replaced by u^α.

Clearly $u^R \le u^\alpha \le u^*$ (for any $\alpha>1$) and $u^\alpha \le u^\beta$ if $1<\alpha\le\beta$. Hence if $z\in B$, $u^*(z)=u^\alpha(z)=u^R(z)$ and consequently,

$$\mathcal{J}_z^R \subset \mathcal{J}_z^\alpha \subset \mathcal{J}_z$$

for any $\alpha>1$ and

$$\mathcal{J}_z^\alpha \subset \mathcal{J}_z^\beta$$

for $\alpha\le\beta$.

For $z\in\partial B$ it is less obvious if the same inclusions of Jensen measures hold, but for interior points, the inclusions above are actually equalities.

Proposition 5.4. *Let $z\in B$. Then (for every $\alpha>1$),*

$$\mathcal{J}_z = \mathcal{J}_z^\alpha = \mathcal{J}_z^R.$$

Proof. Fix $\mu\in\mathcal{J}_z$ and let u be an upper bounded plurisubharmonic function on B. From the proof of Theorem 4.10, we see that

$$u^R(z) \le \int_{\overline{\Omega}} u^R \, d\mu,$$

and hence that $\mu\in\mathcal{J}_z^R$. It follows that $\mathcal{J}_z\subset\mathcal{J}_z^R$. $\square$

Since $\mathcal{J}_z=\mathcal{J}_z^\alpha=\mathcal{J}_z^R$, it would be natural to conjecture that $u^*=u^R$ μ-a.e. for every $\mu\in\mathcal{J}_z$. This conjecture fails dramatically, as shown by the following example.

Example 5.5. Define

$$V(z) = \log\frac{|z_2|^2}{1-|z_1|^2}.$$

Note that on B, $|z_2|^2<1-|z_1|^2$, and hence that $V(z)<\log 1=0$ on B. Also, V is plurisubharmonic on B, since $-\log(1-|\zeta|)$ is minus the log of the distance from $\zeta\in\Delta$ to $\partial\Delta$, and hence subharmonic.

Clearly $V^R(\zeta,0)=-\infty$ for every $\zeta\in\partial\Delta$, since $V(z_1,0)\equiv-\infty$. On the other hand, it is easy to verify that $V^*(\zeta,0)=0$ for every $\zeta\in\partial\Delta$.

These observations show that

$$V^R(z) = \begin{cases} 0, & z_2\neq 0, \\ -\infty, & z_2=0 \end{cases}$$

and

$$V^*(z)=0, \quad z\in\partial B.$$

Hence V^R and V^* disagree on the set $\partial\Delta\times\{0\}$, which is the support of some Jensen measure $\mu\in\mathcal{J}_0$. (Take μ as the 1-dimensional Lebesgue measure on $\partial\Delta\times\{0\}$.)

A more careful calculation shows that $V^\alpha(\zeta,0)=\log(1-1/2\alpha)$ for $\zeta\in\partial\Delta$ and hence that each of the different boundary values differ on a set of full μ-measure.

Remark. Note that this example of a function such that $u^*\neq u^R$, is not a several variable phenomenon. In fact, it is well known that there even exists a bounded harmonic function h on the unit disc in $\mathbf{C}$, such that $h^*\neq h^R$ on a large part of the circle.

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